

Eilenberg lectures - Fall 2023

**Some new geometric structures in the Langlands
program**

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First lecture - Sept 12

1. The local Langlands correspondence

1.1. Notations. Fix a prime number p . We need the following datum

- E is a finite degree extension of \mathbb{Q}_p with residue field \mathbb{F}_q and uniformizer π
- We fix an algebraic closure \overline{E} of E and let

$$\Gamma_E = \text{Gal}(\overline{E}|E)$$

and

$$W_E \subset \Gamma_E$$

be the associated Weil group of elements of Γ_E acting as Frob_q^n for some $n \in \mathbb{Z} \subset \widehat{\mathbb{Z}}$ on the residue field.

- G is a reductive group over E
- We fix some $\ell \neq p$ and consider $\overline{\mathbb{Q}}_\ell$ an algebraic closure of \mathbb{Q}_ℓ

We let

$${}^L G = \widehat{G} \rtimes \Gamma_E$$

be the associated L -group over \mathbb{Z} (seen as a pro-algebraic group). Here \widehat{G} is a split reductive group over \mathbb{Z} equipped with an action of Γ_E factorizing through an open subgroup of Γ_E .

EXAMPLE 1.1. (1) If $G = T$ is a torus then $\widehat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m$ with the Γ_E action deduced from the one on $X^*(T)$

(2) If $G = \text{GL}_{n/E}$ then $\widehat{G} = \text{GL}_n$ with trivial Γ_E action

(3) If $G = \text{SL}_{n/E}$ then $\widehat{G} = \text{PGL}_n$ with trivial Γ_E action

(4) If $K|E$ is a quadratic extension with Galois group $\{\text{Id}, *\}$, $A \in M_n(K)$ is hermitian non-degenerate, i.e. satisfies ${}^t A^* = A$ and $\det(A) \neq 0$, the associated unitary group G such that $G(E) = \{B \in \text{GL}_n(K) \mid BA {}^t B^* = A\}$ satisfies $\widehat{G} = \text{GL}_n$ with the action of Γ_E factorizing through $\text{Gal}(K|E)$ and where the non-trivial element of the Galois

group acts as $g \mapsto w {}^t g^{-1} w$ where $w = \begin{pmatrix} & & 1 \\ & -1 & \\ & 1 & \\ \dots & & \end{pmatrix}$

1.2. The local Langlands correspondence.

1.2.1. *Smooth representations.* Let Λ be a $\mathbb{Z}[\frac{1}{p}]$ -algebra. Recall the following definition.

DEFINITION 1.2. A smooth representation of $G(E)$ with coefficients in Λ is a Λ -module M equipped with a linear action of $G(E)$ such that the stabilizer of any vector is open in $G(E)$. We note

$$\text{Rep}_\Lambda(G(E))$$

for the category of smooth representations with coefficients in Λ .

Let

$$\mathcal{C}(G(E), \Lambda)$$

be the Λ -module of locally constant with compact support functions on $G(E)$ with coefficients in Λ . Let

$$\mathcal{H}_\Lambda(G(E)) = \text{Hom}_\Lambda(\mathcal{C}(G(E), \Lambda), \Lambda)$$

be the Hecke convolution algebra of distributions on $G(E)$ with coefficients in Λ that are smooth with compact support. The choice of a Haar measure μ on $G(E)$ with values in $\mathbb{Z}[\frac{1}{p}]$ defines an isomorphism

$$\begin{aligned} \mathcal{C}(G(E), \Lambda) &\xrightarrow{\sim} \mathcal{H}_\Lambda(G(E)) \\ f &\mapsto f\mu \end{aligned}$$

where the ring structure on $\mathcal{C}(G(E))$ is now given by $(f * g)(x) = \int_{G(E)} f(xy^{-1})g(y)d\mu(y)$. For each $K \subset G(E)$ an open pro- p subgroup there is associated an idempotent

$$e_K \in \mathcal{H}_\Lambda(G(E))$$

given by $\langle e_K, \varphi \rangle = \int_K \varphi$ where, in this formula, the integration on K is with respect to the Haar measure with volume 1. In other words, $e_K = \frac{1}{\mu(K)} \mathbf{1}_K \in \mathcal{C}(G(E))$ via the preceding identification. Then, one has $e_K * e_{K'} = e_K$ if $K \subset K'$ and

$$\mathcal{H}_\Lambda(G(E)) = \bigcup_K \underbrace{e_K * \mathcal{H}(G(E), \Lambda) * e_K}_{\mathcal{H}_\Lambda(K \backslash G(E) / K)}$$

where $\mathcal{H}(K \backslash G(E) / K)$ is the Hecke algebra of K -bi-invariant distributions on $G(E)$ with compact support.

To any $\pi \in \text{Rep}_\Lambda(G(E))$ with associated Λ -module M_π , one can associate a module over $\mathcal{H}(G(E), \Lambda)$ by setting for $m \in M_\pi$ and $T \in \mathcal{H}(G(E), \Lambda)$,

$$T.m = \int_{G(E)} \pi(g).m dT(g).$$

One then has

$$e_K.M_\pi = M_\pi^K$$

as an $\mathcal{H}(K \backslash G(E) / K, \Lambda)$ -module. This induces an equivalence

$$\{\text{smooth rep. of } G(E) \text{ wt. coeff. in } \Lambda\} \xrightarrow{\sim} \{\mathcal{H}_\Lambda(G(E))\text{-modules } M \text{ s.t. } M = \cup_K e_K.M\}.$$

One verifies that if Λ is a field and K is compact open with order invertible in Λ this induces an equivalence

$$\{\pi \in \text{Rep}_\Lambda(G(E)) \text{ irreducible s.t. } \pi^K \neq 0\} \xrightarrow{\sim} \{\text{irreducible } \mathcal{H}_\Lambda(K \backslash G(E)/K)\text{-modules}\}.$$

1.2.2. *Langlands parameters.* The local Langlands correspondence seeks to attach to any irreducible $\pi \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ a *Langlands parameter*

$$\varphi_\pi : W_E \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell).$$

Here the terminology ‘‘Langlands parameter’’ means

- that the composite of φ_π with the projection to Γ_E is the canonical inclusion $W_E \subset \Gamma_E$ i.e. φ_π is given by a 1-cocycle $W_E \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell)$,
- that moreover this cocycle takes values in $\widehat{G}(L)$ where L is a finite degree extension of \mathbb{Q}_ℓ ,
- that this cocycle with values in $\widehat{G}(L)$ is continuous.

