Chapter 3

Symplectic geometry

From now on, we call a pair (P, α) consisting of a $\mathbb{T}$-principal bundle and a connection α of P a *prequantum bundle*. The first chapter is devoted to the automorphism group of prequantum bundles, that is the group of diffeomorphisms φ of P which are $\mathbb{T}$-equivariant and satisfy $\varphi^* \alpha = \alpha$.

3.1 Automorphisms of prequantum bundle

3.1.1 Linear case

Let V be a finite dimensional vector space and $\omega \in \wedge^2 V^*$. Let $\beta \in \Omega^1(V)$ be defined by

$$\beta_x(y) = \frac{1}{2}\omega(x, y), \qquad \forall x, y \in V.$$

Then the differential of β is the constant form equal to ω. Let P be the trivial $\mathbb{T}$-principal principal bundle with base V. Endow P with the connection $\alpha = -\beta + d\theta$.

It is easily seen that the automorphisms of (P, α) are the applications $\varphi : P \to P$ of the form

$$\varphi(x, \theta) = (\varphi_V(x), \theta + f(x))$$

where φ_V is a diffeomorphism of V and f a map from V to $\mathbb{T}$ such that $\varphi_V^* \beta = df + \beta$.

Let $\mathrm{Gl}(V, \omega)$ be the group of linear isomorphisms φ_V of V such that

$$\omega(\varphi_V(x), \varphi_V(y)) = \omega(x, y), \qquad \forall x, y \in V.$$

Observe that such an isomorphism satisfies $\varphi_V^* \beta = \beta$. So the subgroup of $\mathrm{Aut}(P, \alpha)$ consisting of the automorphisms lifting an element of $\mathrm{Gl}(V, \omega)$ is isomorphic to the direct product $\mathrm{Gl}(V, \omega) \times \mathbb{T}$ through the map sending (φ_V, t) to the automorphism $\varphi(x, \theta) = (\varphi_V(x), t + \theta)$.

Replacing the linear automorphism by the translations, we obtain a more interesting group as was already explained in Example 2.5.5. Indeed, the subgroup of $\mathrm{Aut}(P, \alpha)$ consisting of the automorphisms lifting a translations of V is in bijection with the direct product $V \times \mathbb{T}$ through the map sending (u, t) to the automorphism $\varphi_{(u,t)}(x, \theta) = (u + x, t + \theta + \frac{1}{2}\omega(u, x))$. The composition is given by

$$\varphi_{u,t} \circ \varphi_{v,s} = \varphi_{(u+v, t+s+\omega(u,v)/2)}.$$

So we have an exact sequence

$$0 \to \mathbb{T} \to G \to V \to 0, \qquad t \to \varphi_{(0,t)}, \quad \varphi_{(u,t)} \to u$$

When ω does not vanish, this sequence does not split, and we obtain a non trivial extension by $\mathbb{T}$ of the group of translations of V.

3.1.2 Preliminaries on infinitesimal automorphims

Before we study the automorphisms of a prequantum bundle (P, α), let us start with the *infinitesimal automorphism* of (P, α), that is the vector fields Y of P which are $\mathbb{T}$-invariant and satisfy $\mathcal{L}_Y \alpha = 0$. The space $\mathrm{aut}(P, \alpha)$ of infinitesimal automorphisms has to be viewed as the Lie algebra of $\mathrm{Aut}(P, \alpha)$. To understand the meaning of this assertion and which relations we can expect between $\mathrm{Aut}(P, \alpha)$ and $\mathrm{aut}(P, \alpha)$, let us first consider the group $\mathrm{Diff}(M)$ of diffeomorphisms of a manifold M.

As explained by Milnor in [], $\mathrm{Diff}(M)$ can be given the structure of an infinite dimensional manifold, its tangent space at the identity element being $\mathcal{C}^\infty(M, TM)$. Define the smooth curves of $\mathrm{Diff}(M)$ as the 1-parameter families $t \to \varphi_t$ such that the map $(t, x) \to \varphi_t(x)$ is smooth. If (φ_t) is a smooth curve such that φ_0 is the identity element, its tangent vector at $t = 0$ is the vector field X given by

$$X(p) = \frac{d}{dt}\Big|_{t=0} \varphi_t(p), \qquad p \in M.$$

Recall that for a genuine Lie group G with Lie algebra $\mathfrak{g}$, the exponential map $\mathfrak{g} \to G$ is defined in such a way that for any $X \in \mathfrak{g}$, $\exp(tX)$ is the one-parameter subgroup of G such that $\frac{d}{dt}(\exp(tX)) = X$ at $t = 0$. Since the one-parameter subgroups (φ_t) of $\mathrm{Diff}(M)$ are the flows of the complete vector fields of M, we define the exponential of a complete vector field X of M as the flow at time 1 of X. That the exponential map is not defined for all the vector fields is a first difference with the finite dimensional case. When M is compact, any vector field is complete. But even in that case, there are still differences with the finite dimensional case. For instantce, it is not true that any diffeomorphism close to the identity belongs to a one-parameter subgroup, cf [] for a simple example with $M = S^1$.

In the case M is connected, let us show that the appropriate bracket in the Lie algebra of $\mathrm{Diff}(M)$ is the usual bracket of vector fields, up to sign. Recall first the definition for a Lie group G. For any $g \in G$, let $C_g : G \to G$ be the morphism sending h to ghg^{-1}, and let $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ be the differential of C_g. Then

$$[X,Y] := \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)} Y, \qquad X, Y \in \mathfrak{g}.$$

So if $\Phi \in \mathrm{Diff}(M)$ and (Ψ_t) is the flow of a vector field of Y of M, then $Z = \mathrm{Ad}_\Phi Y$ is the vector field given by

$$\begin{aligned} Z(p) =& \frac{d}{dt}\Big|_{t=0} \Phi(\Psi_t(\Phi^{-1}(p))) \\ =& T_{\Phi^{-1}(p)}\Phi(Y(\Phi^{-1}(p))) \end{aligned}$$

Assume now (Φ_t) is the flow of X, then

$$\begin{aligned} [X,Y](p) =& \frac{d}{dt}\Big|_{t=0} T_{\Phi_t^{-1}(p)}\Phi_t(Y(\Phi_t^{-1}(p))) \\ =& \frac{d}{dt}\Big|_{t=0} (T_p\Phi_{-t})^{-1}(Y(\Phi_{-t}(p))) \\ =& -\mathcal{L}_X Y(p) \end{aligned}$$

as was to be proved.

