Eilenberg lectures - Fall 2023

Some new geometric structures in the Langlands program

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Préface

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Chapitre 1

First lecture - Sept 12

1. The local Langlands correspondence

1.1. Notations. Fix a prime number p. We need the following datum

— E is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and uniformizer π

— We fix an algebraic closure $\overline{E}$ of E and let
$$\Gamma_E = \mathrm{Gal}(\overline{E}|E)$$
and
$$W_E \subset \Gamma_E$$
be the associated Weil group of elements of Γ_E acting as Frob_q^n for some $n \in \mathbb{Z} \subset \widehat{\mathbb{Z}}$ on the residue field.

— G is a reductive group over E

— We fix some $\ell \neq p$ and consider $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$

We let
$${}^L G = \widehat{G} \rtimes \Gamma_E$$
be *the associated L-group* over $\mathbb{Z}$ (seen as a pro-algebraic group). Here $\widehat{G}$ is a split reductive group over $\mathbb{Z}$ equipped with an action of Γ_E factorizing through an open subgroup of Γ_E.

Example 1.1. *(1) If $G = T$ is a torus then $\widehat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m$ with the Γ_E action deduced from the one on $X^*(T)$*

(2) If $G = \mathrm{GL}_{n/E}$ then $\widehat{G} = \mathrm{GL}_n$ with trivial Γ_E action

(3) If $G = \mathrm{SL}_{n/E}$ then $\widehat{G} = \mathrm{PGL}_n$ with trivial Γ_E action

(4) If $K|E$ is a quadratic extension with Galois group $\{\mathrm{Id}, *\}$, $A \in M_n(K)$ is hermitian non-degenerate, i.e. satisfies ${}^t A^* = A$ and $\det(A) \neq 0$, the associated unitary group G such that $G(E) = \{B \in \mathrm{GL}_n(K) \mid BA\,{}^t B^* = A\}$ satisfies $\widehat{G} = \mathrm{GL}_n$ with the action of Γ_E factorizing through Gal$(K|E)$ and where the non-trivial element of the Galois group acts as $g \mapsto w\,{}^t g^{-1} w$ where $w = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}$

1.2. The local Langlands correspondence.

1.2.1. *Smooth representations.* Let Λ be a $\mathbb{Z}[\frac{1}{p}]$-algebra. Recall the following definition.

DEFINITION 1.2. *A smooth representation of $G(E)$ with coefficients in Λ is a Λ-module M equipped with a linear action of $G(E)$ such that the stabilizer of any vector is open in $G(E)$. We note*

$$\mathrm{Rep}_\Lambda(G(E))$$

for the category of smooth representations with coefficients in Λ.

Let

$$\mathscr{C}(G(E),\Lambda)$$

be the Λ-module of locally constant with compact support functions on $G(E)$ with coefficients in Λ. Let

$$\mathscr{H}_\Lambda(G(E)) = \mathrm{Hom}_\Lambda(\mathscr{C}(G(E),\Lambda),\Lambda)$$

be *the Hecke convolution algebra* of distributions on $G(E)$ with coefficients in Λ that are smooth with compact support. The choice of a Haar measure μ on $G(E)$ with values in $\mathbb{Z}[\frac{1}{p}]$ defines an isomorphism

$$\begin{array}{ccc}\mathscr{C}(G(E),\Lambda) & \xrightarrow{\sim} & \mathscr{H}_\Lambda(G(E)) \\ f & \longmapsto & f\mu\end{array}$$

where the ring structure on $\mathscr{C}(G(E))$ is now given by $(f*g)(x) = \int_{G(E)} f(xy^{-1})g(y)d\mu(y)$. For each $K \subset G(E)$ an open pro-p subgroup there is associated an *idempotent*

$$e_K \in \mathscr{H}_\Lambda(G(E))$$

given by $\langle e_K, \varphi\rangle = \int_K \varphi$ where, in this formula, the integration on K is with respect to the Haar measure with volume 1. In other words, $e_K = \frac{1}{\mu(K)}\mathbf{1}_K \in \mathscr{C}(G(E))$ via the preceding identification. Then, one has $e_K * e_{K'} = e_K$ if $K \subset K'$ and

$$\mathscr{H}_\Lambda(G(E)) = \bigcup_K \underbrace{e_K * \mathscr{H}(G(E),\Lambda) * e_K}_{\mathscr{H}_\Lambda(K\backslash G(E)/K)}$$

where $\mathscr{H}(K\backslash G(E)/K)$ is the Hecke algebra of K-bi-invariant distributions on $G(E)$ with compact support.

To any $\pi \in \mathrm{Rep}_\Lambda(G(E))$ with associated Λ-module M_π, one can associate a module over $\mathscr{H}(G(E),\Lambda)$ by setting for $m \in M_\pi$ and $T \in \mathscr{H}(G(E),\Lambda)$,

$$T.m = \int_{G(E)} \pi(g).m \; dT(g).$$

One then has

$$e_K.M_\pi = M_\pi^K$$

as an $\mathscr{H}(K\backslash G(E)/K,\Lambda)$-module. This induces an equivalence

$$\{\text{smooth rep. of } G(E) \text{ wt. coeff. in } \Lambda\} \xrightarrow{\sim} \{\mathscr{H}_\Lambda(G(E))\text{-modules } M \text{ s.t. } M = \cup_K e_K.M\}.$$

One verifies that if Λ is a field and K is compact open with order invertible in Λ this induces an equivalence

$$\{\pi \in \mathrm{Rep}_\Lambda(G(E)) \text{ irreducible s.t. } \pi^K \neq 0\} \xrightarrow{\sim} \{\text{irreducible } \mathscr{H}_\Lambda(K\backslash G(E)/K)\text{-modules }\}.$$

1.2.2. *Langlands parameters.* The local Langlands correspondence seeks to attach to any *irreducible* $\pi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ a *Langlands parameter*

$$\varphi_\pi : W_E \longrightarrow {}^LG(\overline{\mathbb{Q}}_\ell).$$

Here the terminology "Langlands parameter" means

— that the composite of φ_π with the projection to Γ_E is the canonical inclusion $W_E \subset \Gamma_E$ i.e. φ_π is given by a 1-cocycle $W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$,

— that moreover this cocycle takes values in $\widehat{G}(L)$ where L is a finite degree extension of $\mathbb{Q}_\ell$,

— that this cocycle with values in $\widehat{G}(L)$ is continuous.

Remark 1.3. *There's a way to make this notion of Langlands parameter independent of the choice of the ℓ-adic topology. In fact, Grothendieck's ℓ-adic monodromy theorem ("any ℓ-adic representation is potentially semi-stable") applies in this context and a Langlands parameter $\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ as before is in fact the same as a couple (ρ, N) where*

— *$\rho : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is a Langlands parameter that is trivial on an open sub-group of W_E,*

— *$N \in \mathfrak{g}_{\overline{\mathbb{Q}}_\ell}(-1)$ is nilpotent and satisfies : $\forall \tau \in W_E,\ \mathrm{Ad}\,\rho(\tau).N = q^{v(\tau)}N$ where τ acts as $\mathrm{Frob}_q^{v(\tau)}$ on the residue field.*

The couples

$$(\rho, N)$$

are the so-called **Weil-Deligne parameters**. *There is a 1-cocycle $t_\ell : W_E \to \mathbb{Z}_\ell(1)$ sending τ to $\big(\tau(\pi^{1/\ell^n})/\pi^{1/\ell^n}\big)_{n\geq 1}$. The correspondence sends (ρ, N) to the parameter φ such that for $\tau \in W_E$,*

$$\varphi(\tau) = \rho(\tau)\exp(t_\ell(\tau)N) \rtimes \tau.$$

Nevertheless, since we fix a prime number ℓ in our work with Scholze we prefer to give a formulation using the ℓ-adic topology. This is justified by the fact that we construct such parameters over $\overline{\mathbb{F}}_\ell$ too and *our correspondence is compatible with mod ℓ reduction.*

One last remark : φ_π is only defined up to $\widehat{G}(\mathbb{Q}_\ell)$-conjugation i.e. we see it as an element of $H^1(W_E, \widehat{G}(\overline{\mathbb{Q}}_\ell))$. Up to now the local Langlands correspondence is a map

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G(E))/\sim\ \longrightarrow\ \{\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)$$

i.e. a map between isomorphism classes of object. We will later see *this correspondence has some categorical flavors* (and this is quite important since at the end we formulate a real categorical local Langlands correspondence with Scholze) but up to now we deal with objects up to isomorphisms

1.2.3. *What to expect from the local Langlands correspondence.* Here is what we expect from the local Langlands correspondence.

(1) **Frobenius semi-simplicity** First, there is one condition on φ_π : this has to be *Frobenius semi-simple* in the sense that the associated couple (ρ, N) has to be such that for all τ, $\rho(\tau)$ is semi-simple (i.e. $\rho(\tau)$ is semi-simple for a τ satisfying $v(\tau) = 1$).

(2) **Finiteness of the L-packets** The fibers of $\{\pi\} \mapsto \{\varphi_\pi\}$ are finite : those are the so-called L-packets

(3) **Description of the image** When G is quasi-split the correspondence $\{\pi\} \mapsto \{\varphi_\pi\}$ should be surjective. For other G, there is so-called relevance condition so that a parameter $\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is isomorphic to some φ_π if and only if as soon as φ factorizes (up to $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy) through some parabolic subgroup ${}^LP(\overline{\mathbb{Q}}_\ell)$ where P is a parabolic subgroup of G^* then P transfers to G.

   For example : if $G = D^\times$ where D is a central division algebra over E with $[D : E] = n^2$ then a Langlands parameter $\varphi : W_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) = \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is relevant if and only if φ, as a linear representation of W_E, is indecomposable.

(4) **Compatibility with local class field theory** If $G = T$ is a torus class field theory gives an isomorphism of groups
$$\mathrm{Hom}(T(E), \overline{\mathbb{Q}}_\ell^\times) \xrightarrow{\sim} H^1(W_E, {}^LT(\overline{\mathbb{Q}}_\ell))$$
   this has to be the cloal Langlands correspondence for tori. Typically, when T is a spli torus, there is an Artin reciprocity isomorphism
$$T(E) \xrightarrow{\sim} W_E^{ab} \otimes_{\mathbb{Z}} X_*(T)$$
   deduced from
$$\mathrm{Art}_E : E^\times \xrightarrow{\sim} W_E^{ab},$$
   and this isomorphism induces the local Langlands correspondence for T.

(5) **Compatibility with the unramified local Langlands correspondence (Satake isomorphism)** If G is unramified, K is hyperspecial, after the choice of a square root of q in $\overline{\mathbb{Q}}_\ell$, there is a *Satake isomorphism* given by a *constant term map*
$$\mathscr{H}(K\backslash G(E)/K) \xrightarrow{\sim} \mathscr{H}(T(\mathcal{O}_E)\backslash T(E)/T(\mathcal{O}_E))^W$$
   where T is an unramified torus coming from an integral model associated to the choice of K. If $A \subset T$ is the maximal split torus inside T then
$$\mathscr{H}(T(\mathcal{O}_E)\backslash T(E)/T(\mathcal{O}_E))^W = \mathscr{H}(A(\mathcal{O}_E)\backslash A(E)/A(\mathcal{O}_E))^W$$
   that is identified with
$$\overline{\mathbb{Q}}_\ell[X_*(A)]^W = \overline{\mathbb{Q}}_\ell\big[X^*(\widehat{A})\big]^W.$$
   If π is such that $\pi^K \neq 0$ then the irreducible module π^K over the spherical Hecke algebra thus defines a character
$$\overline{\mathbb{Q}}_\ell\big[X^*(\widehat{A})\big]^W \longrightarrow \overline{\mathbb{Q}}_\ell$$

that is to say an element of $\widehat{A}(\overline{\mathbb{Q}}_\ell)/W$. One can prove that this is the same as an element of

$$\{\text{unramified (semi-simple) } \varphi : W_E/I_E \to {}^LG(\overline{\mathbb{Q}}_\ell)\} \,/\, \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

(6) **Compatibility with Kazhdan-Lusztig depth 0 local Langlands**

If G is split and I is an Iwahori subgroup of $G(E)$ then the category

$$\mathrm{Rep}^I_{\overline{\mathbb{Q}}_\ell}(G(E))$$

of $\pi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ generated by π^I form a block in $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ in the sense that there is an indecomposable idempotent e in the Bernstein center of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ such that

$$e.\,\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E)) = \mathrm{Rep}^I_{\overline{\mathbb{Q}}_\ell}(G(E)).$$

This is the so-called central block. This category is then identified with the category of modules over the Iwahori-Hecke algebra

$$\mathscr{H}(I\backslash G(E)/I).$$

The identification of this Iwahori-Hecke algebra with the equivariant K-theory of the Steinberg variety has allowed Kazhdan and Lusztig to give a parametrization of irreducible $\mathscr{H}(I\backslash G(E)/I)$-modules as couples (s, N) where $s \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is semi-simple and $N \in \widehat{\mathfrak{g}}_{\overline{\mathbb{Q}}_\ell}$ is nilpotent and satisfies $\mathrm{Ad}(s).N = qN$. We ask that this is the local Langlands correspondence in this case.

(7) **Compatibility up to semi-simplification with parabolic induction**

We say a parameter φ is semi-simple if the associated Weil-Deligne Langlands parameter (ρ, N) is such that $N = 0$. Equivalently, $\varphi_{|I_E}$ is trivial on an open subgroup. For a parameter φ we can define φ^{ss} its semi-simplification. Then, if P is a parabolic subgroup with Levi subgroup M we ask the following : for π an irreducible smooth representation of $M(E)$, if π' is an irreducible subquotient of the finite length representation

$$\mathrm{Ind}_{P(E)}^{G(E)}\pi$$

(normalized parabolic induction), then

$$\varphi^{ss}_{\pi'}$$

is the composite of φ^{ss}_π with the inclusion ${}^LM(\overline{\mathbb{Q}}_\ell) \hookrightarrow {}^LG(\overline{\mathbb{Q}}_\ell)$.

Let us remark that, of course, this is false without the semi-simplification since the Steinberg representation of $\mathrm{GL}_n(E)$ and the trivial one do not have the same Langlands parameters.

(8) **Categorical flavor : description of supercuspdial L-packets**

We are now introducing some categorical flavor inside the Langlands parameters : we are not looking at the set quotient

$$\{\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)$$

but the quotient as a groupoid

$$\big[\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)\big],$$

and thus

$$\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell) = \pi_0\big[\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)\big].$$

Suppose G is quasi-split (we will see later, following the work of Vogan, Kottwitz and Kaletha what to do in the non-quasi-split case). For a parameter φ we define

$$S_\varphi = \{g \in \widehat{G}(\overline{\mathbb{Q}}_\ell) \mid g\varphi g^{-1} = \varphi\}.$$

This is the automorphism group of φ in the preceding groupoid. There is always an inclusion

$$Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E} \subset S_\varphi.$$

We say that φ is cuspidal if it is semi-simple and $S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}$ is finite. We say a packet is supercuspidal if all of its elements are supercuspidal. Then

$$\{\text{supercuspidal L-packets}\} \xrightarrow{\sim} \{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell) \text{ cuspidal }\}/\widehat{G}(\overline{\mathbb{Q}}_\ell).$$

Moreover, the choice of a Whittaker datum defines a bijection for φ a cuspidal parameter

$$\mathrm{Irr}(\underbrace{S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}}_{\text{finite group}}) \xrightarrow{\sim} \text{L-packet associated to } \varphi$$

where the trivial representation should correspond to the unique generic (with respect to the choice of the Whittaker datum) representation of the L-packet.

(9) **Local global compatibility**

Let K be a number field and Π be an algebraic automorphic representation of G where now G is a reductive group over K. Conjecturally, Π_f is defined over a number field as a smooth representation of $G(\mathbb{A}_f)$. Let us fix an embedding of this number field inside $\overline{\mathbb{Q}}_\ell$. Then one should be able to attach to Π an ℓ-adic Langlands parameter

$$\varphi_\Pi : \mathrm{Gal}(\overline{K}|K) \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell).$$

For a place v of K dividing $p \neq \ell$,

$$\varphi_{\Pi\,|\,W_{K_v}}$$

depends only on Π_v and is given up to conjugation by

$$\varphi_{\Pi_v}.$$

2. Background on the global Langlands correspondence and global Langlands parameters

Let G be a reductive group over a number field K. Let Π be an automorphic representation of G i.e. an irreducible sub-quotient of the space of automorphic forms on G. As an abstract representation

$$\Pi \simeq \bigotimes_v \Pi_v$$

where v goes through the places of K. If $v|\infty$, the local Langlands correspondence is known for Π_v is known and we can define

$$\varphi_{\Pi_v} : W_{K_v} \longrightarrow^L G_{\mathbb{C}}.$$

There is a natural morphism

$$\mathbb{C}^\times \longrightarrow W_{K_v}$$

that is an isomorphism if $K_v \simeq \mathbb{C}$ and fits into a non-split exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{K_v} \longrightarrow \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \longrightarrow 1$$

if $K_v \simeq \mathbb{R}$.

Definition 2.1. *An automorphic representation Π of G is algebraic if for all $v|\infty$, $\varphi_{\Pi_v|\mathbb{C}^\times} : \mathbb{C}^\times \longrightarrow \widehat{G}(\mathbb{C})$ is algebraic i.e. is given by an algebraic morphism $\mathbb{S}_{\mathbb{C}} \to \widehat{G}_{\mathbb{C}}$ where $\mathbb{S}$ is Deligne's torus $\mathrm{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m$ via the inclusion $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$.*

It is the same as to ask that for all $v|\infty$, Π_v has the same infinitesimal character as the one of an algebraic irreducible finite dimension representation of the algebraic group $G_{\overline{K_v}}$ with coefficients in $\mathbb{C}$.

Conjecturally there exists a global Langlands group

$$\mathscr{L}_K$$

that is a locally compact topological group sitting in an exact sequence

$$1 \longrightarrow \mathscr{L}_K^\circ \longrightarrow \mathscr{L}_K \longrightarrow \mathrm{Gal}(\overline{K}|K) \longrightarrow 1$$

and with an identification

$$\mathscr{L}_K/(\mathscr{L}_K^\circ)' = W_K$$

the global Weil group. Moreover, on expects the following.

CONJECTURE 2.2. *The following is expected :*

(1) *To each automorphic representation Π of G one can associate a Langlands parameter*
$$\varphi_\Pi : \mathscr{L}_K \longrightarrow {}^L G_{\mathbb{C}}$$
compatibly with the local Langlands correspondence at archimedean places and the unramified one at almost all finite places

(2) *If Π is algebraic then Π_f is defined over a number field inside $\mathbb{C}$ and to the choice of an embedding of such a number field inside $\overline{\mathbb{Q}}_\ell$ is associated an ℓ-adic Langlands parameter*
$$\varphi_{\Pi,\ell} : Gal(\overline{K}|K) \longrightarrow {}^L G_{\overline{\mathbb{Q}}_\ell}$$

(3) *The Tannakian category of continuous representations of $\mathscr{L}_K$ on finite dimensional $\mathbb{C}$-vector spaces that are algebraic is identified with the category of Grothendieck motives for numerical equivalence with $\mathbb{C}$ coefficients.*

This is known for tori when we consider the category of CM-motives for absolute Hodge cycles.

The construction of the ℓ-adic Langlands parameters is know for cohomological automorphic representations of GL_n. Other cases are known using the cohomology of Shimura varieties.

For example, if $f = \sum_{n\geq 1} a_n q^n$ is a normalized weight $k \geq 1$ holomorphic modular form for $\Gamma_0(N)$ that is new and an Hecke eigenvector of the Hecke operators $(T_p)_{p\not| N}$ then one can associate (Shimura, Deligne, Deligne-Serre) a Galois representation
$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$
such that for $p \not| N$, that characteristic polynomial of $\rho_f(\mathrm{Frob}_p)$ is $X^2 - a_p X + p^{k-1}$.

3. What we do with Scholze

We prove the following theorem.

THEOREM 3.1 (F.-Scholze). *For ℓ a good prime with respect to G (any ℓ if $G = \mathrm{GL}_n$, $\ell \neq 2$ for classical groups) there exists a monoidal action of the category of perfect complexes*
$$\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})$$
on
$$D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$$
where $\mathrm{LocSys}_{\widehat{G}} \to Spec(\mathbb{Z}\big[\frac{1}{p}\big])$ is the moduli space of Langlands parameter, an algebraic stack locally complete intersection of dimension 0 over $Spec(\mathbb{Z}\big[\frac{1}{p}\big])$.

As a consequence we can construct the semi-simple local Langlands correspondence
$$\pi \mapsto \varphi_\pi^{ss}$$

for any reductive group over E, over $\overline{\mathbb{F}}_\ell$ and $\overline{\mathbb{Q}}_\ell$ (and compatibly with mod ℓ reduction).

As for now the statement of the local Langlands conjecture is the following.

CONJECTURE 3.2 (Categorical local Langlands). *Suppose G is quasi-split and fix a Whittaker datum (B,ψ). Suppose ℓ is a good prime. There exists an equivalence of stable ∞-categories*

$$\mathcal{D}^b_{coh}(\mathrm{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})_{nilp.ss.supp} \xrightarrow{\sim} \mathcal{D}_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega$$

compatible with the preceding spectral action and sending the structural sheaf $\mathcal{O}$ to the Whittaker sheaf.

The goal of those lectures is to explain how after 20 years of work, starting from the classical local Langlands correspondence in terms of parameters of smooth irreducible representations as in the work of Harris-Taylor, we arrived at such a statement and what are those geometric objects showing up in the preceding statement, starting with the so-called Lubin-Tate spaces continuing with Rapoport-Zink spaces, Hodge-Tate periods, the curve and so on.

Chapitre 2

Second lecture - Sept 19

Problem : construct the local Langlands correspondence for a given group using local-global compatibility + some known cases of the global construction of ℓ-adic parameters via the cohomology of Shimura varieties.

More precisely, if $\Pi \simeq \bigotimes_v \Pi_v$ is a cohomological automorphic representation of G defined over a number field $\mathbb{Q}$ and

$$\Pi \longmapsto r_\mu \circ \varphi_{\Pi|\mathrm{Gal}(\overline{\mathbb{Q}}|L)}$$

via the cohomology of Shimura varieties where

- $\varphi_\Pi : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is the expected global ℓ-adic parameter,
- L is the reflex field associated to the Shimura variety, a number field inside $\mathbb{C}$,
- $r_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|L))$ is an algebraic representation associated to our Shimura datum

one expects that for $p \neq \ell$,

$$\varphi_{\Pi|W_{\mathbb{Q}_p}} = \varphi_{\Pi_p}$$

and thus, if $v|p$ is a place of L associated to the choice of an embedding f L inside $\overline{\mathbb{Q}}_p$,

$$\boxed{r_\mu \circ \varphi_{\Pi|W_{L_v}} = r_\mu \circ \varphi_{\Pi_p|W_{L_v}}}$$

Remark 0.1. *(1) By definition, a cohomological automorphic representation is a particular type of algebraic automorphic representation that shows up in the cohomology of locally symmetric spaces. For example, for* GL_2*, the automorphic representation associated to an holomorphic modular form of weight* $k \geq 1$ *is algebraic but cohomological only when* $k \geq 2$*. The* ℓ*-adic Langlands parameter associated to a weight* ≥ 2 *holomorphic modular forms is obtained inside the intersection cohomology cohomology of modular curves with coefficients in some local systems (Shimura, Deligne).*

For weight 1 *holomorphic modular forms this* ℓ*-adic Langlands parameter is obtained by* ℓ*-adic interpolation from the weight* ≥ 2 *case (Deligne-Serre).*

There is another class of automorphic representation of $\mathrm{GL}_{2/\mathbb{Q}}$ *that are algebraic but not cohomological : the one associated to non-holomorphic Maass forms* f *that satisfy* $\Delta f = \frac{1}{4} f$ *where* $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ *is the hyperbolic Laplacian. We do not know how to construct their* ℓ*-adic Langlands parameter.*

(2) Suppose that $G_{\mathbb{R}}$ *has discrete series, that is to say* $G_{ad,\mathbb{R}}$ *is an inner form of its compact form. This is for example the case if* G *can be enhanced to a Shimura datum. One can prove that one can globalize any supercuspdial representation of* $G(\mathbb{Q}_p)$ *to*

an automorphic representation Π such that Π_∞ is a discrete series representation. Those are cohomological and show up in middle degree in the cohomology of locally symmetric spaces.

(3) *One of the difficulties of the preceding approach is that we can not construct φ_Π but its composition with r_μ where r_μ is a very particular type representation of the Laglands dual since μ is minuscule. This difficulty is removed over function fields over $\mathbb{F}_q$ using general Shtuka moduli spaces but we don't know, even for GL_2, how to define Shimura varieties for non-minuscule μ. We will see later how to remove this difficulty for local Shimura varieties at p.*

We would like to use this type of formula to define φ_{Π_v} after choosing suitable Shimura data giving rise to different representations r_μ, This leads to the question : why, after composing with r_μ, would $\varphi_{\Pi|W_{K_v}}$ depend only on Π_v ? This is the problem of **local-global compatibility**. The answer is that there are **local Shimura varieties** linked to the global one via a process of p-adic uniformization.

1. Shimura varieties

1.1. Hermitian symmetric spaces. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}}\,\mathbb{G}_m$ be Deligne's torus. Recall the the Tannakian category of real Hodge structures is equivalent to $\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})$.

Datum : a couple

$$(G, \{h\})$$

where

(1) G reductive group over $\mathbb{R}$.

(2) $h : \mathbb{S} \to G$ with $G(\mathbb{R})$-conjugacy class $\{h\}$

This is the same as the datum of G together with a $\otimes$-functor

$$\mathrm{Rep}(G) \longrightarrow \mathbb{R}\text{-Hodge structures},$$

i.e. *a G-$\mathbb{R}$-Hodge structure* , such that the composite

$$\mathrm{Rep}(G) \to \mathbb{R}\text{-Hodge structures} \xrightarrow{can} \mathrm{Vect}_{\mathbb{R}}$$

is isomorphic to the canonical fiber functor on $\mathrm{Rep}\,G$.

We note $\mu_h : \mathbb{G}_{m/\mathbb{C}} \to G_{\mathbb{C}}$ for the composite of $h_{\mathbb{C}}$ with $z \mapsto (z,1)$ from $\mathbb{G}_{m/\mathbb{C}} \to \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m/\mathbb{C}} \times \mathbb{G}_{m/\mathbb{C}}$. This defines the Hodge filtration.

Hypothesis :

(1) (**Weight** 0 **adjoint Hodge structure**) $w_h : \mathbb{G}_m \to G$, obtained by composing h with the morphism $\mathbb{G}_m \to \mathbb{S}$ inducing $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on the $\mathbb{R}$-points, is central that is to say the Hodge structure $(\mathfrak{g}, \mathrm{Ad} \circ h)$ is pure of weight 0.

(2) (**Polarization**) Conjugation by $h(i)$ is a Cartan involution on G_{ad} that is to say the Killing form on $\mathfrak{g}_{ad}$ defines a polarization of the weight 0 Hodge structure $(\mathfrak{g}_{ad}, \mathrm{Ad} \circ h)$

(3) (**Griffiths transversality**) $\mu_h : \mathbb{G}_{m/\mathbb{C}} \to G_{\mathbb{C}}$ is minuscule that is to say the weights of $\mathrm{Ad} \circ \mu_h$ on $\mathfrak{g}_{\mathbb{C}}$ are in $\{-1, 0, 1\}$ that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \mathrm{Ad} \circ h)$ is of type $(-1,1), (1,-1), (0,0)$

Under those hypothesis, if K_∞ is the centralizer of $h(i)$ in $G(\mathbb{R})$, a sub-group of $G(\mathbb{R})$ that is compact modulo the center,

$$X = G(\mathbb{R})/K_\infty.$$

More precisely, if $\mathcal{F}$ is the flag manifold defined by μ_h, the map

$$X \longrightarrow \mathcal{F}$$

that sends some h' that is $G(\mathbb{R})$-conjugate to h to the class of $\mu_{h'}$ is an open embedding,

$$X \underset{\text{open}}{\subset} \mathcal{F}.$$

Then, X is a moduli space of rigidified variation of Hodge structures equipped with a G-structure.

More precisely, if S is a smooth complex analytic space then $X(S)$ is the set of equivalence classes of $(\mathscr{F}, \mathrm{Fil}^\bullet \mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S, \eta)$ where

— $\mathscr{F} : \mathrm{Rep}\, G \to \{\mathbb{R} \text{ local systems on S}\}$ is a $\otimes$-functor,

— $\mathrm{Fil}^\bullet \mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S$ is a finite decreasing filtration of the $\otimes$-functor

$$\mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S : \mathrm{Rep}\, G \to \{\text{vector bundles on } S\}$$

satisfying Griffiths transversality : if $\nabla = \mathrm{Id} \otimes d$ then $\nabla \mathrm{Fil}^k \subset \mathrm{Fil}^{k-1} \otimes \Omega^1_S$

— for each $\mathbb{R}$-linear representation (V, ρ) of G and $s \in S$, the complex conjugate of the associated filtration of $V_{\mathbb{C}}$ is $\rho \circ w_h$-opposite to the filtration of $V_{\mathbb{C}}$ and thus defines a weight $\rho \circ w_h$ Hodge structure,

— η is an isomorphism between tensor functors between $\mathscr{F}$ and the canonical functor $(V, \rho) \mapsto \underline{V}$,

— We ask that for each $s \in S$, the associated morphism $\mathbb{S} \to G$ defined by taking the stalk at s of the preceding variation is $G(\mathbb{R})$-conjugated to h.

Thus, $X =$ **moduli of Hodge structures**. We will see later that we can define moduli of p-adic Hodge structures using the curve. But we are first going to treat a particular case : **Lubin-Tate spaces**.

1.2. Shimura variety.

Datum :

(1) G is a reductive group over $\mathbb{Q}$

(2) $h : \mathbb{S} \to G_{\mathbb{R}}$

Hypothesis :

(1) (**Weight** 0 **adjoint Hodge structure**) $w_h : \mathbb{G}_m \to G_{\mathbb{R}}$, obtained by composing h with the morphism $\mathbb{G}_m \to \mathbb{S}$ inducing $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on the $\mathbb{R}$-points, is central that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \mathrm{Ad} \circ h)$ is pure of weight 0.

(2) (**Polarization**) Conjugation by $h(i)$ is a Cartan involution on $G_{\mathbb{R},ad}$ that is to say the Killing form on $\mathfrak{g}_{ad}$ defines a polarization of the weight 0 Hodge structure $(\mathfrak{g}_{\mathbb{R},ad}, \mathrm{Ad} \circ h)$

(3) (**Griffiths transversality**) μ_h is minuscule that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \mathrm{Ad} \circ h)$ is of type $(-1,1),(1,-1),(0,0)$

(4) (**Density of CM points**) For any $\mathbb{Q}$-factor H of G_{ad}, $H(\mathbb{R})$ is not compact.