REMARK 1.3. *There’s a way to make this notion of Langlands parameter independent of the choice of the ℓ -adic topology. In fact, Grothendieck’s ℓ -adic monodromy theorem (‘‘any ℓ -adic representation is potentially semi-stable’’) applies in this context and a Langlands parameter $\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ as before is in fact the same as a couple (ρ, N) where*

- $\rho : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is a Langlands parameter that is trivial on an open sub-group of W_E ,
- $N \in \mathfrak{g}_{\overline{\mathbb{Q}}_\ell}(-1)$ is nilpotent and satisfies : $\forall \tau \in W_E, \text{Ad } \rho(\tau).N = q^{v(\tau)}N$ where τ acts as $\text{Frob}_q^{v(\tau)}$ on the residue field.

The couples

$$(\rho, N)$$

are the so-called **Weil-Deligne parameters**. There is a 1-cocycle $t_\ell : W_E \rightarrow \mathbb{Z}_\ell(1)$ sending τ to $(\tau(\pi^{1/\ell^n})/\pi^{1/\ell^n})_{n \geq 1}$. The correspondence sends (ρ, N) to the parameter φ such that for $\tau \in W_E$,

$$\varphi(\tau) = \rho(\tau) \exp(t_\ell(\tau)N) \rtimes \tau.$$

Nevertheless, since we fix a prime number ℓ in our work with Scholze we prefer to give a formulation using the ℓ -adic topology. This is justified by the fact that we construct such parameters over $\overline{\mathbb{F}}_\ell$ too and *our correspondence is compatible with mod ℓ reduction*.

One last remark : φ_π is only defined up to $\widehat{G}(\mathbb{Q}_\ell)$ -conjugation i.e. we see it as an element of $H^1(W_E, \widehat{G}(\overline{\mathbb{Q}}_\ell))$. Up to now the local Langlands correspondence is a map

$$\text{Irr}_{\overline{\mathbb{Q}}_\ell}(G(E))/\sim \longrightarrow \{\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)$$

i.e. a map between isomorphism classes of object. We will later see *this correspondence has some categorical flavors* (and this is quite important since at the end we formulate a real categorical local Langlands correspondence with Scholze) but up to now we deal with objects

up to isomorphisms

1.2.3. *What to expect from the local Langlands correspondence.* Here is what we expect from the local Langlands correspondence.

- (1) **Frobenius semi-simplicity** First, there is one condition on φ_π : this has to be *Frobenius semi-simple* in the sense that the associated couple (ρ, N) has to be such that for all τ , $\rho(\tau)$ is semi-simple (i.e. $\rho(\tau)$ is semi-simple for a τ satisfying $v(\tau) = 1$).
- (2) **Finiteness of the L-packets** The fibers of $\{\pi\} \mapsto \{\varphi_\pi\}$ are finite : those are the so-called L-packets
- (3) **Description of the image** When G is quasi-split the correspondence $\{\pi\} \mapsto \{\varphi_\pi\}$ should be surjective. For other G , there is so-called relevance condition so that a parameter $\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is isomorphic to some φ_π if and only if as soon as φ factorizes (up to $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ -conjugacy) through some parabolic subgroup ${}^L P(\overline{\mathbb{Q}}_\ell)$ where P is a parabolic subgroup of G^* then P transfers to G .

For example : if $G = D^\times$ where D is a central division algebra over E with $[D : E] = n^2$ then a Langlands parameter $\varphi : W_E \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) = \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is relevant if and only if φ , as a linear representation of W_E , is indecomposable.

- (4) **Compatibility with local class field theory** If $G = T$ is a torus class field theory gives an isomorphism of groups

$$\mathrm{Hom}(T(E), \overline{\mathbb{Q}}_\ell^\times) \xrightarrow{\sim} H^1(W_E, {}^L T(\overline{\mathbb{Q}}_\ell))$$

this has to be the cloal Langlands correspondence for tori. Typically, when T is a split torus, there is an Artin reciprocity isomorphism

$$T(E) \xrightarrow{\sim} W_E^{ab} \otimes_{\mathbb{Z}} X_*(T)$$

deduced from

$$\mathrm{Art}_E : E^\times \xrightarrow{\sim} W_E^{ab},$$

and this isomorphism induces the local Langlands correspondence for T .

- (5) **Compatibility with the unramified local Langlands correspondence (Satake isomorphism)** If G is unramified, K is hyperspecial, after the choice of a square root of q in $\overline{\mathbb{Q}}_\ell$, there is a *Satake isomorphism* given by a *constant term map*

$$\mathcal{H}(K \backslash G(E) / K) \xrightarrow{\sim} \mathcal{H}(T(\mathcal{O}_E) \backslash T(E) / T(\mathcal{O}_E))^W$$

where T is an unramified torus coming from an integral model associated to the choice of K . If $A \subset T$ is the maximal split torus inside T then

$$\mathcal{H}(T(\mathcal{O}_E) \backslash T(E) / T(\mathcal{O}_E))^W = \mathcal{H}(A(\mathcal{O}_E) \backslash A(E) / A(\mathcal{O}_E))^W$$

that is identified with

$$\overline{\mathbb{Q}}_\ell[X_*(A)]^W = \overline{\mathbb{Q}}_\ell[X^*(\widehat{A})]^W.$$

If π is such that $\pi^K \neq 0$ then the irreducible module π^K over the spherical Hecke algebra thus defines a character

$$\overline{\mathbb{Q}}_\ell[X^*(\widehat{A})]^W \longrightarrow \overline{\mathbb{Q}}_\ell$$

that is to say an element of $\widehat{A}(\overline{\mathbb{Q}}_\ell)/W$. One can prove that this is the same as an element of

$$\{\text{unramified (semi-simple) } \varphi : W_E/I_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)\} / \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

(6) **Compatibility with Kazhdan-Lusztig depth 0 local Langlands**

If G is split and I is an Iwahori subgroup of $G(E)$ then the category

$$\text{Rep}_{\overline{\mathbb{Q}}_\ell}^I(G(E))$$

of $\pi \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ generated by π^I form a block in $\text{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ in the sense that there is an indecomposable idempotent e in the Bernstein center of $\text{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ such that

$$e \cdot \text{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E)) = \text{Rep}_{\overline{\mathbb{Q}}_\ell}^I(G(E)).$$

This is the so-called central block. This category is then identified with the category of modules over the Iwahori-Hecke algebra

$$\mathcal{H}(I \backslash G(E) / I).$$

The identification of this Iwahori-Hecke algebra with the equivariant K -theory of the Steinberg variety has allowed Kazhdan and Lusztig to give a parametrization of irreducible $\mathcal{H}(I \backslash G(E) / I)$ -modules as couples (s, N) where $s \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is semi-simple and $N \in \widehat{\mathfrak{g}}_{\overline{\mathbb{Q}}_\ell}$ is nilpotent and satisfies $\text{Ad}(s).N = qN$. We ask that this is the local Langlands correspondence in this case.