Let us return to prequantum bundles. Observe that $\mathrm{aut}(P,\alpha)$ is a Lie subalgebra of $\mathcal{C}^\infty(P,TP)$, and correspondingly $\mathrm{Aut}(P,\alpha)$ is a subgroup of $\mathrm{Diff}(P)$. Furthermore let I be an open interval, $(\varphi_t, t \in I)$ be a smooth curve of $\mathrm{Diff}(P)$ of $\mathrm{Aut}(P,\alpha)$ and $(Y_t \in \mathcal{C}^\infty(P,TP), t \in I)$ be its infinitesimal generator

$$Y_t(\varphi_t(p)) = \frac{d}{dt}(\varphi_t(p)).$$

Assume that for some $t_0 \in I$, $\varphi_{t_0} \in \mathrm{Aut}(P,\alpha)$. Then it is easily seen that $\varphi_t \in \mathrm{Aut}(P,\alpha)$ for any $t \in I$ if and only if $Y_t \in \mathrm{aut}(P,\alpha)$ for any $t \in I$. Because of this property, we may view $\mathrm{aut}(P,\alpha)$ as the Lie algebra of $\mathrm{Aut}(P,\alpha)$.

3.1.3 Infinitesimal automorphisms of prequantum bundles

Consider a prequantum bundle (P,α) with curvature $\omega \in \Omega^2(M,\mathbb{R})$. Recall that any $\mathbb{T}$-invariant vector field of P writes uniquely as $X^{\mathrm{hor}} + (\pi^* f)\partial_\theta$, where X and f are respectively a vector field and a function of M.

Proposition 3.1.1. *The space* $\mathrm{aut}(P,\alpha)$ *consists of the vector fields of the form* $X^{\mathrm{hor}} - (\pi^* f)\partial_\theta$, *where* $X \in \mathcal{C}^\infty(M,TM)$ *and* $f \in \mathcal{C}^\infty(M,\mathbb{R})$ *are such that*

$$\omega(X,\cdot) + df = 0.$$

Furthermore, the Lie bracket is given in terms of this decomposition by

$$[X_1^{\mathrm{hor}} - (\pi^* f_1)\partial_\theta, X_2^{\mathrm{hor}} - (\pi^* f_2)\partial_\theta] = [X_1,X_2]^{\mathrm{hor}} - (\pi^*\omega(X_1,X_2))\partial_\theta. \quad (3.1)$$

So $\mathrm{aut}(P,\alpha)$ is isomorphic to the Lie algebra consisting of the pairs (X, f) such that $\omega(X,\cdot) + df = 0$, the bracket being given by

$$\big[(X_1, f_1), (X_2, f_2)\big] = ([X_1, X_2], \omega(X_1, X_2)).$$

(3.1). Observe that this latter algebra may be defined for any closed 2-form ω of M and does not depend on the choice of the prequantum bundle.

In a pair (X, f) satisfying $\omega(X,\cdot) + df = 0$, f is determined by X up to some locally constant function. When ω is non-degenerate, X is determined by f. In that case, we call X the *Hamiltonian vector field* of f. Furthermore, if $\omega(X_i,\cdot) + df_i = 0$ for $i = 1, 2$, the function

$$\{f_1, f_2\} = \omega(X_1, X_2)$$

is the *Poisson bracket* of f_1, f_2. So when ω is symplectic, $\mathrm{aut}(P,\alpha)$ is isomorphic to $(\mathcal{C}^\infty(M), \{\cdot,\cdot\})$.

Proof. The $\mathbb{T}$-invariant horizontal vector fields of P are the horizontal lifts of the vector fields of M. The $\mathbb{T}$-invariant vertical vector fields have the form $(\pi^* f)\partial_\theta$, with f a function on M. So any $\mathbb{T}$-invariant vector field of P can be decomposed uniquely as

$$Y = X^{\mathrm{hor}} - (\pi^* f)\partial_\theta,$$

for some $X \in \mathcal{C}^\infty(M, TM)$ and $f \in \mathcal{C}^\infty(M,\mathbb{R})$. We have by Cartan formula

$$\mathcal{L}_Y\alpha = \iota_Y d\alpha + d\alpha(Y) = -\iota_Y\pi^*\omega - d\pi^* f = -\pi^*(\omega(X,\cdot) + f)$$

so that $\mathcal{L}_Y\alpha = 0$ if and only if $\omega(X,\cdot) + df = 0$.

Let us compute the Lie bracket. Assume that $\omega(X_i,\cdot) + df_i = 0$ for $i = 1, 2$. Since X_i^{hor} and $\pi^* f_i$ are $\mathbb{T}$-invariant,

$$\begin{aligned}[X_1^{\mathrm{hor}} - (\pi^* f_1)\partial_\theta, X_2^{\mathrm{hor}} - (\pi^* f_2)\partial_\theta] =& [X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}] - \big(X_1^{\mathrm{hor}}.(\pi^* f_2) - X_2^{\mathrm{hor}}.(\pi^* f_1)\big)\partial_\theta \\ =& [X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}] - \pi^*(X_1.f_2 - X_2.f_1)\big)\partial_\theta \\ =& [X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}] - 2\pi^*\omega(X_1, X_2)\partial_\theta.\end{aligned}$$

because $\omega(X_1, X_2) = X_1.f_2 = -X_2.f_1$. We conclude with Proposition 1.2.4. □

Introduce the Hermitian line bundle $L = P \times \mathbb{C}/\mathbb{T}$ associated to P as in section 1.3. Recall that the space of sections of L identifies naturally with a subspace of functions on P. The derivation with respect to a vector field in $\mathrm{aut}(P,\alpha)$ preserves this latter space. We obtain a representation of $\mathrm{aut}(P,\alpha)$ on $\mathcal{C}^\infty(M, L)$.

Proposition 3.1.2. *The map from* $\mathrm{aut}(P,\alpha)$ *to* $\mathrm{End}(\mathcal{C}^\infty(M, L))$*, sending* $X^{\mathrm{hor}} - (\pi^* f)\partial_\theta$ *to the endomorphism*

$$s \to \nabla_X s + 2i\pi f s,$$

is a Lie algebra morphism

Proof. Let E be the isomorphism identifying $\mathcal{C}^\infty(M, L)$ with the subspace of $\mathcal{C}^\infty(P)$ consisting in the functions f satifying $f(\theta.y) = e^{-2i\pi\theta} f(y)$. Then recall that

$$X^{\text{hor}}.E(s) = E(\nabla_X s)$$

and observe that $\partial_\theta E(s) = -2i\pi E(s)$ for any section s of L. □

In the case ω is symplectic, we have a representation of the Poisson algebra of M, which was first introduced by Kostant and Souriau.