EXAMPLE 1.1. *(1) $G = \mathrm{GL}_2$ and $h(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Modular curves case.*

(2) Same as before but $G = D^\times$ with D a quaternion division algebra over $\mathbb{Q}$

(3) $G = \mathrm{Gsp}_{2n}$ associated with the symplectic form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Set $h(a+ib) = \begin{pmatrix} aI_n & bI_n \\ -bI_n & aI_n \end{pmatrix}$. Siegel varieties (modular curves for $n = 1$)

(4) Let K be a CM field and B be a central simple algebra over K equipped with an involution $*$ inducing complex conjugation on K. Let $G = GU(D, *)$ be the associated similitude unitary group. Let Fix an isomorphism $G_{\mathbb{R}} \simeq G(\prod_{\tau \in \Phi} U(p_\tau, q_\tau))$ where $(p_\tau, q_\tau)_{\tau \in \Phi}$ is a set of signatures index by a CM type Φ of K. Then if $h(z) = (h_\tau(z))_{\tau \in \Pi}$ with $h_\tau(z) = \mathrm{diag}(\underbrace{z, \dots, z}_{p_\tau}, \underbrace{\overline{z}, \dots, \overline{z}}_{q_\tau})$ this defines aunitary type Shimura variety.

Shimura variety

$$\mathrm{Sh}_K = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K)$$

for $K \subset G(\mathbb{A}_f)$ compact open "sufficiently small". Writing $G(\mathbf{A}_f) = \coprod_{i \in I} G(\mathbb{Q}) g_i K$ with I finite (finiteness of the class number), one has

$$\mathrm{Sh}_K = \coprod_{i \in I} \Gamma_i \backslash X$$

where $\Gamma_i = G(\mathbb{Q}) \cap g_i K g_i^{-1}$ is an arithmetic subgroup of $G(\mathbb{R})$.

The smooth complex analytic space Sh_K has an interpretation as a moduli of variations of G-$\mathbb{Q}$-Hodge structures. Well, in fact the natural moduli space is not Sh_K but

$$\coprod_{\ker^1(\mathbb{Q}, G)} \mathrm{Sh}_K$$

a finite disjoint union of copies Sh_K. More precisely, if S is a smooth complex analytic space then $\coprod_{\ker^1(\mathbb{Q},G)} \mathrm{Sh}_K(S)$ is the set of equivalence classes of $(\mathscr{F}, \mathrm{Fil}^\bullet \otimes_{\mathbb{Q}} \mathcal{O}_S, \overline{\eta})$ where

— $\mathscr{F} : \mathrm{Rep}\ G \to \{\mathbb{Q} - \text{local systems on } S\}$ is a $\otimes$-functor,

— $\mathrm{Fil}^\bullet \mathscr{F} \otimes_{\mathbb{Q}} \mathcal{O}_S$ is a finite filtration of the $\otimes$-functor

$$\mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S : \mathrm{Rep}\ G \to \{\text{vector bundles on } S\}$$

satisfying Griffiths transversality : if $\nabla = \mathrm{Id} \otimes d$ then $\nabla \mathrm{Fil}^k \subset \mathrm{Fil}^{k-1} \otimes \Omega^1_S$,

— for each $\mathbb{R}$-linear representation (V, ρ) of G and $s \in S$, the complex conjugate of the associated filtration of $V_{\mathbb{C}}$ is $\rho \circ w_h$-opposite to the filtration of $V_{\mathbb{C}}$ and thus defines a weight $\rho \circ w_h$ Hodge structure,

— for each $s \in S$, the associated functor $\mathrm{Rep}\, G_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{R}}$ obtained by taking the stalk at s is trivial and the associated $G_{\mathbb{R}}$-Hodge structure is in the $G(\mathbb{R})$-conjugacy class of h,

— $\bar{\eta}$ is a K^p-orbit of trivialization $\eta : \mathrm{can} \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} \mathscr{F} \otimes_{\mathbb{Q}} \mathbb{A}_f$.

Recall the following. We note L for the reflex field of the Shimura datum (G, X). This is the field of definition of the conjugacy class of μ_h.

Theorem 1.2. *The tower of complex analytic spaces $(\mathrm{Sh}_K)_K$ is a tower of smooth quasi-projective algebraic varieties defined over L. When G is anisotropic modulo its center those are projective smooth algebraic varieties over L.*

Algebraicity as a $\mathbb{C}$-analytic space is due due Baily and Borel where they prove that if one adds a boundary to X by forming X^*, a generalization of $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, whose boundary components are parametrized by conjugacy classes of maximal parabolic subgroups of G, equipped with the so-called Satake topology then $\Gamma_i \backslash X^*$ is a compact normal $\mathbb{C}$-analytic space. The quasi-projectivity assertion is then done by proving that the dualizing sheaf ω on those spaces is ample. This is donne via the construction of Eisentein-Poincaré series that are automorphic forms sections of $\omega^{\otimes n}$ for $n \gg 0$. The co-compact case, i.e. when G is anisotropic modulo its center, was done before by Cartan and is much more simple via the construction of Poincaré series and the realization of X as a bounded domain.

The descent datum from $\mathbb{C}$ to L is first constructed on CM-points via the theory of Shimura and Taniyama and the proof that it extends to an effective descent datum to the entire Shimura variety is “easy” in the Hodge type and more generally abelian type case and delicate, essentially due to Deligne, in the general case.

This is equipped with an action of $G(\mathbb{A}_f)$ when K varies. We can look at

$$\varinjlim_K H^\bullet_{\text{ét}}(\mathrm{Sh}_K \otimes_L \overline{L}, \overline{\mathbb{Q}}_\ell)$$

as a smooth representation of $G(\mathbb{A}_f)$ equipped with a continuous commuting action of $\mathrm{Gal}(\overline{L}|L)$.

Let us now recall the following.

THEOREM 1.3 (Mastushima, Borel, Franke). *For G a reductive group over $\mathbb{Q}$, $K_\infty \subset G(\mathbb{R})$ compact whose neutral connected component is the neutral connected component of a maximal compact subgroup, and $K \subset G(\mathbf{A}_f)$ compact open "sufficiently small", if*

$$X_K = G(\mathbb{Q})\backslash(G(\mathbb{R})/K_\infty A_G(\mathbb{R})^+ \times G(\mathbf{A}_f)/K)$$

as a locally symmetric space, where A_G is the maximal split torus in Z_G, then

(1) If G is anisotropic modulo its center then, as a module over the Hecke algebra $\mathscr{H}(K\backslash G(\mathbb{A}_f/K)$,

$$H^\bullet(X_K,\mathbb{C}) = \bigoplus_{\Pi} m_\Pi . \dim_{\mathbb{C}} H^\bullet(\mathfrak{g}_\infty, K_\infty; \Pi_\infty).\Pi_f^K$$

where
- *Π goes through the set of automorphic representations of G with trivial central character when restricted to $A_G(\mathbb{R})^+$,*
- *m_Π is the multiplicity of Π in the space of automorphic forms,*
- *$H^\bullet(\mathfrak{g}_\infty, K_\infty, \Pi_\infty)$ is a finite dimensional cohomology $\mathbb{C}$-vector space associated to Π_∞.*

In particular this cohomology space is semi-simple as a module over the Hecke algebra $\mathscr{H}(K\backslash G(\mathbb{A}_f)/K)$.

(2) For any G, any constituent of $H^\bullet(X_K,\mathbb{C})$ as a module over the Hecke algebra is automorphic in the sense that is is isomorphic to Π_f^K where Π is a cohomological automorphic representation of G.

2. Harris-Taylor Shimura varieties

2.1. Generic fiber. Let E be a given p-adic field. We are looking to define the local Langlands correspondence for $G = \mathrm{GL}_{n/E}$.

Harris and Taylor have exhibited some PEL-type Shimura datum (G, X) such that

$$G_{\mathbb{R}} \simeq G(U(1,n-1)\times U(n)\times\cdots\times U(n))$$

and

$$G_{\mathbb{Q}_p} \simeq \mathrm{GL}_{n/E}\times\mathbb{G}_m.$$

Moreover, one has

$$\widehat{G} = \mathrm{GL}_n\times\mathrm{GL}_n\times\cdots\times\mathrm{GL}_n\times\mathbb{G}_m$$

with r_μ the standard representation of dimension n on the first GL_n-factor, trivial on the other GL_n factors and all of this is twisted by the standard representation of $\mathbb{G}_m$. We can moreover suppose that G is anisotropic modulo its center.

In fact, G is a similitude unitary group attached to to a division algebra over a CM field equipped with an involution inducing complex conjugation on the CM field.

We get

$$\begin{tikzcd} \mathrm{Sh}_K \arrow[d] \\ \mathrm{Spec}(L) \end{tikzcd}$$

a **proper smooth algebraic variety that is in fact a moduli of abelian varieties equipped with additional structures** like a polarization and an action of a division algebra. Set $L_v = E$ our p-adic field where v is a place of L dividing p.

We are going to analyze the cohomology of $(\mathrm{Sh}_K)_K \otimes_L L_v$ by making a degeneration from $p \neq 0$ to $p = 0$.

2.2. Integral models. If $K_p \subset G(E)$ is compact hyperspecial, $K_p = \mathrm{GL}_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$ ("minimal level at p"), then $\mathrm{Sh}_{K_pK^p}$ degenerates smoothly for any K^p compact open inside $G(\mathbb{A}_f^p)$ there exists a smooth projective model

$$\begin{tikzcd} S_{K^p} \arrow[r, hook] \arrow[d] & \mathrm{Sh}_{K_pK^p} \otimes_L L_v \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_E) \arrow[r, hook] & \mathrm{Spec}(E) \end{tikzcd}$$

with $S_{K^p} \otimes_{\mathcal{O}_E} E = \mathrm{Sh}_{K_pK^p} \otimes_L L_v$. This is a moduli space of abelian schemes with additional structures.

Main point Let

$$\begin{tikzcd} \mathcal{A} \arrow[d] \\ S_{K^p} \end{tikzcd}$$

be the universal abelian scheme. The fact is that the p-divisible group $\mathcal{A}[p^\infty]$ splits as

$$\mathcal{A}[p^\infty] = \mathcal{G} \oplus \mathcal{G}^D$$

where $\mathcal{G}$ is equipped, as an extra additional structure, with an action of $M_n(\mathcal{O}_E)$. The additional structure that is the polarization on $\mathcal{A}[p^\infty]$ is the canonical polarization on $\mathcal{G} \oplus \mathcal{G}^D$. Let $e = \begin{pmatrix} 1 & \\ & \end{pmatrix}$ as an idempotent of $M_n(\mathcal{O}_E)$. Then (Morita equivalence), a p-divisible group such as $\mathcal{G}$ equipped with an action of $M_n(\mathcal{O}_E)$ is the same as a p-divisible group equipped with an action of $\mathcal{O}_E$,

$$H := e.\mathcal{G}$$

in our case. The fact now is that the signature at ∞ of our unitary group

$$(1, n-1) \times (0, n) \times \cdots \times (0, n)$$

transfers at p as the condition that

(1) H is a 1-dimensional p-divisible group with an action of $\mathcal{O}_E$
(2) The action of $\mathcal{O}_E$ on $\operatorname{Lie} H$ is the canonical one via $S_{K^p} \to \mathcal{O}_E$.

We call such an object a **1-dimensional π-divisible $\mathcal{O}_E$-module**.

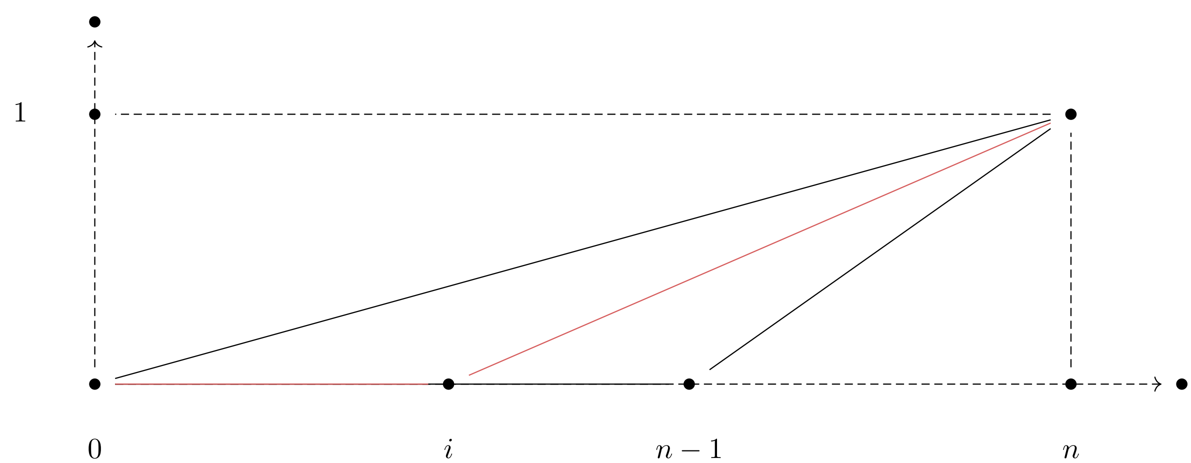
2.3. Newton stratification. Let

$$\overline{S}_{K^p} = S_{K^p} \otimes_{\mathcal{O}_E} \mathbb{F}_q$$

be the reduction modulo π of our Shimura variety. This again forms a tower of Ã©tale coverings equipped with an action of $G(\mathbb{A}_f^p)$ when K^p varies. Let

$$\begin{tikzcd} \overline{H} \arrow[d] \\ \overline{S}_{K^p} \end{tikzcd}$$

be our 1-dimensional π-divisible $\mathcal{O}_E$-modules. Geometrically fiberwise on $\overline{S}_{K^p}$ this has a Newton polygon that is of the following shape in red :



for an integer $i \in \{0, \dots, n-1\}$. In the preceding picture the Hodge polygon has slope 0 with multiplicity $n-1$ and 1 with multiplicity 1. The basic polygon has slope $1/n$. The integer i is the $\mathcal{O}_E$-height of the étale part. More precisely, there is a stratification by locally closed subsets

$$\overline{S}_{K^p}^{(i)}, \quad 0 \leq i \leq n-1$$

where a geometric point x of $\overline{S}_{K^p}$ lies in $\overline{S}_{K^p}^{(i)}$ if an only if

$$0 \longrightarrow \underbrace{\overline{H}_x^{\circ}}_{\substack{\text{1-dim.formal} \\ \text{of } \mathcal{O}_E\text{-height } n-i}} \longrightarrow \overline{H}_x \longrightarrow \underbrace{\overline{H}_x^{\text{ét}}}_{\mathcal{O}_E\text{-height } i} \longrightarrow 0.$$

(1) The closed stratum is $\overline{S}_{K^p}^{(0)}$ that is a finite set of closed points, the so-called basic locus,
(2) The open stratum is $\overline{S}_{K^p}^{(n-1)}$ that is the so-called μ-ordinary locus.

2.4. Level structures at p. We worked before with a level structure at p for which $K_p = \mathrm{GL}_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$. In this case the integral models are smooth. Drinfeld defined a "good" notion of level structures at p for the principal congruence subgroups $K_p = \mathrm{Id} + \pi^m M_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$ when $m \geq 1$. This is very particular to 1-dimensional p-divisible groups. By "good" we mean that the associated integral models

$$S_{m,K^p}$$

are **regular** and the change of level morphism

$$\begin{tikzcd} S_{m,K^p} \arrow[d, "\text{finite flat}"'] \arrow[r, equal] & \text{regular} \\ S_{K^p} \arrow[r, equal] & \text{smooth}/\mathcal{O}_E \end{tikzcd}$$

is **finite flat**. Moreover those morphisms are totally ramified over the points of the basic locus. We obtain a tower

$$(S_{m,K^p})_{m\geq 1}$$

that is equipped at the limit when $m \to +\infty$ with an action of $G(\mathbb{Q}_p)$ and commuting Hecke correspondences associated to elements of $K^p \backslash G(\mathbf{A}_f^p)/K^p$.

2.5. Analysis of the ℓ-adic cohomology at p via nearby cycles.

2.5.1. *Background on nearby cycles. Nearby cycles are a construction that allows us to analyze the cohomology of an algebraic variety via the cohomology of the special fiber of a "1-parameter degeneration" of this algebraic variety i.e. a degeneration parametrized by what we call a trait (the spectrum of a rank 1 valuation ring).*

Let

$$\begin{tikzcd} X \arrow[d] \\ \mathrm{Spec}(V) \end{tikzcd}$$

be finite presentation morphism of schemes where V is an Henselian rank 1 valuation ring. Let $K = \mathrm{Frac}(V)$ and k be the residual field of V. Fix an algebraic closure $\overline{K}$ of K and let $\overline{k}$ be the associated algebraic closure of k. We note $\mathrm{Spec}(k)$, $\overline{s} = \mathrm{Spec}(\overline{k})$, $\eta = \mathrm{Spec}(K)$ and $\bar{\eta} = \mathrm{Spec}(\overline{K})$.

There is a diagram

$$\begin{tikzcd} X_s \arrow[r, hook] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook'] \arrow[d] \\ s \arrow[r, hook] & \mathrm{Spec}(V) & \eta \arrow[l, hook'] \end{tikzcd}$$

Let $\mathscr{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ with ℓ invertible in V. We want to understand

$$H^\bullet(X_{\bar{\eta}}, \overline{\mathbb{Q}}_\ell)$$

with its $\mathrm{Gal}(\overline{K}|K)$ action in terms of the special fiber X_s of our degeneration. There is a "nearby cycle fiber functor"

$$D^b_c(X_\eta,\overline{\mathbb{Q}}_\ell) \xrightarrow{R\Psi_{\bar\eta}} \left\{\begin{matrix}\text{objects in } D^b_c(X_{\bar s},\overline{\mathbb{Q}}_\ell)\text{+action of } \mathrm{Gal}(\overline{K}|K) \\ \text{compatible with the one of } \mathrm{Gal}(\bar k|k)\end{matrix}\right\}$$

such that for any geometric point $\bar x$ of $X_{\bar s}$,

$$R\Psi_{\bar\eta}(\mathscr{F})_{\bar x} = R\Gamma\big(\underbrace{\mathrm{Spec}(\mathcal{O}^{sh}_{\overline{X},\bar x}[\tfrac{1}{\varpi}])}_{\substack{\text{schematical Milnor fiber}\\ \text{over } \bar x}}, \mathscr{F}\big)$$

where ϖ is a pseudo-uniformizer in V and $\overline{X} = X\otimes_V \overline{V}$ with $\overline{V}$ the integral closure of V in $\overline{K}$.

Remark 2.1. *The fiber at geometric points of $R\Psi_{\bar\eta}(\mathscr{F})$ is thus identified with the cohomology complex of those schematical Milnor fibers.* **Grothendieck's construction of the functor $R\Psi_{\bar\eta}$ is a way to take all those cohomology complexes of the different "classical" Milnor fibers and build a sheaf out of it**. *Deligne's theorem says that this complex has constructible cohomology and thus the cohomology of those Milnor fibers "varies constructibly".*

Proper base change then says that if $X\to \mathrm{Spec}(V)$ is proper then

$$R\Gamma(X_{\bar s}, R\Psi_{\bar\eta}(\mathscr{F})) \xrightarrow{\sim} R\Gamma(X_{\bar\eta},\mathscr{F}).$$

We will now use the following very important result that says that the nearby cycles depend only on the formal completion and not the henselelinzation. Suppose that k is perfect.

Theorem 2.2 (Berkovich,Huber). *Let x be a closed point of $X_{\bar s}$ and $\mathfrak{X}_x$ be the formal completion of $X\otimes_V V^{un}$ at x where V^{un} is the integral closure of V in the maximal unramified extension K^{un} of K. This is a formal scheme over $Spf(\widehat{V^{un}})$. Let $\mathfrak{X}^{ad}_x$ be its generic fiber as an adic space over $\mathrm{Spa}(\widehat{K^{un}})$. There is then an isomorphism*

$$\underbrace{R\Psi_{\bar\eta}(\mathscr{F})_x}_{\substack{\textit{cohomology of the}\\ \textit{schematical Milnor fiber}}} \xrightarrow{\sim} \underbrace{R\Gamma_{\text{ét}}\big(\mathfrak{X}_{x,\eta}\hat{\otimes}_{\widehat{K^{un}}}\widehat{\overline{K}},\mathscr{F}^{ad}\big)}_{\substack{\textit{cohomology of the}\\ \textit{rigid analytic Milnor fiber}}}$$

2.6. A localization phenomenon. The geometry of non-basic Newton strata implies the following result. For $m\geq 1$ we note

$$R\Psi_{\bar\eta}(\overline{\mathbb{Q}}_\ell)_{m,K^p} \in D^b_c(\overline{S}_{m,K^p}\otimes\overline{\mathbb{F}}_q,\overline{\mathbb{Q}}_\ell).$$

If $m'\geq m$ and $\Pi_{m',m}:\overline{S}_{m',K^p}\to\overline{S}_{m,K^p}$ then

$$R\Psi_{\bar\eta}(\overline{\mathbb{Q}}_\ell)_{m,K^p} = \Pi_{m',m*}R\Psi_{\bar\eta}(\overline{\mathbb{Q}}_\ell)_{m',K^p}.$$

Moreover if $\mathscr{H}_m = \mathscr{H}(G(E)//\,\mathrm{Id}+\pi^m M_n(\mathcal{O}_E))$,

$$R\Psi_{\bar\eta}(\overline{\mathbb{Q}}_\ell)_{m,K^p}$$

is equipped with an action of $\mathscr{H}_m \otimes \mathscr{H}(K^p\backslash G(\mathbf{A}_f^p)/K^p)$.

THEOREM 2.3 (Harris-Taylor). *For any $m \geq 1$,*

$$\big[R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p}\big]_{supercuspdial\ at\ p} \xrightarrow{\sim} \bigoplus_{x \in \overline{S}^{(0)}_{m,K^p}(\overline{\mathbb{F}}_q)} i_{x*}\big[R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p}\big]_{x,supercuspdial\ at\ p}$$

that is to say the supercuspidal at p-part of the complex of nearby cycles localizes on supersingular points.

3. Lubin-Tate spaces

DEFINITION 3.1. *Let $\mathbb{H}$ be a one dimensional formal p-divisible group over $\overline{\mathbb{F}}_q$ equipped with an action of $\mathcal{O}_E$ such that the action of $\mathcal{O}_E$ on Lie $\mathbb{H}$ is the canonical one. We note*

$$\mathcal{LT} \longrightarrow \mathrm{Spf}(\mathcal{O}_{\breve{E}})$$

for the deformation space of $\mathbb{H}$.

This is a formal scheme (non-canonically) isomorphic to

$$\mathrm{Spf}(\mathcal{O}_{\breve{E}}[\![x_1, \dots, x_{n-1}]\!]).$$

We note

$$\mathcal{LT}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{E}}$$

for its generic fiber as a locally of finite type adic space over $\mathrm{Spa}(\breve{E})$.

On this open ball the Tate module of the universal deformation is an $\mathcal{O}_E$–étale local system of rank n. The moduli of its trivializations defines a tower of rigid analytic spaces with finite étale transition morphisms

$$(\mathcal{LT}_{\eta,K})_{K \subset \mathrm{GL}_n(\mathcal{O}_E)} \longrightarrow \mathcal{LT}_\eta$$

equipped with an action of $\mathrm{GL}_n(E)^1$ at the limit. There is another group that shows up : the group of automorphisms by quasi-isogenies of $\mathbb{H}$, $\mathrm{End}(\mathbb{H})^\times_{\mathbb{Q}}$, that is identified with

$$D^\times$$

where D is a division algebra with invariant $\frac{1}{n}$ over E. At the end the tower $(\mathcal{LT}_{\eta,K})_K$ has a commuting action of $(D^\times \times \mathrm{GL}_n(E))^1$, the subgroup of $D \times \mathrm{GL}_n(E)$ formed by elements (d, g) such that $v(\mathrm{Nrd}(d)) + v(\det(g)) = 0$ where Nrd is the reduced norm.

In fact we prefer to work with

$$\mathcal{M}_K = \mathcal{LT}_{\eta,K} \overset{\mathcal{O}_D^\times}{\times} D^\times$$

that is a $\coprod_{\mathbb{Z}}$ of copies of the Lubin-Tate space. The tower $(\mathcal{M}_K)_K$ has an action of $D^\times \times \mathrm{GL}_n(E)$ and a (non-effective since this shifts everything by $+1$ in the components $\coprod_{\mathbb{Z}}$) descent datum $\mathcal{M}_K^{(\sigma)} \xrightarrow{\sim} \mathcal{M}_K$ from $\breve{E}$ to E. We now define

$$R\Gamma(\mathcal{M}_K \hat{\otimes}_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell) := \bigoplus_{\alpha \in \pi_0(\mathcal{M}_K)} R\Gamma(\mathcal{M}_K^\alpha \hat{\otimes}_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell).$$

This has an action of $D^\times \times W_E$ where the action of $D^\times$ is smooth and a commuting action of $\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(K \backslash \mathrm{GL}_n(E)/K)$.

Remark 3.2. *As for Harris-Taylor Shimura varieties, the notion of Drinfeld level structure allows us to define some regular integral models of $\mathcal{LT}_{\eta,K}$ when $K = \mathrm{Id} + \pi^m M_n(\mathcal{O}_E)$, a principal congruence subgroup. Those are formal spectrum of complete regular Noetherian rings that are finite free over $\mathcal{O}_{\breve{E}}[\![x_1, \ldots, x_{n-1}]\!]$.*

3.1. The basic locus as a zero dimensional locally symmetric space. Let I be the algebraic reductive group over $\mathbb{Q}$ that is the endomorphism by quasi-isogenies of an abelian variety over $\overline{\mathbb{F}}_q$ equipped with its additional structures defining an $\overline{\mathbb{F}}_q$-point of $\overline{S}^{(0)}_{K^p}$. This satisfies

(1) $I(\mathbb{R})$ is compact modulo its center,

(2) $I(\mathbb{Q}_p) = D^\times \times \mathbb{Z}_p^\times$ via the action of an automorphism on the Dieudonné module,

(3) $I(\mathbf{A}_f^p) = G(\mathbf{A}_f^p)$ via the action of an automorphism on the étale cohomology outside p.

In fact I is an inner form of G that is isomorphic to G outside $p\infty$.

The fact is, like for modular curves, that all basic points are in an unique isogeny class. From this we deduce that, after fixing a base point,

$$I(\mathbb{Q}) \backslash (I(\mathbb{Q}_p)/\mathcal{O}_D^\times \times I(\mathbf{A}_f^p)/K^p) \xrightarrow{\sim} \overline{S}^{(0)}_{K^p}(\overline{\mathbb{F}}_q).$$

3.2. Harris-Taylor theorem. From the preceding we obtain that

$$\underbrace{\varinjlim_K R\Gamma(\mathrm{Sh}_K \otimes_L \overline{L}, \overline{\mathbb{Q}}_\ell)_{|W_E, \text{cusp at } p}}_{\text{expressed in terms of aut. rep. of } G} \xrightarrow{\sim} \underbrace{\mathcal{A}(I)}_{\substack{\text{expressed in} \\ \text{terms of} \\ \text{aut.rep. of } I}} \otimes^{\mathbb{L}}_{\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(D^\times)} \varinjlim_K R\Gamma(\mathcal{M}_K \hat{\otimes}_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell)_{\text{cusp at } p}$$

Via a comparison between automorphic representations on the two inner forms I and G (global Jacquet-Langlands) obtained via a comparison of Arthur trace formulas Harris and Taylor prove the following result. This result is obtained via global methods using the fact that any supercuspdial representation globalizes to an automorphic representation that is a discrete series at ∞.

Theorem 3.3. *The cuspidal part of the middle degree cohomology*

$$\varinjlim_K H^{n-1}(\mathcal{M}_K \hat{\otimes}_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell)$$

is, up to a Tate twist, of the form

$$\bigoplus_{\substack{\pi \\ supercuspidal}} \mathrm{JL}^{-1}(\pi) \otimes \pi \otimes \varphi_\pi$$

where φ_π is an n-dimensional $\overline{\mathbb{Q}}_\ell$-representation of W_E. The correspondence $\pi \mapsto \varphi_\pi$ defines a local Langlands correspondence for $\mathrm{GL}_{n/E}$.

Chapitre 3

Third lecture - Sept 26

We are now dealing with **period morphisms for p-divisible groups**.

1. Some general thoughts on period morphisms

For $p = \infty$ there is only on period morphism and this is a $G(\mathbb{R})$-equivariant embedding

$$G(\mathbb{R}) \circlearrowright X \xhookrightarrow{\text{open}} \mathscr{F}_{\mu_h} \circlearrowleft G(\mathbb{C})$$

where G is a reductive group over $\mathbb{R}$, X is an hermitian symmetric space defined by the $G(\mathbb{R})$-conjugacy class of $h : \mathbb{S} \to G$, and $\mathscr{F}_{\mu_h}$ is the complex flag manifold defined by μ_h. This embedding is nothing else than the map that sends a Hodge structure to the Hodge filtration.

Moreover, the image of this embedding is easy to describe. In fact, the complex conjugate of μ_h is $\mu_h^c = w_h.\mu_h^{-1}$ with $w_h : \mathbb{G}_m \to G$ central and thus complex conjugation defines $\overline{(-)} : \mathscr{F}_{\mu_h} \xrightarrow{\sim} \mathscr{F}_{\mu_h^{-1}}$, and

$$\boxed{\begin{array}{rcl} X & \underset{\text{open/closed}}{\subset} & \{z \in \mathscr{F}_{\mu_h} \mid z \text{ and } \overline{z} \text{ are opposite parabolic subgroups}\} \\ & = & \underbrace{\{z \in \mathscr{F}_{\mu_h} \mid \operatorname{inv}(z, \overline{z}) = 1\}}_{\text{Deligne-Luztig variety at } \infty} \end{array}}$$

where here $P_{\mu_h^{-1}}$ is opposite to P_{μ_h} and

$$\operatorname{inv} : G_{\mathbb{C}}/P_{\mu_h} \times G_{\mathbb{C}}/P_{\mu_h^{-1}} \longrightarrow P_{\mu_h^{-1}} \backslash G_{\mathbb{C}}/P_{\mu_h}.$$

Here the open/closed condition defining X is that for z satisfying $\operatorname{inv}(x, \overline{z}) = 1$, one has an associated $h_z : \mathbb{G}_m \to G$ and we ask this is $G(\mathbb{R})$-conjugated to h.

EXAMPLE 1.1. *(1) Consider $G = \mathrm{Gsp}_{2n}$. Then, $\mathscr{F}_{\mu_h}$ is the variety of Lagrangians in $\mathbb{C}^{2n}$ equipped with the standard symplectic structure. Moreover, for a Lagrangian subspace $L \subset \mathbb{C}^{2n}$, the condition defining our open subset is that $L \cap \overline{L} = (0)$. It is clear that if $L \cap \big(\mathbb{C}^n \oplus (0)\big) \neq (0)$ then L is not in our open subset. The subset of $\mathscr{F}_{\mu_h}$ formed by Lagrangian subspace L satisfying $L \cap \big(\mathbb{C}^n \oplus (0)\big) = \emptyset$ is identified with the affine space of symmetric matrices $A \in M_n(\mathbb{C})$, ${}^tA = A$. To such a matrix A one associated the image of $\mathbb{C}^n \oplus (0)$ by $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$. Now, the associated Lagrangian subspace L satisfies $L \cap \overline{L} = (0)$ iff $\mathrm{Im}(A)$ (imaginary part) is invertible. Our open subset has* *thus n connected components given by the signature of the symmetric non-singular matrix* $\mathrm{Im}(A)$.