(7) **Compatibility up to semi-simplification with parabolic induction**

We say a parameter φ is semi-simple if the associated Weil-Deligne Langlands parameter (ρ, N) is such that $N = 0$. Equivalently, $\varphi|_{I_E}$ is trivial on an open subgroup. For a parameter φ we can define φ^{ss} its semi-simplification. Then, if P is a parabolic subgroup with Levi subgroup M we ask the following : for π an irreducible smooth representation of $M(E)$, if π' is an irreducible subquotient of the finite length representation

$$\text{Ind}_{P(E)}^{G(E)} \pi$$

(normalized parabolic induction), then

$$\varphi_{\pi'}^{ss}$$

is the composite of $\varphi_{\pi'}^{ss}$ with the inclusion ${}^L M(\overline{\mathbb{Q}}_\ell) \hookrightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$.

Let us remark that, of course, this is false without the semi-simplification since the Steinberg representation of $\text{GL}_n(E)$ and the trivial one do not have the same Langlands parameters.

(8) **Categorical flavor : description of supercuspidal L-packets**

We are now introducing some categorical flavor inside the Langlands parameters : we are not looking at the set quotient

$$\{\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)\} / \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

but the quotient as a groupoid

$$[\{\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)],$$

and thus

$$\{\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell) = \pi_0[\{\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)].$$

Suppose G is quasi-split (we will see later, following the work of Vogan, Kottwitz and Kaletha what to do in the non-quasi-split case). For a parameter φ we define

$$S_\varphi = \{g \in \widehat{G}(\overline{\mathbb{Q}}_\ell) \mid g\varphi g^{-1} = \varphi\}.$$

This is the automorphism group of φ in the preceding groupoid. There is always an inclusion

$$Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E} \subset S_\varphi.$$

We say that φ is cuspidal if it is semi-simple and $S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}$ is finite. We say a packet is supercuspidal if all of its elements are supercuspidal. Then

$$\{\text{supercuspidal L-packets}\} \xrightarrow{\sim} \{\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell) \text{ cuspidal}\}/\widehat{G}(\overline{\mathbb{Q}}_\ell).$$

Moreover, the choice of a Whittaker datum defines a bijection for φ a cuspidal parameter

$$\underbrace{\text{Irr}(S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E})}_{\text{finite group}} \xrightarrow{\sim} \text{L-packet associated to } \varphi$$

where the trivial representation should correspond to the unique generic (with respect to the choice of the Whittaker datum) representation of the L-packet.

(9) Local global compatibility

Let K be a number field and Π be an algebraic automorphic representation of G where now G is a reductive group over K . Conjecturally, Π_f is defined over a number field as a smooth representation of $G(\mathbb{A}_f)$. Let us fix an embedding of this number field inside $\overline{\mathbb{Q}}_\ell$. Then one should be able to attach to Π an ℓ -adic Langlands parameter

$$\varphi_\Pi : \text{Gal}(\overline{K}|K) \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell).$$

For a place v of K dividing $p \neq \ell$,

$$\varphi_{\Pi|W_{K_v}}$$

depends only on Π_v and is given up to conjugation by

$$\varphi_{\Pi_v}.$$

2. Background on the global Langlands correspondence and global Langlands parameters

Let G be a reductive group over a number field K . Let Π be an automorphic representation of G i.e. an irreducible sub-quotient of the space of automorphic forms on G . As an abstract representation

$$\Pi \simeq \bigotimes_v \Pi_v$$

where v goes through the places of K . If $v|\infty$, the local Langlands correspondence is known for Π_v is known and we can define

$$\varphi_{\Pi_v} : W_{K_v} \longrightarrow {}^L G_{\mathbb{C}}.$$

There is a natural morphism

$$\mathbb{C}^\times \longrightarrow W_{K_v}$$

that is an isomorphism if $K_v \simeq \mathbb{C}$ and fits into a non-split exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{K_v} \longrightarrow \text{Gal}(\mathbb{C}|\mathbb{R}) \longrightarrow 1$$

if $K_v \simeq \mathbb{R}$.

DEFINITION 2.1. *An automorphic representation Π of G is algebraic if for all $v|\infty$, $\varphi_{\Pi_v|\mathbb{C}^\times} : \mathbb{C}^\times \longrightarrow \widehat{G}(\mathbb{C})$ is algebraic i.e. is given by an algebraic morphism $\mathbb{S}_{\mathbb{C}} \rightarrow \widehat{G}_{\mathbb{C}}$ where \mathbb{S} is Deligne's torus $\text{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m$ via the inclusion $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$.*

It is the same as to ask that for all $v|\infty$, Π_v has the same infinitesimal character as the one of an algebraic irreducible finite dimension representation of the algebraic group $G_{\overline{K_v}}$ with coefficients in \mathbb{C} .

Conjecturally there exists a global Langlands group

$$\mathcal{L}_K$$

that is a locally compact topological group sitting in an exact sequence

$$1 \longrightarrow \mathcal{L}_K^\circ \longrightarrow \mathcal{L}_K \longrightarrow \text{Gal}(\overline{K}|K) \longrightarrow 1$$

and with an identification

$$\mathcal{L}_K / (\mathcal{L}_K^\circ)' = W_K$$

the global Weil group. Moreover, one expects the following.

CONJECTURE 2.2. *The following is expected :*

- (1) *To each automorphic representation Π of G one can associate a Langlands parameter*

$$\varphi_{\Pi} : \mathcal{L}_K \longrightarrow^L G_{\mathbb{C}}$$

compatibly with the local Langlands correspondence at archimedean places and the unramified one at almost all finite places

- (2) *If Π is algebraic then Π_f is defined over a number field inside \mathbb{C} and to the choice of an embedding of such a number field inside $\overline{\mathbb{Q}}_{\ell}$ is associated an ℓ -adic Langlands parameter*

$$\varphi_{\Pi, \ell} : \text{Gal}(\overline{K}|K) \longrightarrow^L G_{\overline{\mathbb{Q}}_{\ell}}$$

- (3) *The Tannakian category of continuous representations of \mathcal{L}_K on finite dimensional \mathbb{C} -vector spaces that are algebraic is identified with the category of Grothendieck motives for numerical equivalence with \mathbb{C} coefficients.*

This is known for tori when we consider the category of CM-motives for absolute Hodge cycles.

The construction of the ℓ -adic Langlands parameters is known for cohomological automorphic representations of GL_n . Other cases are known using the cohomology of Shimura varieties.

For example, if $f = \sum_{n \geq 1} a_n q^n$ is a normalized weight $k \geq 1$ holomorphic modular form for $\Gamma_0(N)$ that is new and an Hecke eigenvector of the Hecke operators $(T_p)_{p \nmid N}$ then one can associate (Shimura, Deligne, Deligne-Serre) a Galois representation

$$\rho_f : \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \text{GL}_2(\overline{\mathbb{Q}}_{\ell})$$

such that for $p \nmid N$, that characteristic polynomial of $\rho_f(\text{Frob}_p)$ is $X^2 - a_p X + p^{k-1}$.

3. What we do with Scholze

We prove the following theorem.

