

Recall from last time

$$E \begin{cases} \rightarrow \mathbb{F}_q((\pi)) \\ \rightarrow [E: \mathbb{Q}_p] L + \infty \end{cases} \quad \mathbb{C}_{E/\pi} = \mathbb{F}_q$$

$$\overline{\mathbb{F}_q}/\mathbb{F}_q \quad \hat{E} = \hat{E}^{\text{un}} \circ \sigma$$

\*  $S \in \text{Perf}_{\mathbb{F}_q} \rightsquigarrow X_S = Y_S / \varphi^{\mathbb{Z}}$

$$Y_S = D_S^* = \{0 < |u| < 1\} \subset \text{Al}_S^{\mathbb{Z}} \text{ if } E = \mathbb{F}_q((\pi))$$

$G$   
 $\varphi = \text{Frob}_S$

$$Y_S = \text{Spa}(A, A^+) \setminus V(\pi \cdot [0, R]) \quad \text{if } S = \text{Spa}(R, R^+) \text{ affinoid perfectoid}$$

$$A = W_{0, \mathbb{F}_q}(R^{\circ}) \quad \text{and } E/\mathbb{Q}_p$$

$$A^+ = [R^+] + \pi W_{0, \mathbb{F}_q}(R^{\circ})$$

\* If  $S/\overline{\mathbb{F}_q}$   $\varphi\text{-Mod}_{\mathbb{F}_q} \rightarrow \text{Bun}_{X_S}$   
 $(D, \varphi) \mapsto \mathcal{E}(D, \varphi)$

Th: S geo. point  $\Rightarrow \mathcal{E}(-)$  is essentially surjective.

$\rightarrow \forall \mathcal{E} \in \text{Bun } X_S$

$$\mathcal{E} \simeq \bigoplus_i \mathcal{O}_{X_S}(\lambda_i) \quad \lambda_i \in \mathbb{Q}$$

stable of slope  $\lambda_i$

$\hookrightarrow$  what does it mean?

$\rightarrow$  need the schematic curve: declare  $\mathcal{O}(1)$  is ample

$S = \text{Spa}(F)$   $F$  perf. field.

$$\text{Set } P = \bigoplus_{d \geq 0} \underbrace{H^0(X_S, \mathcal{O}(d))}_{\mathcal{O}(Y_S)^{\mathcal{O}_F = \pi^d}}$$

Th:  $\text{Proj}(P)$  is a Dedekind scheme that is "Complete":  $\forall f \in \mathbb{E}(X_F^{\text{sch}})^{\times}$   
 $\deg(\text{div } f) = 0$  ( $\deg \kappa = 1$  for  $\kappa$  closed points if  $F$  alg. closed.)  
 $\Leftrightarrow$  good notion of degree of a v.b. /  $X_F^{\text{sch}}$

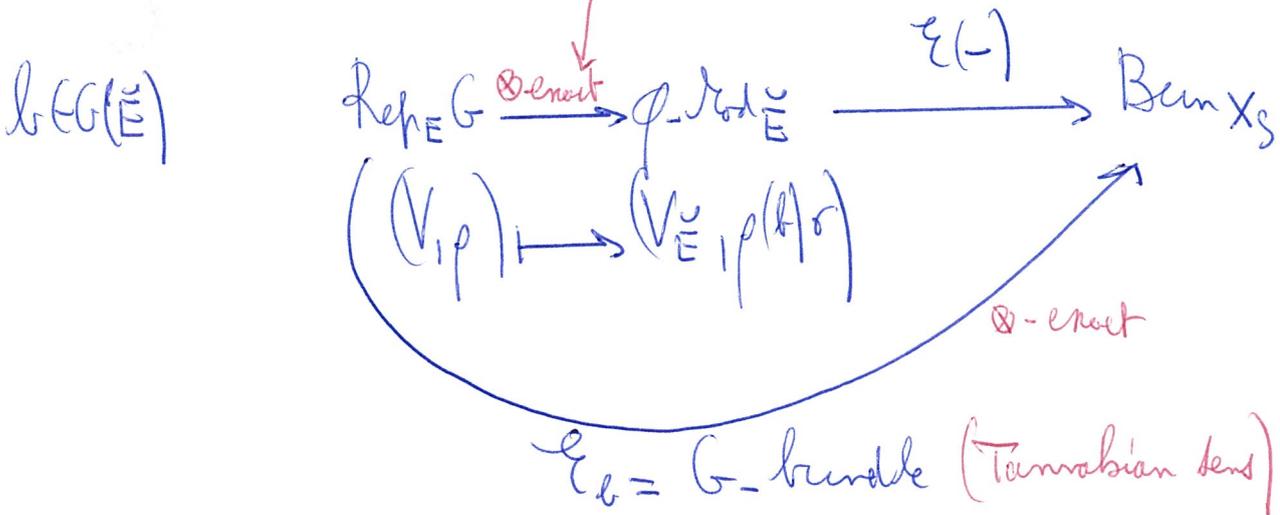
$\exists$  morphism of reduced spaces  $X_F \xrightarrow{\alpha} X_F^{\text{alg}}$

GAGA: Th (Kedlaya-Liu):  $\alpha: \text{Bun}_{X_F^{\text{sch}}} \xrightarrow{\sim} \text{Bun}_{X_F}$

- Harder-Narasimhan reduction theory for vector bundles.