The open/closed subspace X is the union of the two connected components that correspond to the signatures $(n,0)$ and $(0,n)$ that is to say $\mathrm{Im}(A)$ *or* $-\mathrm{Im}(A)$ *is positive definite. This is $\pm\mathcal{H}_n$ where $\mathcal{H}_n$ is Siegel upper half space.*

(2) *Let $G = GU(1,n-1)$ with $h(z) = \mathrm{diag}(z,\overline{z},\dots,\overline{z})$. One has $\mathscr{F}_{\mu_h} = \mathbb{P}^{n-1}(\mathbb{C})$ and our open subset is $\big\{[z_1 : \dots : z_n] \mid |z_1|^2 - \sum_{i=2}^n |z_i|^2 \neq 0\big\}$. This has two connected components : the first one is an open ball $\{[1 : z_2 : \dots : z_n] \mid \sum_{i=2}^n |z_i|^2 < 1\} \subset \mathbb{C}^{n-1}$ and the other one is $\{[1 : z_2 : \dots : z_n] \mid \sum_{i=2}^n |z_i|^2 > 1\} \cup \{[0 : z_2 : \dots : z_n] \in \mathbb{P}^{n-2}(\mathbb{C})\}$. The space X is the first connected component identified with an open ball.*

For $p \neq \infty$ the story is different :

(1) There are *two period maps* and *two groups acting*
(2) Those are linked to the *two cohomology theories* : crystalline cohomology and p-adic étale cohomology. For $p = \infty$ we only have Betti cohomology.
(3) Those two period maps correspond to the *two spectral sequences* : the Hodge to de Rham spectral sequence and the Hodge-Tate spectral sequence
(4) The period maps aren't embeddings in general.

2. The case of Lubin-Tate spaces

2.1. The Lubin-Tate tower. Take $E = \mathbb{Q}_p$ to simplify. Let

$$\mathbb{H}$$

be a one dimensional 1-dimensional formal p-divisible group over $\overline{\mathbb{F}}_p$ (such an $\mathbb{H}$ is unique up to a non-unique isomorphism). This can be seen, after fixing a coordinate $\mathrm{Spf}(\overline{\mathbb{F}}_p[\![T]\!]) \xrightarrow{\sim} \mathbb{H}$ as a one dimensional formal group law $\mathfrak{F} \in \overline{\mathbb{F}}_p[\![X,Y]\!]$ that gives the addition : $X \underset{\mathfrak{F}}{+} Y = \mathfrak{F}(X,Y)$.

Let n be the height of $\mathbb{H}$ that is to say $[p]_{\mathfrak{F}} = aT^{p^n} + \dots$ with $a \neq 0$.

Definition 2.1. *The moduli space of deformations of $\mathbb{H}$ over complete local $W(\overline{\mathbb{F}}_p)$-algebras is the Lubin-Tate space*

$$\begin{array}{c} \mathcal{LT} \\ \downarrow \\ \mathit{Spf}(W(\overline{\mathbb{F}}_p)). \end{array}$$

This is non-canonically isomorphic to

$$\mathrm{Spf}\big(W(\overline{\mathbb{F}}_p)[\![x_1,\dots,x_{n-1}]\!]\big).$$

Let D be a division algebra with invariant $\frac{1}{n}$ over $\mathbb{Q}_p$, $D = \mathbb{Q}_{p^n}[\Pi]$ where $\mathbb{Q}_{p^n}$ is the degree n unramified extension of $\mathbb{Q}_p$, $\Pi^n = p$ and if σ is the Frobenius of $\mathbb{Q}_{p^n}|\mathbb{Q}_p$ then $\Pi x \Pi^{-1} = x^\sigma$.

One has an identification

$$\mathcal{O}_D = \mathrm{End}(\mathbb{H})$$

where $\mathcal{O}_D$ is the maximal order in D, $\mathcal{O}_D = \mathbb{Z}_{p^n}[\Pi]$.

There is an evident action of $\mathcal{O}_D^\times$ on $\mathcal{LT}$

$$\begin{tikzcd} \mathcal{LT} \arrow[loop below, "\mathcal{O}_D^\times"] \end{tikzcd}$$

Definition 2.2. *Let $\mathcal{LT}_\eta$ be the generic fiber of $\mathcal{LT}$ as a locally of finite type adic space over* $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$.

After fixing some formal coordinates

$$\mathcal{LT}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p}$$

that is again equipped with an action of $\mathcal{O}_D^\times$. The Tate module of the universal deformation defines an étale $\mathbb{Z}_p$-local system T of rank n on $\mathcal{LT}_\eta$.

Definition 2.3. *For $K \subset \mathrm{GL}_n(\mathbb{Z}_p)$ we note*

$$\mathcal{LT}_{\eta,K}$$

the moduli space of trivializations mod K of the $\mathbb{Z}_p$-local system T.

$\rightarrow$ we force the monodromy of our local system to leave in K. This means

$$\mathcal{LT}_{\eta,K} = \underline{(K/\,\mathrm{Id} + p^m M_n(\mathbb{Z}_p))}\backslash \underline{\mathrm{Isom}}\big((\underline{Z/p^m\mathbb{Z}})^n, T/p^m T\big)$$

for $m \gg 0$.

We obtain a tower

$$\begin{tikzcd} \big(\mathcal{LT}_{\eta,K}\big)_K \arrow[loop right, "\mathrm{GL}_n(\mathbb{Q}_p)^1"] \arrow[loop below, "\mathcal{O}_D^\times"] \end{tikzcd}$$

where
— the action of $\mathcal{O}_D^\times$ is horizontal,
— the action of $\mathrm{GL}_n(\mathbb{Q}_p^1$ is vertical : for $g \in \mathrm{GL}_n(\mathbb{Q}_p)^1$, $g : \mathcal{LT}_{\eta,K} \xrightarrow{\sim} \mathcal{LT}_{\eta,gKg^{-1}}$,
— both actions commute.

Here the action of $\mathrm{GL}_n(\mathbb{Z}_p)$ is the evident one. To extend it to an action of $\mathrm{GL}_n(E)^1$ we have to go back to some integral models of $\mathcal{LT}_{\eta,K}$ for K a principal congruence subgroup, $K = \mathrm{Id} + p^m M_n(\mathbb{Z}_p)$, $m \geq 1$. This is given by this notions of Drinfeld level structure that defines a an integral model $\mathrm{Spf}(R_m)$ where R_m is a complete regular $W(\overline{\mathbb{F}}_p)$-algebra. We now use the following two elementary results;

(1) if S is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ and H a one dimensional height n formal p-divisible group over S equipped with a level m Drinfel structure
$$\eta : \underline{(\mathbb{Z}/p^m)^n} \longrightarrow H[p^m]$$
then any subgroup M of $(\mathbb{Z}/p^m\mathbb{Z})^n$ defines a finite flat closed subgroup scheme $G \subset H[p^m]$ such that $\eta_{|M} : \underline{M} \to G$.

(2) if S is a reduced $\mathbb{F}_p$-scheme and $f : H \to H'$ is a height 0 quasi-isogeny between one dimensional formal p-divisible groups then f is an isomorphism.

At the end we obtain an action of $(\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times)^1$ on our tower.

2.2. The de Rham period morphism. Let
$$\boldsymbol{D} = \mathbb{D}(\mathbb{H})$$
be the covariant rational Dieudonné module of $\mathbb{H}$. This is an n-dimensional $\breve{Q}_p$-vector space equipped with a crystalline Frobenius φ,
$$\boldsymbol{D} \circlearrowleft \varphi \,.$$
The matrix of the associated Verschiebung $p\varphi^{-1}$ is given in a suitable basis by
$$p\varphi^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma^{-1}$$

We now use the following property. Recall that a quasi-isogeny between p-divisible groups H and H' over a quasi-compact scheme S is an element of $f \in \mathrm{Hom}(H, H')\left[\frac{1}{p}\right]$ such that there exists $g \in \mathrm{Hom}(H', H)\left[\frac{1}{p}\right]$ satisfying $g \circ f = \mathrm{Id}$ and $f \circ g = \mathrm{Id}$.

Lemma 2.4 (rigidity of quasi-isogenies). *Let $S_0 \hookrightarrow S$ be a nilpotent closed immersion of schemes and H, H' be p-divisible groups over S. Then, reduction to S_0 induces an isomorphism*
$$\mathrm{Qisog}(H, H') \xrightarrow{\sim} \mathrm{Qisog}(H \times_S S_0, H' \times_S S_0).$$

We now use **the crystalline nature of the Dieudonné crystal of a p-divisible group**. Let R be a p-adic ring, H a p-divisible group over $\mathrm{Spf}(R)$ and H_0 be a p-divisible group over $\mathrm{Spec}(R/pR)$. Suppose given a quasi-isogeny
$$\rho : H_0 \to H \otimes_R R/pR.$$
Let $\mathscr{E}$ be the covariant Dieudonné crystal of H on $(\mathrm{Spec}(R)/\mathrm{Spec}(\mathbb{Z}_p))_{crys}$ and $\mathscr{E}_0$ be the one of H_0 on $(\mathrm{Spec}(R/pR)/\mathrm{Spec}(\mathbb{Z}_p))_{crys}$. This gives rise to an isomorphism
$$\rho_* : \mathscr{E}_{0, R \twoheadrightarrow R/pR}\left[\tfrac{1}{p}\right] \xrightarrow{\sim} \mathscr{E}_{R \twoheadrightarrow R}\left[\tfrac{1}{p}\right].$$
From this and the rigidity of quasi-isogenies we decue the following result.

PROPOSITION 2.5. *Let $(\mathscr{E},\nabla)$ be the convergent isorcystal on $\mathcal{LT}_\eta$ associated to the universal deformation as an $\mathcal{O}_D^\times$-equivariant vector bundle equipped with an integrable connexion. There is a canonical $\mathcal{O}_D^\times$-equivariant isomorphism*

$$\big(\boldsymbol{D}\otimes_{\breve{\mathbb{Q}}_p}\mathcal{O}_{\mathcal{LT}_\eta},\mathrm{Id}\otimes d\big)\xrightarrow{\sim}(\mathscr{E},\nabla)$$

and thus $(\mathscr{E},\nabla)$ is generated by its horizontal sections that are identified with D,

$$\boldsymbol{D}\xrightarrow{\sim}\mathscr{E}^{\nabla=0}.$$

The rank n vector bundle $\mathscr{E}$ can be though of as being the $(\mathcal{H}^1_{dR})^\vee$ of the universal deformation. There is an **Hodge filtration**

$$\mathrm{Fil}\,\mathscr{E}\subset\mathscr{E}$$

that is identified with $\omega_{H^D}\left[\frac{1}{p}\right]$ where H is the universal deformation and fits into the Hodge exact sequence

$$0\longrightarrow\underbrace{\omega_{H^D}\left[\tfrac{1}{p}\right]}_{\text{rk. }n-1}\longrightarrow\mathscr{E}\longrightarrow\underbrace{\omega_H\left[\tfrac{1}{p}\right]}_{\text{rk. }1}\longrightarrow 0$$

DEFINITION 2.6 (de Rham period morphism for Lubin-Tate spaces). *We note*

$$\pi_{dR}:\mathcal{LT}_\eta\longrightarrow\mathbb{P}(\boldsymbol{D})\simeq\mathbb{P}^{n-1}$$

for the $\mathcal{O}_D^\times$-equivariant morphism defined by the Hodge filtration and Proposition 2.5.

Grothendieck-Messing theory says that to deform a p-divisible group is the same as to deform its Hodge filtration. From this the following basic result is elementary.

PROPOSITION 2.7. *The de Rham period morphism π_{dR} satisfies the following :*

(1) It is (partially proper) étale,

(2) Its geometric fibers are the Hecke orbits

The following result is quite deep and will be later reinterpreted in terms of the curve.

THEOREM 2.8 (Gross-Hopkins). *The de Rham period morphism*

$$\pi_{dR}:\mathcal{LT}_\eta\longrightarrow\mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$$

is surjective.

At the end we thus have an étale cover

$$\mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p}\longrightarrow\mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$$

with infinite discrete fibers.

The following result can be verified in an elementary way. We note $\mathbb{Q}_p^{cyc}:=\widehat{\cup_{n\geq1}\mathbb{Q}_p(\zeta_n)}$.

Proposition 2.9. *The projective limit*

$$\mathcal{LT}_{\eta,\infty} := \varprojlim_K \mathcal{LT}_{\eta,K}$$

makes sense as a $\mathbb{Q}_p^{cyc}$-perfectoid space.

At the end we obtain the following picture.

$$\begin{tikzcd}
\mathcal{LT}_{\eta,\infty} \arrow[d] \arrow[d, phantom, bend left=60, "\underline{\mathrm{GL}_n(\mathbb{Z}_p)}"'] \arrow[dd, phantom, bend right=90, "\underline{\mathrm{GL}_n(\mathbb{Q}_p)^1}"] \\
\mathcal{LT}_\eta \arrow[d, "\pi_{dR}"] \\
\mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}
\end{tikzcd}$$

where the torsors are pro-étale torsors.

2.3. The Hodge-Tate period morphism. We now come to the other period morphism in the game.

Recall that if G is a (commutative) finite locally free group scheme over a scheme there is a morphism of fppf sheaves

$$\begin{array}{rcl} G = \mathscr{H}om(G^D, \mathbb{G}_m) & \longrightarrow & \omega_{G^D} \\ f & \longmapsto & f^* \dfrac{dT}{T}. \end{array}$$

from G toward the fppf sheaf associated to the coherent sheaf ω_{G^D}.

Let now H be a p-divisible group over $\mathrm{Spec}(R)$ where R is a p-torsion free p-adic ring. Suppose moreover that R is integrally closed in $R\left[\frac{1}{p}\right]$. The preceding construction applied to the collection $(H[p^n])_{h\geq 1}$ defines a $\mathbb{Z}_p$-linear morphism

$$\boxed{\alpha_H : \underbrace{\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H_\eta)}_{\mathbb{Z}_p\text{-module}} \underbrace{=}_{\substack{R \text{ int. closed} \\ \text{in } R\left[\frac{1}{p}\right]}} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H) \longrightarrow \underbrace{\omega_{H^D}}_{\substack{\text{projective of} \\ \text{finite type} \\ R\text{-module} \\ \text{of rk.}\, \mathrm{ht}(H)-\dim(H)}}}$$

where H_η is the étale p-divisible group $H \otimes_R R\left[\frac{1}{p}\right]$. We note $\alpha_H \otimes 1$ for its linearization

$$\alpha_H \otimes 1 : \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H_\eta) \otimes_{\mathbb{Z}_p} R \longrightarrow \omega_{H^D}.$$

The key result is now the following.

Proposition 2.10 (Faltings, F.). *If $R = \mathcal{O}_C$ with $C|\mathbb{Q}_p$ a complete algebraically closed extension of $\mathbb{Q}_p$ then the preceding induces a complex*

$$0 \longrightarrow \omega_H^\vee(1) \xrightarrow{(\alpha_{H^D}\otimes 1)^\vee(1)} T_p(H) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{\alpha_H \otimes 1} \omega_{H^D} \longrightarrow 0$$

whose cohomology is killed by $p^{\frac{1}{p-1}}$ if $p \neq 2$ and 4 if $p = 2$. In particular one has an Hodge-Tate exact sequence

$$0 \longrightarrow \omega_H^\vee(1)\left[\tfrac{1}{p}\right] \xrightarrow{(\alpha_{H^D}\otimes 1)^\vee(1)} V_p(H) \otimes_{\mathbb{Q}_p} C \xrightarrow{\alpha_H \otimes 1} \omega_{H^D}\left[\tfrac{1}{p}\right] \longrightarrow 0$$

Let us remark that $\frac{1}{p-1} = v_p(2i\pi)$ in the preceding proposition. Using this result we can construct a morphism

$$\pi_{HT} : \mathcal{LT}_{\eta,\infty} \longrightarrow \check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p}$$

that is $\mathrm{GL}_n(\mathbb{Q}_p)^1$-equivariant and $\mathcal{O}_D^\times$-invariant. Here $\check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p}$ is the dual projective space classifying rank $n-1$ quotients of $\mathcal{O}^n$. Let us fix the isomorphism $\check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$ given by the identification of $(\mathcal{O}^n)^\vee$ and $\mathcal{O}^n$ deduced from the dual of the canonical basis. This commutes with the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ twisted by $g \mapsto {}^t g^{-1}$.

Theorem 2.11 (Faltings, F.). *The image of*

$$\pi_{HT} : \mathcal{LT}_{\eta,\infty} \longrightarrow \check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$$

is Drinfeld's space Ω. Moreover, $\mathcal{LT}_{\eta,\infty} \to \Omega$ is a pro-étale $\underline{\mathcal{O}_D}^\times$-torsor that is identified with Drinfeld-tower.

At the end we obtain the following diagram.

$$\begin{tikzcd}
& \mathcal{LT}_{\eta,\infty} \arrow[loop above, "\underline{[\mathrm{GL}_n(\mathbb{Q}_p)\times D^\times]^1}"] \arrow[dl, "\pi_{dR}"'] \arrow[dr, "\pi_{HT}"] \arrow[dl, bend right=40, no head, "\underline{\mathrm{GL}_n(\mathbb{Q}_p)^1}"'] \arrow[dr, bend left=40, no head, "\underline{\mathcal{O}_D}^\times"] & \\
\mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p} \arrow[loop below, "\underline{\mathcal{O}_D^\times}"] & & \Omega_{\breve{\mathbb{Q}}_p} \arrow[loop below, "\underline{\mathrm{GL}_n(\mathbb{Q}_p)^1}"]
\end{tikzcd}$$

3. Rapoport-Zink spaces

3.1. Integral models in hyperspecial level. Rapoport-Zink spaces are generalizations of Lubin-Tate and Drinfeld spaces. We only explain the $G = \mathrm{GL}_n$-case.

Let $\mathbb{H}$ be a p-divisible group over $\overline{\mathbb{F}}_p$ of dimension n and dimension d. Let $(\boldsymbol{D}, \varphi)$ be its covariant rational Dieudonné isocrystal. We note :

(1) $G = \mathrm{GL}_n$,
(2) G_b the reductive algebraic group over $\mathbb{Q}_p$ whose R-points are $\mathrm{Aut}(D \otimes_{\mathbb{Q}_p} R, \varphi \otimes \mathrm{Id})$.

Here the $b \in G(\breve{\mathbb{Q}}_p)$ refers to the matrix of Frobenius in a basis of $\boldsymbol{D}$, in which case φ can be identified with $b\sigma \in G(\breve{\mathbb{Q}}_p) \rtimes \sigma$. Then, G_b is identified with the twisted centralizer of b,

$$G_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) \mid gb\sigma = b\sigma g\}$$

that is to say

$$gbg^{-\sigma} = b.$$

If $(\lambda_1, \ldots, \lambda_r)$ are the slopes of $(\boldsymbol{D}, \varphi)$ with respective multiplicities $(m_1, \ldots, m_r)$, then

$$G_b \simeq \prod_{i=1}^{r} \mathrm{GL}_{m_i}(D_{-\lambda_i})$$

where D_λ is the division algebra with invariant λ over $\mathbb{Q}_p$.

Definition 3.1. *We note $\mathcal{M}$ for the functor on $W(\overline{\mathbb{F}}_p)$-schemes on which p is locally nilpotent such that*

$$\mathcal{M}(S) = \{(H, \rho)\}/\sim$$

where
— *H is a p-divisible group over $\overline{\mathbb{F}}_p$,*
— *$\rho : \mathbb{H} \times_{\overline{\mathbb{F}}_p} (S \bmod p) \longrightarrow H \times_S (S \bmod p)$ is quasi-isogeny.*

The $\overline{\mathbb{F}}_p$-points of this moduli are identified via Dieudonné theory with

$$\mathcal{M}(\overline{\mathbb{F}}_p) = \{M \subset \boldsymbol{D} \text{ a lattice s.t. } pM \subset \varphi(M) \subset M\}.$$

This can be rewritten in the following way. Let $\mu : \mathbb{G}_m \to G$ be the Hodge cocharacter

$$\mu(z) = (\underbrace{z, \ldots, z}_{d \text{ times}}, \underbrace{1, \ldots, 1}_{n-d \text{ times}}).$$

Then we have

$$\boxed{\mathcal{M}(\overline{\mathbb{F}}_p) = \left\{g \in G\big(W(\overline{\mathbb{F}}_p)[\tfrac{1}{p}]\big)/G\big(W(\overline{\mathbb{F}}_p)\big) \;\middle|\; \mathrm{inv}(bg^\sigma, g) = \{\mu\}\right\}}$$

where

$$\begin{array}{ccc} G\big(W(\overline{\mathbb{F}}_p)\big[\tfrac{1}{p}\big]\big)/G\big(W(\overline{\mathbb{F}}_p)\big) \times G\big(W(\overline{\mathbb{F}}_p)\big[\tfrac{1}{p}\big]\big)/G\big(W(\overline{\mathbb{F}}_p)\big) & & ([g_1],[g_2]) \\ \Big\downarrow \text{inv} & & \big\downarrow \\ G\big(W(\overline{\mathbb{F}}_p)\big)\backslash G\big(W(\overline{\mathbb{F}}_p)\big[\tfrac{1}{p}\big]\big)/G\big(W(\overline{\mathbb{F}}_p)\big) & & [g_1^{-1}g_2] \end{array}$$

that is identified with $\mathrm{Hom}(\mathbb{G}_m, G)/G$-conjugacy via $\mu \mapsto [\mu(p)]$.

Thus, **the $\overline{\mathbb{F}}_p$-points of $\mathcal{M}$ can be identified with an affine Deligne-Lusztig set** (we say "set" instead of "variety" because à priori we don't know how to put a geometric structure on this now).

Further more, for any $x \in \mathcal{M}(\overline{\mathbb{F}}_p)$ if H_x is the associated p-divisible group, there is an identification

$$\boxed{\widehat{\mathcal{M}}_{/x} = \mathrm{Def}(H_x)}$$

that is representable by a formal scheme isomorphic to

$$\mathrm{Spf}(W(\overline{\mathbb{F}}_p)[\![x_1, \ldots, x_{d(n-d)}]\!]).$$

The moduli space $\mathcal{M}$ is a much subtler version on the naive formal scheme

$$\coprod_{x \in \mathcal{M}(\overline{\mathbb{F}}_p)} \mathrm{Def}(H_x).$$

We have in fact the following theorem.

THEOREM 3.2 (Rapoport-Zink). *The functor $\mathcal{M}$ is a representable by a Spf$(W(\overline{\mathbb{F}}_p))$-formal scheme locally formally of finite type that is to say locally isomorphic to*

$$Spf\big(W(\overline{\mathbb{F}}_p)[\![X_1, \ldots, X_s]\!]\langle Y_1, \ldots, Y_t\rangle / \,\mathrm{Ideal}\big).$$

Moreover the irreducible components of $\mathcal{M}_{red}$ are projective algebraic varieties over $\overline{\mathbb{F}}_p$.

The action of $G_b(\mathbb{Q}_p)$ on the quasi-isogeny ρ defines a continuous action of $G_b(\mathbb{Q}_p)$ on $\mathcal{M}$,

$$\mathcal{M} \circlearrowleft G_b(\mathbb{Q}_p)$$

EXAMPLE 3.3. *From the fact that any degree 0 quasi-isogeny between 1-dimensional formal p-divisible groups over $\overline{\mathbb{F}}_p$ we deduce that in the Lubin-Tate case*

$$\mathcal{M} = \mathcal{LT} \overset{\mathcal{O}_D^\times}{\times} D^\times$$

that is (non-canonically) isomorphic to $\coprod_{\mathbb{Z}} \mathcal{LT}$ where the action of $\mathcal{O}_D^\times$ on the factor $\mathcal{LT}$ associated to $k \in \mathbb{Z}$ is the canonical one twisted by $d \mapsto \Pi^k d \Pi^{-k}$.

REMARK 3.4. *Although $\mathcal{M}$ is formally smooth, in general $\mathcal{M}_{red}$ is not smooth.*

3.2. The tower. Let

$$\mathcal{M}_\eta \circlearrowleft G_b(\mathbb{Q}_p)$$

be the generic fiber of $\mathcal{M}$ as a locally of finite type adic space over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$. As before wit the Lubin-Tate tower one obtains a tower

$$\boxed{\begin{tikzcd} \left(\mathcal{M}_{\eta,K}\right)_K \arrow[loop right, "G(\mathbb{Q}_p)"] \arrow[loop below, "G_b(\mathbb{Q}_p)"] \end{tikzcd}}$$

where K goes through the set of compact open subgroups of $G(\mathbb{Q}_p)$ and both actions commute. The definition of the action of $G(\mathbb{Q}_p)$ is more subtle than in the Lubin-Tate case since there is no "good notion" of integral level structures like this is the case for one dimensional p-divisible groups according to Drinfeld.

This relies on **Raynaud's flatification by blow-ups** : if S is a quasi-compact quasi-separated scheme, $G \to S$ is a finite locally free group scheme, $U \subset S$ is an open subset and $H \subset G \times_S U$ is a closed finite locally free sub-group scheme then after a blow-up supported on $S \setminus U$ we can suppose that H extends to a closed subgroup scheme of G finite locally free over S.

Example 3.5. *For the Lubin-Tate tower, the associated RZ tower is*

$$(\mathcal{M}_{\eta,K})_K = (\mathcal{LT}_{\eta,K})_K \overset{\left(\mathrm{GL}_n(\mathbb{Q}_p)\times D^\times\right)^1}{\times} \mathrm{GL}_n(\mathbb{Q}_p) \times D^\times.$$

3.3. Period morphisms. As for Lubin-Tate spaces, if $(\mathscr{E}, \nabla)$ is the convergent isocrystal associated to the universal deformation H on $\mathcal{M}$, the universal quasi-isogeny ρ induces an isomorphism

$$(\boldsymbol{D} \otimes_{\breve{\mathbb{Q}}_p} \mathcal{O}_{\mathcal{M}_\eta}, \mathrm{Id} \otimes d) \xrightarrow{\sim} (\mathscr{E}, \nabla).$$

The Hodge filtration then defines a $G_b(\mathbb{Q}_p)$-equivariant morphism

$$\pi_{dR} : \mathcal{M}_\eta \longrightarrow \mathscr{F}_\mu$$

where $\mathscr{F}_\mu$ is the rigid analytic flag manifold associated to μ. This satisfies :
— This is étale and thus in particular its image is open,
— Its geometric fibers are the Hecke orbits.

The image of the étale morphism π_{dR},

$$\mathscr{F}_\mu^a := \mathrm{Im}(\pi_{dR}),$$

is the so-called **admissible open subset** of $\mathscr{F}_\mu$. This is a partially proper open subset inside the flag manifold $\mathscr{F}_\mu$. Little is known in general about it outside of the fact that
— there is an inclusion

$$\mathscr{F}_\mu^a \subset \mathscr{F}_\mu^{wa}$$

where $\mathscr{F}_\mu^{wa}$ is the so-called **weakly admissible open subset**, a very concrete open subset that is of the form

$$\mathscr{F}_\mu \setminus \bigcup_{\text{profinite}} \text{Schubert varities.}$$

— For $[K : \breve{\mathbb{Q}}_p] < +\infty$,

$$\mathscr{F}_\mu^a(K) = \mathscr{F}_\mu^{wa}(K)$$

that is to say $\mathscr{F}_\mu^a$ and $\mathscr{F}_\mu^{wa}$ have the same Tate classical points.

— There is a complete characterization of when $\mathscr{F}_\mu^a = \mathscr{F}_\mu^{wa}$.

The picture at this point for the Hodge-Tate period morphism is more difficult to describe since we first need to give a meaning to

$$\mathcal{M}_{\eta,\infty} := \varprojlim_K \mathcal{M}_{\eta,K}.$$

The fact is that this is a perfectoid space (if $\mathbb{H}$ is not étale) but we can make a sense out of it using integral models and blow-ups as in the Lubin-Tate case. At the end there is picture

(1)
$$
\begin{tikzcd}
 & \mathcal{M}_{\eta,\infty} \arrow[loop above, "\underline{G(\mathbb{Q}_p)\times G_b(\mathbb{Q}_p)}"] \arrow[dl, "\pi_{dR}"'] \arrow[dr, "\pi_{HT}"] \arrow[dl, bend right=40, dash, "\underline{G(\mathbb{Q}_p)}"'] \arrow[dr, bend left=40, dash, "G_b(\mathbb{Q}_p)"] & & \\
\mathscr{F}_\mu^a \arrow[loop below, "\underline{G_b(\mathbb{Q}_p)}"] & & \operatorname{Im}(\pi_{HT}) \arrow[r, hook] \arrow[loop below, "\underline{G(\mathbb{Q}_p)}"] & \mathscr{F}_{\mu^{-1}}
\end{tikzcd}
$$

where $\operatorname{Im}(\varphi_{HT}) \subset \mathscr{F}_{\mu^{-1}}$ is not an open subset in general and is well defined in general only as a locally spatial diamond. When b is basic i.e. the isocrystal $(\boldsymbol{D}, \varphi)$ is isoclinic then $\operatorname{Im}(\pi_{HT})$ is open inside the dual flag manifold $\mathscr{F}_{\mu^{-1}}$ and this is a classical rigid analytic open subset.

3.4. Cohomology. As for Lubin-Tate spaces one can use the cohomology spaces

$$H_c^\bullet(\mathcal{M}_K \hat{\otimes}_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$$

as representations of $\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(K\backslash G(\mathbb{Q}_p)/K)$ and $G_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ to define **a kernel for the local Langlands correspondence**. More precisely, we look at the correspondence

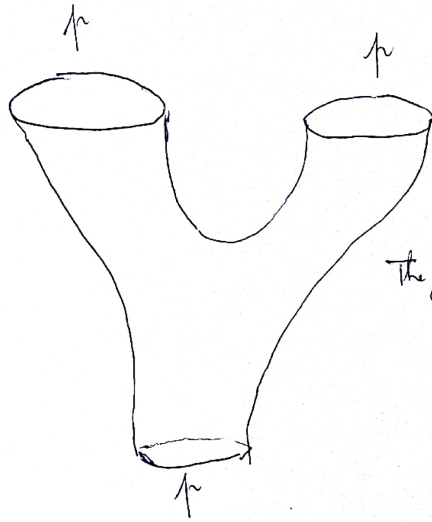
$$\underbrace{\pi}_{\text{smooth rep.of } G_b(\mathbb{Q}_p)} \longmapsto \underbrace{\varinjlim_K \operatorname{Ext}^\bullet_{G_b(\mathbb{Q}_p)}(H_c^\bullet(\mathcal{M}_K \hat{\otimes}_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell), \pi)}_{\text{smooth}\times\text{continuous representation of } G(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}} \quad . \tag{2}$$

4. Final thoughts

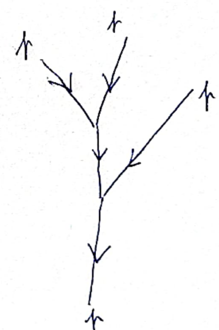
Diagram (1) has been a great motivation for the geometrization conjecture of the local Langlands correspondence with relation with the correspondence given by the Hecke stack

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd}$$

The "cohomological kernel" of equation (2) given by the cohomology of Rapoport-Zink spaces is even a reminder that the preceding correspondence should be upgraded to a cohomological correspondence.