THEOREM 3.1 (F.-Scholze). *For ℓ a good prime with respect to G (any ℓ if $G = \text{GL}_n$, $\ell \neq 2$ for classical groups) there exists a monoidal action of the category of perfect complexes*

$$\text{Perf}(\text{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_{\ell}})$$

on

$$\text{Dis}(\text{Bun}_G, \overline{\mathbb{Z}}_{\ell})$$

where $\text{LocSys}_{\widehat{G}} \rightarrow \text{Spec}(\mathbb{Z}[\frac{1}{p}])$ is the moduli space of Langlands parameter, an algebraic stack locally complete intersection of dimension 0 over $\text{Spec}(\mathbb{Z}[\frac{1}{p}])$.

As a consequence we can construct the semi-simple local Langlands correspondence

$$\pi \mapsto \varphi_{\pi}^{\text{ss}}$$

for any reductive group over E , over $\overline{\mathbb{F}}_\ell$ and $\overline{\mathbb{Q}}_\ell$ (and compatibly with mod ℓ reduction).

As for now the statement of the local Langlands conjecture is the following.

CONJECTURE 3.2 (Categorical local Langlands). *Suppose G is quasi-split and fix a Whittaker datum (B, ψ) . Suppose ℓ is a good prime. There exists an equivalence of stable ∞ -categories*

$$\mathcal{D}_{coh}^b(\text{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})_{nilp.ss.supp} \xrightarrow{\sim} \mathcal{D}_{lis}(\text{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega$$

compatible with the preceding spectral action and sending the structural sheaf \mathcal{O} to the Whittaker sheaf.

The goal of those lectures is to explain how after 20 years of work, starting from the classical local Langlands correspondence in terms of parameters of smooth irreducible representations as in the work of Harris-Taylor, we arrived at such a statement and what are those geometric objects showing up in the preceding statement, starting with the so-called Lubin-Tate spaces continuing with Rapoport-Zink spaces, Hodge-Tate periods, the curve and so on.

Second lecture - Sept 19

Problem : construct the local Langlands correspondence for a given group using local-global compatibility + some known cases of the global construction of ℓ -adic parameters via the cohomology of Shimura varieties.

More precisely, if $\Pi \simeq \bigotimes_v \Pi_v$ is a cohomological automorphic representation of G defined over a number field \mathbb{Q} and

$$\Pi \longmapsto r_\mu \circ \varphi_{\Pi|\text{Gal}(\overline{\mathbb{Q}}|L)}$$

via the cohomology of Shimura varieties where

- $\varphi_\Pi : \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is the expected global ℓ -adic parameter,
 - L is the reflex field associated to the Shimura variety, a number field inside \mathbb{C} ,
 - $r_\mu \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G} \rtimes \text{Gal}(\overline{\mathbb{Q}}|L))$ is an algebraic representation associated to our Shimura datum
- one expects that for $p \neq \ell$,

$$\varphi_{\Pi|W_{\mathbb{Q}_p}} = \varphi_{\Pi_p}$$

and thus, if $v|p$ is a place of L associated to the choice of an embedding f of L inside $\overline{\mathbb{Q}}_p$,

$$r_\mu \circ \varphi_{\Pi|W_{L_v}} = r_\mu \circ \varphi_{\Pi_p|W_{L_v}}$$

REMARK 0.1. (1) *By definition, a cohomological automorphic representation is a particular type of algebraic automorphic representation that shows up in the cohomology of locally symmetric spaces. For example, for GL_2 , the automorphic representation associated to an holomorphic modular form of weight $k \geq 1$ is algebraic but cohomological only when $k \geq 2$. The ℓ -adic Langlands parameter associated to a weight ≥ 2 holomorphic modular forms is obtained inside the intersection cohomology cohomology of modular curves with coefficients in some local systems (Shimura, Deligne).*

For weight 1 holomorphic modular forms this ℓ -adic Langlands parameter is obtained by ℓ -adic interpolation from the weight ≥ 2 case (Deligne-Serre).

There is another class of automorphic representation of GL_2/\mathbb{Q} that are algebraic but not cohomological : the one associated to non-holomorphic Maass forms f that satisfy $\Delta f = \frac{1}{4}f$ where $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the hyperbolic Laplacian. We do not know how to construct their ℓ -adic Langlands parameter.

- (2) *Suppose that $G_{\mathbb{R}}$ has discrete series, that is to say $G_{ad,\mathbb{R}}$ is an inner form of its compact form. This is for example the case if G can be enhanced to a Shimura datum. One can prove that one can globalize any supercuspidal representation of $G(\mathbb{Q}_p)$ to*

an automorphic representation Π such that Π_∞ is a discrete series representation. Those are cohomological and show up in middle degree in the cohomology of locally symmetric spaces.

- (3) One of the difficulties of the preceding approach is that we can not construct φ_Π but its composition with r_μ where r_μ is a very particular type representation of the Langlands dual since μ is minuscule. This difficulty is removed over function fields over \mathbb{F}_q using general Shtuka moduli spaces but we don't know, even for GL_2 , how to define Shimura varieties for non-minuscule μ . We will see later how to remove this difficulty for local Shimura varieties at p .

We would like to use this type of formula to define φ_{Π_v} after choosing suitable Shimura data giving rise to different representations r_μ . This leads to the question : why, after composing with r_μ , would $\varphi_{\Pi|W_{K_v}}$ depend only on Π_v ? This is the problem of **local-global compatibility**. The answer is that there are **local Shimura varieties** linked to the global one via a process of p -adic uniformization.

1. Shimura varieties

1.1. Hermitian symmetric spaces. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m$ be Deligne's torus. Recall the the Tannakian category of real Hodge structures is equivalent to $\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})$.

Datum : a couple

$$(G, \{h\})$$

where

- (1) G reductive group over \mathbb{R} .
- (2) $h : \mathbb{S} \rightarrow G$ with $G(\mathbb{R})$ -conjugacy class $\{h\}$

This is the same as the datum of G together with a \otimes -functor

$$\mathrm{Rep}(G) \longrightarrow \mathbb{R}\text{-Hodge structures,}$$

i.e. a G - \mathbb{R} -Hodge structure , such that the composite

$$\mathrm{Rep}(G) \rightarrow \mathbb{R}\text{-Hodge structures} \xrightarrow{\mathrm{can}} \mathrm{Vect}_{\mathbb{R}}$$

is isomorphic to the canonical fiber functor on $\mathrm{Rep} G$.

We note $\mu_h : \mathbb{G}_{m/\mathbb{C}} \rightarrow G_{\mathbb{C}}$ for the composite of $h_{\mathbb{C}}$ with $z \mapsto (z, 1)$ from $\mathbb{G}_{m/\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m/\mathbb{C}} \times \mathbb{G}_{m/\mathbb{C}}$. This defines the Hodge filtration.