$G/E$  reductive group  $S/\mathbb{F}_q$

$$B(G) = G(\mathbb{F}) / G\text{-conjugacy} = \{G\text{-isocrystals}\} / \sim$$



Th: Sgebr. point - then

$$B(G) \xrightarrow{\sim} \{G\text{-bundles}/X_S\} / \sim$$

$$[b] \mapsto [\xi_b]$$

The stack  $\text{Bun}_G$ : Equip  $\text{Perf}_{\mathbb{F}_q}$  with the pro-étale topology

Consequence of descent results for v. l. / pro-étale coverings

Def:  $\text{Bun}$  is the stack on  $\text{Perf}_{\mathbb{F}_q}$  such that (Ser, Scholze)

$$\text{Bun}(S) = \{G\text{-bundles}/X_S\}$$

Rem:  $X =$  preperfectoid  $E$ -adic space like  $X_S$

(1) By def. a v.b./ $X$  is a locally free of finite rank sheaf of  $\mathcal{O}_X$ -modules/ $X$

If  $X = \text{Spa}(R, R^+)$  is affinoid pre-perfectoid

Th (Kedlaya-Liu):  $H^0(X, -)$ : v.b./ $X \cong$  Projective  $R$ -modules of finite type

$\Rightarrow$  v.b. on  $\text{Spa}(R, R^+) =$  v.b. on  $\text{Spa}(R, R^\circ)$

easy  $\Rightarrow$  if  $S = \text{Spa}(R, R^+)$  aff. perf. /  $\mathbb{A}^1$

v.b. on  $X_{R, R^+} =$  v.b. on  $X_{R, R^\circ}$

i.e.  $\text{Bun}(R, R^+) = \text{Bun}(R, R^\circ)$

$\Rightarrow$  Bun is partially proper

(without boundary in Berberich's terminology)  
- overconvergent in Tate & Spies terminology)

$\rightarrow$  don't care about  $R^+$  in the following.

(2) Prop:  $G^{\text{ad}}$ -torsors on  $X$  locally trivial for the étale topology  $\leadsto$  Tambaraian  $G$ -bundles on  $X$

$\text{Rep}_E G \xrightarrow{\cong \text{ exact}} \text{Bun}_X$

$\tau \mapsto [(V, \rho) \rightarrow \tau_{G, \rho}^X V]$

$$\Rightarrow \pi_0(\text{Bun}(S)) = H_{\text{ét}}^1(X_S, G^{\text{ad}})$$

$$= H_{\text{ét}}^1(X_F^{\text{sch}}, G)$$

↑ if  $S = \text{Spa}(F)$  -  $F$  perf. field.

(3)  $S = \text{Spa}(R, R^+)$

$$Y_{R, R^+} \xrightarrow{\text{Spa}(\rightarrow)_a} Y_{R, R^+} = \text{Spa}(A, A^+) \setminus V(\pi, [\omega_R])$$

↑ Compactification by 2 divisors at  $\infty$

$$Y = Y \cup V(\bar{u}) \cup V([\omega_R])$$

$G$   
 properly discontinuously  
 invariant under  $\varphi$

Set  $R_R := \varinjlim_{Y \supset U \supset V(\bar{u})} \mathcal{O}(U \setminus V(\bar{u})) =$  germs of hol. sect outside  $V(\bar{u})$

↑  
 open neighborhood of  $V(\bar{u})$

Bezout ring if  $R = \mathbb{F}$  field but not in general

projective of finite type +  $\varphi$ -linear automorphism

$$\text{Bun}/X_S \xrightarrow{\sim} \varphi\text{-Mod } R_R$$

$$\mathcal{E} \text{ mod } \varinjlim_{Y \supset U \supset V(\bar{u})} \Gamma(U \setminus V(\bar{u}), \beta^* \mathcal{E}) \quad \beta: Y_S \rightarrow X_S = Y_S / \varphi^2$$

$\Rightarrow G\text{-bundles}/X_S = G\text{-torsors } T \text{ on } \text{Spec}(\mathbb{R}_R)_{\text{ét}} \left. \begin{array}{l} \\ + \text{ isomorphism } \varphi^* T \cong T \end{array} \right\} \text{Concrete}$