The fusion of two copies of p



$$\mathrm{Div}^1 \xhookrightarrow{\mathrm{diag}} \mathrm{Div}^1 \times \mathrm{Div}^1$$

$$\mathrm{Div}^1 = \mathrm{Spa}(\mathbb{Q}_p)^\diamond / \varphi^{\mathbb{Z}}$$

The fusion of 3 copies of p

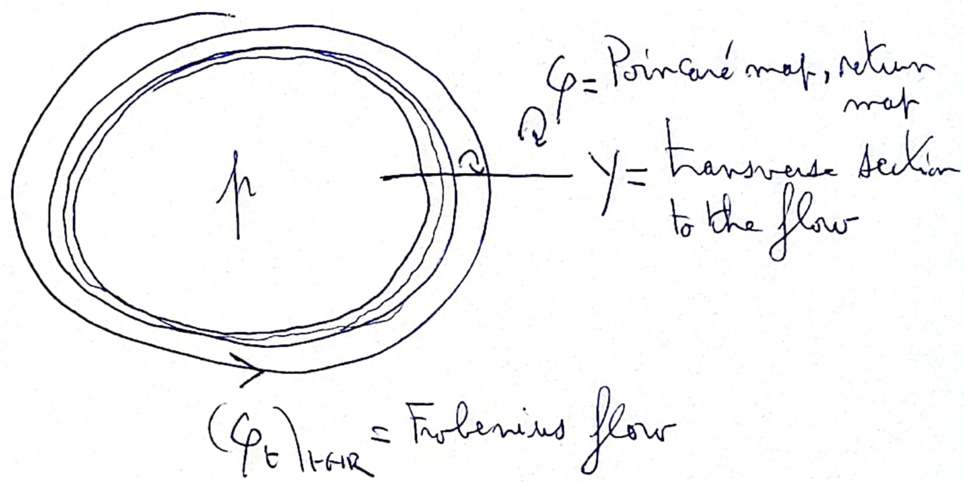
$$\mathrm{Div}^1 \hookrightarrow \mathrm{Div}^1 \times \mathrm{Div}^1 \hookrightarrow \mathrm{Div}^1 \times \mathrm{Div}^1 \times \mathrm{Div}^1$$

$$x \mapsto (x,x)$$

$$(y,z) \mapsto (y,y,z)$$

Chapitre 4

Fourth lecture - October 10



Frobenius flow: limit cycle of length log p for each prime number p.

1. Holomorphic functions of the variable p

Let E be a finite degree extension of $\mathbb{Q}$ with residue field $\mathbb{F}_q$. **Contrary to the "classical case", the curve "X" does not exists absolutely over** $\mathbb{F}_q$, it exists only after pull-back to an $\mathbb{F}_q$-perfectoid field F i.e. "X" makes no sense but X_F makes sense for each such F. Let us thus fix an $\mathbb{F}_q$-perfectoid field F. This is nothing else than a perfect, complete with respect to a non-trivial rank 1 valuation, non-archimedean field. One may, for example, want to consider $F = \mathbb{F}_q((T^{1/p^\infty}))$ or $F = \widehat{\overline{\mathbb{F}_q((T))}}$.

Definition 1.1. *We note* $A_{\inf} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ *equipped with its Frobenius* φ *lifting* Frob_q *modulo* π.

One has

$$A_{\inf} \underbrace{=}_{\substack{\text{unique}\\ \text{writting}}} \Big\{ \sum_{n\geq 0} [a_n]\pi^n \mid a_n \in \mathcal{O}_F \Big\}$$

and

$$\varphi\Big(\sum_{n\geq 0}[a_n]\pi^n\Big) = \sum_{n\geq 0}[a_n^q]\pi^n.$$

We think of $A_{\inf}$ as being a ring of holomorphic functions where π is the variable and the coefficients are in $\mathcal{O}_F$. In fact, we want to define an open punctured disk of the variable π over F. This is the space Y_F that will come. For this space Y_F, the ring $A_{\inf}$ is the subring of $\mathcal{O}(Y_F)$ formed by holomorphic functions that are holomorphic at $\pi = 0$ and bounded by 1. We fix a pseudo-uniformizer ϖ of F.

Definition 1.2. *We note* $Y_F = \mathrm{Spa}(A_{\inf}, A_{\inf}) \setminus V(\pi.[\varpi])$ *equipped with its Frobenius* φ.

Let us begin by saying the following to remove any doubt.

Theorem 1.3. *The following is satisfied :*

(1) *Y_F is sous-perfectoid in the sense that for any $K|E$ perfectoid, $Y_F\hat{\otimes}_E K$ is a K-perfectoid space with tilting* $\mathrm{Spa}(F) \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(K^\flat)$ *where φ is identified with* $\mathrm{Frob}_q \times \mathrm{Id}$.

(2) *(Kedlaya) Y_F is strongly Noetherian.*

In particular, via point (1) or (2), Huber's presheaf of holomorphic functions on $|Y_F|$ is a sheaf.

Remark 1.4. *We will define later Y_S for any $\mathbb{F}_q$-perfectoid space S. Property (1) is still valid in this context but property (2) does not hold anymore in general.*

There is a radius continuous function

$$\begin{aligned} \rho : |Y_F| &\longrightarrow \;]0,1[\\ y &\longmapsto q^{-\frac{v(\pi(y^{max}))}{v([\varpi](y^{max}))}} \end{aligned}$$

where y^{max} is the maximal generalization of y seen as a Berkovich point that is to say a valuation with values in $\mathbb{R}$. This extends to a continuous function

$$|\mathrm{Spa}(A_{\inf}, A_{\inf})| \longrightarrow [0,1].$$

where $\rho = 0$ corresponds to the Cartier divisor $\pi = 0$ and $\rho = 1$ to $[\varpi] = 0$. Those two divisors are fixed by φ and one has the formula

$$\rho(\varphi(y)) = \rho(y)^{1/q}.$$

In particular, φ acts properly discontinuously without fixed points on $|Y_F|$.

For any compact interval $I \subset]0,1[$ of the form $[a,b]$ with $a, b \in q^{\mathbb{Q}}$, the annulus

$$Y_{F,I} = \{y \mid \rho(y) \in I\}$$

is a rational domain and in particular affinoid (even affinoid sous-perfectoid). One has

$$Y_F = \bigcup_{\substack{0<a\leq b<1 \\ a,b\in q^{\mathbb{Q}}}} \underbrace{Y_{F,[a,b]}}_{\substack{\text{affinoid} \\ \text{annulus}}}.$$

The main difficulty (and this is one of the main reasons why "p-adic Hodge theory is difficult") is that $\mathcal{O}(Y_F)$ is defined as a (Frechet) completion of $A_{\text{inf}}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ and there is no explicit formula, typically as a power series expansion, for elements in this ring.

2. Newton polygons and Weierstrass factorization

A Key definition is the following.

Definition 2.1. *An element $\xi = \sum_{n\geq 0}[a_n]\pi^n \in A_{\text{inf}}$ is distinguished of degree $d \geq 1$ if*
- $a_0, \dots, a_{d-1} \in \mathfrak{m}_F$,
- $a_0 \neq 0$,
- $a_d \in \mathcal{O}_F^\times$.

The product of a degree d and degree d' distinguished elements is a degree $d+d'$ distinguished element. If ξ is distinguished of degree d and $u \in A_{\text{inf}}^\times$ then $u\xi$ is distinguished of degree d.

Another key property if the following. Let us normalize the valuation v on F such that $v(\varpi) = 1$. For any $r > 0$ and $f = \sum_{n\geq 0}[a_n]\pi^n \in A_{\text{inf}}$, the formula

$$v_r(f) = \inf_{n\geq 0} v(a_n) + rn$$

defines a Gauss valuation $Gauss_r \in |Y_F|$ with $\rho(Gauss_r) = q^{-r}$. The function $r \mapsto v_r(f)$ is a concave polygon and using a process of (inverse) Legendre transform we can deduce from it a Newton polygon. More precisely :

For any interval $I \subset]0,1[$ with extremities in $q^{\mathbb{Q}}$ and any $f \in \mathcal{O}(Y_{F,I}) \setminus \{0\}$, one can define naturally a **Newton polygon** $\text{Newt}_I(f)$ with breakpoints at integral x-coordinates and whose slopes are in $-\log_q I$ in such a way that

(1) For $f = \sum_{n\gg-\infty}[a_n]\pi^n \in A_{\text{inf}}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$, $\text{Newt}_{]0,1[}(f)$ is the convex envelope of $(v(a_n), n)_{n\in\mathbb{Z}}$,

(2) $\text{Newt}_I(fg)$ is obtained by concatenation from $\text{Newt}_I(f)$ and $\text{Newt}_I(g)$.

Here is the main factorization result we obtained with Fontaine.

THEOREM 2.2. *The following is satisfied :*

(1) *For* $\xi \in A_{\text{inf}}$ *distinguished irreducible of degree* d, $K_\xi = A_{\text{inf}}[\frac{1}{\pi}]/\xi$ *is a perfectoid field and the map* $x \mapsto \big([x^{1/p^n}] \bmod \xi)_{n\geq 0}$ *induces an embedding* $F \hookrightarrow K_\xi^\flat$ *such that*
$$[K_\xi^\flat : F] = d.$$
(2) *If* F *is algebraically closed then any irreducible distinguished element* ξ *is of degree* 1. *We thus has*
$$K_\xi^\flat = F.$$
Moreover $\xi = u.(\pi - [a])$ *with* $a \in \mathfrak{m}_F \setminus \{0\}$ *and* $u \in A_{\text{inf}}^\times$.

(3) *For any* $I \subset]0,1[$ *with extremities in* $q^{\mathbb{Q}}$, *for any* $f \in \mathcal{O}(Y_{F,I}) \setminus \{0\}$, *and any slope* λ *of* $\text{Newt}_I(f)$, *there exists a factorization*
$$f = g.\xi$$
where $g \in \mathcal{O}(Y_{F,I})$, ξ *is distinguished irreducible with* $\text{Newt}_{]0,1[}(\xi)$ *a line with slope* λ *between* 0 *and* $\deg(\xi)$.

EXAMPLE 2.3 (Weierstrass factorization). *If* F *is algebraically closed and* ξ *is distinguished of degree* d *one can write*
$$\xi = u(\pi - [a_1]) \times \cdots \times (\pi - [a_d])$$
where u *is a unit and* $v(a_1), \ldots, v(a_d)$ *are the slopes of* $\text{Newt}_{]0,1[}(\xi)$.

DEFINITION 2.4. *A point* $y \in |Y_F|$ *of the form* $V(\xi)$ *with* ξ *distinguished irreducible is called a* **classical point** *of* Y_F. *By definition,* $\deg(y) := \deg(\xi)$.

Thus, for $y \in |Y_F|^{cl}$, $K(y)$ is a perfectoid field with
$$[K(y)^\flat : F] = \deg(y).$$
This is a form of the point of view that one may think of Y_F as a moduli of untilts of the perfectoid field F.

3. The adic curve

We finally arrive to the curve.

DEFINITION 3.1. *We note*
$$X_F = Y_F/\varphi^{\mathbb{Z}}$$
as a quasi-compact quasi-separated E*-adic space.*

This is thus strongly Noetherian sous-perfectoid with
$$(X_F \hat{\otimes}_F K)^\flat = (\text{Spa}(F) \times_{\text{Spa}(\mathbb{F}_q)} \text{Spa}(K^\flat))/\varphi^{\mathbb{Z}} \times \text{Id}\,.$$

This is a curve because of the following. This uses heavily the preceding factorization results.

THEOREM 3.2. *For any compact interval $I \subset]0,1[$ with extremities in $p^{\mathbb{Q}}$, the Banach E-algebra $\mathcal{O}(Y_{F,I})$* **is a P.I.D.** *with an identification*

$$\mathrm{Spm}(\mathcal{O}(Y_{F,I})) = |Y_{F,I}|^{cl}.$$

One deduces from this result that for any $U \subset Y_F$ an affinoid open subset, $\mathcal{O}(U)$ is a P.I.D. and thus X_F **is a curve**. In particular one has the following : for any $x \in |X_F|^{cl}$,

- $\mathcal{O}_{X,x}$ is an Henselian D.V.R. such that if $y \mapsto x$ with $y \in |Y_F|^{cl}$, $y = V(\xi)$, $\mathcal{O}_{X_F,x} \xrightarrow{\sim} \mathcal{O}_{Y_F,y}$,
- in particular **the residue field at x, $K(x)$, is perfectoid**,
- and one has

$$\widehat{\mathcal{O}}_{X_F,x} \xrightarrow{\sim} B_{dR}^+(K(x))$$

as complete D.V.R..

3.1. The schematical curve. The adic curve X_F does not come alone. It is in fact equipped with an "ample" line bundle.

DEFINITION 3.3. *We note $\mathcal{O}_{X_F}(1)$ for the line bundle on X_F associated to the automorphy factor $\varphi \mapsto \pi^{-1}$ on Y_F equipped with its action of $\varphi^{\mathbb{Z}}$.*

This means that the pullback of $\mathcal{O}_{X_F}(1)$ to Y_F is trivialized and the descent datum along the cover $Y_F \to X_F$ is given by $\varphi \mapsto \pi^{-1}$.

Let us define

$$\mathbb{B}(F) := \mathcal{O}(Y_F)$$

as a Frechet E-algebra equipped with the continuous automorphism φ. One has for any $d \in \mathbb{Z}$,

$$H^0(X_F, \mathcal{O}(d)) = \underbrace{\mathbb{B}(F)^{\varphi=\pi^d}}_{\{f \in \mathbb{B}(F) \mid \varphi(f) = \pi^d f\}}$$

that is

- 0 if $d < 0$,
- E if $d = 0$,
- an infinite dimension E-Banach space if $d > 0$.

REMARK 3.4. *Suppose $E = \mathbb{Q}_p$. If $y \in |Y_F|^{cl}$ there is an inclusion*

$$\bigcap_{n \geq 0} \varphi^n\big(B_{cris}^+(\mathcal{O}_{K(y)}/p)\big) \subset \mathbb{B}(F)$$

that induces for all $d \in \mathbb{Z}$ an identification

$$B^+_{cris}(\mathcal{O}_{K(y)}/p)^{\varphi=p^d} \xrightarrow{\sim} \mathbb{B}(F)^{\varphi=p^d}.$$

We now declare that $\mathcal{O}(1)$ is ample.

DEFINITION 3.5. *We define*

$$P_F = \bigoplus_{d\geq 0} H^0(X_F, \mathcal{O}_{X_F}(d))$$

as a graded E-algebra and

$$\mathfrak{X}_F = \mathrm{Proj}(P_F)$$

as an E-scheme.

One of the main structure results for the graded algebra P_F is the following.

THEOREM 3.6. *Suppose that F is algebraically closed. The graded E-algebra P_F is graded factorial in the sense that the commutative monoid*

$$\coprod_{n\geq 0} P_{F,n} \smallsetminus \{0\}/E^\times$$

is commutative free on degree 1 non-zero elements up to $E^\times$.

In other terms, for any $f \in P_{F,d} \smallsetminus \{0\}$, one can write

$$f = t_1 \dots t_d$$

where $t_1, \dots t_d \in P_{F,1} \smallsetminus \{0\}$ are uniquely determined up to multiplication by an element of $E^\times$. The proof of this theorem relies on two facts :

(1) Using the preceding results on the factorization of elements and Newton polygons one defines

$$\mathrm{Div}^+(Y_F) = \Big\{ \sum_{y\in|Y_F|^{cl}} a_y[y] \;\Big|\; a_y \in \mathbb{N}, \{y \mid a_y \neq 0\} \text{ is locally finite} \Big\}$$

and an injection of monoids

$$\mathrm{div} : \mathcal{O}(Y_F) \smallsetminus \{0\}/E^\times \hookrightarrow \mathrm{Div}^+(Y_F)$$

given by "the divisor of an holomorphic function". In particular, this defines an injection

$$\coprod_{n\geq 0} P_{F,n} \smallsetminus \{0\}/E^\times \hookrightarrow \mathrm{Div}^+(Y_F)^{\varphi=\mathrm{Id}}$$

where the right hand side is the free commutative monoid on $\{\sum_{n\in\mathbb{Z}}[\varphi^n(y)] \in \mathrm{Div}^+(Y_F)\ y \in |Y_F|^{cl} \bmod \varphi^{\mathbb{Z}}\}$.

(2) For any $y \in |Y_F|^{cl}$ one can construct (this is where the hypothesis F alg. closed shows up) some $t \in P_{F,1} \smallsetminus \{0\}$ such that $\mathrm{div}(t) = \sum_{n\in\mathbb{Z}}[\varphi^n(y)]$. In fact, when $E = \mathbb{Q}_p$, it suffices to take $t =$ Fontaine's $2i\pi$ associated to the algebraically closed field $K(y)|\mathbb{Q}_p$.

THEOREM 3.7. *The scheme $\mathfrak{X}_F$ is a Dedekind scheme.*

One can go further into the structure of $\mathfrak{X}_F$ using GAGA. More precisely, for any $t \in P_{F,1} \setminus \{0\}$, one has $D^+(t) = \mathrm{Spec}(B_{e,t})$ with

$$B_{e,t} = \mathbb{B}(F)[\tfrac{1}{t}]^{\varphi=\mathrm{Id}}$$

that is identified with $P_F[\frac{1}{t}]_0$. The morphism

$$B_{e,t} \hookrightarrow \mathbb{B}(F)[\tfrac{1}{t}] \to \mathcal{O}(Y_F \setminus V(t))$$

induces a morphism of ringed spaces $(Y_F \setminus V(t))/\varphi^{\mathbb{Z}} \to D^+(t)$. When t varies this defines a GAGA morphism of ringed spaces

$$X_F \longrightarrow \mathfrak{X}_F.$$

One then has the following result.

THEOREM 3.8. *Consider the GAGA morphism $X_F \to \mathfrak{X}_F$.*

(1) It induces a bijection $|X_F|^{cl} \xrightarrow{\sim} |\mathfrak{X}_F|^{closed}$ (closed points).

(2) For any $x \in |X_F|^{cl}$, if $x \mapsto x' \in |\mathfrak{X}_F|$, the morphism of D.V.R. $\mathcal{O}_{\mathfrak{X}_F,x'} \to \mathcal{O}_{X_F,x}$ induces an isomorphism

$$\widehat{\mathcal{O}}_{\mathfrak{X}_F,x'} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_F,x} = B^+_{dR}(K(x)).$$

In particular the residue fields at closed points of $\mathfrak{X}_F$ are perfectoid fields.

Let us note for x a closed point of $\mathfrak{X}_F$

$$\deg(x) = [K(x)^\flat : F].$$

We can now dig a little bit deeper into the structure of $\mathfrak{X}_F$.

THEOREM 3.9. *(1)* **The curve is complete :** *for any $f \in E(\mathfrak{X}_F)^\times$,*

$$\deg(\mathrm{div}(f)) = 0.$$

(2) If F is algebraically closed then for any $t \in P_{F,1} \setminus \{0\}$, $V^+(t)$ is one closed point ∞_t and $\boldsymbol{\mathfrak{X}_F \setminus \{\infty_t\} = \mathrm{Spec}(B_{e,t})}$ **with** $\boldsymbol{B_{e,t}}$ **a P.I.D.**. *In other words,*

$$\mathrm{Pic}^0(\mathfrak{X}_F) = 0.$$

(3) If F is algebraically closed one has

$$\boldsymbol{H^1(\mathfrak{X}_F, \mathcal{O}) = 0}$$

and

$$\boldsymbol{H^1(\mathfrak{X}_F, \mathcal{O}(-1)) \neq 0}.$$

Said in another way, for the stathme $\deg_t := -\mathrm{ord}_{\infty_t} : B_{e,t} \to \mathbb{N} \cup \{-\infty\}$, **the couple $\boldsymbol{(B_{e,t}, \deg_t)}$ is not euclidean but almost euclidean** *: for any $a, b \in B_{e,t}$ with $b \neq 0$ we can write $a = bx + y$ with $\deg_t(y) \leq \deg_t(b)$ but not $\deg_t(y) < \deg_t(b)$ in general.*

4. GAGA

THEOREM 4.1. *The GAGA morphism $X_F \to \mathfrak{X}_F$ induces an equivalence of categories*

$$\{\text{vector bundles on } \mathfrak{X}_F\} \xrightarrow{\sim} \{\text{vector bundles on } X_F\}.$$

At the heart of the preceding theorem is the following result due to Kedlaya : for any vector bundle $\mathscr{E}$ on X_F, for $n \gg 0$ one has

— $H^1(X_F, \mathscr{E}(n)) = 0$,

— $\mathscr{E}(n)$ is generated by its global sections.

Chapitre 5

5th course - October 17

1. Vector bundles on the curve

1.1. Isocrystals. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. We note $\breve{E} = \widehat{E^{un}}$ with its Frobenius σ lifting Frob_q. Recall the following definition.

DEFINITION 1.1. *An isocrystal is a pair (D, φ) where D is a finite dimensional $\breve{E}$-vector space and φ a σ-linear automorphism of D.*

Those are classified by Dieudonné-Manin in terms of slopes : the category of isocrystals is semi-simple with a unique isoclinic of slope λ object for each $\lambda \in \mathbb{Q}$. If $\lambda = \frac{d}{h}$ with $h \geq 1$ and $(d, h) = 1$ then the associated simple object has dimension h over $\breve{E}$ and in a suitable basis φ is given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \pi^d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma.$$

More precisely, one has an orthogonal decomposition

$$\mathrm{Isoc} = \overset{\perp}{\bigoplus_{\lambda \in \mathbb{Q}}} \underbrace{\mathrm{Isoc}^{\lambda}}_{\substack{\text{Slope } \lambda \\ \text{isoclinic isocrystals}}}$$

where Isoc^{λ} has a unique simple object as described before.

1.2. A simple construction. Let $F|\overline{\mathbb{F}}_q$ be a perfectoid field. We have a morphism

$$\begin{tikzcd} Y_F \arrow[loop right, "\varphi"] \arrow[d] \\ \mathrm{Spa}(\breve{E}) \arrow[loop right, "\sigma"] \end{tikzcd}$$

By pullback this induces a functor from σ-equivariant vector bundles on $\mathrm{Spa}(\breve{E})$, i.e. isocrystals, to φ-equivariant vector bundles on Y_F, i.e. vector bundles on X_F.

Definition 1.2. *(1) We note* $\boldsymbol{\mathscr{E}(D,\varphi)}$ *the vector bundle*

$$Y_F \overset{\varphi^{\mathbb{Z}}}{\times} D$$

on X_F *associated to the isocrystal* (D,φ).

(2) For $\lambda \in \mathbb{Q}$ *we note*

$$\boldsymbol{\mathcal{O}_{X_F}(\lambda)}$$

for $(D,\varphi) = (\breve{E}^h, \varphi)$ *where* $\lambda = \frac{d}{h}$ *with* $h \geq 1$, $(d,h) = 1$, *and*

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & \pi^{-d} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma.$$

The global sections of $(\mathscr{E},\varphi)$ are given by

$$H^0(X_F, \mathscr{E}(D,\varphi)) = \big(D \otimes_{\breve{E}} \mathbb{B}(F)\big)^{\varphi = \mathrm{Id}}$$

where "φ" here means $\varphi \otimes \varphi$ acting on $D \otimes_{\breve{E}} \mathbb{B}(F)$. In particular

$$H^0(X_F, \mathcal{O}(\lambda)) = \mathbb{B}(F)^{\varphi^h = \pi^d}.$$

We use the same notations for the associated vector bundle on $\mathfrak{X}_F$ via GAGA (Theorem 4.1). In fact we have the following formula : if

$$M(D,\varphi) = \bigoplus_{d \geq 0} \big(D \otimes_{\breve{E}} \mathbb{B}(F)\big)^{\varphi = \pi^d}$$

as a graded P_F-module then

$$\mathscr{E}(D,\varphi) = \widetilde{M(D,\varphi)}$$

on $\mathfrak{X}_F = \mathrm{Proj}(P_F)$.

At the end there is a **$\otimes$-exact functor** between monoidal categories

$$\mathscr{E}(-) : \mathrm{Isoc} \xrightarrow{\otimes} \{\text{vector bundles on } \mathfrak{X}_F\}. \tag{3}$$

1.3. Cohomology. For $n \geq 1$ let $E_n|E$ be the degree n unramified extension of E inside $\breve{E}$. There is an identification

$$\begin{aligned} X_{F,E} \otimes_E E_n &= X_{F,E_n} \\ \mathfrak{X}_{F,E} \otimes_E E_n &= \mathfrak{X}_{F,E_n} \end{aligned}$$

where X_{F,E_n} and $\mathfrak{X}_{F,E_n}$ are defined using $Y_{F,E_n} = Y_{F,E}$ but where the Frobenius has been replace by φ^n. At the end the finite Galois cover

$$\begin{tikzcd} Y_F/\varphi^{n\mathbb{Z}} \arrow[d] \arrow[d, bend right=60, dash, "\mathbb{Z}/n\mathbb{Z}"'] \\ Y_F/\varphi^{\mathbb{Z}} \end{tikzcd}$$

is identified with

$$\begin{tikzcd} X_{F,E_n} \arrow[r, equal] \arrow[d] \arrow[d, bend right=60, dash, "\mathrm{Gal}(E_n|E)"'] & X_{F,E} \otimes_E E_n \\ X_{F,E}. \end{tikzcd}$$

Let us note π_n this finite étale morphism. One easily verifies that if $\lambda = \frac{d}{h}$ as before then

$$\mathcal{O}_{X_{F,E}}(\lambda) = \pi_{h*}\mathcal{O}_{X_{F,E_h}}(d).$$

Using this, up to replacing E by a finite unramified extension, one deduces using Theorem 3.9 the following for $\lambda \in \mathbb{Q}$ and F algebraically closed :

- $H^0(X_F, \mathcal{O}(\lambda)) = \begin{cases} 0 \text{ if } \lambda < 0 \\ E \text{ if } \lambda = 0 \\ \text{an infinite dim. } E\text{-Banach space if } \lambda > 0 \end{cases}$
- $H^1(X_F, \mathcal{O}(\lambda)) = \begin{cases} \text{an infinite dim. } E\text{-Banach space if } \lambda < 0 \\ 0 \text{ if } \lambda \geq 0 \end{cases}$

1.4. Upgrade of the construction. Let G be a reductive group over E. By definition, an isocrystal with a G-structure is a $\otimes$-functor

$$\mathrm{Rep}(G) \xrightarrow{\otimes} \mathrm{Isoc}\,.$$

Another way to phrase it is to consider the Dieudonné gerbe

$$\begin{tikzcd} \mathfrak{D} \arrow[d] \\ \mathrm{Spec}(E) \end{tikzcd}$$

of fiber functors on Isoc seen as a stack over Spec(E) that is a cofiltered limit of algebraic stacks. More precisely, if $\mathbb{D}$ is the slope pro-torus with $X^*(\mathbb{D}) = \mathbb{Q}$ then Isoc is banded by $\mathbb{D}$ via the equivalence

$$\mathrm{Isoc} \otimes_E E^{un} \xrightarrow{\sim} \{\mathbb{Q}\text{-graded } E^{un}\text{-vector spaces}\}$$

given by the functor

$$(D,\varphi) \longmapsto \bigoplus_{\lambda\in\mathbb{Q}} \bigcup_{n\gg 1} D^{\varphi^n=\pi^{n\lambda}}.$$

One has $\mathrm{Isoc} = \cup_{n\geq 1} \mathrm{Isoc}_n$ where Isoc_n is the Tannakian category of isocrystals with slopes in $\frac{1}{n}\mathbb{Z}$. Then,

$$\mathfrak{D} = 2 - \varprojlim_{n\geq 1} \underbrace{\mathfrak{D}_n}_{\substack{\text{algebraic stack,}\\ \text{gerbe banded by } \mathbb{G}_m\\ \text{neutral over } E_n}}$$

There is then an identification

$$\{\text{Isocrystals with a } G\text{-structure}\} \xrightarrow{\sim} \{\text{étale } G\text{-torsors on } \mathfrak{D}\}.$$

Definition 1.3. *We note* $\boldsymbol{B(G)}$ *for the set of isomorphism classes of isocrystals equipped with a G-structure.*

One thus has

$$B(G) = H^1_{\text{ét}}(\mathfrak{D}, G).$$

According to Steinberg, $H^1(\breve{E}, G)$ is trivial. From this one deduces that

$$B(G) = G(\breve{E})/\sim$$

where $\sim$ is the σ-conjugacy relation,

$$b \sim gbg^{-\sigma}.$$

To $b \in G(\breve{E})$ one associates the G-isocrystal that sends $(V,\rho) \in \mathrm{Rep}(G)$ to the isocrystal $(V \otimes_E \breve{E}, \rho(b)\sigma)$.

The functor (3) from isocrystals to vector bundles on the curve $\mathscr{E}(-)$ defines a morphism of stacks

$$\begin{tikzcd}
\mathfrak{X}_F \arrow[dr, "\text{induced by } \mathscr{E}(-)"] \arrow[dd, "\text{structural morphism}"'] & \\
& \mathfrak{D} \arrow[dl] \\
\mathrm{Spec}(E) &
\end{tikzcd}$$

and thus by pullback a map

$$\{G\text{-isocrystals}\} \longrightarrow \{\text{étale } G\text{-torsors on } \mathfrak{X}_F\}$$

inducing

$$\boxed{B(G) \longrightarrow H^1_{\text{ét}}(\mathfrak{X}_F, G).}$$

DEFINITION 1.4. *For $b \in G(\breve{E})$ we note $\boldsymbol{\mathscr{E}_b}$ the associated G-bundle on $\mathfrak{X}_F$.*

2. Semi-stability

2.1. Vector bundles. Since "$\mathfrak{X}_F$ is complete", there is a "nice" degree function

$$\deg : \mathrm{Pic}(\mathfrak{X}_F) \longrightarrow \mathbb{Z}$$

simply defined by the formula $\deg(\mathscr{L}) = \deg(\mathrm{div}(s))$ where s is any rational section of $\mathscr{L}$, $s : E(\mathfrak{X}_F) \xrightarrow{\sim} \mathscr{L}_\eta$. This allows us to define the degree of a vector bundle $\mathscr{E}$ via the formula

$$\deg(\mathscr{E}) := \deg(\det(\mathscr{E})).$$

Its main property is that if $u : \mathscr{E} \to \mathscr{E}'$ is a morphism between vector bundles that is generically an isomorphism then $\deg(\mathscr{E}) \leq \deg(\mathscr{E}')$ with equality if and only if u is an isomorphism. This property implies the existence and uniqueness of Harder-Narasimhan filtrations for the slope function

$$\mu = \frac{\deg}{\mathrm{rk}}.$$

EXAMPLE 2.1. *For any $\lambda \in \mathbb{Q}$ the vector bundle $\mathcal{O}(\lambda)$ is semi-stable with slope λ. In fact $\mathcal{O}(\lambda)$ is the pushforward via a finite étale morphism of a semi-stable vector bundle : the direct sum of line bundles of the same degree (the direct sum of two semi-stable vector bundles with same slope is semi-stable).*

2.2. Principal G-bundles. Here we suppose G is quasi-split to simplify (in fact $\mathfrak{X}_F \times G$ is a quasi-split reductive group scheme over $\mathfrak{X}_F$ for any G). If $\mathscr{E}$ is an étale G-torsor recall that $\mathscr{E}$ is **semi-stable** if for any parabolic subgroup P of G, for any reduction $\mathscr{E}_P$ of $\mathscr{E}$ to P,

$$\deg(s^* T_{P \backslash \mathscr{E}}) \geq 0$$

(tangent bundle of $P\backslash\mathscr{E} \to \mathfrak{X}_F$) where s is the section

$$\begin{tikzcd} P\backslash\mathscr{E} \arrow[d] \\ \mathfrak{X}_F \arrow[u, bend left, "s"] \end{tikzcd}$$

corresponding to the reduction $\mathscr{E}_P$ i.e. the P-torsor $\mathscr{E}_P$ is the pullback by s of the étale P-torsor $\mathscr{E} \to P\backslash\mathscr{E}$.