Hypothesis :

- (1) (**Weight 0 adjoint Hodge structure**) $w_h : \mathbb{G}_m \rightarrow G$, obtained by composing h with the morphism $\mathbb{G}_m \rightarrow \mathbb{S}$ inducing $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on the \mathbb{R} -points, is central that is to say the Hodge structure $(\mathfrak{g}, \text{Ad} \circ h)$ is pure of weight 0.
- (2) (**Polarization**) Conjugation by $h(i)$ is a Cartan involution on G_{ad} that is to say the Killing form on \mathfrak{g}_{ad} defines a polarization of the weight 0 Hodge structure $(\mathfrak{g}_{ad}, \text{Ad} \circ h)$
- (3) (**Griffiths transversality**) $\mu_h : \mathbb{G}_{m/\mathbb{C}} \rightarrow G_{\mathbb{C}}$ is minuscule that is to say the weights of $\text{Ad} \circ \mu_h$ on $\mathfrak{g}_{\mathbb{C}}$ are in $\{-1, 0, 1\}$ that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \text{Ad} \circ h)$ is of type $(-1, 1), (1, -1), (0, 0)$

Under those hypothesis, if K_∞ is the centralizer of $h(i)$ in $G(\mathbb{R})$, a sub-group of $G(\mathbb{R})$ that is compact modulo the center,

$$X = G(\mathbb{R})/K_\infty.$$

More precisely, if \mathcal{F} is the flag manifold defined by μ_h , the map

$$X \longrightarrow \mathcal{F}$$

that sends some h' that is $G(\mathbb{R})$ -conjugate to h to the class of $\mu_{h'}$ is an open embedding,

$$X \underset{\text{open}}{\subset} \mathcal{F}.$$

Then, X is a moduli space of rigidified variation of Hodge structures equipped with a G -structure.

More precisely, if S is a smooth complex analytic space then $X(S)$ is the set of equivalence classes of $(\mathcal{F}, \text{Fil}^\bullet \mathcal{F} \otimes_{\mathbb{R}} \mathcal{O}_S, \eta)$ where

- $\mathcal{F} : \text{Rep } G \rightarrow \{\mathbb{R} \text{ local systems on } S\}$ is a \otimes -functor,
- $\text{Fil}^\bullet \mathcal{F} \otimes_{\mathbb{R}} \mathcal{O}_S$ is a finite decreasing filtration of the \otimes -functor

$$\mathcal{F} \otimes_{\mathbb{R}} \mathcal{O}_S : \text{Rep } G \rightarrow \{\text{vector bundles on } S\}$$

satisfying Griffiths transversality : if $\nabla = \text{Id} \otimes d$ then $\nabla \text{Fil}^k \subset \text{Fil}^{k-1} \otimes \Omega_S^1$

- for each \mathbb{R} -linear representation (V, ρ) of G and $s \in S$, the complex conjugate of the associated filtration of $V_{\mathbb{C}}$ is $\rho \circ w_h$ -opposite to the filtration of $V_{\mathbb{C}}$ and thus defines a weight $\rho \circ w_h$ Hodge structure,
- η is an isomorphism between tensor functors between \mathcal{F} and the canonical functor $(V, \rho) \mapsto \underline{V}$,
- We ask that for each $s \in S$, the associated morphism $\mathbb{S} \rightarrow G$ defined by taking the stalk at s of the preceding variation is $G(\mathbb{R})$ -conjugated to h .

Thus, $X =$ **moduli of Hodge structures**. We will see later that we can define moduli of p -adic Hodge structures using the curve. But we are first going to treat a particular case : **Lubin-Tate spaces**.

1.2. Shimura variety.

Datum :

- (1) G is a reductive group over \mathbb{Q}
- (2) $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$

Hypothesis :

- (1) (**Weight 0 adjoint Hodge structure**) $w_h : \mathbb{G}_m \rightarrow G_{\mathbb{R}}$, obtained by composing h with the morphism $\mathbb{G}_m \rightarrow \mathbb{S}$ inducing $\mathbb{R}^{\times} \hookrightarrow \mathbb{C}^{\times}$ on the \mathbb{R} -points, is central that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \text{Ad} \circ h)$ is pure of weight 0.
- (2) (**Polarization**) Conjugation by $h(i)$ is a Cartan involution on $G_{\mathbb{R},ad}$ that is to say the Killing form on \mathfrak{g}_{ad} defines a polarization of the weight 0 Hodge structure $(\mathfrak{g}_{\mathbb{R},ad}, \text{Ad} \circ h)$
- (3) (**Griffiths transversality**) μ_h is minuscule that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \text{Ad} \circ h)$ is of type $(-1, 1), (1, -1), (0, 0)$
- (4) (**Density of CM points**) For any \mathbb{Q} -factor H of G_{ad} , $H(\mathbb{R})$ is not compact.

EXAMPLE 1.1. (1) $G = \text{GL}_2$ and $h(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Modular curves case.

(2) Same as before but $G = D^{\times}$ with D a quaternion division algebra over \mathbb{Q}

(3) $G = \text{GSp}_{2n}$ associated with the symplectic form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Set $h(a+ib) = \begin{pmatrix} aI_n & bI_n \\ -bI_n & aI_n \end{pmatrix}$.
Siegel varieties (modular curves for $n = 1$)

(4) Let K be a CM field and B be a central simple algebra over K equipped with an involution $*$ inducing complex conjugation on K . Let $G = \text{GU}(D, *)$ be the associated similitude unitary group. Let Fix an isomorphism $G_{\mathbb{R}} \simeq G(\prod_{\tau \in \Phi} U(p_{\tau}, q_{\tau}))$ where $(p_{\tau}, q_{\tau})_{\tau \in \Phi}$ is a set of signatures index by a CM type Φ of K . Then if $h(z) = (h_{\tau}(z))_{\tau \in \Pi}$ with $h_{\tau}(z) = \text{diag}(\underbrace{z, \dots, z}_{p_{\tau}}, \underbrace{\bar{z}, \dots, \bar{z}}_{q_{\tau}})$ this defines a unitary type Shimura variety.

Shimura variety

$$\text{Sh}_K = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K)$$

for $K \subset G(\mathbb{A}_f)$ compact open “sufficiently small”. Writing $G(\mathbb{A}_f) = \prod_{i \in I} G(\mathbb{Q}) g_i K$ with I finite (finiteness of the class number), one has

$$\text{Sh}_K = \prod_{i \in I} \Gamma_i \backslash X$$

where $\Gamma_i = G(\mathbb{Q}) \cap g_i K g_i^{-1}$ is an arithmetic subgroup of $G(\mathbb{R})$.