An *infinitesimal automorphism* of (M, ω) is a vector field X of M such that $\mathcal{L}_X \omega = 0$, equivalently such that $\omega(X, \cdot)$ is closed. The space $\text{aut}(M, \omega)$ of infinitesimal automorphisms of (M, ω) is a Lie subalgebra of the Lie algebra of vector fields of M.

Proposition 3.1.3. *Assume M is connected. Then we have a Lie algebra exact sequence*

$$0 \to \mathbb{R} \xrightarrow{\Phi_1} \text{aut}(P, \alpha) \xrightarrow{\Phi_2} \text{aut}(M, \omega) \xrightarrow{\Phi_3} H^1(M, \mathbb{R})$$

where $\mathbb{R}$ and $H^1(M, \mathbb{T})$ are commutative Lie algebras, and the Lie algebra morphisms Φ_1, Φ_2 and Φ_3 are defined by

$$\Phi_1(c) = c\partial_\theta, \quad \Phi_2\big(X^{\text{hor}} - (\pi^* f)\partial_\theta\big) = X, \quad \Phi_3(X) = [\omega(X, \cdot)].$$

Furthermore, if ω is non-degenerate, Φ_3 is onto.

The proof is easy and a good exercice. Introduce the Lie subalgebra of $\text{aut}(M, \omega)$

$$\text{ham}(M, \omega) = \text{Im}\,\Phi_2 = \ker \Phi_3 = \big\{ X \in \Gamma(M, TM);\ \omega(X, \cdot) \text{ is exact } \big\}$$

In the case, ω is non degenerate, the elements of $\text{aut}(M, \omega)$ and $\text{ham}(M, \omega)$ are called respectively the *symplectic* and the *Hamiltonian vector fields*.

3.1.4 Automorphisms

Consider again a prequantum bundle (P, α) with curvature $\omega \in \Omega^2(M, \mathbb{R})$. In this section, we establish the group properties corresponding to the Lie algebra properties given in the previous section. The flow of any infinitesimal automorphism of (P, α) is a one-parameter group of $\text{Aut}(P, \alpha)$. The following proposition describes this flow in terms of the decomposition $Y = X^{\text{hor}} - (\pi^* f)\partial_\theta$ given in proposition 3.1.1.

Proposition 3.1.4. *Let X be a complete vector field of M and $f \in \mathcal{C}^\infty(M, \mathbb{R})$ be such that $\omega(X, \cdot) + df = 0$. Then $Y = X^{\text{hor}} - (\pi^* f)\partial_\theta$ is complete and its flow φ_t lifts the flow φ_M^t of X. Furthermore*

$$\varphi_t(y) = \theta_t(\pi(y)).\mathcal{T}_t(y)$$

where

- *$\mathcal{T}_t$ is the $\mathbb{T}$-equivariant diffeomorphism of P lifting φ_t^M and such that for any $x \in M$, $\mathcal{T}_t : P_x \to P_{\varphi_t^M(x)}$ is the parallel transport along the path $s \in [0,t] \to \varphi_s^M(x)$.*

- *for any $x \in M$, $\theta_t(x) = -tf(x)$ modulo $\mathbb{Z}$.*

Proof. We first prove that the flow of X^{hor} is $\mathcal{T}_t$. Since $t \to \mathcal{T}_t(y)$ is an horizontal lift, $\alpha(\dot{\mathcal{T}}(y)) = 0$. Since $\pi(\mathcal{T}_t(y)) = \varphi_t^M(\pi(y))$, $\dot{\mathcal{T}}_t(y)$ projects to $X(\varphi_t(\pi(y)))$. This shows that

$$\dot{\mathcal{T}}_t(y) = X^{\mathrm{hor}}(\mathcal{T}_t(y)).$$

Because $\omega(X,\cdot) + df = 0$, f is constant on the trajectories of X, so that

$$\dot{\theta}_t(x) = -f(x) = -f(\varphi_t(x)).$$

This implies with $x = \pi(y)$ that

$$\begin{aligned}\frac{d}{dt}\big(\theta_t(x).\mathcal{T}_t(y)\big) &= -\, f(\varphi_t(x))\partial_\theta + T_y\ell_{\theta_t(x)}(X^{\mathrm{hor}}(\mathcal{T}_t(y)) \\ &= -\, f(\varphi_t(x))\partial_\theta + X^{\mathrm{hor}}(\theta_t(x).\mathcal{T}_t(y)) \\ &= Y(\theta_t(x).\mathcal{T}_t(y))\end{aligned}$$

which ends the proof. □

Let $\mathrm{Aut}(M,\omega)$ be the group of diffeomorphism of M preserving ω. Any automorphism φ of (P,α) lifts an element φ_M of $\mathrm{Aut}(M,\omega)$. Introduce the subgroup $\mathrm{Ham}(M,\omega)$ of $\mathrm{Aut}(M,\omega)$ consisting of the diffeomorphisms of M which are the flow at time 1 of a time dependent vector field X_t of M satisfying $\omega(X_t,\cdot) + df_t = 0$ for some smooth familly f_t of $\mathcal{C}^\infty(M,\mathbb{R})$. In the case $\omega = 0$, $\mathrm{Ham}(M,\omega)$ is the group of diffeomorphisms of M isotopic to the identity. In the case ω is symplectic, Ham is the Hamiltonian group of symplectic geometry.

Denote by $\mathrm{Aut}^0(M,\omega)$ the subgroup of $\mathrm{Aut}(M,\omega)$ consisting of the elements isotopic to the identity through a smooth curve of $\mathrm{Aut}(P,\alpha)$.

Proposition 3.1.5. $\mathrm{Ham}(M,\omega)$ *consists in the diffeomorphisms of M which are lifted by an element in* $\mathrm{Aut}^0(P,\alpha)$

Proof. We can adapt the proof of proposition 3.1.4, working with a time dependent vector field $Y_t = X_t^{\mathrm{hor}} - (\pi^* f_t)\partial_\theta$. To integrate Y_t, we consider as before the parallel transport along the integral curves of X, and we multiply by the function θ_t given by

$$\theta_t(x) = -\int_0^t f(\varphi_s^M(x))\, ds$$

where φ_t^M is the smooth curve of $\mathrm{Diff}(M)$ generated by X_t. □

Let us consider the exact sequence exponentiating the Lie algebra exact sequence of Proposition 3.1.3.