One can then prove that for any $\mathscr{E}$ there exists (up to $G(E)$-conjugacy) a unique parabolic subgroup P and a reduction $\mathscr{E}_P$ of $\mathscr{E}$ to E satisfying :

(1) $\mathscr{E}_P \overset{P}{\times} \underbrace{P/R_uP}_{\text{Levi quotient}}$ is semi-stable,

(2) for any $\chi \in X^*(P/Z_G) \smallsetminus \{0\} \cap \mathbb{N}.\Delta$ we have $\deg \chi_* \mathscr{E} > 0$ where Δ is the set of simple roots.

This is the so-called **canonical reduction of $\boldsymbol{\mathscr{E}}$**.

3. Vector and numerical invariants

3.1. *B(G)*. There is an exact sequence of pointed sets

$$1 \to \underbrace{H^1(E,G)}_{\substack{\text{unit root}\\ G\text{-isocrystal}}} \longrightarrow B(G) \longrightarrow \Big[\mathrm{Hom}(\mathbb{D}_{\overline{E}}, G_{\overline{E}}) \underbrace{/}_{\substack{\text{action by}\\ \text{conjugation}}} G(\overline{E})\Big]^{\Gamma_E}$$

that is identified with a low degree Hochschild-Serre spectral sequence

$$1 \to H^1(E,G) \longrightarrow H^1_{\text{ét}}(\mathfrak{D}, G) \longrightarrow H^0\big(\Gamma_E, H^1_{\text{ét}}(\mathfrak{D}_{\overline{E}}, G)\big).$$

Here by "unit root" we mean slope 0. For $[b] \in B(G)$ we note $[\nu_b]$ for the class of the associated morphism $\mathbb{D}_{\overline{E}} \xrightarrow{\nu_b} G_{\overline{E}}$.

Let us suppose, to simplify, that G is quasi-split. Let $A \subset T \subset B$ be the inclusion of a maximal split torus inside a maximal torus inside a Borel subgroup. Then,

$$\boxed{[\nu_b] \in X_*(A)^+_{\mathbb{Q}}}$$

is the **generalized Newton polygon of [*b*]**.

There is a second invariant associated to $[b]$,

$$\boxed{\kappa(b) \in \underbrace{\pi_1(G)}_{\substack{\text{Borovoi}\\ \text{fund. group}}}{}_\Gamma.}$$

that is the generalization of the endpoint of the Newton polygon of an isocrystal. In fact, the images of $[\nu_b]$ and $\kappa(b)$ in $\pi_1(G)_\Gamma \otimes \mathbb{Q}$ are equal.

The abelian group $\pi_1(G)$ is Borovoi's fundamental group,

$$\pi_1(G) = X_*(T)/\langle\check{\Phi}\rangle$$

where $\check{\Phi}$ is the set of coroots. Its profinite completion is identified with Grothendieck's étale fundamental group :

$$\widehat{\pi_1(G)} \underset{\circlearrowleft\,\Gamma_E}{=} \pi_1^{\text{ét}}(G_{\overline{E}}).$$

This is defined via an **abelianization map**

$$B(G) = H^1(\sigma^{\mathbb{Z}}, G(\breve{E})) \longrightarrow \underbrace{H^1\big(\sigma^{\mathbb{Z}}, \underbrace{[G_{sc}(\breve{E}) \to G(\breve{E})]}_{\text{crossed module}}\big)}_{\substack{\text{group coho. with coeff.}\\ \text{in a crossed module}}}.$$

Here G_{sc} is the universal cover of the derived subgroup G_{der} and G acts on G_{sc} via the morphism $G \longrightarrow G_{ad} \xrightarrow{\text{conj. action}} \mathrm{Aut}(G_{sc})$. If T_{sc} is the pullback of $T \cap G_{der}$ to G_{sc} the morphism of crossed modules

$$[T_{sc} \to T] \longrightarrow [G_{sc} \to G]$$

is a quasi-isomorphism and thus induces a bijection

$$H^1(\sigma^{\mathbb{Z}}, T_{sc}(\breve{E}) \to T(\breve{E})) \xrightarrow{\sim} H^1\big(\sigma^{\mathbb{Z}}, [G_{sc}(\breve{E}) \to G(\breve{E})]\big).$$

We deduce an exact sequence

$$B(T_{sc}) \longrightarrow B(T) \longrightarrow H^1(\sigma^{\mathbb{Z}}, T_{sc}(\breve{E}) \to T(\breve{E})) \longrightarrow 0.$$

One key result due to Kottwitz is that for a torus S, there is a canonical (in S) identification

$$\boxed{B(S) \xrightarrow{\sim} X_*(S)_\Gamma}$$

such that $B(\mathbb{G}_m) = \mathbb{Z}$ is given by $\breve{E}^\times \ni x \mapsto v(x)$. We deduce our κ map

$$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma.$$

3.2. Principal G-bundles. Suppose again that G is quasi-split. Let $\mathscr{E}$ be an étale G-torsor on $\mathfrak{X}_F$ and let $\mathscr{E}_P$ be its canonical reduction where P is a standard parabolic subgroup with respect to the choice of B as before. The morphism

$$\begin{array}{rcl} X^*(P) & \longrightarrow & \mathbb{Z} \\ \chi & \longmapsto & \deg(\chi_* \mathscr{E}_P) \end{array}$$

can be seen as an element of $X_*(A)_{\mathbb{Q}}$. Moreover the second condition in the definition of the canonical reduction of $\mathscr{E}$ implies this is an element of $X_*(A)^+_{\mathbb{Q}}$. We note it

$$\boxed{[\nu_{\mathscr{E}}] \in X_*(A)^+_{\mathbb{Q}}}$$

and we think about it as a **generalized Harder-Narasimhan polygon**.

As before there is an abelianization map

$$H^1_{\text{ét}}(\mathfrak{X}_F, G) \longrightarrow H^1_{\text{ét,ab}}(\mathfrak{X}_F, G) := H^1_{\text{ét}}(\mathfrak{X}_F, \underbrace{[G_{sc} \to G]}_{\substack{\text{crossed module} \\ \text{on } \mathfrak{X}_{F,\text{ét}}}}).$$

One can prove that when F is algebraically closed then for a torus S over E

$$B(S) = H^1_{\text{ét}}(\mathfrak{D}, S) \xrightarrow{\sim} H^1_{\text{ét}}(\mathfrak{X}_F, S)$$

(the proof is reduced to the $\mathbb{G}_m$-case where one of the key ingredients is to prove that $\text{Br}(\mathfrak{X}_F) = 0$; we will later see that this isomorphism is true for any reductive group G but one can give a simpler proof for a torus) and thus

$$H^1_{\text{ét,ab}}(\mathfrak{D}, G) \xrightarrow{\sim} H^1_{\text{ét,ab}}(\mathfrak{X}_F, G).$$

At the end this allows us to define

$$c_1(\mathscr{E}) \in \pi_1(G)_\Gamma$$

the first Chern class of $\mathscr{E}$.

4. Classification of G-isocrystals

Recall the following definition that generalizes the definition of an isoclinic isocrystal.

Definition 4.1. *The element $[b] \in B(G)$ is basic if ν_b is central.*

One of the first basic results in the domain is the following.

Proposition 4.2. *Kottwitz κ map induces a bijection*

$$\kappa_{|B(G)_{bsc}} : B(G)_{bsc} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

Remark 4.3. *This result that seems mysterious at first will be fully understood later : any connected component of* Bun_G *contains a unique semi-stable point. For any $[b]$, the basic element associated to $\kappa(b)$ in $B(G)$ will correspond to the maximal generalization of the point of* Bun_G *associated to $[b]$.*

We still suppose that G is quasi-split and we fix $A \subset T \subset B$. Then, if M_b is is the Standard Levi subgroup that is the centralizer of the slope morphism $[\nu_b] \in X_*(A)^+_{\mathbb{Q}}$, **$[b]$ has a canonical basic reduction $[b_{M_b}] \in B(M_b)_{\mathrm{basic}}$.**

Finally, one can prove that the map $[b] \mapsto ([\nu_b], \kappa(b))$ is an injection

$$B(G) \hookrightarrow \pi_1(G)_\Gamma \times X_*(A)^+_{\mathbb{Q}}.$$

One can describe its image but this is not useful for what we do. Let us just remark that the injectivity of this map will later be reinterpreted as saying that on any connected component $\mathcal{C}$ of Bun_G, the map given by $[b] \mapsto [\nu_b]$ induces an injection $|\mathcal{C}| \hookrightarrow X_*(A)^+_{\mathbb{Q}}$.

5. Classification of principal G-bundles

5.1. Vector bundles. The following classification result is a difficult very important result in the domain.

Theorem 5.1. *Suppose F is algebraically closed. There is a bijection*

$$\begin{aligned} \big\{\lambda_1 \geq \cdots \geq \lambda_r \mid r \in \mathbb{N},\ \lambda_i \in \mathbb{Q}\big\} &\xrightarrow{\sim} \big\{v.b. \text{ on } \mathfrak{X}_F\big\}/\sim \\ (\lambda_1, \ldots, \lambda_r) &\longmapsto \Big[\bigoplus_{i=1}^r \mathcal{O}(\lambda_i)\Big]. \end{aligned}$$

In terms of reduction theory, this can be split in two parts :

(1) Slope λ semi-stable vector bundles are isomorphic to directs sums of $\mathcal{O}(\lambda)$,

(2) The Harder-Narasimhan filtration of a vector bundle is (non-canonically) split.

Point (2) is an immediate consequence of point (1) since

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = H^1(\mathfrak{X}_F, \underbrace{\mathcal{O}(-\lambda) \otimes \mathcal{O}(\mu)}_{\substack{\text{finite direct} \\ \text{sum of } \mathcal{O}(\mu-\lambda)}})$$

is zero if $\lambda \leq \mu$ when F is algebraically closed.

5.2. Principal G-bundles. Here is the main result.

THEOREM 5.2. *When F is algebraically closed there is a bijection of pointed sets*

$$\begin{array}{ccc} B(G) & \xrightarrow{\sim} & H^1_{\text{ét}}(\mathfrak{X}_F, G) \\ [b] & \longmapsto & [\mathscr{E}_b]. \end{array}$$

Via this theorem we have the following **dictionary between arithmetic and geometry** :

(1) $\underbrace{[b] \text{ is basic}}_{\substack{\text{arithmetic condition} \\ \text{on the } p\text{-adic} \\ \text{valuations of the eigenvalues} \\ \text{of Frob}}} \Leftrightarrow \underbrace{\mathscr{E}_b \text{ is semi-stable}}_{\substack{\text{geometric semi-stability} \\ \text{condition}}}$

(2) $\underbrace{[\nu_b]}_{\text{Newton polygon}} = w.(- \underbrace{[\nu_{\mathscr{E}_b}]}_{\text{HN polygon}})$ where w is the longest element in the Weyl group

(3) $\underbrace{\kappa(b)}_{\substack{\text{terminal point} \\ \text{of Newton}}} = - \underbrace{c_1(\mathscr{E}_b)}_{\text{first Chern class}}$.

6. On the proof of the classification theorem

6.1. Background on Beauville-Laszlo. Let $\infty \in |\mathfrak{X}_F|$ be a degree 1 closed point with residue field K. We note $B^+_{dR} := B^+_{dR}(K) = \widehat{\mathcal{O}}_{\mathfrak{X}_F,\infty}$ with uniformizer t.

A modification of a G-bundle $\mathscr{E}$ at ∞ is the data given by a a G-bundle $\mathscr{E}'$ together with an isomorphism

$$\mathscr{E}_{|\mathfrak{X}_F \setminus \{\infty\}} \xrightarrow{\sim} \mathscr{E}'_{|\mathfrak{X}_F \setminus \{\infty\}}.$$

When $G = \mathrm{GL}_n$, Beauville-Laszlo tells us that such a modification is the same as the datum of a B^+_{dR}-lattice in $\widehat{\mathscr{E}}_\infty[\frac{1}{t}]$ where here we see $\mathscr{E}$ as a vector bundle. In general, this is the same as an étale G-torsor $\mathscr{F}$ on $\mathrm{Spec}(B^+_{dR})$ together with an isomorphism

$$\mathscr{E} \times_{\mathfrak{X}_F} \mathrm{Spec}(B_{dR}) \xrightarrow{\sim} \mathscr{F} \times_{\mathrm{Spec}(B^+_{dR})} \mathrm{Spec}(B_{dR}).$$

Since B_{dR}^{+} is complete and the residue field of K contains $\overline{\mathbb{F}}_q$ any étale G-torsor on $\mathrm{Spec}(B_{dR}^{+})$ is trivial (Steinberg, Hensel). We deduce that this is the same as an element of

$$\mathscr{E}(B_{dR})/G(B_{dR}^{+}).$$

where $\mathscr{E}(B_{dR})$ is the set of sections

$$\begin{tikzcd} & \mathscr{E} \arrow[d] \\ \mathrm{Spec}(B_{dR}) \arrow[r] \arrow[ur, bend left, "\text{section}"] & \mathfrak{X}_F \end{tikzcd}$$

6.2. Some piece of the Hecke groupoid.

Let us now remark that **for any $\boldsymbol{b \in G(\breve{E})}$ there is a canonical trivialization of**

$$\mathscr{E}_b \times_{\mathfrak{X}_F} \mathrm{Spec}(B_{dR}^{+}).$$

Suppose now that $\mathscr{E} = \mathscr{E}_b$ and $\mathscr{E}' = \mathscr{E}_{b'}$. Then, modifications

$$\mathfrak{M} : \mathscr{E}_b \dashrightarrow \mathscr{E}_{b'}$$

at ∞ are given by an element of

$$p_1(\mathfrak{M}) \in G(B_{dR})/G(B_{dR}^{+})$$

and its inverse

$$p_2(\mathfrak{M}) := p_1(\mathfrak{M}^{-1}) \in G(B_{dR})/G(B_{dR}^{+}).$$

The images of $p_1(\mathfrak{M})$ and of $p_2(\mathfrak{M})^{-1} \in G(B_{dR}^{+})\backslash G(B_{dR})$ in

$$G(B_{dR}^{+})\backslash G(B_{dR})/G(B_{dR}^{+})$$

are equal. We call this the type of the modification.

Suppose that G is split to simplify. There is then a bijection

$$\begin{array}{rcl} X_*(A)^{+} & \xrightarrow{\sim} & G(B_{dR}^{+})\backslash G(B_{dR})/G(B_{dR}^{+}) \\ \mu & \longmapsto & G(B_{dR}^{+})\mu(t)G(B_{dR}^{+}). \end{array}$$

We equip $X_*(A)^{+}$ with the order $\mu \leq \mu'$ if $\mu' - \mu \in \mathbb{N}.\Delta^{\vee}$.

Definition 6.1. *For G split over E and $\{\mu\}$ a conjugacy class of cocharaters of G we define for $F|\overline{\mathbb{F}}_q$ a perfectoid field*

(1)
$$\mathrm{Sh}(G, b, b', \mu)(F)$$
as the set of a degree 1 point $\infty \in |\mathfrak{X}_F|$ and a modification at ∞
$$\mathscr{E}_b \dashrightarrow \mathscr{E}_{b'}$$
of type $\leqslant \mu$.

(2)
$$\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F)$$
as the set of a degree 1 closed point ∞ on $\mathfrak{X}_F$ with residue field K and an element of $G(B_{dR}(K))/G(B^+_{dR}(K))$ whose image in $G(B^+_{dR}(K))\backslash G(B_{dR}(K))/G(B^+_{dR}(K))$ is $\leqslant \mu$.

We thus have two maps

$$\begin{tikzcd} & \mathrm{Sh}(G, b, b', \mu)(F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F) & & \mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu^{-1}}(F) \end{tikzcd}$$

6.3. Modifications of vector bundles associated to p-divisible groups. The following is the starting point of the link between Rapoport-Zink spaces and modification of vector bundles on the curve.

Proposition 6.2. *Let $\mathcal{M}$ be the deformation space by quasi-isogenies of the p-divisible group $\mathbb{H}$ over $\overline{\mathbb{F}}_p$ as defined by Rapoport-Zink. Let $C|\breve{\mathbb{Q}}_p$ be algebraically closed and consider an element $x \in \mathcal{M}(\mathcal{O}_C)$. Let*
- *V be the rational Tate module of the universal deformation specialized at x,*
- *(D, φ) be covariant isocrystal of $\mathbb{H}$,*
- *$\mathrm{Fil}\, D_C$ be the Hodge filtration.*

There is a canonical exact sequence of coherent sheaves on $\mathfrak{X}_{C^\flat}$
$$0 \longrightarrow V \otimes \mathcal{O}_{\mathfrak{X}_{C^\flat}} \longrightarrow \mathscr{E}(D, p^{-1}\varphi) \longrightarrow i_{\infty *} D_C/\,\mathrm{Fil}\, D_C \longrightarrow 0$$
where $\infty \in |\mathfrak{X}_{C^\flat}|$ is the closed point associated to the untilt C of $C^\flat$.

This is a rewriting in terms of the curve of Fontaine/Faltings comparison theorems :
$$V \otimes_{\mathbb{Q}_p} \mathbb{B}(C^\flat)[\tfrac{1}{t}] \xrightarrow{\sim} D \otimes_{\breve{\mathbb{Q}}_p} \mathbb{B}(C^\flat)[\tfrac{1}{t}].$$
where
$$\mathrm{Id} \otimes \varphi \leftrightarrow p^{-1}\varphi \otimes \varphi.$$

Define now for $F|\overline{\mathbb{F}}_p$ algebraically closed

$$\mathcal{M}^{\diamond}_{\eta,\infty}/\varphi^{\mathbb{Z}}(F)$$

as the set of

- an untilt C of F over E up to a power of Frobenius (i.e. the identification between F and $C^{\flat}$ is taken up to a power of Frobenius),
- an object of $\mathcal{M}(\mathcal{O}_C)$,
- an infinite level structure on this object i.e. a base of the associated rational Tate module.

Let d be the dimension of $\mathbb{H}$ and set $\mu(z) = \mathrm{diag}(\underbrace{z,\dots,z}_{d\text{-times}},1,\dots,1)$ for $G = \mathrm{GL}_n$ over $\mathbb{Q}_p$.

Set $G = \mathrm{GL}_n$ where n is the height of $\mathbb{H}$. The preceding proposition defines a map for F algebraically closed

$$\boxed{\begin{tikzcd}
& \mathcal{M}^{\diamond}_{\eta,\infty}/\varphi^{\mathbb{Z}}(F) \arrow[d] \arrow[ddl, bend right, "\pi_{dR}"'] \arrow[ddr, bend left, "\pi_{HT}"] & \\
& \mathrm{Sh}(G,b,1,\mu)(F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F) & & \mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu^{-1}}(F)
\end{tikzcd}}$$

where here there is an identification between $\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F)$ and $\mathscr{F}^{\diamond}_{\mu}/\varphi^{\mathbb{Z}}(F)$ since μ is minuscule.

6.4. Application to the classification. As a consequence of the study of de Rham and Hodge-Tate periods of Lubin-Tate spaces one deduces the following from the preceding construction

$$\boxed{\{p\text{-divisible groups}/\mathcal{O}_C\} \longrightarrow \{\text{modifications of vector bundles }/\mathfrak{X}_{C^{\flat}}\}}$$

For F algebraically closed :

- (*Surjectivity of the de Rham period morphism for L.T. spaces*) For any exact sequence of coherent sheaves on $\mathfrak{X}_F$
$$0 \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}(\tfrac{1}{n}) \longrightarrow \mathscr{F} \longrightarrow 0,$$
where $\mathscr{F}$ is a torsion coherent sheaf of degree 1, one has
$$\mathscr{E} \simeq \mathcal{O}^n.$$
- (*Computation of the image of π_{HT} for L.T. spaces*) For any exact sequence of coherent sheaves on $\mathfrak{X}_F$
$$0 \longrightarrow \mathcal{O}^n \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0$$

where $\mathscr{F}$ is a torsion coherent sheaf of degree 1, one has

$$\mathscr{E} \simeq \mathcal{O}^{n-r} \oplus \mathcal{O}(\tfrac{1}{r})$$

for some integer $1 \leq r \leq n$.

Using those two results about degree 1 modifications of vector bundles on the $\mathfrak{X}_F$ one can obtain by elementary manipulations the classification theorem 5.1.

6.5. Equivalence. Reciprocally one can prove the following result (using the classification of vector bundles on the curve).

THEOREM 6.3 (F., Scholze-Weinstein). *The following is satisfied :*

(1) For F algebraically closed the map

$$\mathcal{M}^{\diamond}_{\eta,\infty}/\varphi^{\mathbb{Z}}(F) \longrightarrow \mathrm{Sh}(G, b, 1, \mu)(F)$$

is a bijection.

(2) The functor that sends a p-divisible group over $\mathcal{O}_C$ to the corresponding modification of vector bundles on $\mathfrak{X}_{C^\flat}$

$$\mathscr{E} \dashrightarrow \mathscr{E}',$$

with $\mathscr{E}$ a trivial vector bundle, together with a lattice in $H^0(\mathfrak{X}_{C^\flat}, \mathscr{E})$, is an equivalence of categories.

Using this result together with some arguments bout Banach-Colmez spaces one can prove that in infinite level Rapoport-Zink spaces are perfectoid and are identified as a moduli of modifications of vector bundles.

7. p-adic Hodge structures and local Shtukas

The following result says that a (geometric) p-adic Hodge structure is the same as a local Shtukas.

THEOREM 7.1 (F.). *Let $C|E$ be algebraically closed. Let ξ be a generator of the kernel of $\theta : A_{\mathrm{inf}} := W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$. There is an equivalence of categories between*

- **local Shtukas** *i.e. couples (M, φ) where M is a free A_{inf}-module of finite type and φ an isomorphism*

$$\varphi : M[\tfrac{1}{\varphi^{-1}(\xi)}] \xrightarrow{\sim} M[\tfrac{1}{\xi}]$$

- **geometric p-adic Hodge structures** *i.e. modifications of vector bundles*

$$\mathscr{E} \dashrightarrow \mathscr{E}'$$

at ∞ where $\mathscr{E}$ is a trivial vector bundle.

Remark 7.2. *This last result is the starting point of $A_{\inf}$-cohomology. In fact if X is a proper smooth algebraic variety over K where $[K:\mathbb{Q}_p] < +\infty$ and $C = \widehat{\overline{K}}$, the "classical" comparison theorems (Fontaine, Fontaine-Messing, Tsuji, Faltings) associates to any cohomologycal degree $i \in \mathbb{N}$ a modification of vector bundles $\mathscr{E} \dashrightarrow \mathscr{E}'$ on $\mathfrak{X}_{C^\flat}$ where*

- $\mathscr{E} = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}$
- $\mathscr{E}' = \mathscr{E}(D, \varphi)$ *where* $D = H^i_{\text{cris}}(X_{\overline{k_K}}, W)\left[\frac{1}{p}\right]$ *with its crystalline Frobenius.*

The lattice

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)/\textit{torsion}$$

then gives rise by application of the preceding theorem to a φ-module over $A_{\inf}$. One of the starting point of $A_{\inf}$-cohomology is to refine this construction by construction a cohomology complex in the derived category of $A_{\inf}$-modules to take into account torsion.

8. Some final thoughts

Theorem 6.3 has been a great motivation for the introduction of the geometrization conjecture. Already in my joint work with Fontaine, in the proof we gave of weakly admissible implies admissible, modification of vector bundles and B^+_{dR}-lattices showed up in an essential way. The appearance of Kottwitz set already, strongly linked to the mod p geometry of Shimura varieties, is a great sign.

Chapitre 6

6th lecture - October 24

1. What is a diamond ?

- Schemes are obtained by gluing affine schemes for the Zariski topology.
- Algebraic spaces are obtained by gluing affine schemes for the étale topology.
- Usually one adds a coherence condition in the definition of an algebraic space, one typically assumes that they are quasi-separated to remove pathological objects like $\mathbb{G}_{a,\mathbb{C}}/\mathbb{Z}$ (action by translations) that is not quasi-separated since the étale topology is not coarse enough contrary to the analytic topology where $\mathbb{C}/\mathbb{Z}$ is a nice good object as a complex analytic space.
- There is an Artin criterion for algebraic spaces.

The theory of diamonds follows the same path by replacing schemes by $\mathbb{F}_p$-perfectoid spaces and the étale topology by the pro-étale topology. The nice coherence condition that one adds to make them look like analytic adic spaces is called **the spatialness condition**. There is even an analog of **Artin's criterion**.

Historically there has been different sources of diamonds :

(1) The first one comes from he theory of **finite dimensional Banach spaces in the sense of Colmez**. The first, from the historical point of view, typical question being to describe geometrically $\mathbb{B}^{\varphi=p^2}$ that, contrary to $\mathbb{B}^{\varphi=p}$ that is a 1-dimensional open ball, is a quotient of a 2-dimensional ball by a pro-étale pro-p equivalence relation but is not represented by a perfectoid space.

(2) The second one is the remark that points of the curve correspond to **untilts** up to Frobenius, this has lead to the introduction of $\mathbf{Spa}(\mathbb{Q}_p)^\diamond$.

(3) The third one is the remark due to Faltings and Colmez that **pro-étale locally analytic adic spaces are perfectoid** ; this is typically the remark due to Faltings that "the Frobenius of $\overline{R}/p\overline{R}$ is surjective". This has lead later to the construction of $X^\diamond$ where X is an analytic adic space.

(4) The fourth one come from the desire to put **a geometric structure on the set of** B_{dR}^+**-latticed in** B_{dR}^n, a construction that already showed up in the curve proof of weakly admissible implies admissible.

2. Background on the pro-étale and the v-topology

Let **Perf** be the category of perfectoid spaces.

Here, and this is essential for our work, see Remark 2.2, we consider affinoid perfectoid algebras that may not contain a field. They are classified as couples $((R,R^+),I)$ where

- (R, R^+) is an $\mathbb{F}_p$-affinoid perfectoid algebra,
- $I \subset W(R^+)$ is an ideal generated by a degree 1 distinguished element i.e. an element of the form $\sum_{n\geq 0}[a_n]p^n$ where $a_0 \in R^{\circ\circ}$ and $a_1 \in (R^+)^\times$ (such elements are regular and such an ideal I is a Cartier divisor).

This correspondence is given by the following rules :

- To (A, A^+) affinoid perfectoid we associate
$$\big((A^\flat, A^{\flat,+}), \ker\theta\big)$$
where $\theta : W(A^{\flat,+}) \to A^+$.
- In the other direction, to $((R, R^+), I)$ we associate
$$\big(W(R^+)/I[\tfrac{1}{[\varpi]}], W(R^+)/I\big).$$

If (A, A^+) contains a field and corresponds to $((R, R^+), (\xi))$ with $\xi = \sum_{n\geq 0}[a_n]p^n$ then either $a_0 = 0$ i.e. A contains $\mathbb{F}_p$, either $a_0 \in R^\times$ i.e. A contains $\mathbb{Q}_p$.

Example 2.1. *If we take*
$$(R, R^+) = (K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle)$$
with K a characteristic p perfectoid field and
$$I = ([T] + p)$$
then the corresponding perfectoid space $S = \mathrm{Spa}(A, A^+)$ satisfies $|S| = |\mathbb{B}^1_K|$ that is connected. The open subset $|\mathbb{B}^1_K \smallsetminus \{0\}|$ is a $\mathbb{Q}_p$-perfectoid space and the origin $\{0\} \subset |\mathbb{B}^1_K|$ is $\mathrm{Spa}(K)$ that is an $\mathbb{F}_p$-perfectoid space.

Remark 2.2. *From this example we deduce a quotient map*
$$|\mathbb{B}^1_K| \to \underbrace{|\mathrm{Spa}(\mathbb{Z}_p)^\diamond|}_{\substack{\text{top. space}\\ \text{associated}\\ \text{to a small}\\ v\text{-sheaf}\\ \text{(see later)}}} \underbrace{=}_{\substack{\text{as a}\\ \text{set}}} \{s, \eta\}$$
where the image of $|\mathbb{B}^1_K \smallsetminus \{0\}|$ is η and the one of $\{0\}$ is s. This implies that $\eta \geqslant s$ and thus $|\mathrm{Spa}(\mathbb{Z}_p)^\diamond| = \{s, \eta\}$ with $\eta \geqslant s$ as a topological space. This fact is crucial for the proof of the geometric Satake correspondence where we use a degeneration of the B_{dR}-affine Grassmanian from $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ to $\mathrm{Spa}(\mathbb{F}_p)^\diamond$ via $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ to the usual Witt vector affine Grassmanian where we can apply some classical arguments using the decomposition theorem.

The category Perf is equipped with three natural Grothendieck topologies.

2.1. The étale topology. This is the usual étale topology on perfectoid spaces. One of its main properties is that it is compatible with the tilting equivalence : if S is a perfectoid space, via the equivalence

$$(-)^\flat : \mathrm{Perf}_S \xrightarrow{\sim} \mathrm{Perf}_{S^\flat},$$

$T \to S$ is étale if and only if $T^\flat \to S^\flat$ is étale. This is part of the so-called purity theorem. Among its elementary properties is the fact that any étale morphism is open.

In general the étale site of a perfectoid space is considered as a small site.

2.2. The pro-étale topology.

2.2.1. *Definition.* One of the great features of perfectoid spaces, compared to "classical Noetherian analytic adic spaces" is that some operations that do not exist in the Noetherian world make a sense for perfectoid spaces. Typically, if $(S_i)_i$ is a cofiltered projective system of affinoid perfectoid spaces, $S_i = \mathrm{Spa}(R_i, R_i^+)$, then

$$\varprojlim_i S_i$$

is well defined, and affinoid perfectoid, as $\mathrm{Spa}(R_\infty^+[\frac{1}{\varpi}], R_\infty^+)$ where R_∞^+ is the ϖ-adic completion of $\varinjlim_i R_i^+$ and ϖ is the image of some pseudo-uniformizer in R_i for some index i.

Recall the following definition.

Definition 2.3. *A morphism $T \to S$ of perfectoid spaces is* **pro-étale** *if it can be written locally on T and S as*

$$T = \varprojlim_{i \geq i_0} S_i \longrightarrow S_{i_0} = S$$

where $(S_i)_i$ is a cofiltered projective system of affinoid perfectoid spaces with étale transition morphisms.

The pro-étale topology has to be manipulated carefully for the following reason : contrary to étale morphisms of perfectoid spaces, **in general pro-étale morphisms are not open**. This is for example the case for any $s \in S$ where

$$\mathrm{Spa}(K(s), K(s)^+) = \varprojlim_{U \ni s} U \hookrightarrow S$$

is pro-étale not open. This may still be the case for surjective morphisms of affinoid perfectoid spaces, typically $S \coprod \mathrm{Spa}(K(s), K(s)^+) \to S$.