The smooth complex analytic space Sh_K has an interpretation as a moduli of variations of G - \mathbb{Q} -Hodge structures. Well, in fact the natural moduli space is not Sh_K but

$$\prod_{\ker^1(\mathbb{Q}, G)} \text{Sh}_K$$

a finite disjoint union of copies Sh_K . More precisely, if S is a smooth complex analytic space then $\coprod_{\ker^1(\mathbb{Q}, G)} \text{Sh}_K(S)$ is the set of equivalence classes of $(\mathcal{F}, \text{Fil}^\bullet \otimes_{\mathbb{Q}} \mathcal{O}_S, \bar{\eta})$ where

- $\mathcal{F} : \text{Rep } G \rightarrow \{\mathbb{Q} - \text{local systems on } S\}$ is a \otimes -functor,
- $\text{Fil}^\bullet \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{O}_S$ is a finite filtration of the \otimes -functor

$$\mathcal{F} \otimes_{\mathbb{R}} \mathcal{O}_S : \text{Rep } G \rightarrow \{\text{vector bundles on } S\}$$

satisfying Griffiths transversality : if $\nabla = \text{Id} \otimes d$ then $\nabla \text{Fil}^k \subset \text{Fil}^{k-1} \otimes \Omega_S^1$,

- for each \mathbb{R} -linear representation (V, ρ) of G and $s \in S$, the complex conjugate of the associated filtration of $V_{\mathbb{C}}$ is $\rho \circ w_h$ -opposite to the filtration of $V_{\mathbb{C}}$ and thus defines a weight $\rho \circ w_h$ Hodge structure,
- for each $s \in S$, the associated functor $\text{Rep } G_{\mathbb{R}} \rightarrow \text{Vect}_{\mathbb{R}}$ obtained by taking the stalk at s is trivial and the associated $G_{\mathbb{R}}$ -Hodge structure is in the $G(\mathbb{R})$ -conjugacy class of h ,
- $\bar{\eta}$ is a K^p -orbit of trivialization $\eta : \text{can } \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} \mathcal{F} \otimes_{\mathbb{Q}} \mathbb{A}_f$.

Recall the following. We note L for the reflex field of the Shimura datum (G, X) . This is the field of definition of the conjugacy class of μ_h .

THEOREM 1.2. *The tower of complex analytic spaces $(\text{Sh}_K)_K$ is a tower of smooth quasi-projective algebraic varieties defined over L . When G is anisotropic modulo its center those are projective smooth algebraic varieties over L .*

Algebraicity as a \mathbb{C} -analytic space is due due Baily and Borel where they prove that if one adds a boundary to X by forming X^* , a generalization of $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, whose boundary components are parametrized by conjugacy classes of maximal parabolic subgroups of G , equipped with the so-called Satake topology then $\Gamma_i \backslash X^*$ is a compact normal \mathbb{C} -analytic space. The quasi-projectivity assertion is then done by proving that the dualizing sheaf ω on those spaces is ample. This is done via the construction of Eisenstein-Poincaré series that are automorphic forms sections of $\omega^{\otimes n}$ for $n \gg 0$. The co-compact case, i.e. when G is anisotropic modulo its center, was done before by Cartan and is much more simple via the construction of Poincaré series and the realization of X as a bounded domain.

The descent datum from \mathbb{C} to L is first constructed on CM-points via the theory of Shimura and Taniyama and the proof that it extends to an effective descent datum to the entire Shimura variety is “easy” in the Hodge type and more generally abelian type case and delicate, essentially due to Deligne, in the general case.

This is equipped with an action of $G(\mathbb{A}_f)$ when K varies. We can look at

$$\varinjlim_K H_{\text{ét}}^\bullet(\text{Sh}_K \otimes_L \bar{L}, \bar{\mathbb{Q}}_\ell)$$

as a smooth representation of $G(\mathbb{A}_f)$ equipped with a continuous commuting action of $\text{Gal}(\bar{L}|L)$.

Let us now recall the following.

THEOREM 1.3 (Mastushima, Borel, Franke). *For G a reductive group over \mathbb{Q} , $K_\infty \subset G(\mathbb{R})$ compact whose neutral connected component is the neutral connected component of a maximal compact subgroup, and $K \subset G(\mathbb{A}_f)$ compact open “sufficiently small”, if*

$$X_K = G(\mathbb{Q}) \backslash (G(\mathbb{R}) / K_\infty A_G(\mathbb{R})^+ \times G(\mathbb{A}_f) / K)$$

as a locally symmetric space, where A_G is the maximal split torus in Z_G , then

- (1) *If G is anisotropic modulo its center then, as a module over the Hecke algebra $\mathcal{H}(K \backslash G(\mathbb{A}_f) / K)$,*

$$H^\bullet(X_K, \mathbb{C}) = \bigoplus_{\Pi} m_{\Pi} \cdot \dim_{\mathbb{C}} H^\bullet(\mathfrak{g}_\infty, K_\infty; \Pi_\infty) \cdot \Pi_f^K$$

where

- Π goes through the set of automorphic representations of G with trivial central character when restricted to $A_G(\mathbb{R})^+$,
- m_{Π} is the multiplicity of Π in the space of automorphic forms,
- $H^\bullet(\mathfrak{g}_\infty, K_\infty, \Pi_\infty)$ is a finite dimensional cohomology \mathbb{C} -vector space associated to Π_∞ .

In particular this cohomology space is semi-simple as a module over the Hecke algebra $\mathcal{H}(K \backslash G(\mathbb{A}_f) / K)$.

- (2) *For any G , any constituent of $H^\bullet(X_K, \mathbb{C})$ as a module over the Hecke algebra is automorphic in the sense that it is isomorphic to Π_f^K where Π is a cohomological automorphic representation of G .*

2. Harris-Taylor Shimura varieties

2.1. Generic fiber. Let E be a given p -adic field. We are looking to define the local Langlands correspondence for $G = \mathrm{GL}_n/E$.

Harris and Taylor have exhibited some PEL-type Shimura datum (G, X) such that

$$G_{\mathbb{R}} \simeq G(U(1, n-1) \times U(n) \times \cdots \times U(n))$$

and

$$G_{\mathbb{Q}_p} \simeq \mathrm{GL}_n/E \times \mathbb{G}_m.$$

Moreover, one has

$$\widehat{G} = \mathrm{GL}_n \times \mathrm{GL}_n \times \cdots \times \mathrm{GL}_n \times \mathbb{G}_m$$

with r_μ the standard representation of dimension n on the first GL_n -factor, trivial on the other GL_n factors and all of this is twisted by the standard representation of \mathbb{G}_m . We can moreover suppose that G is anisotropic modulo its center.

In fact, G is a similitude unitary group attached to a division algebra over a CM field equipped with an involution inducing complex conjugation on the CM field.

We get

$$\begin{array}{c} \mathrm{Sh}_K \\ \downarrow \\ \mathrm{Spec}(L) \end{array}$$

a **proper smooth algebraic variety that is in fact a moduli of abelian varieties equipped with additional structures** like a polarization and an action of a division algebra. Set $L_v = E$ our p -adic field where v is a place of L dividing p .