Proposition 3.1.6. *Let (P, α) be a prequantum bundle with a connected base M and curvature ω. Then we have a group exact sequence*

$$0 \to \mathbb{T} \xrightarrow{\Phi_1} \mathrm{Aut}^0(P, \alpha) \xrightarrow{\Phi_2} \mathrm{Aut}^0(M, \omega) \xrightarrow{\Phi_3} \mathrm{Mor}^0(H_1(M), \mathbb{T})$$

where

- *$\Phi_1(\theta)$ is the action of θ on P,*
- *$\Phi_2(\varphi) = \varphi_M$ if φ lifts φ_M,*
- *$\Phi_3(\varphi_M)$ is the morphism $H_1(M) \to \mathbb{T}$ sending $[\gamma]$ into $\mathrm{hol}(\varphi_M \circ \gamma) - \mathrm{hol}(\gamma)$.*

Furthermore, if ω is non-degenerate and M compact, Φ_3 is onto.

The morphism Φ_3 may be called a flux morphism. By Proposition 3.1.5, the kernel of Φ_3 is $\mathrm{Ham}(M, \omega)$.

Proof. Exactness at $\mathrm{Aut}^0(P, \alpha)$: an equivariant diffeomorphism φ of P lifting the identity, has the form $y \to f(\pi(y)).y$ with f a smooth function on M. Furthermore, $\varphi^*\alpha = \alpha + \pi^* df$, so that φ preserves the connection iff $df = 0$. M being connected, f is constant.

Φ_3 is well-defined: if $\gamma = \partial D$, then

$$\mathrm{hol}(\varphi_M(\gamma)) = \int_D \varphi_M^* \omega = \int_D \omega = \mathrm{hol}(\gamma),$$

because $\varphi_M \in \mathrm{Aut}(M, \omega)$. So the morphism sending $\gamma \in Z_1(M)$ to $\mathrm{hol}(\varphi_M(\gamma)) - \mathrm{hol}(\gamma)$ factors through a morphism from $H_1(M)$ to $\mathbb{T}$.

Φ_3 is a group morphism: Let ψ_M and φ_M in $\mathrm{Aut}(M, \omega)$. Since φ_M is isotopic to the identity of M, for any $\gamma \in Z_1(M)$, $\varphi_M(\gamma)$ is homologue to γ. So

$$\mathrm{hol}(\psi_M(\gamma)) - \mathrm{hol}(\gamma) = \mathrm{hol}(\psi_M(\varphi_M(\gamma))) - \mathrm{hol}(\varphi_M(\gamma))$$

which implies that

$$\mathrm{hol}(\psi_M(\varphi_M(\gamma))) - \mathrm{hol}(\gamma) = \mathrm{hol}(\psi_M(\gamma)) - \mathrm{hol}(\gamma) + \mathrm{hol}(\varphi_M(\gamma)) - \mathrm{hol}(\gamma)$$

showing that $\Phi_3(\psi_M \circ \varphi_M) = \Phi_3(\psi_M) + \Phi_3(\varphi_M)$.

Exactness at $\mathrm{Aut}^0(M, \omega)$: for any diffeomorphism φ_M of M, the holonomy of γ in the prequantum bundle $(\varphi_M^* P, \varphi_M^* \alpha)$ is $\mathrm{hol}(\varphi_M(\gamma))$. So if $\Phi_3(\varphi_M) = 0$, (P, α) and $(\varphi_M^* P, \varphi_M^* \alpha)$ have the same holonomy. By Theorem 2.3.1, (P, α) and $(\varphi_M^* P, \varphi_M^* \alpha)$ are isomorphic, through an isomorphism $P \to \varphi_M^* P$ lifting the identity of M. Composing this isomorphism with the natural map $\varphi_M^* P \to P$, we obtain an automorphim of (P, α) lifting φ_M.

Φ_3 is onto if ω is symplectic and M compact: first, we define the connected component of $\mathrm{Mor}(H_1(M), \mathbb{T})$ as the set of morphisms χ such that there exists a continuous familly $(\chi_t, t \in [0, 1])$ with $\chi_0 = 0$ and $\chi_1 = \chi$. Here continuous means that for any $[\gamma] \in H_1(M)$, $\chi_t([\gamma])$ depends continuously on t.

We claim that $\mathrm{Mor}^0(H_1(M), \mathbb{T})$ consists in the morphisms which lift from $H_1(M)$ to $\mathbb{R}$. Indeed, lifting continuously $\chi_t([\gamma]) \in \mathbb{T}$ to $\tilde{\chi}_t([\gamma]) \in \mathbb{R}$, for any cycle γ, we obtain a morphism $\tilde{\chi}_t$. Conversely, if $\tilde{\chi} \in \mathrm{Mor}(H_1(M), \mathbb{R})$ and p is the projection from $\mathbb{R}$ to $\mathbb{T}$, then the familly $p \circ (t\tilde{\chi})$ connects continuously $p \circ \tilde{\chi}$ with 0.

So let $\chi \in \mathrm{Mor}(H_1(M), \mathbb{R})$. By the universal coefficient theorem,

$$\mathrm{Mor}(H_1(M), \mathbb{R}) = H^1(M, \mathbb{R}).$$

Then by de Rham theorem, there exists a closed 1-form β on M such that $\chi = [\beta]$. Since ω is non-degenerate, there exists a vector field X of M such that $\omega(X, \cdot) = \beta$. Let φ_M^t be the flow of X, which is well-defined since M is compact. Let us prove that $\Phi_3(\varphi_M^1) = p \circ \chi$, where p is the projection $\mathbb{R} \to \mathbb{T}$.

By Proposition 2.2.2, for any loop $\gamma : S^1 \to M$,

$$\mathrm{hol}(\varphi_M^t(\gamma)) - \mathrm{hol}(\gamma) = \int_{[0,t]\times S^1} \xi^* \omega \mod \mathbb{Z}$$

with $\xi : \mathbb{R} \times S^1 \to M$ the map sending (t, x) into $\varphi_M^t(\gamma(x))$. Derivating with respect to t the right-hand side, we get

$$\int_{S^1} (\xi^*\omega)(\partial_t, \cdot) = \int_\gamma (\varphi_M^t)^* \omega(X, \cdot) = \int_\gamma \omega(X, \cdot) = \chi([\gamma])$$

because φ_M^t leaves ω invariant. Consequently, integrating from 0 to 1, we obtain

$$\mathrm{hol}(\varphi_M^1(\gamma)) - \mathrm{hol}(\gamma) = \chi([\gamma])$$

which concludes the proof. □

Let (L, ∇) be the Hermitian line bundle associated to P. An automorphism of (L, ∇) is a line bundle automorphism of L preserving the metric and the connection ∇ in the sense that $\varphi^* \nabla s = \nabla \varphi^* s$ for any smooth section s of M. We have a natural isomorphism between $\mathrm{Aut}(M, \omega)$ and the group of automorphism of (L, ∇). To see this, identify P with $U(L)$ and extend any automorphism φ of (P, α) to L in such a way that for any $x \in M$, φ restricts to a linear map from L_x into $L_{\varphi^M(x)}$.