Points:  $|Bun| := \left( \coprod_{\substack{F/\mathbb{F}_q \\ \text{perf. field.}}} Bun(F) \right) / \sim$ . Preceding theorem  
 $\Downarrow$   
Geometry of Bun  $|Bun_{\mathbb{F}_q}| = B(G)$

## Connected Components

$$K: B(G) \rightarrow \pi_1(G)_\Gamma \quad \Gamma = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$$

Kottwitz map

Borovoi  $\pi_1 = X_*(T) / \langle \sum \sigma \rangle$

For  $G = GL_n$ ,  $K = -\text{degree of a v.b.}$  In general  $K = -c_1^G \leftarrow G\text{-equivariant first Chern class}$

Th:  $K$  is locally constant on  $Bun$ .

$\rightarrow$  easy when  $G^{der}$  is simply connected since then

$$K: B(G) \rightarrow B(G/G^{der}) = \pi_1(G)_\Gamma$$

$\underbrace{\hspace{10em}}_{\text{torus}}$

$\Rightarrow$  reduced to the case of  $G = \text{torus}$

reduced to the case of  $G = G_m \rightarrow$  easy.

$\rightarrow$  when  $G^{der}$  not s.c. one needs the following theorem.

$S \in \text{Perf}_{\mathbb{F}_q}$

$\tau: (X_S)_{\text{ét}} \rightarrow S_{\text{ét}}$  morphism of étale sites

given by  $\underbrace{T/S}_{\text{étale}} \mapsto \underbrace{X_T/X_S}_{\text{étale}}$

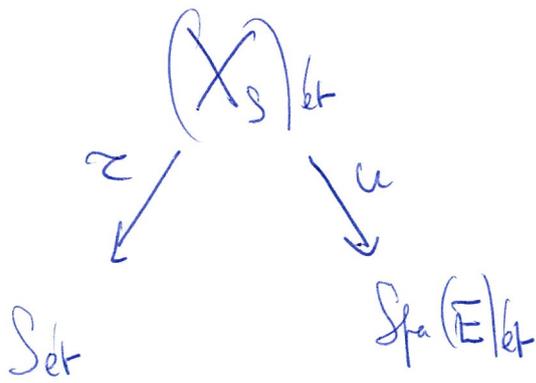
There is no morphism  $X_S \rightarrow S$   $\text{étale}$  but there is one at the level of étale sites or analytic sites too.

↓  
∃ continuous application  $|X_S| \rightarrow |S|$   
that is closed and open  
↑ "X<sub>S</sub>/S is proper"      ↑ "X<sub>S</sub>/S is smooth"

Th 1.1.1

~~$\mathbb{R}^2 \times \mathbb{Z}/m\mathbb{Z}$~~   $\mathbb{R}^2 \times \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$   
↑ given by  $c_1(\mathcal{O}(1))$   
fundamental class of local class field.

\* more generally



Künneth formula

$$R\Gamma(X_S, \tau^* \mathcal{F} \otimes_{\mathbb{Z}/m\mathbb{Z}} u^* \mathcal{G}) = R\Gamma(S, \mathcal{F}) \otimes_{\mathbb{Z}/m\mathbb{Z}} R\Gamma(\text{Spa}(E), \mathcal{G})$$

$\mathcal{F}, \mathcal{G}$   $\mathbb{Z}/m\mathbb{Z}$ -étale local systems

galois theory

$$* \quad \text{Bun} = \coprod_{\alpha \in \pi_1(G)} \text{Bun}^\alpha$$

open/closed subsch.

Conjecturally  $\text{Bun}^\alpha$  connected

# H.N. stratification

G quasi-split  
TCB



$$B(G) \longrightarrow X_* (\pi)_Q^+ = \text{Hom} \left( \mathbb{D}, G_{\mathbb{E}} \right) / G_{\mathbb{E}} \text{-conj.}$$

$\uparrow$  slope pro-torus  $X_*(\mathbb{D}) = Q$

$$[b] \longmapsto [Y_b]$$

$\uparrow$  slope morphism gives Dieudonné-Manin slope decomposition

$b$  basic  $\iff E_b$  is semi-stable as a  $G$ -bundle

$\uparrow$  v.b. central generalization of isoclinic for  $GL_n$

(Atiyah-Bott generalization of H.N. reduction theory to  $G$ -bundles)

$\downarrow$   
 $E$   $G$ -bundle /  $X_F$  is semi-stable if  $\text{ad } E$  is semi-stable