We are going to analyze the cohomology of $(\mathrm{Sh}_K)_K \otimes_L L_v$ by making a degeneration from $p \neq 0$ to $p = 0$.

2.2. Integral models. If $K_p \subset G(E)$ is compact hyperspecial, $K_p = \mathrm{GL}_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$ ("minimal level at p "), then $\mathrm{Sh}_{K_p K^p}$ degenerates smoothly for any K^p compact open inside $G(\mathbb{A}_f^p)$ there exists a smooth projective model

$$\begin{array}{ccc} S_{K^p} & \hookrightarrow & \mathrm{Sh}_{K_p K^p} \otimes_L L_v \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_E) & \hookrightarrow & \mathrm{Spec}(E) \end{array}$$

with $S_{K^p} \otimes_{\mathcal{O}_E} E = \mathrm{Sh}_{K_p K^p} \otimes_L L_v$. This is a moduli space of abelian schemes with additional structures.

Main point Let

$$\begin{array}{c} \mathcal{A} \\ \downarrow \\ S_{K^p} \end{array}$$

be the universal abelian scheme. The fact is that the p -divisible group $\mathcal{A}[p^\infty]$ splits as

$$\mathcal{A}[p^\infty] = \mathcal{G} \oplus \mathcal{G}^D$$

where \mathcal{G} is equipped, as an extra additional structure, with an action of $M_n(\mathcal{O}_E)$. The additional structure that is the polarization on $\mathcal{A}[p^\infty]$ is the canonical polarization on $\mathcal{G} \oplus \mathcal{G}^D$.

Let $e = \begin{pmatrix} 1 & \\ & \end{pmatrix}$ as an idempotent of $M_n(\mathcal{O}_E)$. Then (Morita equivalence), a p -divisible group such as \mathcal{G} equipped with an action of $M_n(\mathcal{O}_E)$ is the same as a p -divisible group equipped with an action of \mathcal{O}_E ,

$$H := e.\mathcal{G}$$

in our case. The fact now is that the signature at ∞ of our unitary group

$$(1, n-1) \times (0, n) \times \cdots \times (0, n)$$

transfers at p as the condition that

- (1) H is a 1-dimensional p -divisible group with an action of \mathcal{O}_E
- (2) The action of \mathcal{O}_E on $\text{Lie } H$ is the canonical one via $S_{K^p} \rightarrow \mathcal{O}_E$.

We call such an object a **1-dimensional π -divisible \mathcal{O}_E -module**.

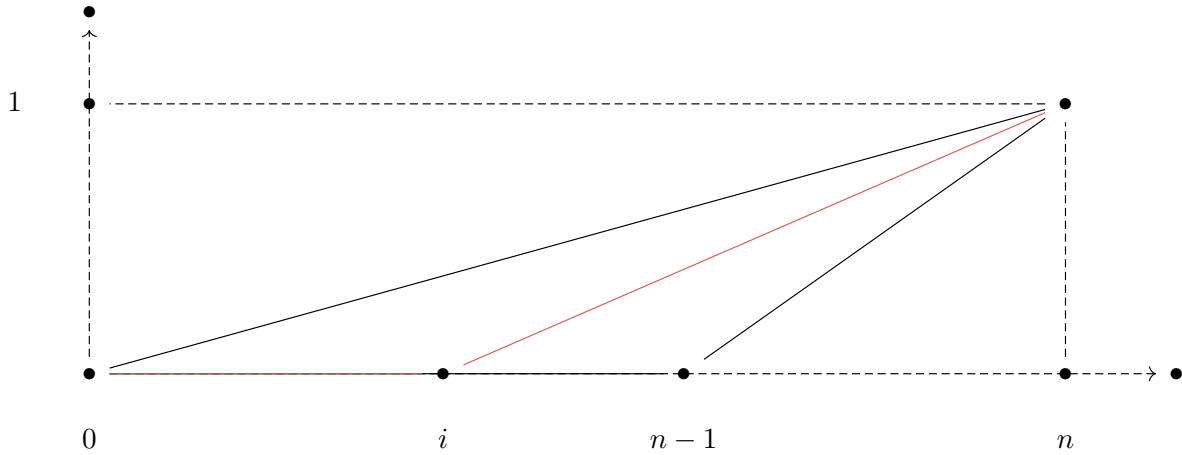
2.3. Newton stratification. Let

$$\overline{S}_{K^p} = S_{K^p} \otimes_{\mathcal{O}_E} \mathbb{F}_q$$

be the reduction modulo π of our Shimura variety. This again forms a tower of \tilde{A} - \mathbb{C} -étale coverings equipped with an action of $G(\mathbb{A}_f^p)$ when K^p varies. Let

$$\begin{array}{c} \overline{H} \\ \downarrow \\ \overline{S}_{K^p} \end{array}$$

be our 1-dimensional π -divisible \mathcal{O}_E -modules. Geometrically fiberwise on \overline{S}_{K^p} this has a Newton polygon that is of the following shape in red :



for an integer $i \in \{0, \dots, n-1\}$. In the preceding picture the Hodge polygon has slope 0 with multiplicity $n-1$ and 1 with multiplicity 1. The basic polygon has slope $1/n$. The integer i is the \mathcal{O}_E -height of the étale part. More precisely, there is a stratification by locally closed subsets

$$\overline{S}_{K^p}^{(i)}, \quad 0 \leq i \leq n-1$$

where a geometric point x of \overline{S}_{K^p} lies in $\overline{S}_{K^p}^{(i)}$ if and only if

$$0 \longrightarrow \underbrace{\overline{H}_x^\circ}_{\substack{\text{1-dim. formal} \\ \text{of } \mathcal{O}_E\text{-height } n-i}} \longrightarrow \overline{H}_x \longrightarrow \underbrace{\overline{H}_x^{\text{ét}}}_{\mathcal{O}_E\text{-height } i} \longrightarrow 0.$$

- (1) The closed stratum is $\overline{S}_{K^p}^{(0)}$ that is a finite set of closed points, the so-called basic locus,
- (2) The open stratum is $\overline{S}_{K^p}^{(n-1)}$ that is the so-called μ -ordinary locus.

2.4. Level structures at p . We worked before with a level structure at p for which $K_p = \mathrm{GL}_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$. In this case the integral models are smooth. Drinfeld defined a “good” notion of level structures at p for the principal congruence subgroups $K_p = \mathrm{Id} + \pi^m M_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$ when $m \geq 1$. This is very particular to 1-dimensional p -divisible groups. By “good” we mean that the associated integral models

$$S_{m,K^p}$$

are **regular** and the change of level morphism

$$\begin{array}{ccc} S_{m,K^p} & \xlongequal{\quad} & \text{regular} \\ \text{finite flat} \downarrow & & \\ S_{K^p} & \xlongequal{\quad} & \text{smooth}/\mathcal{O}_E \end{array}$$

is **finite flat**. Moreover those morphisms are totally ramified over the points of the basic locus. We obtain a tower

$$(S_{m,K^p})_{m \geq 1}$$

that is equipped at the limit when $m \rightarrow +\infty$ with an action of $G(\mathbb{Q}_p)$ and commuting Hecke correspondences associated to elements of $K^p \backslash G(\mathbf{A}_f^p) / K^p$.