We obtain a representation of $\mathrm{Aut}(P, \alpha)$ on the space of sections of L, where we let $\varphi \in \mathrm{Aut}(P, \alpha)$ act by pull-back

$$(\varphi^* s)(x) = \varphi^{-1}(s(\varphi_M(x))), \qquad \forall s \in \mathcal{C}^\infty(M, L)$$

Here φ_M is the diffeomorphism of M lifted by φ. The corresponding infinitesimal representation is the Kostant-Souriau Lie algebra representation of Proposition 3.1.2, in the sense that if (φ_t) is the flow of $X^{\mathrm{hor}} - (\pi^* f)\partial_\theta$, then

$$\frac{d}{dt}\Big|_{t=0} (\varphi_t^* s)(p) = \nabla_X s(p) + 2i\pi f(p) s(p). \tag{3.2}$$

This follows from the identification between sections of L and functions of P satisfying $f(\theta.y) = e^{-2i\pi y} f(y)$.

3.2 Lie group action and reduction

3.2.1 Preliminaries

Let us first recall some basic facts on Lie group action. Let M be a manifold and G be a Lie group acting on M. We will always assume that the action is smooth, meaning that the map $G \times M \to M$ sending (g, x) into $g.x$ is smooth. For any $g \in G$, we denote by ℓ_g the action of g on M. For any ξ in the Lie algebra $\mathfrak{g}$ of M, introduce the vector field of M

$$\xi_M(p) = \frac{d}{dt}\Big|_{t=0} e^{t\xi}.p, \qquad \forall p \in M,$$

called the infinitesimal representation of ξ. The flow at time t of ξ_M is $\ell_{\exp(t\xi)}$. Denote by Ad the adjoint representation of G.

Lemme 3.2.1. *For any $\xi, \eta \in \mathfrak{g}$ and $g \in G$, we have*

$$T_x\ell_g(\xi_M(x)) = (\mathrm{Ad}_g\, \xi)_M(g.x) \quad \text{ and } \quad [\xi_M, \eta_M] = -[\xi, \eta]_M.$$

Proof. First, $T_x\ell_g(\xi_M(x))$ is the derivative at $t = 0$ of $t \to \ell_g(e^{t\xi}.x)$. Since

$$\ell_g(e^{t\xi}.x) = ge^{t\xi}g^{-1}.gx = e^{t\,\mathrm{Ad}_g\,\xi}.gx,$$

we obtain $T_x\ell_g(\xi_M(x)) = (\mathrm{Ad}_g\, \xi)_M(gx)$.

Since the flow of ξ_M at times t is $\ell_{e^{t\xi}}$, we have

$$[\xi_M, \eta_M](x) = \frac{d}{dt}\Big|_{t=0} (T_x\ell_{e^{t\xi}})^{-1}(\eta_M(e^{t\xi}x))$$

By the first part of the proposition,

$$(T_x\ell_{e^{t\xi}})^{-1}(\eta_M(e^{t\xi}x)) = (\mathrm{Ad}_{e^{-t\xi}}\, \eta)_M(x)$$

Taking the derivative, we obtain $[\xi_M, \eta_M] = -[\xi, \eta]_M$. □

So the linear map from $\mathfrak{g}$ to $\mathcal{C}^\infty(M, TM)$ sending ξ into ξ_M is a Lie algebra anti-morphism. This map may be viewed as the differential of the group morphism from G into $\mathrm{Diff}(M)$ sending g into ℓ_g. If G is simply connected and H is another Lie group, it is a well-known result that any Lie algebra morphism from $\mathfrak{h}$ to the Lie algebra of H is the differential of a Lie group morphism from H to G. The extension of this result to the case H is the group of diffeomorphisms of M is the Palais' theorem.

Theorem 3.2.2. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Let ρ be a Lie algebra anti-morphism from $\mathfrak{g}$ to the Lie algebra of vector fields of a manifold M. Assume that $\rho(\xi)$ is complete for any $\xi \in \mathfrak{g}$. Then there exist a unique left action of G on M with infinitesimal representation ρ.*

Consider again a Lie group G acting on a manifold M. Let ω be a G-invariant 2-form and μ be an equivariant map from M to $\mathfrak{g}^*$. Here the equivariance is with respect to the coadjoint action, that is

$$\mu(g.x) = \mathrm{Ad}^*_{g^{-1}} \mu(x), \qquad \forall x \in M, g \in G$$

Equivalently, for any $\xi \in \mathfrak{g}$,

$$\langle \mu(g.x), \xi \rangle = \langle \mu(x), \mathrm{Ad}_{g^{-1}} \xi \rangle.$$

We call (ω, μ) an *equivariant 2-form.* We say that (ω, μ) is closed if the following equations hold

$$d\omega = 0, \qquad \omega(\xi_M, \cdot) + d\mu(\xi) = 0, \quad \forall \xi \in \mathfrak{g}. \tag{3.3}$$

As the name suggest it, the closed equivariant 2-forms are the closed 2-cochains of the complex of equivariant differential forms, defined as

$$\Omega^k_G(M) = \sum_{2i+j=k} \Big(S^i \mathfrak{g}^* \otimes \Omega^j(M) \Big)^G, \qquad d_G = d + \iota_{\xi_M}$$

In the sequel we will only consider $\Omega^2_G(M)$. In symplectic geometry, a map μ satisfying the second equation of (3.3) is called a *momentum* of the action. An action that admits a momentum is called a *Hamiltonian* action. Here we will use this terminology even when ω is degenerate.