One thus has to add the following condition in the definition of a pro-étale cover :

Definition 2.4. *A family of morphisms of perfectoid spaces $(T_i \to S)_{i\in I}$ is a pro-étale cover if for any quasi-compact open subset U in S there exists $I' \subset I$ finite and for each $i \in I'$ a quasi-compact open subset $V_i \subset T_i$ such that*

$$U = \bigcup_{i\in I'} \mathrm{Im}(V_i \to T).$$

If for all indices $i \in I$, $T_i \to S$ is open this "strong surjectivity condition" is equivalent to "the weak one" saying that $\coprod_{i\in I} |T_i| \to |S|$ is surjective. But as we said before this is not true in general. The pro-étale site is seen as a big site.

2.2.2. *Pro-étale local structure of perfectoid spaces.* One of the most important results is the following structure of perfectoid spaces pro-étale locally. In fact, recall the following definition. We use the fact that for any qc qs perfectoid space X there is a morphism

$$X \longrightarrow \underline{\pi_0(X)}$$

whose fibers are the connected components of X (that are perfectoid spaces). Here

$$\pi_0(X) = \overbrace{\pi_0(\underbrace{|X|}_{\substack{\text{spectral}\\ \text{space}}})}^{\text{profinite}}.$$

Remark 2.5. *Here we use the following construction. If T is a topological space then we define $\underline{T}$ as a functor on* Perf *via the formula*

$$\underline{T}(S) = \mathscr{C}(|S|, T).$$

This defines a pro-étale (and even a v)-sheaf on Perf*.*

Definition 2.6. *A perfectoid qc qs space X is* **strictly totally discontinuous** *if it satisfies the following equivalent properties :*

(1) *Every connected components of X contains a unique closed point i.e. is of the form $\mathrm{Spa}(K, K^+)$ with (K, K^+) an affinoid perfectoid field. We moreover ask that all residue fields are algebraically closed i.e. any connected component is of the form $\mathrm{Spa}(C, C^+)$ with C algebraically closed.*
(2) *Any étale cover of X splits i.e. admits a section.*

Strictly totally disconnected perfectoid spaces can be though of as a amalgamations of collections Spec$(C(x), C(x)^+)$ with $C(x)$ algebraically closed when x goes along a profinite set.

The following says that pro-étale locally any perfectoid space if a disjoint union of strictly totally disconnected perfectoid spaces.

PROPOSITION 2.7 (**Pro-étale local structure of perfectoid spaces**). *For any qc qs perfectoid space X there exists an open pro-étale surjective morphism*

$$\widetilde{X} \longrightarrow X$$

with $\widetilde{X}$ strictly totally discontinuous.

EXAMPLE 2.8. *For any perfectoid space X, if $X_\bullet \to X$ is an hypercover by $\coprod$ strictly totally disconnected perfectoid spaces then*

$$\{\text{\'etale sheaves on } X\} \xrightarrow{\sim} \{\text{cartesian sheaves on } |X_\bullet|\}.$$

From this point of view étale cohomology of perfectoid spaces is simpler than étale cohomology of schemes : everything is reduced to cartesian sheaves on simplicial topological spaces.

2.2.3. *A geometric fiberwise criterion to be pro-étale pro-étale locally.* **Pro-étale morphisms do not satisfy descent for the pro-étale topology**. This problems has lead to the following.

PROPOSITION 2.9. *A morphism of perfectoid spaces $X \to S$ is pro-étale pro-étale locally on S if and only if for all its geometric fibers, $X \times_S \mathrm{Spa}(C, C^+)$ for $\mathrm{Spa}(C, C^+) \to S$, are locally profinite, i.e. locally of the form $\underline{P} \times \mathrm{Spa}(C, C^+)$ for a profinite set P.*

This has lead to the definition of **quasi-pro-étale morphisms** and his a very useful criterion for application to morphisms of moduli spaces for which computing the geometric fibers is usually easy.

EXAMPLE 2.10. *Let $T \to S$ be a morphism of qc qs perfectoid spaces such that $|T| \to |S|$ is surjective (i.e. this is a v-cover) and such that for all $s : \mathrm{Spa}(C, C^+) \to S$, $T_s \simeq \underline{P} \times \mathrm{Spa}(C, C^+)$ with P a profinite set. Then, up to replacing S by a pro-étale cover, $T \to S$ is a pro-étale cover. From this we deduce that $T \to S$ is a surjective morphism of pro-étale sheaves. This is for example the case for the Kummer map* $\underbrace{\mathbb{B}_K^{1,1/p^\infty}}_{\substack{\textit{perfectoid}\\ \textit{ball}}} \xrightarrow{z \mapsto z^n} \mathbb{B}_K^{1,1/p^\infty}$ *when K is a perfectoid field.*

2.3. The v-topology. The v-topology is an analog of the fpqc topology for schemes. This is a big site on perfectoid spaces where we take the same definition for covers as for the pro-étale topology but by taking any morphism of perfectoid spaces instead of the pro-étale one. This is the most general topology we use. **It is subcanonical** : the functor defined by a perfectoid space is a v-sheaf. It moreover satisfies some nice descent properties. For example :

(1) Vector bundles satisfy descent for the v-topology.

(2) Separated étale morphisms satisfy descent for the v-topology.

This last (difficult) result is used all the times.

Example 2.11. *Let G be a locally profinite group and $T \to S$ be a $\underline{G}$ torsor for the v-topology where S is a perfectoid space. One has, as v-sheaves,*

$$T \xrightarrow{\sim} \varprojlim_{K} \underline{K}\backslash T$$

where K goes through the set of compact open subgroups of G. Since v-locally $\underline{K}\backslash T \to S$ is separated étale, one deduces that $\underline{K}\backslash T \to S$ is representable by a separated étale morphism of perfectoid spaces. In particular, $T \to S$ is a pro-étale morphism of perfectoid spaces,

$$T = \varprojlim_{K' \subset K} \underline{K'}\backslash T \underbrace{\longrightarrow}_{\substack{\text{pro-} \\ \text{étale finite}}} \underline{K}\backslash T \underbrace{\longrightarrow}_{\substack{\text{étale} \\ \text{separated}}} S,$$

and thus a pro-étale torsor and we have

$$H^1_{\text{pro-ét}}(S, \underline{G}) \xrightarrow{\sim} H^1_v(S, \underline{G}).$$

Example 2.12. *Let $\mathbb{Q}_p^{cyc} = \cup_{n \geq 1} \mathbb{Q}_p(\zeta_n)$. Then, $\widehat{\mathbb{Q}_p^{cyc}}$ is a perfectoid field. The morphism*

$$\mathrm{Spa}(\mathbb{C}_p) \longrightarrow \mathrm{Spa}(\widehat{\mathbb{Q}_p^{cyc}})$$

is a v (and even pro-étale) cover. Let $H = Gal(\overline{\mathbb{Q}}_p | \mathbb{Q}_p^{cyc})$. The preceding morphism is an $\underline{H}$-torsor. The fact that vector bundles descend along this morphisms is then equivalent to (Sen)

$$\underbrace{\mathrm{Vect}_{\widehat{\mathbb{Q}_p^{cyc}}}}_{\substack{\text{finite dim.} \\ \widehat{\mathbb{Q}_p^{cyc}}\text{-v.s.}}} \xrightarrow{\sim} \underbrace{\mathrm{Rep}_{\mathbb{C}_p}(H)}_{\substack{\text{semi-linear} \\ \text{rep. of } H \\ \text{on finite dim.} \\ \mathbb{C}_p\text{-v.s.}}}$$

3. Diamonds

3.1. Definition and elementary results. As we said, diamonds are algebraic spaces for the pro-étale topology. One of the ides of the theory is to push everything in characteristic p.

DEFINITION 3.1. *A diamond is a pro-étale sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ such that there exists an $\mathbb{F}_p$-perfectoid space $\widetilde{X}$ and an equivalence relation $R \subset \widetilde{X} \times \widetilde{X}$*

- *that is representable by a perfectoid space,*
- *such that both maps $R \rightrightarrows \widetilde{X}$ are pro-étale,*
- *and we have*
$$X \simeq \widetilde{X}/R$$
(quotient as pro-étale sheaves).

As is well known (Gabber), any algebraic space is an fppf sheaf. The same holds for diamonds : one can prove that **any diamond is a v-sheaf**.

The category of diamonds is very behaved : it has fibered products and finite products.

3.2. Spatial diamonds. Let X be a v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$. Suppose it is small in the sense that there exists a perfectoid space S and a surjection $S \to X$. One can then define

$$|X| = \big\{\mathrm{Spa}(K, K^+) \to X \mid (K, K^+) \text{ affinoid perf. field}\big\}/\sim$$

where two morphisms $\mathrm{Spa}(K_1, K_1^+) \to X$ and $\mathrm{Spa}(K_2, K_2^+) \to X$ are equivalent if there exists a diagram

$$\begin{tikzcd}
& \mathrm{Spa}(K_1, K_1^+) \arrow[dr] & \\
\mathrm{Spa}(K_3, K_3^+) \arrow[ur] \arrow[dr] & & X \\
& \mathrm{Spa}(K_2, K_2^+) \arrow[ur] &
\end{tikzcd}$$

where $\mathrm{Spa}(K_3, K_3^+) \to \mathrm{Spa}(K_1, K_1^+)$ and $\mathrm{Spa}(K_3, K_3^+) \to \mathrm{Spa}(K_2, K_2^+)$ sends the closed point to the closed point. This is equipped with the structure of a topological space where the open subsets are the subsets

$$|U| \subset |X|$$

where $U \hookrightarrow X$ is a morphisms of v-sheaves representable by an open immersion. One can verify that if

$$S_1 \rightrightarrows S_0 \longrightarrow X$$

is a 1-v-hypercover by perfectoid spaces then

$$\underbrace{\mathrm{coeq}}_{\substack{\text{coeq in}\\ \text{the cat. of}\\ \text{top. spaces}}} \big(|S_1| \rightrightarrows |S_0| \big) \underbrace{\xrightarrow{\sim}}_{\text{homeomorphism}} |X|.$$

Here is the "good notion" of diamonds we use.

DEFINITION 3.2. *A diamond X is spatial if X is qc qs and each point of $|X|$ has a basis of neighborhoods formed of qc open subsets.*

One can verify that in fact the topological condition is equivalent to saying that $|X|$ **is a spectral space**. This makes spatial diamonds look like qc qs analytic adic spaces. This throws out some pathological objects that are not related to rigid analytic geometry like

$$\underline{T} \times \mathrm{Spa}(K)$$

where T is a compact Hausdorff space and K is a perfectoid field. In fact, this last object is a diamond with topological space T. More precisely, if βT_{disc} is the Stone-Chech compactification of T_{disc} there is a surjective quotient map

$$\underbrace{\beta T_{disc}}_{\text{profinite}} \longrightarrow T$$

sending an ultrafilter to its limit. This shows that $\underline{T} \times \mathrm{Spa}(K)$ is a diamond with topological space T.

This spatialness notion is extremely flexible, giving rise to a new geometry. For example, if $(X_i)_i$ is a cofiltered projective system of spatial diamonds then $\varprojlim_i X_i$ is a spatial diamond with $|\varprojlim_i X_i| = \varprojlim_i |X_i|$ as spectral spaces.

Maybe one of the greatest features of the geometry of spatial diamonds is the following. If X is a v-sheaf and $Z \subset |X|$ a subset then Z defines a sub-v-sheaf of X via the formula

$$Z(S) = \{S \to X \mid \mathrm{Im}(|S| \to |X|) \subset Z\}.$$

We have the following result that is a consequence of the fact that *if X is a strictly totally disconnected perfectoid space then any pro-constructible generalizing subset of $|X|$ is representable by a perfectoid space.*

Proposition 3.3. *Let X be a spatial diamond and let $Z \subset |X|$ be pro-constructible generalizing subset. Then Z defines a spatial diamond with topological space Z equipped with a qc injection inside X.*

The geometry of (locally) spatial diamonds is much more flexible than the geometry of classical rigid spaces à la Tate.

3.3. Some abstract construction : tilting anything. Let X be a v-sheaf on Perf.

Definition 3.4 (**Tilting anything**). *We note $X^\diamond$ for the v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$ whose value on S is given by the datum $(S^\sharp, \iota, s)$ where*
- *$S^\sharp$ is a perfectoid space,*
- *$\iota : S \xrightarrow{\sim} (S^\sharp)^\flat$,*
- *s is an element of $X(S^\sharp)$.*

Of course, if X is a perfectoid space then

$$X^\diamond = X^\flat.$$

This abstract construction allows us to tilt anything in characteristic p. If S is any v-sheaf then

$$(-)^\diamond : \mathrm{Perf}\,/S \xrightarrow{\sim} \mathrm{Perf}_{\mathbb{F}_p}\,/S^\diamond$$

that is a generalized form of the tilting equivalence. This extends to equivalences of topoï (étale, pro-étale or v)

$$\widetilde{\mathrm{Perf}/S} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_p}/S^\diamond}$$

For example, $\mathbb{Q}_p$-perfectoid spaces are the same as $\mathbb{F}_p$-perfectoid spaces sitting over $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$.

3.4. First example : $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$. The v-sheaf $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ is a spatial diamond. This is the moduli of untilts of a characteristic p perfectoid space in characteristic 0. In fact,

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}.$$

REMARK 3.5. *One has to be careful that* $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ *is not a diamond. In fact, one can prove that any sub-v-sheaf of a diamond is a diamond, but* $\mathrm{Spa}(\mathbb{F}_p)^\diamond$ *is not a diamond.*

3.5. $X^\diamond$ for X an analytic adic space. The starting point of this is the following due to Colmez : *If R is a uniform complete Tate Huber ring then there exists a filtered inductive system $(R_i)_{i\geq i_0}$ of complete uniform Tate Huber rings with $R_{i_0} = R$ such that all transition morphisms are finite étale and*

$$\widehat{\varinjlim_i R_i}$$

is perfectoid. From this one deduces the following.

PROPOSITION 3.6. *If X is an analytic adic space then $X^\diamond$ is a locally spatial diamond with $|X^\diamond| = |X|$.*

EXAMPLE 3.7. *If X is characteristic p then $X^\diamond$ is a perfectoid space equal to X^{1/p^∞}.*

EXAMPLE 3.8 (Faltings). *Let R be a p-torsion free p-adic integral normal domain. Let $K = \mathrm{Frac}(R)$ and $\overline{K}$ an algebraic closure of K. Let $\overline{R}$ be the integral closure of R in the maximal extension of K inside $\overline{K}$ that is étale over $R[\frac{1}{p}]$ i.e. $\mathrm{Aut}_R(\overline{R}) = \pi_1(Spec(R[\frac{1}{p}], \overline{x})$ with $\overline{x}$ given by the choice of $\overline{K}$. Then* $\widehat{\overline{\boldsymbol{R}}}$ **is perfectoid**. *In fact, if $x \in \overline{R}$ then the polynomial $P(T) = T^p + pT - x$ is separable over $\overline{R}[\frac{1}{p}]$. A zero of this polynomial in $\overline{K}$ is then an element of $\overline{R}$ whose p-power is congruent to x modulo p.*

Here is how to explicitly construct some perfectoid charts on $X^\diamond$. Let

$$\widetilde{X} \to X$$

be a pro-étale cover with $\widetilde{X}$ perfectoid. Let

$$\widetilde{X} \times_X \widetilde{X}$$

be the categorical product in the category of X-perfectoid spaces. This product exists since (locally on X and $\widetilde{X}$ that we can suppose affinoid) if $\widetilde{X} = \varprojlim_{i \geq i_0} X_i$ with $X_{i_0} = X$ and finite étale transition morphisms then the purity theorem says that for all indices i,

$$\widetilde{X} \times_X X_i$$

is perfectoid. On can then take

$$\widetilde{X} \times_X \widetilde{X} = \varprojlim_i \widetilde{X} \times_X X_i.$$

Then one has

$$\boxed{X^\diamond = \mathrm{coeq}\big(\ (\widetilde{X} \times_X \widetilde{X})^\flat \rightrightarrows \widetilde{X}^\flat \ \big).}$$

REMARK 3.9. *One has to be careful that the product $\widetilde{X} \times_X \widetilde{X}$ should not be taken in the category of adic spaces. For example, $\mathbb{C}_{\boldsymbol{p}} \hat{\otimes}_{\mathbb{Q}_{\boldsymbol{p}}} \mathbb{C}_{\boldsymbol{p}}$* **is not perfectoid** *contrary to $\mathscr{C}(Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p), \mathbb{C}_p)$.*

EXAMPLE 3.10. *One has*
$\mathrm{Spa}\big(\mathbb{Q}_p\langle T, T^{-1}\rangle, \mathbb{Z}_p\langle T, T^{-1}\rangle\big)^\diamond = \mathrm{Spa}\big(\mathbb{C}_p^\flat\langle T^{\pm 1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}_p^\flat}\langle T^{\pm 1/p^\infty}\rangle\big)/\underline{\mathbb{Z}_p(1) \rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}$.

Let us conclude with the following.

THEOREM 3.11. *The functor $X \mapsto X^\diamond$ satisfies the following :*

(1) It is fully faithful from the category of Noetherian **normal** *analytic adic spaces to the category of locally spatial diamonds,*

(2) For X a Noetherian analytic adic space one has an equivalence of topoÃ⁻

$$\widetilde{X^\diamond}_{\text{ét}} \xrightarrow{\sim} \widetilde{X}_{\text{ét}}$$

and in particular one can compute the étale cohomology of X in terms of the one of $X^\diamond$.

4. Some final thoughts

The theory of diamonds gives access to some new geometry. For example, if X is an analytic adic space and $Z \subset |X|$ a subset that is locally on X pro-constructible generalizing then Z defines a sub-locally spatial diamond of $X^\diamond$ that is not attached to a classical analytic adic space in general. For example, the étale cohomology of diamonds allows us to define the étale cohomology of such a Z.

Chapitre 7

7th lecture - October 31

1. The linear objects of the category of diamonds : BC spaces

1.1. The relative curve. For S an $\overline{\mathbb{F}}_q$-perfectoid space we can define

$$X_S = Y_S/\varphi^{\mathbb{Z}}$$

the **relative curve associated to** S as an E-adic space. This is defined as when S is the spectrum of a perfectoid field. More precisely, if $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid then we can define

$$Y_S = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \setminus V(\pi[\varpi])$$

with its Frobenius φ. This construction glues and lead to the definition of Y_S for any S. The preceding constructions when $S = \mathrm{Spa}(F)$ extends :

(1) For S affinoid perfectoid Y_S is sous-perfectoid and thus **Huber's structural presheaf of holomorphic functions is in fact a sheaf**. That being said, in general this is not a Noetherian adic space.

(2) If $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid we can define the associated schematical curve $\mathfrak{X}_{R,R^+}$ as before and GAGA theorem extends :

$$\{\text{vector bundles on } \mathfrak{X}_{R,R^+}\} \xrightarrow{\sim} \{\text{vector bundles on } X_{R,R^+}\}.$$

The construction $S \mapsto X_S$ is functorial in S and one can thought of X_S as being "$X \times S$" although "X" does not exist.

Here is a computation.

PROPOSITION 1.1. *One has*

$$X_S^\diamond = \big(S \times \mathrm{Spa}(E)^\diamond\big)/\varphi^{\mathbb{Z}} \times \mathrm{Id}$$

as a locally spatial diamond over $\mathrm{Spa}(E)^\diamond$.

This result is reduced to the computation of $Y_S^\diamond$ together with its Frobenius action. One has to compute morphisms $\mathrm{Spa}(A, A^+) \to Y_S$ for (A, A^+) an affinoid perfectoid E-algebra. Suppose that $S = \mathrm{Spa}(R, R^+)$. Such a morphism is given by a morphism

$$W_{\mathcal{O}_E}(R^+) \longrightarrow A^+ \tag{4}$$

such that the image of $[\varpi]$ is a pseudo-uniformizer of A.

We now use the adjunction

$$\text{Perfect } \mathbb{F}_q\text{-algebras} \underset{(-)^\flat}{\overset{W_{\mathcal{O}_E}(-)}{\rightleftarrows}} p\text{-adically separated complete } \mathcal{O}_E\text{-algebras}$$

where the adjunction maps are given by $x \mapsto ([x^{1/p^n}])_{n\geq 0}$ and Fontaine's θ map.

From this adjunction we deduce that to give oneself a morphism as in Equation (4) is the same as a morphism

$$R^+ \longrightarrow A^{\flat,+}$$

sending ϖ to a pseudo-uniformizer in A. The result is easily deduced.

1.2. BC spaces and their families. Affine spaces and their twisted versions (vector bundles) are the natural linear objects showing up in the "classical case" as the relative cohomology of vector bundles. The linear objects of the category of diamonds are the Banach-Colmez spaces.

Before beginning let us recall that **there is a good notion of vector bundles on analytic adic spaces like** X_S. More precisely we have the following. Let (A, A^+) be stably uniform complete Tate Huber ring (for example (A, A^+) is sous-perfectoid). Then

$$(-)^{\text{adification}} : \{\underbrace{\text{vector bundles on } \operatorname{Spec}(A)}_{\substack{\text{projective finite type} \\ A\text{-modules}}}\} \xrightarrow{\sim} \{\underbrace{\text{vector bundles on } \operatorname{Spa}(A, A^+)}_{\substack{\text{locally free} \\ \mathcal{O}\text{-modules}}}\}$$

This sends a projective finite type A-module P to $P \otimes_A \mathcal{O}_{\operatorname{Spa}(A,A^+)}$. Moreover for any such vector bundle $\mathscr{E}$ on $\operatorname{Spa}(A, A^+)$ one has

$$H^i(\operatorname{Spa}(A, A^+), \mathscr{E}) = 0 \text{ for } i > 0.$$

Proposition 1.2 (**Relative cohomology of vector bundles**). *If S is an $\overline{\mathbb{F}}_q$-perfectoid space and $\mathscr{E}$ a vector bundle on X_S then the functors on S-perfectoid spaces*

(1) $T \mapsto H^0(X_T, \mathscr{E}_{|X_T})$,

(2) the pro-étale sheaf associated to the presheaf $T \mapsto H^1(X_T, \mathscr{E}_{|X_T})$

are locally spatial diamonds.

Example 1.3. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two vector bundles on X_S. Then the v-sheaf on* Perf_S

$$T \longmapsto \operatorname{Isom}(\mathscr{E}_{1|X_T}, \mathscr{E}_{2|X_T})$$

is representable by a locally spatial diamond as an open sub-diamond of $T \mapsto H^0(X_T, \mathscr{E}^\vee_{1|X_T} \otimes \mathscr{E}_{2|X_T})$. This means that the diagonal of the stack Bun *of vector bundles on the curve (see later) is representable in locally spatial diamonds.*

THEOREM 1.4 (Le Bras). *When $S = \mathrm{Spa}(C^\flat)$ with $C|E$ algebraically closed one can prove that this relative cohomology construction gives an equivalence between*

(1) *objects $\mathscr{E} \in D^{[-1,0]}(\mathcal{O}_{X_{C^\flat}})$ satisfying the perversity conditions*
   - *$\mathcal{H}^{-1}(\mathscr{E})$ is a vector bundle with < 0 H.N. slopes,*
   - *$\mathcal{H}^{0}(\mathscr{E})$ is a coherent sheaf with $\geqslant 0$ H.N slopes,*

(2) *the sub-abelian category of the category of pro-étale sheaves on $\mathrm{Spa}(C)$ of $\underline{E}$-vector spaces that is the smallest one that*
   - *contains $\underline{E}$*
   - *contains $\mathbb{G}_a$,*
   - *is stable under extensions.*

2. Artin criterion

To go further and give new examples of diamonds we will need the following.

THEOREM 2.1 (Artin criterion for spatial diamonds). *Let X be v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$. This is a spatial diamond if and only if*

(1) *it is small,*
(2) *it is spatial i.e. X is qc qs and $|X|$ is spectral,*
(3) *for any $x \in |X|$, $X_x := \varprojlim_{U \ni x} U$ is a diamond that is to say isomorphic to $\mathrm{Spa}(C, C^+)/\underline{G}$ where $G \subset \mathrm{Aut}(C, C^+)$ is a profinite subgroup.*

Like the classical Artin criterion :

	Diamonds	Algebraic spaces
Global hypothesis	spatial v-sheaf	finite presentation fppf sheaf
Local hypothesis	$\forall x \in \lvert X\rvert, X_x$ is a diamond	the formal completion at each point is representable by a formal scheme

The way we are going to apply the preceding result is the following. We will first prove that X is spatial using the following elementary result.

LEMMA 2.2. *Let X be a spectral space and $R \subset X \times X$ be a pro-constructible equivalence relation such that both maps $R \rightrightarrows X$ are open. Then, X/R is a spectral space.*

Here is the corollary we will use. We use the notion of ℓ-cohomological smoothness (to be seen later) : the only thing to know is that

$$\ell\text{-cohomologically smooth} \implies \text{open}.$$

PROPOSITION 2.3. *Let X be a spatial diamond and $R \subset X \times X$ be an equivalence relation such that R is a spatial diamond. Suppose both maps $R \rightrightarrows X$ are ℓ-cohomologically smooth for some $\ell \neq p$. Then, X/R is a spatial v-sheaf.*

EXAMPLE 2.4. *Let $X \to S$ be a morphism of spatial diamonds and G a spatial diamond that is group over S action on X. Suppose that $G \to S$ is ℓ-cohomologically smooth for some $\ell \neq p$. Then X/G is a spatial v-sheaf.*

The second step is the following one we know that X is a spatial v-sheaf. We will exhibit a finite stratification

$$|X| = \bigcup_i Z_i,$$

where Z_i is locally closed generalizing, such that for all indices i, Z_i is a diamond. This will prove that X is a spatial diamond.

Let's put this in a corollary.

COROLLARY 2.5. *Let X be a qc qs v-sheaf that is ℓ-cohomologically smooth locally a spatial diamond and such that $|X| = \bigcup_i Z_i$ with $Z_i \subset |X|$ locally closed generalizing that is a diamond. Then X is a spatial diamond.*

3. Schubert cells in the B_{dR}-affine Grassmanian

3.1. The B_{dR}-affine Grassmanian. Let G be our reductive group over E. We can consider the v-sheaf or filtered E-algebras

$$\begin{array}{c} \mathbb{B}_{dR}^{+} \\ \downarrow \\ \mathrm{Spa}(E)^{\diamond}. \end{array}$$

This sends (R, R^+) an $\mathbb{F}_p$-perfectoid algebra to

- an untilt $(R^\sharp, R^{\sharp,+})$ over E,
- an element of the completion of $W_{\mathcal{O}_E}(R^+)\left[\frac{1}{p}\right]$ for the $\ker\theta$-adic topology where

$$\theta : W_{\mathcal{O}_E}(R^+)\left[\tfrac{1}{p}\right] \longrightarrow R^{\sharp,+}\left[\tfrac{1}{p}\right].$$

We note $\mathbb{B}_{dR}$ for the localization of $\mathbb{B}_{dR}^{+}$ obtained after inverting a generator of $\ker\theta$.

DEFINITION 3.1. *We note*

$$\begin{array}{c}\mathrm{Gr}_G^{B_{dR}}\\ \big\downarrow\\ \mathrm{Spa}(E)^\diamond\end{array}$$

for $G(\mathbb{B}_{dR})/G(\mathbb{B}_{dR}^+)$ (étale quotient).

3.2. Schubert cells. Suppose G is split to simplify. Fix $T \subset B$ a maximal torus inside a Borel subgroup. For each $\mu \in X_*(T)^+$ there is defined an open Schubert cell inside a closed one

$$\mathrm{Gr}_{G,\mu}^{B_{dR}} \underbrace{\subset}_{\text{open}} \mathrm{Gr}_{G,\leqslant\mu}^{B_{dR}}.$$

This is defined via a pointwise condition for each morphism $\mathrm{Spa}(C, C^+) \to \mathrm{Gr}_G^{B_{dR}}$. The fact that the inclusion is an open immersion is not completely evident. Nevertheless, there is a Bialynicki-Birula morphism

$$BL_\mu : \mathrm{Gr}_{G,\mu}^{B_{dR}} \longrightarrow \mathscr{F}_\mu^\diamond$$

where $\mathscr{F}_\mu$ is the flag variety associated to μ. This morphism is an iterated étale fibration in $(\mathbb{A}^1)^\diamond$ and we deduce that the open Schubert cell is a diamond. As an application of Artin's criterion, using the stratification by open Schubert cells, one can prove the following.

THEOREM 3.2 (Scholze). *For all μ, the closed Schubert cell $\mathrm{Gr}_{G,\leqslant\mu}^{B_{dR}}$ is a spatial diamond.*

4. Punctured absolute Banach-Colmez spaces

Let

$$* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$$

be the final object of the v-topos. For each $\lambda \in \mathbb{Q}_{>0}$ let us note

$$\mathrm{BC}(\mathcal{O}(\lambda)) \longrightarrow *$$

for the v-sheaf

$$S \mapsto H^0(X_S, \mathcal{O}(\lambda)).$$

We call this **an absolute Banach-Colmez space**. Here the terminology "absolute" refers to

When $\lambda \in]0,1]$, this is represented by a formal scheme isomorphic to

$$\mathrm{Spf}(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!])$$

where $\lambda = \frac{d}{h}$ with $(d,h) = 1$. More precisely, this is the universal cover of a dimension d and height h π-divisible $\mathcal{O}_E$-module. This is clearly not represented by a perfectoid space or even a diamond. Nevertheless,

$$BC(\mathcal{O}(\lambda)) \smallsetminus \{0\} = \mathrm{Spa}\big(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!], \overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!]\big) \smallsetminus V(x_1, \dots, x_d)$$

that is a qc qs perfectoid space.

For $\lambda < 0$ we can consider similarly

$$\mathrm{BC}(\mathcal{O}(\lambda)[1]) \longrightarrow *$$

that is the sheaf whose value on S affinoid perfectoid is

$$H^1(X_S, \mathcal{O}(\lambda)).$$

On then has the following result.

Theorem 4.1 (F.-Scholze). *For all $\lambda \in \mathbb{Q}_{>0}$, the punctured absolute Banach-Colmez spaces*

$$\mathrm{BC}(\mathcal{O}(\lambda)) \smallsetminus \{0\}$$

and

$$\mathrm{BC}(\mathcal{O}(-\lambda)[1]) \smallsetminus \{0\}$$

are spatial diamonds.

Those last spatial diamonds are not associated to any classical usual objects like Noetherian analytic adic spaces or formal schemes. They are among the most "original" and new objects showing up in our work and are completely unrelated to any usual classical object.

5. Some final thoughts

Those last objects, the negative punctured absolute Banach-Colmez spaces, are a key ingredient in our joint work with Scholze. They allow use to construct some very particular charts of the stack of G-bundles on the curve, the so-called "$\mathcal{M}_b$". The spatialness of $\mathrm{BC}(\mathcal{O}(-\lambda)[1]) \smallsetminus \{0\}$ is one of the reason why we consider

$$\begin{array}{c} \mathrm{Bun}_G \\ \big\downarrow \\ * \end{array}$$

absolutely and not by replacing $*$ by $\mathrm{Spa}(C)$ where C is an algebraically closed $\mathbb{F}_p$-perfectoid field. Working absolutely over $\mathrm{Spa}(\overline{\mathbb{F}}_q)$ is an essential point.