$\longleftarrow$   
 Adjoint v.b. via  
 $\text{Ad}: G \rightarrow GL(\text{Lie } G)$

$$\text{HN: } |\text{Bun}| \longrightarrow X_* (\pi)_Q^+$$

$\cong$   
 $B(G)$

$\cong$   
 $\text{HN}(\text{Lie } G)$

$\text{HN} = \text{w. } (-\gamma)$

$\uparrow$  biggest length element in Weyl group

$\rightarrow (\mathbb{D}, \varphi)$  isoclinic slope  $\lambda \Rightarrow E(\mathbb{D}, \varphi)$  semi-stable slope  $-\lambda$   
 $\cong O(-\lambda)^m$

$HN$  is semi-continuous on  $|Bun|$ . (order on the  $\geq 0$  Weyl chamber)

In particular  $Bun^{\text{ss}} \subset Bun$  is open

\* Kottwitz:  $X_{|B(G)|_{\text{basic}}} : B(G)_{\text{basic}} \xrightarrow{\sim} \pi_1(G)_{\mathbb{F}}$

Geometric translation:  $\forall \alpha \in \pi_1(G)_{\mathbb{F}} \quad |Bun^{\alpha, \text{ss}}| = \text{one point}$

Rem: For other  $\mathcal{O} \in X_{\#}(T)_{\alpha}^+$   $|Bun^{\alpha, HN=\mathcal{O}}| = \begin{cases} \emptyset \\ \text{or one point} \end{cases}$

Description of the semi-stable locus

$[h] \in B(G)$  - Suppose  $h$  basic. Then  $J_h = \sigma$ -centralizer of  $G$  is an inner form of  $G$ .

$\hookrightarrow$  extended pure inner form.

$H^1(E, G) \subset B(G)_{\text{basic}}$  is given by  $H^1_{\text{ét}}(\text{Spec } E, G) \rightarrow H^1_{\text{ét}}(X, G)$

(unit root  $G$ -isocrystals)

$\hookrightarrow$  gives rise to pure inner forms à la Vogan.

Then the sheaf  $\mathcal{P}erf_{\overline{\mathbb{F}}_q} \rightarrow \text{groups}$   
 $S \mapsto \text{Aut}(E_b/X_S)$

is  $\underline{J}_b(E) : S \mapsto \mathcal{L}(|S|, J_b(E))$   
 ↑ constant pro-finite sheaf associated to a locally profinite set

Th:  $b$  basic such that  $K(b) = \alpha \in \pi_1(G/\Gamma)$  then  $E_b$  defines an isomorphism  
 classifying stacks of pro-finite covers.

$$\boxed{[\text{Spa}(\overline{\mathbb{F}}_q) / \underline{J}_b(E)] \xrightarrow{\sim} \text{Bun}_{\overline{\mathbb{F}}_q}^{\alpha, \Delta}}$$

has  $E/X_S$  with  $S/\overline{\mathbb{F}}_q$  s.t.  $\forall \bar{s} = \text{geo. point of } S^1 \quad E|_{X_{b(\bar{s})}} \cong E_b$

Then  $\exists \tilde{S} \rightarrow S$  pro-finite covering s.t.  $E|_{X_{\tilde{S}}} \cong E_b$ .

Rem: For  $b$  non-basic  $\tilde{J}_b : S \mapsto \text{Aut}(E_b/X_S)$  is not zero dimensional:  $\pi_0(\tilde{J}_b) = \underline{J}_b(E)$  ← inner form of  $G_0(\mathbb{R})$  Levi of  $G$

$$\tilde{J}_b \cong \tilde{J}_b^{\circ} \times \underline{J}_b(E)$$

Unipotent diamond  $(B_{\text{cns}}^+)^{p=1}$  ↓ Connected B.C. space

→  $G = GL_n \quad b = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$

$$\tilde{J}_b \cong \begin{pmatrix} \mathbb{Q}_p^{\times} & H^0(\mathcal{O}(1)) \\ 0 & \mathbb{Q}_p^{\times} \end{pmatrix}$$

# Geometric structure on Bun

"Bun is a perfectoid stack" analogous notion to "algebraic stack"

2 ways to do it:

(1) Via Beauville - Laszlo uniformization of Bun by

$\mathcal{G}_m^{\text{Bar}}$  ← Scholze's Bar-affine Grassmannian

$\mathcal{G}_m \rightarrow \text{Bun} \times \text{Div}^1$

(2) Via the Quot diamond:

Th:  $\exists$  a v.b. on  $X_S$ ,  $S \in \text{Perf}_{\mathbb{F}_q}$ . Then the functor

$$\begin{array}{ccc} \text{Perf}_S & \longrightarrow & \text{Sets} \\ T/S & \longmapsto & \left\{ \begin{array}{c} \mathcal{E}|_{X_T} \longrightarrow \mathcal{F} \\ \uparrow \text{v.b. on } X_T \end{array} \right\} / \sim \end{array}$$

is representable by a diamond, the quot. diamond