Lemme 3.2.3. *Let (ω, μ) be a closed equivariant 2-form. Then for any $\xi, \eta \in \mathfrak{g}$, $\langle \mu, [\xi, \eta] \rangle + \omega(\xi_M, \eta_M) = 0$*

Proof. By the momentum equation, we have that $\omega(\xi_M, \eta_M) = \xi_M.\langle \mu, \eta \rangle$. By definition of the infinitesimal action corresponding to ξ,

$$(\xi_M.\langle \mu, \eta \rangle)(p) = \frac{d}{dt}\Big|_{t=0} \langle \mu, \eta \rangle (e^{t\xi}.p)$$

μ being equivariant, we have

$$\langle \mu(e^{t\xi}.p), \eta \rangle = \langle \mu(p), \mathrm{Ad}_{e^{-t\xi}} \eta \rangle$$

and the derivative with respect to t at $t = 0$ is equal to $\langle \mu(p), -[\xi, \eta] \rangle$. □

3.2.2 Action on prequantum bundle

We call a prequantum bundle endowed with an action of a Lie group G by prequantum bundle automorphisms, a *G-prequantum bundle.* Next proposition shows that the actions on prequantum bundles lift Hamiltonian actions.

Proposition 3.2.4. *Let (P, α) be a G-prequantum bundle with projection π, base M and curvature ω. Then the map $\mu : M \to \mathfrak{g}^*$, defined by*

$$\pi^* \mu(\xi) + \alpha(\xi_P) = 0, \qquad \forall \xi \in \mathfrak{g},$$

is a momentum of the induced action on (M, ω). Furthermore ω is G-invariant.

Proof. Since G acts by prequantum bundle automorphisms, the infinitesimal action of any $\xi \in \mathfrak{g}$ is an infinitesimal automorphism of (P, α). By Proposition 3.1.1, we have that

$$\xi_P = \xi_M^{\mathrm{hor}} - (\pi^* \mu^\xi) \partial_\theta$$

for a unique function μ^ξ satisfying $\omega(\xi_M, \cdot) + d\mu^\xi = 0$. This defines the momentum μ. Let us check the equivariance condition. By lemma 3.2.1, the tangent linear map to the action of g on P sends ξ_P into $(\mathrm{Ad}_g \xi)_P$. So

$$\begin{aligned} \langle \alpha|_{gx}, \xi_P(gx) \rangle &= \langle \alpha|_{gx}, T_x \ell_g ((\mathrm{Ad}_{g^{-1}} \xi)_P(x)) \rangle \\ &= \langle \alpha|_x, (\mathrm{Ad}_{g^{-1}} \xi)_P(x) \rangle \end{aligned}$$

because the action of G preserves α. □

Let us adress the converse question. Let us start with a closed equivariant 2-form (ω, μ). Assume that ω is the curvature of a prequantum bundle (P, α). Then the momentum defines a linear map from $\mathfrak{g}$ to $\mathrm{aut}(P, \alpha)$

$$\xi \in \mathfrak{g} \to \xi_M^{\mathrm{hor}} - \pi^* \mu^\xi \partial_\theta \in \mathrm{aut}(P, \alpha)$$

By Proposition 3.1.1 and Lemma 3.2.3, this map is Lie algebra anti-morphism. Can we integrate this in a G-action ? A necessary condition is that these vector fields are complete. This is guaranted by Proposition 3.1.4. When G is simply connected, we deduce from Theorem 3.2.2 the following

Proposition 3.2.5. *Let (P, α) be a prequantum bundle with base M and curvature ω. Assume G is a connected and simply connected Lie group acting on (M, ω) with an equivariant momentum μ. Then there exists a unique lift to $\mathrm{Aut}(P, \alpha)$ of the action on M such that the corresponding momentum is μ.*

More generally, assume G is connected but not necessarily simply connected. Recall that the universal covering group of G is a connected Lie group $\tilde{G}$, together with a surjective homomorphism $p : \tilde{G} \to G$, such that $\ker p$ is a discrete subgroup of $\tilde{G}$ which is canonically isomorphic to the fundamental group $\pi_1(G)$. Since the differential of π is an isomorphism, we identify the Lie algebra of $\tilde{G}$ with $\mathfrak{g}$.

Assume G acts on (M, ω) with an equivariant momentum, then the same holds for $\tilde{G}$ and by Proposition 3.2.5, the action of $\tilde{G}$ lifts to $\mathrm{Aut}(P, \alpha)$. Then by Proposition 3.1.6, $\pi_1(G)$ acts by $\mathbb{T}$, in the sense that for each $g \in \pi_1(G)$, there exists $\theta_g \in \mathbb{T}$ such that the action of g is the multiplication by θ_g. In the case $\pi_1(G)$ acts trivially, the action $\tilde{G} \to \mathrm{Aut}(P, \alpha)$ factors to an action of G.

Example 3.2.6. Let S^2 be the unit sphere of $\mathbb{R}^3$ equipped the $SO(3)$-invariant volume form ω such that $\int_{S^2} \omega = V$. Consider the circle action by rotations around the z-axis. Then, the function $\mu = Vz/2$ is a momentum of this action. Here we identify with $\mathbb{R}$ the dual of the Lie algebra $\mathbb{R}$ of $\mathbb{R}/\mathbb{Z}$.

Assume that V is an integer and introduce a prequantum bundle (P, α) over M with curvature ω. Then by proposition 3.2.5, we obtain an action ρ :

subsequence if necessary, we deduce from the properness assumption that (g_n) converges. Its limit g satisfies $g.p = p$, and the action being free, $g = 1$. This contradicts the fact that φ is into on a neighborhood of $(1, p)$.

Using Proposition 3.2.1, we deduce from the first condition that φ is a local diffeomorphism. So its image is open. Since it is into, it is a diffeomorphism onto its image. □

Such a submanifold is called a *slice* at p. This lemma has many consequences. First, observe that the orbit are closed submanifolds of M, diffeomorphic to G. Furthermore the quotient M/G, endowed with the quotient topology, is Hausdorff. Also the orbit space M/G inherits a manifold structure as explained in the next proposition. Denote by p_M the projection from M onto M/G.

Proposition 3.2.8. *If the action is free and proper, then M/G, endowed with the quotient topology, has a unique differential structure such that for any slice S, the map from S to $p_M(S)$ sending x to $p_M(x)$ is a diffeomorphism.*

Proof. For any slice S, $p_M(S)$ is an open set of M/G and the restriction of p_M to S is a homeomorphism from S to $p_M(S)$. So we have a chart

$$\mathcal{C}_S = (p_M(S), (p_M|_S)^{-1} : p_M(S) \to S).$$

One checks easily that these charts are compatible. Then we endow M/G with the maximal atlas containing these charts. □

Observe that the projection p_M is a smooth submersion and for any $x \in M$, the kernel of the tangent linear map to p_M is given by

$$\ker(T_x p_M) = T_x(G.x) = \{\xi_M(x) / \ \xi \in \mathfrak{g}\}.$$

A form on M which is the pull-back of a form on M is called a basic form. The following characterization of the basic form is proved exactly as Lemma 1.2.2.