The geometry of locally spatial diamonds is much more flexible than the usual one of Noetherian analytic adic spaces ; typically the fact that any locally proconstructible generalizing subset of $|X|$, X locally spatial, defines a sub-locally spatial diamond is extremely useful.

As a final remark : for any small v-sheaf (and even any small v-stack) X and Λ a torsion ring we can define

$$D_{\text{ét}}(X, \Lambda)$$

via a descent procedure : this is

$$\pi_0 \varprojlim_{S \to X} \underbrace{\mathcal{D}(|S|, \Lambda)}_{\text{stable } \infty\text{-category}}$$

where S is a strictly totally disconnected perfectoid space. More concretely, if

$$S_\bullet \longrightarrow X$$

is a v-hypercover by $\coprod$ of strictly totally disconnected perfectoid spaces then $D_{\text{ét}}(X,\Lambda)$ is identified with the derived category of cartesian sheaves of Λ-modules on $|S_\bullet|$. This makes the category $D_{\text{ét}}(X,\Lambda)$ quite abstract, in particular the functor Rf_*, when f is a morphism of small v-sheaves, is not explicit : if we have a diagram

$$\begin{tikzcd} S'_\bullet \arrow[r, "g"] \arrow[d] & S_\bullet \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

then Rf_* is computed as

$$D_{\text{cart}}(|S'_\bullet|,\Lambda) \xrightarrow{R|g|_*} D(|S_\bullet|,\Lambda) \xrightarrow{\substack{\text{cartesianification}\\ \text{functor}}} D_{\text{cart}}(|S_\bullet|,\Lambda)$$

where the cartesianification functor is not explicit.

Nevertheless, if X is a locally spatial diamond, it has a nice étale site $X_{\text{ét}}$ of locally separated étale morphisms ; this is a consequence of the fact that separated étale morphisms descend for the v-topology and one has

$$\underbrace{D^+(X_{\text{ét}},\Lambda)}_{\substack{\text{concrete usual}\\ \text{derived category of}\\ \text{sheaves on } X_{\text{ét}}}} \xrightarrow{\sim} \underbrace{D^+_{\text{ét}}(X,\Lambda)}_{\text{abstractly defined via descent}}$$

with

$$\underbrace{Rf_{\text{ét}*}}_{\substack{\text{concrete}\\ \text{usual derived functor}\\ \text{of étale pushfoward}}} = \underbrace{Rf_*}_{\substack{\text{not explicit}\\ \text{in general}}}$$

when f is a morphism of locally spatial diamonds.

Chapitre 8

8th lecture - November 14

1. The moduli stack of G-bundles on the curve

Recall the following. We have $E|\mathbb{Q}_p$ with $\mathcal{O}_E/\pi = \mathbb{F}_q$. We let

$$* = \mathrm{Spa}(\overline{\mathbb{F}}_q)^\diamond$$

be the final object of $(\mathrm{Perf}_{\overline{\mathbb{F}}_q})^{\sim}_v$, the v-topos. For each $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ we have, functorially in S,

$$X_S$$

an E-adic sous-perfectoid space that can be thought of as "$X \times S$" allthought X does not exist. Being sous-perfectoid, there is a good notion of vector bundles on it.

Definition 1.1. *For any S as before, a G-bundle on X_S is a faithful tensor functor*

$$\mathrm{Rep}\, G \xrightarrow{\otimes} \{\textit{vector bundles on } X_S\}.$$

When $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, if $\mathfrak{X}_{R,R^+}$ is the schematical curve as an E-scheme, there is a GAGA equivalence

$$\big\{G\text{-bundles on } X_{R,R^+}\big\} \xrightarrow{\sim} \big\{\text{étale } G\text{-torsors on } \mathfrak{X}_{R,R^+}\big\}.$$

Here is the main object of our study.

Definition 1.2. *We note $\mathbf{Bun}_{\boldsymbol{G}}$ the fibered groupoid over* $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$S \longmapsto \underbrace{\{G\text{-bundles on } X_S\ \}}_{groupoid}.$$

The first basic result is the following.

Proposition 1.3. *The fibered groupoid* Bun_G *is a stack for the v-topology on* $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

This is easily derived from the fact that the fibered groupoid on Perf

$$T \longmapsto \{ \text{ vector bundles on } T \ \}$$

is a v-stack i.e. vector bundles satisfy descent for the v-topology.

2. Six operations

2.1. Small v-stacks. We need a definition to start with.

DEFINITION 2.1. *(1) A small v-stack is a v-stack X on $\mathrm{Perf}_{\mathbb{F}_p}$ such that there exists a v-surjective morphism*

$$\underbrace{S}_{\text{perf. space}} \longrightarrow X$$

and

$$\underbrace{T}_{\text{perf. space}} \longrightarrow \underbrace{S \times_X S}_{v\text{-sheaf}}.$$

(2) A morphism $X \to Y$ of small v-stacks is 0-truncated if for any

$$\underbrace{S}_{\text{perf. space}} \longrightarrow Y$$

the stacky fibered product

$$X \times_Y S$$

is a v-sheaf i.e. $X \times_Y S \xrightarrow{\sim} \underbrace{\pi_0(X \times_Y S)}_{\text{coarse moduli}}$.

The point is that for any small v-stack X there is a v-hypercovering

$$\underbrace{S_\bullet}_{\substack{\text{simplicial}\\ \text{perf. space}}} \longrightarrow X.$$

2.2. $D_{\text{ét}}(X, \Lambda)$ for X a small v-stack. Let Λ be a prime to p torsion ring. We now would like to define

$$D_{\text{ét}}(X, \Lambda)$$

for any small v-stack X. The way we define it is via descent : we want, functorially in X,

$$D_{\text{ét}}(X, \Lambda) = \underbrace{\mathrm{Ho}}_{\substack{\text{homotopy}\\ \text{category}}} \underbrace{\mathcal{D}_{\text{ét}}(X, \Lambda)}_{\substack{v\text{-hypersheaf}\\ \text{of presentable stable } \infty\text{-cat.}}}$$

where the v-hypersheaf condition means that if

$$S_\bullet \longrightarrow X$$

is a v-hypercover of X by perfectoid spaces then

$$\mathcal{D}_{\text{ét}}(X, \Lambda) \xrightarrow{\sim} \varprojlim_{[n] \in \Delta} \mathcal{D}_{\text{ét}}(S_n, \Lambda)$$

where the limit is taken in the ∞-category of presentable stable ∞-categories. The key remark is now the following.

LEMMA 2.2. *The correspondence $S \mapsto \mathcal{D}(S, \Lambda)$ from the category of spectral spaces equipped with qc generalizing morphisms is an hypersheaf.*

This is a consequence of the fact that if $T \to S$ is a qc generalizing map between spectral spaces then it is a quotient map. Let us now recall that if S is a strictly totally disconnected perfectoid space then étale sheaves on S are the same as sheaves on $|S|$. Coupled with the preceding lemma we can thus define the following.

Definition 2.3. *For X a small v-stack we set*

(1)
$$\mathcal{D}_{\text{ét}}(X, \Lambda) = \varprojlim_{S \to X} \mathcal{D}(|S|, \Lambda)$$
where S is a strictly totally disconnected perfectoid space.

(2)
$$D_{\text{ét}}(X, \Lambda) = \operatorname{Ho} \mathcal{D}_{\text{ét}}(X, \Lambda).$$

One can compute the following that does not require any ∞-categories : there are morphisms of sites
$$\mathrm{Perf}_{\mathbb{F}_p, v} \xrightarrow{\lambda} \mathrm{Perf}_{\mathbb{F}_p, \text{pro-ét}} \xrightarrow{\nu} \mathrm{Perf}_{\mathbb{F}_p, \text{ét}} .$$
Then on can prove that for X a perfectoid space

$$D_{\text{ét}}(X, \Lambda) \overset{\nu^*}{\hookrightarrow} \underbrace{D_{\text{pro-ét}, \blacksquare}(X, \Lambda)}_{\substack{\text{solid pro-étale} \\ \text{sheaves}}} \overset{\lambda^*}{\hookrightarrow} D_v(X, \Lambda).$$

As a consequence one can check that for X a small v-stack

(1) $D_{\text{ét}}(X, \Lambda) = \{A \in D_v(X, \Lambda) \mid \forall S \to X, S \text{ s.t.d. perf. space}, A_{|S} \in D(|S|, \Lambda)\}$.
(2) If $S_\bullet \to X$ is a v-hypercover by s.t.d. perfectoid spaces then
$$D_{\text{ét}}(X, \Lambda) \xrightarrow{\sim} D_{\text{cart}}(|S_\bullet|, \Lambda).$$

2.3. 4 **operations.** It is now easy to define a formalism of 4 operations for $D_{\text{ét}}(-, \Lambda)$
$$(f^*, Rf_*, R\mathscr{H}om_\Lambda(-,-), \otimes^{\mathbb{L}}_\Lambda)$$
where here f is a 0-truncated morphism of small v-stacks. Here f^* and $\otimes^{\mathbb{L}}_\Lambda$ are explicit but Rf_* and $R\mathscr{H}om(-,-)$ are not explicit in general, they are constructed as adjoints of explicit functors.

Since separated étale morphisms descend for the v-topology and thus the pro-étale one, for any locally spatial diamond X there is a "good" small étale site $X_{\text{ét}}$ whose objects are locally separated étale morphisms of locally spatial diamonds $X' \to X$. One then has
$$\underbrace{\widehat{D}(X_{\text{ét}}, \Lambda)}_{\substack{\text{left completion of } D(X_{\text{ét}}, \Lambda) \\ = \operatorname{Ho} \varprojlim_{n \geq 0} \mathcal{D}^{\geq -n}(X_{\text{ét}}, \Lambda)}} \xrightarrow{\sim} D_{\text{ét}}(X, \Lambda)$$

where he the process of left completion corresponds to the fact that, in general without any finite cohomological assumption, Postnikov towers of an object may not converge to the original object. Then, for $f : X \to Y$ a morphism of locally spatial diamonds the preceding operations are explicit and are the usual one, for example

$$Rf_* = \underbrace{Rf_{\text{ét}*}}_{\substack{\text{usual derived}\\ \text{functor extended}\\ \text{via left completion}}}.$$

2.4. 6 **operations.** It remains to define the two other operations $(Rf_!, Rf^!)$. For this we need to upgrade our assumptions and add some technical hypothesis : f is not only 0-truncated but *representable in locally spatial diamonds, compactifiable of finite geometric transcendance degree*. We don't enter into the details, let's just say that those hypothesis are satisfied for all the morphisms we consider. One the obtains a formalism of 6 operations

$$\boxed{(f^*, Rf_*, Rf_!, Rf^!, R\mathscr{H}om(-,-), \otimes^{\mathbb{L}}_{\Lambda})}$$

where $Rf^!$ is formally defined as a right adjoint to $Rf_!$.

3. Cohomological smoothness

By definition, a morphism is cohomologically smooth if a version of relative Poincaré duality, in the sense of Verdier, is satisfied. More precisely, let

$$f : X \longrightarrow Y$$

be a morphism of small v-stacks that is representable in locally spatial diamonds, compactifiable of finite dim. trg.. By formal adjunction properties there is always a natural transformation

$$Rf^!(\Lambda) \otimes^{\mathbb{L}}_{\Lambda} f^*(-) \longrightarrow Rf^!(-)$$

between functors $D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda)$.

Definition 3.1. *The morphism f is cohomologically smooth if for any $\ell \neq p$, for any $S \to Y$ with S a strictly totally disconnected perfectoid space, if*

$$f_S : X \times_Y S \longrightarrow S$$

then

(1) $Rf^!_S(\mathbb{F}_\ell) \otimes^{\mathbb{L}}_{\Lambda} f^*_S(-) \xrightarrow{\sim} Rf^!_S(-)$ *as a natural transformation between functors from* $D_{\text{ét}}(Y, \mathbb{F}_\ell)$ *to* $D_{\text{ét}}(X, \mathbb{F}_\ell)$,

(2) $Rf^!_S(\mathbb{F}_\ell)$ *is invertible i.e. étale locally isomorphic to* $\mathbb{F}_\ell[2d]$ *for some* $d \in \frac{1}{2}\mathbb{Z}$.

One has to be careful that this has to be checked after any base change i.e. we force the cohomological smoothness property to be stable under base change. Reciprocally, if f is cohomologically smooth then

$$Rf^!(\Lambda) \otimes^{\mathbb{L}}_{\Lambda} f^*(-) \xrightarrow{\sim} Rf^!(-)$$

and $Rf^!(\Lambda)$ is invertible.

4. Smooth charts on Bun_G

We now have a setup in which we can speak about smooth charts on Bun_G.

THEOREM 4.1. *The v-stack* Bun_G *is an Artin v-stack in the sense that*

(1) Its diagonal is representable in locally spatial diamonds,

(2) There exists a locally spatial diamond U together with a cohomologically smooth surjective morphism $U \longrightarrow \mathrm{Bun}_G$.

It is moreover cohomologically smooth of dimension 0 in the sense that one can choose such a $U \to \mathrm{Bun}_G$ satisfying : $U \to *$ is cohomologically smooth of dimension the dimension of $U \to \mathrm{Bun}_G$.

Moreover one can prove that the dualizing complex of Bun_G is (non-canonically) isomorphic to $\Lambda[0]$.

One way to construct such charts is to use the following result that is an analog of a result by Drinfeld and Simpson.

THEOREM 4.2 (F.). *Let F ba an algebraically closed $\overline{\mathbb{F}}_q$-perfectoid field, $\infty \in |\mathfrak{X}_F|$ a closed point and $\mathscr{E}$ a G-bundle on $\mathfrak{X}_F$. Then*

$$\mathscr{E}_{|\mathfrak{X}_F \setminus \{\infty\}}$$

is trivial.

Using this and Beaville-Laszlo gluing one constructs a v-surjective morphism

$$\mathrm{Gr}_G^{B_{dR}} \longrightarrow \mathrm{Bun}_G$$

that allows us to prove that Bun_G is an Artin v-stack.

5. Points of Bun_G

For any small v-stack X we can define

$$|X|$$

as a topological space. As a consequence of the classification of G-bundles on X_F when F is an algebraically closed perfectoid field one obtains the following.

THEOREM 5.1. *We have an identification*

$$B(G) \xrightarrow{\sim} |\mathrm{Bun}_G|$$

as sets.

6. The topology on $|\mathrm{Bun}_G|$

6.1. Connected components. The following theorem says, for example, that for $\mathcal{G} = \mathrm{GL}_n$ the degree of a vector bundle is a locally constant function and the corresponding open/closed sub-stack Bun_n^d of degree d rank n vector bundles is connected.

THEOREM 6.1. *The function*

$$c_1 : |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_\Gamma$$

is locally constant with connected fibers.

We thus have a decomposition in connected components

$$\mathrm{Bun}_G = \coprod_{c\in\pi_1(G)_\Gamma} \mathrm{Bun}_G^c.$$

6.2. HN stratification. Suppose G is quasi-split to simplify. Let $A \subset T \subset B$ be as usual. The HN polygon defines a map

$$\mathrm{HN} : |\mathrm{Bun}_G| \longrightarrow X_*(A)_{\mathbb{Q}}^+.$$

THEOREM 6.2.
(1) *The map*

$$\mathrm{HN} : |\mathrm{Bun}_G| \longrightarrow X_*(A)_{\mathbb{Q}}^+$$

is semi-continuous in the sense that if $X_*(A)_{\mathbb{Q}}^+$ is equipped with the order $\nu_1 \leq \nu_2 \Leftrightarrow \nu_2 - \nu_1 \in \mathbb{Q}_+.\check{\Phi}$ then $\{[b] \mid [\nu_b] \geq \nu\}$ is open.

(2) *In fact the embedding $B(G) \hookrightarrow \pi_1(G)_\Gamma \times X_*(A)_{\mathbb{Q}}^+$ defines the topology of $|\mathrm{Bun}_G|$ in the sense that for $[b_1],[b_2] \in B(G)$, $[b_1] \leq [b_2]$ in $|\mathrm{Bun}_G|$ if and only if*

$$\begin{cases} \kappa(b_1) = \kappa(b_2), \\ \nu_{b_1} \leq \nu_{b_2}. \end{cases}$$

Recall (Kottwitz) that

$$\kappa_{|B(G)_{bsc}} : B(G)_{bsc} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

This is translated geometrically in the following statement : **any connected component of** Bun_G **has a unique semi-stable point that is thus open**.

EXAMPLE 6.3. *Consider $G = \mathrm{GL}_2$.*

(1) *The unique semi-stable point of Bun_2^0 is $\mathcal{O}^2$. There is then a chain of specializations in Bun_2^0*

$$\mathcal{O}^2 \geqslant \mathcal{O}(1) \oplus \mathcal{O}(-1) \geqslant \mathcal{O}(2) \oplus \mathcal{O}(-2) \geqslant \dots$$

(2) *The unique semi-stable point of Bun_2^1 is $\mathcal{O}(\frac{1}{2})$. There is then a chain of specializations of Bun_2^1*

$$\mathcal{O}(\tfrac{1}{2}) \geqslant \mathcal{O}(2) \oplus \mathcal{O}(-1) \geqslant \mathcal{O}(3) \oplus \mathcal{O}(-2) \geqslant \dots.$$

7. Some "nice charts" on Bun_G

Let us consider the case of GL_2 and more precisely the connected component of degree 0 rank 2 vector bundles, Bun_2^0. For $d \geq 0$ let

$$\mathcal{M}_d$$

be the moduli stack that sends S to extensions

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{E} \longrightarrow \mathscr{L}' \longrightarrow 0$$

where

(1) $\mathscr{E}$ is a rank 2 vector bundle on X_S,
(2) $\mathscr{L}$ is a degree $-d$ line bundle on X_S,
(3) $\mathscr{L}$ is a degree d line bundle on X_S.

We thus consider "anti-Harder-Narasimhan filtrations" of a given rank 2 vector bundle $\mathscr{E}$. The evident morphism

$$\mathcal{M}_d \longrightarrow \mathrm{Bun}_2^0$$

is ℓ-cohomologically smooth. Moreover, its image (that is open as it is ℓ-coho. smooth) is the set of generalizations of $\mathcal{O}(d) \oplus \mathcal{O}(-d)$ i.e. the $\mathcal{O}(d') \oplus \mathcal{O}(-d')$ with $0 \leq d' \leq d$.

The Picard stack, Bun_1^d is isomorphic to $[*/\underline{E}^\times]$ by sending $\mathscr{L}$ to the pro-étale torsor of isomorphisms between $\mathcal{O}(d)$ and $\mathscr{L}$. Let

$$\widetilde{\mathcal{M}}_d = \mathrm{BC}(\mathcal{O}(-2d)[1])$$

be the absolute Banach-Colmez space that is the moduli of extensions of $\mathcal{O}(d)$ by $\mathcal{O}(-d)$. One has

$$\mathcal{M}_d = \left[\mathrm{BC}(\mathcal{O}(-2d)[1])/\underline{E}^\times \times \underline{E}^\times\right].$$

We have the more general following theorem for any G that uses the so-called Jacobian criterion of smoothness.

THEOREM 7.1. *For any $[b] \in B(G)$ one can define a diagram*

$$\begin{tikzcd} \mathcal{M}_b \arrow[r, "\substack{\ell\text{-coho.}\\ \text{smooth}}", "\pi_b"'] \arrow[d] & \mathrm{Bun}_G \\ {[\,*\,/\underline{G_b(E)}\,]} \arrow[u, bend left] & \end{tikzcd}$$

where G_b is the σ-centralizer of b and such that if $\widetilde{\mathcal{M}}_b$ is defined via the cartesian diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}}_b \arrow[r] \arrow[d] & \mathcal{M}_b \arrow[d] \\ * \arrow[r] \arrow[u, bend left] & {[\,*\,/\underline{G_b(E)}\,]} \arrow[u, bend right] \end{tikzcd}$$

then

$$\widetilde{\mathcal{M}}_b \setminus \{*\}$$

is a spatial diamond. Moreover the image of $\mathcal{M}_b \to \mathrm{Bun}_G$ is the set of generalizations pf $[b]$.

One of the main point of the preceding result is the spatialness of $\widetilde{\mathcal{M}}_b \setminus \{*\}$. *This is the main reason why we consider* Bun_G *"absolutely" over* $*$ and not its pullback to $\mathrm{Spa}(C)$ for some algebraically closed $\overline{\mathbb{F}}_q$-perfectoid field C since the pullback to $\mathrm{Spa}(C)$ of $\widetilde{\mathcal{M}}_b \setminus \{*\}$ is only locally spatial non quasi-compact.

For $K \subset G_b(E)$ compact open pro-p we can consider

$$f_K^b : [\widetilde{\mathcal{M}}_b/\underline{K}] \longrightarrow \mathrm{Bun}_G$$

that is thus ℓ-cohomologically smooth and set

$$A_K^b := Rf_{K!}^b Rf_K^{b!}\Lambda \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda).$$

The collection of objects $(A_K^b)_{[b],K}$ is a generalization of the "classical set of compact generators"

$$(\text{c-Ind}_K^{G(E)}\,\Lambda)_K$$

of the category of smooth representations of $G(E)$ with coefficients in Λ.

THEOREM 7.2. *The category $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is compactly generated with $(A_K^b)_{[b],K}$ a set of compact generators.*

Chapitre 9

9th lecture - November 28

1. The moduli of degree 1 divisors on the curve

Let

$$\mathrm{Spa}(\breve{E})^\diamond \longrightarrow *.$$

If S is an $\overline{\mathbb{F}}_q$-perfectoid space then any untilt of S over $\breve{E}$, $S^\sharp$, defines a Cartier divisor

$$S^\sharp \hookrightarrow Y_S.$$

In fact, if $S = \mathrm{Spa}(R, R^+)$, an untilt over $\breve{E}$ is given by an ideal $I \subset W_{\mathcal{O}_E}(R^+)$ generated by a degree 1 distinguished element ξ (that is automatically a regular element). This defines our Cartier divisor

$$V(\xi) \subset Y_S$$

via the embedding of $W_{\mathcal{O}_E}(R^+)$ inside $\mathcal{O}(Y_{R,R^+})$. One verifies that composing with the projection defines a degree 1 Cartier divisor

$$S^\sharp \hookrightarrow X_S.$$

This defines a morphism

$$\mathrm{Spa}(\breve{E})^\diamond \longrightarrow \mathrm{Div}^1$$

where we take the following definition of a relative Cartier divisor.

Definition 1.1. *We note* $\mathrm{Div}^1(S)$ *the set of equivalence classes of couples* $(\mathscr{L}, u)$ *where* $\mathscr{L}$ *is a degree* 1 *line bundle on* X_S *and* $u \in H^0(X_S, \mathscr{L})$ *satisfies*

$$\forall s \in S,\ u_{|X_{K(s),K(s)^+}} \neq 0$$

as an element of $H^0\big(X_{K(s),K(s)^+}, \mathscr{L}_{|X_{K(s),K(s)^+}}\big)$.

This morphism is $\varphi^{\mathbb{Z}}$-invariant and induces an isomorphism.

Proposition 1.2. *The preceding morphism induces an isomorphism*

$$\mathrm{Spa}(\breve{E})/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1.$$

Thus, contrary to the “classical case”, Div^1 is note the curve itself. Nevertheless we have the following remark.

Remark 1.3. *We thus have for any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$,*

$$X_S^\diamond = (S \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\breve{E})^\diamond)/\varphi^{\mathbb{Z}} \times \mathrm{Id}$$

and

$$\mathrm{Div}_S^1 = (S \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\breve{E})^\diamond)/\,\mathrm{Id} \times \varphi^{\mathbb{Z}}$$

and thus

$$|X_S| = |\mathrm{Div}_S^1|$$

and even equivalences of étale sites $(X_S^\diamond)_{\text{ét}} \simeq (\mathrm{Div}_S^1)_{\text{ét}}$. For example, although X_S sits over $\mathrm{Spa}(E)$ and but not over S, there is still a continuous generalizing map of locally spectral spaces

$$|X_S| = |\mathrm{Div}_S^1| \longrightarrow |S|$$

"as if X_S were sitting over S".

Remark 1.4. *One has to be careful that although Div^1 is a qc diamond it is not spatial since not qs. Nevertheless $\mathrm{Div}^1 \to *$ is representable in locally spatial diamonds proper ℓ-cohomologically smooth.*

2. Drinfeld Lemma

The following is our verion of Drinfeld lemma whose proof is simpler than the classical one. Let us note there is a natural morphism

$$\mathrm{Div}^1 \longrightarrow [*/\underline{W_E}]$$

defined by the $\underline{W_E}$-torsor

$$\begin{tikzcd}
\mathrm{Spa}(\widehat{\overline{E}}^\flat) \arrow[d, "\underline{I_E}"] \arrow[dd, bend right=60, swap, "\underline{W_E}"] \\
\mathrm{Spa}(\breve{E})^\diamond \arrow[d] \\
\mathrm{Div}^1 = \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}.
\end{tikzcd}$$

Proposition 2.1 ("Drinfeld lemma"). *For any finite set I there is an equivalence*

$$\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E}]^I, \Lambda) \xrightarrow{\sim} \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda).$$

There is moreover an equivalence

$$\underbrace{\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)}_{\substack{\text{stable } \infty\text{-cat.} \\ \text{upgraded to a} \\ \text{condensed} \\ \text{stable } \infty\text{-cat.}}}{}^{\overbrace{BW_E^I}^{\substack{\text{condensed} \\ \infty\text{-groupoid}}}} \xrightarrow{\sim} \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E}]^I, \Lambda).$$

Here the condensation is to take into account the topology of W_E that is seen as a condensed group and the classifying stack BW_E^I as a condensed ∞-groupoid that is to say an $(\infty, 0)$-category in the topos of condensed sets. More precisely, the functor

$$\begin{array}{rcl} \{\text{profinite sets}\} & \longrightarrow & \text{stable } \infty\text{-categories} \\ P & \longrightarrow & \mathcal{D}_{\text{ét}}(\text{Bun}_G \times \underline{P}, \Lambda) \end{array}$$

sends disjoint unions of extremally disconnected sets to products and is thus an hypersheaf of stable ∞-categories on profinite sets. This is what we call the "condensed upgrade" of the usual stable ∞-category $\mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda)$. It is a condensed infinite sub-category of the evident condensed infinite category

$$\mathcal{D}_{\text{pro-ét},\blacksquare}(\text{Bun}_G, \Lambda).$$

The condensed ∞-groupoid BW_E^I is the hypersheaf of ∞-groupoids

$$\begin{array}{rcl} \{\text{profinite sets}\} & \longrightarrow & \text{stable } \infty\text{-categories} \\ P & \longrightarrow & B(\underbrace{W_E^I(P)}_{\mathscr{C}(P, W_E^I)}). \end{array}$$

Remark 2.2. *Here we use the notation $\mathcal{D}^{\mathcal{C}}$ for the infinite category of ∞-functors from $\mathcal{C}$ to $\mathcal{D}$, and more generally for the hypersheaf of ∞-categories of ∞-functors between hypersheaves of ∞-categories whose value on an object of our topos U is*

$$\lim_{W \to V \to U} \mathcal{D}(W)^{\mathcal{C}(V)}.$$

When X is a topos, Λ a ring in X and G a group in X, one has an identification

$$\underbrace{\mathcal{D}(X, \Lambda)}_{\substack{\textit{hypersheaf of} \\ \infty\textit{-categories} \\ X \ni U \mapsto \mathcal{D}(U, \Lambda)}}^{\overbrace{BG}^{\substack{\textit{hypersheaf of} \\ \infty\textit{-cat.} \\ X \ni U \mapsto BG(U)}}} = \underbrace{\mathcal{D}(\overbrace{BG}^{\substack{\textit{classifying} \\ \textit{stack} \\ \textit{in } X}}, \Lambda)}_{\substack{\textit{hypersheaf} \\ \textit{of } \infty\textit{-cat.} \\ X \ni U \mapsto \mathcal{D}(BG \times U, \Lambda)}}.$$

as hypersheaves of stable ∞-categories on X.

Remark 2.3. *For $p = \infty$ the analog of the preceding is the following. Let us consider the Twister projective line*

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}}/z \sim -\tfrac{1}{\bar{z}}.$$

This can be described as

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}/W_{\mathbb{R}}$$

where $W_{\mathbb{R}}$ is the Weil group. The corresponding torsor $\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\} \to \widetilde{\mathbb{P}}^1_{\mathbb{R}}$ defines a morphism of analytic stacks

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} \longrightarrow [*/\underline{W_{\mathbb{R}}}].$$

3. What we want to do

For each finite set I we equip the infinite category of ∞-functors

$$\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$$

with a monoidal structure by setting

$$u \otimes v := u(-) \otimes_\Lambda^{\mathbb{L}} v(-).$$

The purpose now is to define a monoidal functor between monoidal stable ∞-categories

$$\boxed{F_I : \left(\mathrm{Rep}_\Lambda({}^L G)^I, \otimes\right) \longrightarrow \Big(\underbrace{\mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}\big)}_{\infty\text{-cat. of } \infty\text{-functors}}, \otimes\Big)}$$

where

$$\mathrm{Rep}_\Lambda({}^L G)^I$$

is (the ∞-upgrade of) the category of representations of $({}^L G)^I$ on finite type projective Λ-modules that are algebraic when restricted to $\widehat{G}^I$ and discrete when restricted to W_E^I. The ∞-upgrade is nothing else than the stable ∞-category of perfect complexes of such objects.

We ask moreover that

- (**Factorization property**) This is functorial in the finite set I in the sense that if $I \to I'$ is a map of finite sets then

$$\begin{array}{ccc} \mathrm{Rep}_\Lambda({}^L G)^I & \xrightarrow{F_I} & \mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}\big) \\ \downarrow & & \downarrow \\ \mathrm{Rep}_\Lambda({}^L G)^{I'} & \xrightarrow{F_{I'}} & \mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{BW_E^{I'}}\big) \end{array}$$

  commutes where the left vertical map is induced by the morphism $({}^L G)^{I'} \to ({}^L G)^I$ and the right vertical one by $W_E^{I'} \to W_E^I$.
- (**Linearity**) This is linear over $\mathrm{Rep}_\Lambda W_E^I$ in the sense that if $W \in \mathrm{Rep}_\Lambda W_E^I$ then

$$F_I(W) = - \otimes_\Lambda^{\mathbb{L}} W.$$

EXAMPLE 3.1. *If $I = \{1, 2\}$ and $I' = \{1\}$ the preceding factorization property is the following "**fusion property**". Let $W \in (\mathrm{Rep}_\Lambda {}^L G)^2$. We note $\Delta^* W$ its restriction to the diagonal, for example*

$$\Delta^*(W_1 \boxtimes W_2) = W_1 \otimes W_2.$$

Then, $\mathrm{Res}_{W_E}^{W_E^2} F_{1,2}(W) = F_1(\Delta^* W)$ via the restriction of the W_E^2-action to W_E embedded diagonally inside W_E^2.

REMARK 3.2. *The factorization property implies that after forgetting the action of* W_E^I *the functor*

$$\mathrm{Rep}_\Lambda({}^LG)^I \longrightarrow \mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)\big)$$

factorizes through the restriction to the diagonal $\mathrm{Rep}_\Lambda({}^LG)^I \longrightarrow \mathrm{Rep}_\Lambda {}^LG$.