Proposition 3.2.9. *Assume the action is free and proper so that M/G is a manifold. Then the pull-back $p^* : \Omega(M/G) \to \Omega(M)$ is into and its image consists in the forms $\omega \in \Omega(M)$ which are G-invariant and such that $\iota_{\xi_M} \omega = 0$ for any $\xi \in \mathfrak{g}$.*

3.2.4 Quotient of prequantum bundles

Let $\pi : B \to M$ be a $\mathbb{T}$-principal bundle equipped with a G-action by automorphisms, that is an action of G commuting with the $\mathbb{T}$-action. This action lifts an action of G on the base M. Assume this latter action is free and proper. Then we have the following

Lemme 3.2.10. *The quotient B/G is the total space of a $\mathbb{T}$-principal bundle with base M/G. The projection is the map sending the class of $y \in P$ to the class of $\pi(y)$. The action of $\theta \in \mathbb{T}$ sends $[y]$ into $[\theta.y]$.*

$\mathbb{R} \to \mathrm{Aut}(P, \alpha)$ with momentum μ. The subgroup $\mathbb{Z}$ acts through $\mathbb{T}$. This latter action is easy to compute at fixed points of S^2, for instance at the north pole, by using Proposition 3.1.4. We obtain that $\rho(n) = Vn/2 \mod \mathbb{Z}$. So ρ factors to a morphisms from $\mathbb{R}/\mathbb{Z}$ to $\mathrm{Aut}(P, \alpha)$ if and only if V is even. Observe that we can shift the momentum by a constant without changing the action. Furthermore the action lifts to the prequantum bundle with corresponding momentum $Vz/2 + C$ if and only if $V/2 + C$ is integer.

Consider now the $SO(3)$ action on S^2. Identify the Lie algebra of $SO(3)$ with $\mathbb{R}^3$ so that the vector (x, y, z) corresponds to the matrix

$$2\pi \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Then the action has the momentum $\mu(u) = (V/2).u$. By proposition 3.2.5, this action lifts to a morphism $SU(2) \to \mathrm{Aut}(P, \alpha)$. This latter action descends to a morphism $SO(3) \to \mathrm{Aut}(P, \alpha)$ if and only if V is even.

3.2.3 Quotient by a free and proper action

Consider a Lie group G acting on a manifold M. Recall that the action is said to be *proper* if the map from $G \times M$ to $M \times M$ sending (g, m) into $(m, g.m)$ is proper.

Lemme 3.2.7. *Assume that the action is free and proper. Then for any point $p \in M$, there exists a submanifold S of M containing p such that $G.S$ is open in M and the map*

$$G \times S \to G.S, \qquad (g, x) \to g.x$$

is a diffeomorphism.

Proof. Since the action is free, it is locally free, meaning that for any $x \in M$, the linear map

$$\mathfrak{g} \to T_x M, \qquad \xi \to \xi_M(x)$$

is into. So the subset $\mathcal{D}$ of TM consisting of the $\xi_M(x)$, $\xi \in \mathfrak{g}, x \in M$ is a subbundle of TM, with rank the dimension of G. Consider any submanifold S of M containig x and such that $\mathcal{D}_p \oplus T_p S = T_p M$. Replacing S by $S \cap V$, where V is a neighborhood of p, we have that

1. for any $x \in S$, $\mathcal{D}_x \oplus T_x S = T_x M$
2. the map $\varphi : S \times G \to M$ sending (x, g) into $g.x$ is into.

Let us prove the second condition by contradiction. Observe first that the map is into on a neighborhood of $(p, 1)$ by the local inversion theorem. Assume that there exists sequences (x_n), (x'_n), (g_n) and (g'_n) of S and G respectively such that $x'_n \neq x_n$, $g_n x_n = g'_n x'_n$ and (x_n), (x'_n) converge to p. Then replacing g_n by $(g'_n)^{-1} g_n$ we may assume that $g'_n = 1$ for any n. Next, replacing (g_n) by a

Proof. Since the G-action on M is free and proper, the same holds for the G-action on B, so the quotient B/G has a natural manifold structure. Since the projection from B to M is G-equivariant, it descends to a smooth map from B/G to M/G. The $\mathbb{T}$-action commuting with the G-action, it descends to a smooth $\mathbb{T}$-action on B/G.

Since the G-action on M is free, for any $x \in M$, the projection p_B from B to B/G restricts to a $\mathbb{T}$-equivariant bijection

$$p_{B,x} : B_x \to (B/G)_{[x]}.$$

This has the first consequence that the $\mathbb{T}$-orbits of B/G are the fibers of the projection $B/G \to M/G$. Let us contruct local trivialisations. Consider a slice S of the G action on M, such that $B|_S$ is isomorphic to $S \times \mathbb{T}$. Then $B|_S$ is a slice of the G-action on B. We then have the $\mathbb{T}$-principal bundle isomorphisms

$$(B/G)|_{p_M(S)} \simeq B|_S \simeq S \times \mathbb{T} \simeq p_M(S) \times \mathbb{T},$$

which ends the proof. □

Assume now B has a connection α and that the G-action preserves it.

Lemme 3.2.11. *There exists $\alpha^{B/G} \in \Omega^1(B/G)$ such that $p_B^* \alpha^{B/G} = \alpha$ if and only if the momentum associated to the G-action is identically null. In the case it exists, $\alpha^{B/G}$ is unique and is a connection of B/G.*

Proof. By Proposition 3.2.9, α is basic if and only if the momentum vanishes identically. □

To summarize we have proved the following result.

Proposition 3.2.12. *Let G be a Lie group acting on a prequantum bundle (P, α) by prequantum bundle automorphisms. Assume that the corresponding action on the base M is free, proper and its momentum vanishes. Then P/G is the total space of a prequantum bundle with base M/G and connection form $\alpha^{B/G}$ such that*

$$p_B^* \alpha^{B/G} = \alpha,$$

with p_B the projection from B to B/G. Furthermore the curvature $\omega^{B/G}$ of $\alpha^{B/G}$ satisfies $p_B^* \omega^{B/G}$ where ω is the curvature of α.

Observe that the pull-back of a G-prequantum bundle by a G-equivariant map is a G-prequantum bundle. In particular the restriction of a G-prequantum bundle to a G-invariant submanifold is a G-prequantum bundle.