4. From local to global

To construct our functor F_I we consider **the global Hecke stack**

$$\begin{tikzcd} & \mathrm{Hecke}_I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{tikzcd}$$

where for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, $\mathrm{Hecke}_I(S)$ is the groupoid of quadruples $(\mathscr{E}_1, \mathscr{E}_2, (D_i)_{i\in I}, u)$ where

- $\mathscr{E}_1$ and $\mathscr{E}_2$ are G-bundles on X_S,
- $(D_i)_{i\in I}$ is a collection of degree Cartier divisors on X_S,
- and

  $$u : \mathscr{E}_{1|X_S \smallsetminus \cup_{i\in I} D_i} \xrightarrow{\sim} \mathscr{E}_{2|X_S \smallsetminus \cup_{i\in I} D_i}$$

  that is *meromorphic along the Cartier divisor* $\sum_{i\in I} D_i$.

REMARK 4.1. *When* $S = \mathrm{Spa}(R, R^+)$, *via GAGA, we have that, if* $\mathfrak{X}_{R,R^+}$ *is the schematical curve,* $\mathrm{Hecke}_I(S)$ *is the set of quadruples as before where* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are étale* G*-torsors,* $(D_i)_{i\in I}$ *is a collection of Cartier divisors on* $\mathfrak{X}_{R,R^+}$ *that remain Cartier divisors when pulled-back to* $\mathfrak{X}_{K(s),K(s)^+}$ *for any* $s \in S$, *and* $u : \mathscr{E}_{1|\mathfrak{X}_{R,R^+} \smallsetminus \cup_{i\in I} D_i} \xrightarrow{\sim} \mathscr{E}_{2|\mathfrak{X}_{R,R^+} \smallsetminus \cup_{i\in I} D_i}$.

We want to **upgrade this correspondence to a cohomological one**. This is done in the following way. Let

$$\mathcal{H}ecke_I \longrightarrow (\mathrm{Div}^1)^I$$

be the so-called *local Hecke stack*. This is obtained in the same way as the global Hecke stack but **by replacing** X_S **by its formal completion along the divisor** $\sum_{i\in I} D_i$. Here is a formal definition.

DEFINITION 4.2. *The local Hecke stack is the functor on affinoid perfectoid* $\overline{\mathbb{F}}_q$*-algebras that sends* (R, R^+) *to quadruples* $(\mathscr{E}_1, \mathscr{E}_2, (D_i)_{i\in I}, u)$ *where*

(1) $(D_i)_{i\in I}$ *is as before a collection of "relative" Cartier divisors on* $\mathfrak{X}_{R,R^+}$,

(2) $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are étale* G*-torsors on the formal completion of* $\mathfrak{X}_{R,R^+}$ *along* $\sum_{i\in I} D_i$,

(3) u *is a meromorphic isomorphism between* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *outside the special fiber of the formal completion.*

There is thus a morphism **from global to local**

$$\begin{tikzcd}
 & \mathrm{Hecke}_I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, "loc"] & & \mathcal{H}ecke_I \arrow[dr] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \arrow[rr, "\text{proj.}"] & & (\mathrm{Div}^1)^I
\end{tikzcd}$$

The advantage of the local Hecke stack is that *it has an interpretation in terms of loop groups.*

Definition 4.3. (1) *We note*

$$\mathbb{B}^+_{dR,I}, \ \textit{resp. } \mathbb{B}_{dR,I},$$

for the v-sheaf of E-algebras over $(Div^1)^I$ *that sends* (R, R^+) *to the algebra of formal functions on the formal completion of the curve along* $\sum_{i\in I} D_i$*, resp. the algebra of formal meromorphic functions.*

(2) *We note*

$$L_I^+G, \ \textit{resp. } L_IG$$

for the v-sheaves of groups over $(\mathrm{Div}^1)^I$ *equal to*

$$L_I^+(G) = G(\mathbb{B}^+_{dR,I}), \ \textit{resp. } L_IG = G(\mathbb{B}_{dR,I}).$$

One thus has for $* : S \to (\mathrm{Div}^1)^I$ given by $(D_i)_{i\in I}$,

$$\mathbb{B}^+_{dR,I}(S) \times_{(\mathrm{Div}^1)^I(S)} * = \Gamma\Big(X_S, \varprojlim_{k\geq 0} \mathcal{O}_{X_S}/\prod_{i\in I} \mathscr{I}^k_{D_i}\Big)$$

and

$$\mathbb{B}_{dR,I}(S) \times_{(\mathrm{Div}^1)^I(S)} * = \Gamma\Big(X_S, \varinjlim_{l\geq 0} \varprojlim_{k\geq 0} \prod_{i\in I} \mathscr{I}^{-l}_{D_i}/\prod_{i\in I} \mathscr{I}^k_{D_i}\Big)$$

The following result is easy.

Lemma 4.4. *One has an equality of small v-stacks*

$$\mathcal{H}ecke_I = \big[\ L_I^+G\backslash L_IG/L_I^+G\ \big].$$

5. The Satake correspondence

We now want to use the local Hecke stack to define our functor

$$\boxed{F_I : \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)}$$

via the formula

$$F_I(W) = Rp_{2*}\big(p_1^*(-) \otimes^{\mathbb{L}}_{\Lambda} loc^* S_W\big)$$

for $W \in \mathrm{Rep}_\Lambda({}^LG)^I$ and where

$$S_W \in D_{\text{ét}}(\mathcal{H}ecke_I, \Lambda)^b$$

is the so-called **Satake sheaf associated to** W (where the upperscript "b" means bounded i.e. with quasi-compact support on the B_{dR}-affine Grassmanian that is to say supported on a finite union of closed Schubert cells).

More precisely, we want to define a monoidal functor

$$(\mathrm{Rep}_\Lambda({}^LG)^I, \otimes) \xrightarrow{\otimes} (D_{\text{ét}}(\mathcal{H}ecke_I, \Lambda)^b, *)$$

where the monoidal structure on the right is the one given by the composition of cohomological correspondences that is to say **the convolutions product**

$$A * B = Rb_*(a^* A \boxtimes^{\mathbb{L}}_{\Lambda} B)$$

where

$$\begin{tikzcd}
& {[\, L_I^+G \backslash L_IG \overset{L_I^+G}{\times} L_IG/L_I^+G \,]} \arrow[dl, "a"'] \arrow[dr, "b"] & \\
{[\, L_I^+G \backslash L_IG/L_I^+G \,] \times [\, L_I^+G \backslash L_IG/L_I^+G \,]} & & {[\, L_I^+G \backslash L_IG/L_I^+G \,]}
\end{tikzcd}$$

This is given by the following theorem. Here we suppose that Λ is a $\mathbb{Z}_\ell[q^{1/2}]$-algebra.

THEOREM 5.1 (Geometric Satake equivalence). *Let* $\mathrm{Sat}_I(G, \Lambda)$ *be the category of* **bounded perverse flat ULA sheaves** *on* $\mathcal{H}ecke_I$.

(1) This is stable under the convolution product $*$ *and functorial in* I.

(2) There is an equivalence of monoidal categories

$$\mathbf{Sat}_{\boldsymbol{I}}\boldsymbol{(G, \Lambda)} \xrightarrow{\sim} \mathbf{Rep}_{\boldsymbol{\Lambda}}\boldsymbol{({}^LG)^I}.$$

(3) This equivalence is **functorial in** I**, and linear over** $\mathrm{Rep}_\Lambda W_E^I$ *via the identification between* $\mathrm{Rep}_\Lambda W_E^I$ *and the category of étale local systems of* Λ*-modules on* $(\mathrm{Div}^1)^I$.

(4) For any $\mu \in X_*(T)^I = X^*(\widehat{T})^I$, *if* $\bar{\mu}$ *is the* Γ_E*-orbit of* μ *and* $W_{\bar{\mu}}$ *is the associated highest weight irreducible representation of* $({}^LG)^I$, *then* $W_{\bar{\mu}}$ *corresponds to*

$$\underbrace{j_{\bar{\mu}!*}\Lambda\big[\sum_i \langle \mu_i, 2\rho\rangle\big]}_{\substack{\textit{intersection cohomology} \\ \textit{complex of the Schubert cell}}}$$

where $j_{\bar{\mu}}$ *is the inclusion of the open Schubert cell defined by* $\bar{\mu}$ *inside the closed one.*

Chapitre 10

10th lecture - December 5

Here are the tools used for the geometric Satake equivalence :

(1) The notion of ULA complexes,
(2) Hyperbolic localization,
(3) Fusion,
(4) Degeneracy of the B_{dR}-affine Grassmanian to a "classical" Witt vectors affine Grassmanian.

Here the coefficients Λ are torsion to simplify.

1. ULA complexes

1.1. The classical case. Classically, if $f : X \to S$ is a finite presentation morphism of schemes, we have a good notion of f-ULA complexes in $D_{\text{ét}}(X, \Lambda)$ where here Λ is a Noetherian ring killed by a power of ℓ invertible on S. More precisely, those are the étale complexes "universally without vanishing cycles" i.e. the

$$A \in \underbrace{D^b_{\text{ét,c}}(X, \Lambda)}_{\substack{\text{bounded with constructible} \\ \text{cohomology sheaves}}}$$

such that

$$\forall \ \mathrm{Spec}(V) \longrightarrow S$$

where V is a rank 1 valuation ring, one has

$$R\Phi_{\bar{\eta}}\Big(A_{|X \times_S \mathrm{Spec}(V)}\Big) = 0$$

where $\bar{\eta}$ is a geometric point over the generic point of $\mathrm{Spec}(V)$.

Remark 1.1. *Said roughly this means that for any morphism*

$$\underbrace{\mathcal{C}}_{a\ \text{"curve"}} \longrightarrow S$$

the étale complex $A_{|X \times_S \mathcal{C}}$ is without vanishing cycles relatively to $X \times_S \mathcal{C} \to \mathcal{C}$. Thus, the condition is tested universally for all "curves" mapping to the target S.

One can prove, following Gaitsgory, that this is equivalent to A behaving well with respect to Verdier duality : A is f-ULA if and only if *universally over S*,

$$\boxed{\forall B \in D_{\text{ét}}(S, \Lambda),\ \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^* B \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(A, Rf^! B).}$$

One can moreover prove that if A is f-ULA then it is bidual with respect to Verdier duality :

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

1.2. The diamond case. Let $f : X \longrightarrow Y$ be a morphism of locally spatial diamonds (compactifiable of finite dim. trg.). Let $A \in D_{\text{ét}}(X, \Lambda)$. We define a good notion of A to be f-ULA.

Definition 1.2. *$A \in D_{\text{ét}}(X, \Lambda)$ is f-ULA if f-ULA if for any $j : U \to X$ (separated) étale with composite $U \to X \to Y$ quasicompact then $R(f \circ j)_! A$ is a perfect constructible complex when restricted to each quasicompact open subset of Y.*

Here, when Y is a spatial diamond the constructibility condition has to be though of differently from the usual case of algebraic varieties. Perfect constructible complexes are étale complexes of Λ-modules that differ from local systems only via non-overconvergence i.e. a perfect constructible étale complex of Λ-modules is étale locally constant if and only if it is overconvergent.

One can prove that all properties of "classical" algebraic étale local systems adapt in this situation, typically the nice behaviour with respect to Verdier duality.

1.3. Perverse ULA sheaves on $\mathrm{Gr}_{G,I}$. Suppose G is split to simplify and let $T \subset B$ be a maximal torus inside a Borel subgroup. There is a stratification of the local Hecke stack indexed by $(X_*(T)^+)^I$.

Definition 1.3. *For $(\mu_i)_{i \in I} \in (X_*(T)^+)^I$ we note*

$$\mathrm{Gr}_{G,I,(\mu_i)_{i\in I}}$$

the associated open Schubert cell and

$$\mathscr{H}ecke_{G,I,(\mu_i)_{i\in I}} = [\ L_I^+ G \backslash \mathrm{Gr}_{I,(\mu_i)_{i\in I}}\ ].$$

the associated stratum.

One has (for whatever definition of the dimension : Krull or cohomological) :

$$\dim \mathrm{Gr}_{G,I,(\mu_i)_{i\in I}} / (\mathrm{Div}^1)^I = \sum_{i \in I} \langle \mu_i, 2\rho \rangle$$

(relative dimension).

EXAMPLE 1.4. *For $I = \{1,2\}$, if $\Delta : \mathrm{Div}^1 \hookrightarrow (\mathrm{Div}^1)^2$ is the diagonal,*

$$\begin{tikzcd}
\big(\mathrm{Gr}_{G,\mu_1} \times \mathrm{Gr}_{G,\mu_2}\big) \times_{(\mathrm{Div}^1)^2} (\mathrm{Div}^1)^2 \setminus \Delta \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{G,I,(\mu_1,\mu_2)} \arrow[d] & \mathrm{Gr}_{G,\mu_1+\mu_2} \arrow[l, hook'] \arrow[d] \\
(\mathrm{Div}^1)^2 \setminus \Delta \arrow[r, hook] & (\mathrm{Div}^1)^2 & \mathrm{Div}^1. \arrow[l, hook', "\Delta"']
\end{tikzcd}$$

with cartesian squares.

The following lemma is easy. It is a consequence of the fact that the morphism $\mathscr{H}ecke_{I,(\mu_i)_{i\in I}} \to (\mathrm{Div}^1)^I$ is a gerbe banded by a connected diamond group.

LEMMA 1.5. *One has*

$$\mathcal{D}(\Lambda)^{BW_E^I} = \mathcal{D}_{\text{ét}}((\mathrm{Div}^1)^I, \Lambda) \xrightarrow{\sim} D_{\text{ét}}(\mathscr{H}ecke_{I,(\mu_i)_{i\in I}}, \Lambda).$$

Thus, étale sheaves on the open Schubert cells are given by the (derived) category of discrete representations of W_E^I on Λ-modules.

DEFINITION 1.6. *Let*

$$D := D^{\mathrm{ULA}}_{\text{ét}}(\mathscr{H}ecke_I, \Lambda)^b$$

be the category of $A \in D_{\text{ét}}(\mathscr{H}ecke_I, \Lambda)$ with **qc support** *and that are* **ULA** *relative to the morphism $\mathscr{H}ecke_I \to (\mathrm{Div}^1)^I$.*

We define

$${}^pD^{\leq 0} = \{A \in D \mid \forall x : \mathrm{Spa}(C, C^+) \to \mathscr{H}ecke_I,\ x^*A \in D^{\leq -\sum_{i\in I}\langle \mu_i(x), 2\rho\rangle}(\Lambda)\}$$

where $\mu_i(x) \in X_*(T)^+$, $i \in I$, gives the relative position at x, and

$${}^pD^{\geq 0} = \{A \in D \mid \mathbb{D}(A) \in {}^pD^{\leq 0}\}.$$

One verifies that this defines a t-structure with heart the abelian category

$$\mathrm{Perv}^{\mathrm{ULA}}(\mathscr{H}ecke_I, \Lambda).$$

DEFINITION 1.7. *The Satake category*

$$\mathrm{Sat}_I(G, \Lambda)$$

is the category of $A \in \mathrm{Perv}^{\mathrm{ULA}}(\mathscr{H}ecke_I, \Lambda)$ that are flat perverse in the sense that for all finite presentation Λ-module M, $A \otimes^{\mathbb{L}}_{\Lambda} M$ is perverse.

2. Mirkovick Vilonen cycles and the constant term functor

Suppose G is quasi-split.

Chapitre 11

11th lecture - December 12

1. Background on infinite categories

We fix a "sufficiently large" regular cardinal κ. All our categories and sets are κ-small. Here :

- an ∞-category means an $(\infty,1)$-category i.e. a quasi-category, which is nothing else than a particular type of simplicial set : the *weak Kan simplicial sets.*
- an ∞-groupoid or $(\infty,0)$-category means a *Kan simplicial set.* The basic example of an ∞-groupoid is
$$BG$$
where G is a group. This is the nerve of the category with one object with automorphisms G, $(BG)_n = G^n$.

Example 1.1. *If $\mathcal{C}$ is a "usual" 1-category then its nerve $N\mathcal{C}$ with $(N\mathcal{C})_n = Hom([n],\mathcal{C})$ is an ∞-category.*

If C is an ∞-category we note
$$\underbrace{\mathrm{Ho}(C)}_{\text{homotopy category}}$$
for the category whose objects are C_0 and if $C_0 \overset{d_0}{\underset{d_1}{\rightrightarrows}} C_1$ then
$$\mathrm{Hom}_{\mathrm{Ho}(C)}(x,y) = \{f \in C_1 \mid d_0(f) = x,\ d_1(f) = y\}/\sim$$
where $\sim$ is the equivalence relation
$$f \sim g \Leftrightarrow \exists z \in C_2 \begin{cases} d_2(z) = s_0(x) \\ d_1(z) = f \\ d_0(z) = g. \end{cases}$$
The ∞-category C is an ∞-groupoid if and only if $\mathrm{Ho}(C)$ **is a groupoid.**

If C is an ∞-category :

- we call the elements of C_0 **the objects of** C
- if $x, y \in C_0$ **the maps from** x **to** y are by definition the $f \in C_1$ satisfying $d_0 f = x$, $d_1 f = y$.

By definition :

- an **∞-functor** between to infinity categories C and D is a morphism of simplicial sets from C to D
- a **natural transformation between ∞-functors** $F, G : C \to D$ is a diagram

- If C and D are infinity categories then the simplicial set
$$\underbrace{\underline{\mathrm{Hom}}(C,D)}_{\substack{\text{internal maps} \\ \text{in the category of simplicial sets}}}$$
is a weak Kan complex that we call the ∞-category of functors from F to D. Its objects are ∞-functors as defined earlier and the morphisms are natural transformations as defined before.
- if x, y are two objects of the ∞-category C then
$$\mathrm{Hom}(x, y)$$
is the sub-simplicial complex
$$\mathrm{Hom}(x,y)_n = \{c \in C_{n+1} \mid v_0 c = \cdots = v_n c = x,\ v_{n+1}c = y\}$$
where $v_i : C_n \to C_0$ is the i-th vertex corresponding to the inclusion $[0] \hookrightarrow [n+1]$ sending 0 to i. This is in fact a Kan complex, **the ∞-groupoid of morphisms from x to y** sometimes called **the mapping space from x to y** when we see it as a Kan complex up to (weak) homotopy. Let us note that there exists other versions of this Kan complex but they are all homotopy equivalent. Moreover, the composition
$$\mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \longrightarrow \mathrm{Hom}(x,z)$$
as a morphism of Kan simplicial sets is only well defined up to homotopy. This point of view leads to the one of ∞-categories as categories enriched in the category of spaces (i.e. the category of Kan simplical sets up to homotopy) but this is not the one we use.

  Let us finally note that
$$\mathrm{Hom}_{\mathrm{Ho}(C)}(x,y) = \pi_0 \mathrm{Hom}(x,y).$$

1.1. Homotopy coherent nerve. There is a construction ([**2**, Definition 1.1.5.5], [**3**]) called **homotopy coherent nerve**
$$\mathrm{N}^{\mathrm{hc}} : \underbrace{\underline{\mathrm{sSet}\text{-}\mathrm{Cat}}}_{\substack{\text{categories enriched} \\ \text{in simplicial sets}}} \longrightarrow \underbrace{\underline{\mathrm{sSet}}}_{\substack{\text{simplicial} \\ \text{sets}}}$$
where a category enriched in simplicial sets is such that for any objects x, y, $\mathrm{Hom}(x, y)$ has a structure of simplicial set and the composition
$$\mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \to \mathrm{Hom}(x,z)$$
is a morphism of simplicial sets. Here we recall that if S and T are simplicial sets, $S \times T$ is such that $(S \times T)_n = S_n \times T_n$ i.e. $S \times T$ is defined using the diagonal map $\Delta \to \Delta \times \Delta$, Δ being the simplex category. For basic facts about simplicial objects we advise to look at [**1**].

This construction is such that if for all $x, y \in \mathrm{Ob}\,\mathcal{C}$, the simplicial set $\mathrm{Hom}(x, y)$ is a Kan simplicial set then $\mathrm{N}^{\mathrm{hc}}\,\mathcal{C}$ is a weak Kan complex. We thus have a construction

$$\boxed{\mathrm{N}^{hc} : \underbrace{\underline{\text{Kan-sSet-Cat}}}_{\substack{\text{categories enriched in} \\ \text{Kan simplicial sets}}} \longrightarrow (\infty, 1)\text{-categories.}}$$

If $\mathcal{C}$ is a category enriched in Kan simplicial sets then for $x, y \in \mathrm{Ob}(\mathcal{C})$, the Kan simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \;\text{ and }\; \mathrm{Hom}_{\mathrm{N}^{hc}(\mathcal{C})}(x,y)$$

are homotopy equivalent.

This construction is done via a functor

$$\mathcal{P} : \underbrace{\Delta}_{\substack{\text{simplex}\\ \text{category}}} \longrightarrow \text{sSet-Cat}$$

called **the path category**. Then,

$$\mathrm{N}^{\mathrm{hc}}(C)_n = \mathrm{Hom}_{\text{sSet-Cat}}(\mathcal{P}([n]), C).$$

It satisfies :

(1) $\mathrm{N}^{\mathrm{hc}}(C)_0 = C_0$,

(2) For $x, y \in C_0$,

$$\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(C)}(x,y)_0 = \mathrm{Hom}_C(x,y)_0$$

and there is a natural homotopy equivalence between

$$\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(C)}(x,y) \;\text{ and }\; \mathrm{Hom}_C(x,y)$$

i.e. a canonical isomorphism betwwen those two Kan simplicial sets in the ∞-category of Kan simplicial sets up to homotopy.

(3) There is an equivalence

$$C_0 \xrightarrow{\sim}$$

Example 1.2. *(1) The category of simplicial sets is enriched in simplicial sets : if S and T are simplicial sets, $\underline{Hom}(S,T)$ is the simplical set $[n] \mapsto Hom_{sSet}(X \times \Delta_n, Y)$.*

(2) The category of topological spaces is enriched in simplicial sets by setting, for X and Y two topological spaces,

$$Hom(X,Y)_n = Hom(X \times |\Delta_n|, Y).$$

(3) If $\mathcal{C}$ is a dg-category then this gives rise the the following category enriched in simplicial sets. Recall the Dold-Kan correspondence given by the simplicialization functor

$$\Gamma : \mathrm{CoCh}_{\leq 0}(\mathrm{Ab}) \xrightarrow{\sim} s\,\mathrm{Ab}$$

We can then set for $X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$Hom(X,Y) = \Gamma\,\tau_{\leq 0} Hom_{\mathcal{C}}(X,Y).$$

This defines a morphism of 2-categories

$$\text{dg-Cat} \longrightarrow \textit{sSet-Cat}.$$

Composed with the preceding homotopy coherent nerve construction we obtain a construction

$$\text{dg-Cat} \longrightarrow \infty\text{-Cat}\,.$$

(4) If $\mathcal{A}$ is an abelian category then the category of cochaine complexes of elements of $\mathcal{A}$ is naturally a dg-category. The associated ∞-category is the one of cochain complexes up to homotopy i.e. its objects are cochain complexes and its morphisms are morphisms of cochain complexes with

$$\pi_0 Hom(A^\bullet, B^\bullet) = Hom(A^\bullet, B^\bullet)/\,\text{homotopy}\,.$$

(5) *If S and T are weak Kan simplicial sets then $\underline{Hom}(S,T)$ is a weak Kan simplicial set. Thus, à priori*

$$N(\textit{the simplicially enriched cat. of weak Kan complexes}) = \text{“}\textit{an } (\infty,2)\textit{-category}\text{”}.$$

Nevertheless, there is a construction

$$\mathrm{Core} : \infty\text{-categories} \longrightarrow \infty\text{-groupoids}$$

that sends a weak Kan complex to the biggest sub-Kan complex (i.e. we only keep the 1-morphisms that are isomorphisms). Then, if we take the simplicial category whose objects are the weak Kan complexes with morphisms between S and T

$$\mathrm{Core}\,\underline{Hom}(S,T),$$

its homotopy coherent nerve is what we call **the ∞-category of ∞-categories**.

1.2. Dwyer-Kan localization. There is another construction. If C is an ∞-category and $S \subset C_1$ a set of maps one can define its **Dwyer-Kan localization**

$$S^{-1}C = \text{an } \infty\text{-category}.$$

One has

$$\mathrm{Ho}(\underbrace{S^{-1}C}_{\substack{\text{Dwyer-Kan}\\ \text{localization}}}) = \underbrace{S^{-1}\,\mathrm{Ho}(C)}_{\substack{\text{Gabriel-Zisman}\\ \text{localization}}}.$$

Example 1.3. *By definition, if $f : C \to D$ is a*

1.3. Homotopy limits and colimits.

2. The animation slogan

Recall that if I is a small category then I is said to be **filtered** if colimits (with values in usual 1-categories) indexed by I *commute with finite limits*. This is well known to be equivalent to :

(1) for any $i, j \in I$ there exists $k \in I$ with $\mathrm{Hom}(i,k) \neq \emptyset$ and $\mathrm{Hom}(j,k) \neq \emptyset$,

(2) for two morphisms $i \underset{v}{\overset{u}{\rightrightarrows}} j$ in I there exist a morphism $f : j \to k$ in I such that $f \circ u = f \circ v$.

By definition, a small category is **1-sifted** if colimits (with values in usual 1-categories) indexed by I *commute with finite products*. This is of course the case if I is filtered. Another example is the case of **reflexive co-equalizers** which correspond to the diagram $\tau_{\leq 1}\Delta$

$$\begin{tikzcd} i \arrow[r, shift left=2, "u"] \arrow[r, shift right=2, "v"'] & j \arrow[l, dashed, "s" description] \end{tikzcd}$$

where s is a joint section of u and v i.e. $u \circ s = \mathrm{Id} = v \circ s$. In fact, for a finite collection of morphisms $\left(X_\alpha \rightrightarrows Y_\alpha \right)_\alpha$, the morphism

$$\mathrm{coeq}\Big(\textstyle\prod_\alpha X_\alpha \rightrightarrows \prod_\alpha Y_\alpha \Big) \longrightarrow \displaystyle\prod_\alpha \mathrm{coeq}\left(X_\alpha \rightrightarrows Y_\alpha \right)$$

has an explicit inverse induced by $\prod_\alpha s_\alpha$ if $s_\alpha : Y_\alpha \to X_\alpha$ is a joint section of $X_\alpha \rightrightarrows Y_\alpha$.

REMARK 2.1. *In any category of "algebraic objects" defined from the category of Sets using finite products, typically the category of groups or rings (a set A with two maps $A \times A \xrightarrow{+} A$ and $A \times A \xrightarrow{\times} A$ satisfying some properties), sifted colimits exist. For example,*

(1) if $A \overset{u}{\underset{v}{\rightrightarrows}} B$ *are two morphisms of rings admitting a joint section then* $\mathrm{Im}(u-v)$ *is an ideal of* B*,*

(2) if $G \overset{u}{\underset{v}{\rightrightarrows}} H$ *are two morphisms of groups admitting a joint section then the subgroup generated by the* $u(g)v(g)^{-1}, g \in G$*, is distinguished.*

Animation slogan : If $\mathcal{C}$ is a category

(1) admitting small filtered colimits,

(2) generated under small sifted colimits by its **compact projective** objects,

DEFINITION 2.2. *For W a group we note $\mathcal{C}_W$ for the category of couples $(I, F(I) \to W)$ where*

(1) I is a finite set,

(2) $F(I)$ is the free group on I and $F(I) \to W$ is a morphism of groups,

(3) morphisms between $(I, F(I) \to W)$ and $(I', F(I') \to W)$ are given by morphisms of groups $F(I) \to F(I')$ such that the diagram

$$\begin{tikzcd} F(I) \arrow[d] \arrow[dr] & \\ F(I') \arrow[r] & W \end{tikzcd}$$

commutes.

LEMMA 2.3. *The category $\mathcal{C}_W$ is sifted.*

$\to$ For two objects $F(I) \xrightarrow{u} W$ and $v : F(I') \xrightarrow{v} W$ of $\mathcal{C}_W$ there is a diagram

$$\begin{tikzcd} F(I) \arrow[dr] \arrow[drr, bend left, "u"] & & \\ & F(I \coprod I') \arrow[r, "u*v"] & W \\ F(I') \arrow[ur] \arrow[urr, bend right, "v"'] & & \end{tikzcd}$$

Thus, for $x, y \in \mathrm{Ob}\,\mathcal{C}_W$ one can find $z \in \mathrm{Ob}\,\mathcal{C}_W$ such that $\mathrm{Hom}(x,z) \neq \emptyset$ and $\mathrm{Hom}(y,z) \neq \emptyset$. If we have two morphisms

$$\begin{tikzcd} F(I) \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr] & \\ & W \\ F(I') \arrow[ur] & \end{tikzcd}$$

the image of $F(I)$ in $F(I') \times_W F(I')$ is a finite type subgroup of the free group $F(I) \times F(I)$. It is thus isomorphic to $F(J)$ for a finite set J. From this we deduce that for two morphisms $x \rightrightarrows y$ in $\mathcal{C}_W$, there is a factorization of those two morphisms

$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend left] \arrow[rr, bend right] & z \arrow[r, shift left] \arrow[r, shift right] & y \end{tikzcd}$$

where $z \rightrightarrows y$ is a reflexive coequalizer.
Those two properties prove that $\mathcal{C}_W$ is sifted.

For a finite set I we note $\Sigma I = \vee_{i \in I} S^1$ as an ∞-groupoid.

Corollary 2.4. *We have*

$$\underset{(I, F(I) \to W) \in \mathcal{C}_W}{\mathrm{colim}} \Sigma I \xrightarrow{\sim} BW$$

in the ∞-category of ∞-groupoids i.e. **the ∞-groupoid BW is a sifted colimit of ΣI for finite sets I.**

3. The moduli space of Langlands parameters

Proposition 3.1. *There is an isomorphism of derived stacks over Spec($\mathbb{Z}[\frac{1}{p}]$),*

$$\mathrm{LocSys}_{\widehat{G}} \xrightarrow{\sim} \varprojlim_{(I, F(I) \to W) \in \mathcal{C}_W} [\ \widehat{G}^I \ / \ \widehat{G}\ ]$$

where if $\tau : I \to W$, the action of $\widehat{G}$ on $\widehat{G}^I$ is given by $g.(h_i)_{i \in I} = (g h_i g^{-\tau_i})_{i \in I}$.

4. A conjecture

CONJECTURE 4.1. *The following is conjectured :*

(1) *There exists a locally complete intersection algebraic stack*

$$\begin{array}{c}\mathfrak{X}\\ \big\downarrow\\ \mathit{Spec}(\mathbb{Z})\end{array}$$

satisfying :

(a) $\mathfrak{X}\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{p}\right]$ *is the preceding stack of Langlands parameters* $[Z^1(W_E,\widehat{G})/\widehat{G}]$,

(b) *the* p*-adic completion of* $\mathfrak{X}$,

$$\begin{array}{c}\widehat{\mathfrak{X}}\\ \big\downarrow\\ \mathit{Spf}(\mathbb{Z}_p),\end{array}$$

is the Emerton-Gee stack.

(2) *There is a monoidal action of* $\mathrm{Perf}(\mathfrak{X})$ *on*

$$D_{\mathrm{mot}}(\mathrm{Bun}_G,\mathbb{Z}).$$

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