3.2.5 Symplectic reduction

Proposition 3.2.13. *Let (M, ω) be a symplectic manifold. Let G be a Lie group acting on M in a Hamiltonian way with momentum $\mu : M \to \mathfrak{g}^*$. Assume that the restriction of the action to $\mu^{-1}(0)$ is free and proper. Then*

- *μ is a submersion onto* 0*, so that $\mu^{-1}(0)$ is a submanifold of M.*
- *the quotient $M/\!\!/G$ of $\mu^{-1}(0)$ by G has a natural symplectic form $\omega^{M/\!\!/G}$ satisfying*
$$p^*\omega^{M/\!\!/G} = j^*\omega$$
where j is the embedding of $\mu^{-1}(0)$ into M and p is the projection from $\mu^{-1}(0)$ to $\mu^{-1}(0)/G$

The quotient $\mu^{-1}(0)/G$ is called the *symplectic reduction* of M by G and is denoted by $M/\!\!/G$.

Proof. For any $x \in M$, let $\mathfrak{g}_x = \{\xi \in \mathfrak{g}/ \ \xi_M(x) = 0\}$ and $\mathcal{D}_x = \{\xi_M(x)/ \ \xi \in \mathfrak{g}\}$. $\mathfrak{g}_x$ is the Lie algebra of the isotropy group G_x of x. But for the proof, we only need to know that $\mathfrak{g}_x = \{0\}$ if $G_x = (0)$. Because of the momentum equation, the adjoint of the linear map

$$\mathfrak{g} \to T_x^*M, \qquad \xi \to \omega(\xi_M(x), \cdot)$$

is the tangent linear map $T_x\mu : T_xM \to \mathfrak{g}^*$. So the kernel of $T_x\mu$ is the symplectic orthogonal of $\mathcal{D}_x$ and the image of $T_x\mu$ is the orthogonal of $\mathfrak{g}_x$. Since the action on $\mu^{-1}(0)$ is free, for any $x \in \mu^{-1}(0)$, $\mathfrak{g}_x = 0$ so that $T_x\mu$ is onto. This shows that $\mu^{-1}(0)$ is a submanifold of M, its tangent space at x being the kernel of $T_x\mu$.

The action on $\mu^{-1}(0)$ being free and proper, the quotient of $\mu^{-1}(0)$ has a natural differential structure. Furthermore, the tangent linear map to the projection $p : \mu^{-1}(0) \to \mu^{-1}(0)/G$ induces an isomorphism

$$T_{p(x)}(\mu^{-1}(0)/G) \simeq T_x(\mu^{-1}(0))/\mathcal{D}_x$$

As we have seen, the symplectic orthogonal of $T_x(\mu^{-1}(0)) = \ker(T_x\mu)$ is $\mathcal{D}_x$. The inclusion $\mathcal{D}_x \subset (T_x(\mu^{-1}(0)))^{\perp_\omega}$ has the consequence that the restriction of ω to $\mu^{-1}(0)$ is basic. So it descends to $\omega^{M/\!\!/G} \in \Omega^2(M/G)$. ω being closed, $\omega^{M/\!\!/G}$ is closed. Furthermore, since $(T_x(\mu^{-1}(0)))^{\perp_\omega} \subset \mathcal{D}_x$, $\omega^{M/\!\!/G}$ is non-degenerate. □

The quotient $M/\!\!/\!\!/G$ is called the *symplectic reduction* of M by G. As a consequence of the results of the previous section, we can consider symplectic reduction of prequantum bundle.

Proposition 3.2.14. *Let (P, α) be a G-prequantum bundle with curvature a non-degenerate form and with corresponding momentum $\mu : M \to \mathfrak{g}^*$. Assume that the restriction of the action to $\mu^{-1}(0)$ is free and proper, so that $\mu^{-1}(0)$ is a submanifold of M. Denote by j the injection of $\mu^{-1}(0)$ into M. Then the quotient by G of the restriction of (P, α) to $\mu^{-1}(0)$ is a prequantum bundle with base $M/\!\!/G$ and curvature $\omega^{M/\!\!/G}$.*

To end this section, we revisit the example of the dual of the the tautological bundle of $\mathbb{CP}(n)$.

Example 3.2.15. Consider as in Section 1.4 the dual of the tautological bundle of $\mathbb{CP}(n)$. The associated circle prequantum bundle is $S^{2n+1} \to \mathbb{CP}^n$ with action and connection given by

$$\theta.y = (e^{-2i\pi\theta}y_0, \ldots, e^{-2i\pi\theta}y_n), \qquad \alpha = \frac{i}{4\pi}\sum_{j=0,\ldots,n}(y_j d\bar{y}_j - \bar{y}_j dy_j).$$

We will show that this prequantum bundle can be obtained by a symplectic reduction from $M = \mathbb{C}^{n+1}$. Let P be the trivial $\mathbb{T}$-principal bundle with base $\mathbb{C}^{n+1}$ and connection α^P given by

$$\alpha^P = d\theta - \frac{i}{4\pi}\sum_j (y_j d\bar{y}_j - \bar{y}_j dy_j).$$

Consider the group $\mathbb{T}$ acting on $\mathbb{C}^{n+1}$ by

$$\rho_M(\theta)(y) = e^{2i\pi\theta}y = (e^{2i\pi\theta}y_0, \ldots, e^{2i\pi\theta}y_n)$$

This action has the momentum $\mu = |y|^2 - 1$. The corresponding lift to P is the action

$$\rho_P(\theta)(y, t) = (e^{2i\pi\theta}y, t + \theta).$$

Observe that $\mu^{-1}(0) = S^{2n+1}$ and so $P|_{\mu^{-1}(0)} = S^{2n+1} \times \mathbb{T}$. The diffeomorphism

$$P|_{\mu^{-1}(0)} \to S^{2n+1} \times \mathbb{T}, \qquad (y, t) \to (ye^{-2i\pi t}, t)$$

intertwins the ρ_P action with the action given by $\theta.(y, t) = (y, t + \theta)$. So the map

$$p : P|_{\mu^{-1}(0)} \to S^{2n+1}, \qquad (y, t) \to ye^{-2i\pi t}$$

induces a diffeomorphism from the quotient of $P|_{\mu^{-1}(0)}$ by ρ_P to S^{2n+1}. Furthermore,

$$p(y, \theta + t) = e^{-2i\pi\theta}p(y, t), \quad p^*\alpha = \alpha^P$$

which shows that we recover the $\mathbb{T}$-action on S^{2n+1} and the connection α.