2.5. Analysis of the ℓ -adic cohomology at p via nearby cycles.

2.5.1. *Background on nearby cycles.* Nearby cycles are a construction that allows us to analyze the cohomology of an algebraic variety via the cohomology of the special fiber of a “1-parameter degeneration” of this algebraic variety i.e. a degeneration parametrized by what we call a *trait* (the spectrum of a rank 1 valuation ring).

Let

$$\begin{array}{c} X \\ \downarrow \\ \mathrm{Spec}(V) \end{array}$$

be finite presentation morphism of schemes where V is an Henselian rank 1 valuation ring. Let $K = \mathrm{Frac}(V)$ and k be the residual field of V . Fix an algebraic closure \overline{K} of K and let \overline{k} be the associated algebraic closure of k . We note $\mathrm{Spec}(k)$, $\overline{s} = \mathrm{Spec}(\overline{k})$, $\eta = \mathrm{Spec}(K)$ and $\overline{\eta} = \mathrm{Spec}(\overline{K})$.

There is a diagram

$$\begin{array}{ccccc} X_s & \longleftrightarrow & X & \longleftrightarrow & X_\eta \\ \downarrow & & \downarrow & & \downarrow \\ s & \longleftrightarrow & \mathrm{Spec}(V) & \longleftrightarrow & \eta \end{array}$$

Let $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ with ℓ invertible in V . We want to understand

$$H^\bullet(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$$

with its $\mathrm{Gal}(\overline{K}|K)$ action in terms of the special fiber X_s of our degeneration. There is a “nearby cycle fiber functor”

$$D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell) \xrightarrow{R\Psi_{\bar{\eta}}} \left\{ \begin{array}{l} \text{objects in } D_c^b(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell) + \text{action of } \text{Gal}(\overline{K}|K) \\ \text{compatible with the one of } \text{Gal}(\overline{k}|k) \end{array} \right\}$$

such that for any geometric point \bar{x} of $X_{\bar{s}}$,

$$R\Psi_{\bar{\eta}}(\mathcal{F})_{\bar{x}} = R\Gamma\left(\underbrace{\text{Spec}(\mathcal{O}_{\overline{X}, \bar{x}}^{sh}[\frac{1}{\varpi}])}_{\text{schematic Milnor fiber over } \bar{x}}, \mathcal{F} \right)$$

where ϖ is a pseudo-uniformizer in V and $\overline{X} = X \otimes_V \overline{V}$ with \overline{V} the integral closure of V in \overline{K} .

REMARK 2.1. *The fiber at geometric points of $R\Psi_{\bar{\eta}}(\mathcal{F})$ is thus identified with the cohomology complex of those schematic Milnor fibers. Grothendieck's construction of the functor $R\Psi_{\bar{\eta}}$ is a way to take all those cohomology complexes of the different "classical" Milnor fibers and build a sheaf out of it. Deligne's theorem says that this complex has constructible cohomology and thus the cohomology of those Milnor fibers "varies constructibly".*

Proper base change then says that if $X \rightarrow \text{Spec}(V)$ is proper then

$$R\Gamma(X_{\bar{s}}, R\Psi_{\bar{\eta}}(\mathcal{F})) \xrightarrow{\sim} R\Gamma(X_{\bar{\eta}}, \mathcal{F}).$$

We will now use the following very important result that says that the nearby cycles depend only on the formal completion and not the henselization. Suppose that k is perfect.

THEOREM 2.2 (Berkovich, Huber). *Let x be a closed point of $X_{\bar{s}}$ and \mathfrak{X}_x be the formal completion of $X \otimes_V V^{un}$ at x where V^{un} is the integral closure of V in the maximal unramified extension K^{un} of K . This is a formal scheme over $\text{Spf}(\widehat{V^{un}})$. Let \mathfrak{X}_x^{ad} be its generic fiber as an adic space over $\text{Spa}(\widehat{K^{un}})$. There is then an isomorphism*

$$\underbrace{R\Psi_{\bar{\eta}}(\mathcal{F})_x}_{\text{cohomology of the schematic Milnor fiber}} \xrightarrow{\sim} \underbrace{R\Gamma_{\text{ét}}(\mathfrak{X}_{x,\eta} \hat{\otimes}_{\widehat{K^{un}}} \widehat{K}, \mathcal{F}^{ad})}_{\text{cohomology of the rigid analytic Milnor fiber}}$$

2.6. A localization phenomenon. The geometry of non-basic Newton strata implies the following result. For $m \geq 1$ we note

$$R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m, K^p} \in D_c^b(\overline{S}_{m, K^p} \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_\ell).$$

If $m' \geq m$ and $\Pi_{m', m} : \overline{S}_{m', K^p} \rightarrow \overline{S}_{m, K^p}$ then

$$R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m, K^p} = \Pi_{m', m*} R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m', K^p}.$$

Moreover if $\mathcal{H}_m = \mathcal{H}(G(E) // \text{Id} + \pi^m M_n(\mathcal{O}_E))$,

$$R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m, K^p}$$

is equipped with an action of $\mathcal{H}_m \otimes \mathcal{H}(K^p \backslash G(\mathbf{A}_f^p)/K^p)$.

THEOREM 2.3 (Harris-Taylor). *For any $m \geq 1$,*

$$[R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p}]_{\text{supercuspidal at } p} \xrightarrow{\sim} \bigoplus_{x \in \overline{\mathcal{S}}_{m,K^p}^{(0)}(\overline{\mathbb{F}}_q)} i_{x*} [R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p}]_{x,\text{supercuspidal at } p}$$

that is to say the supercuspidal at p -part of the complex of nearby cycles localizes on supersingular points.

3. Lubin-Tate spaces

DEFINITION 3.1. *Let \mathbb{H} be a one dimensional formal p -divisible group over $\overline{\mathbb{F}}_q$ equipped with an action of \mathcal{O}_E such that the action of \mathcal{O}_E on $\text{Lie } \mathbb{H}$ is the canonical one. We note*

$$\mathcal{LT} \longrightarrow \text{Spf}(\mathcal{O}_{\check{E}})$$

for the deformation space of \mathbb{H} .

This is a formal scheme (non-canonically) isomorphic to

$$\text{Spf}(\mathcal{O}_{\check{E}}[[x_1, \dots, x_{n-1}]])$$

We note

$$\mathcal{LT}_\eta \simeq \mathbb{B}_{\check{E}}^{n-1}$$

for its generic fiber as a locally of finite type adic space over $\text{Spa}(\check{E})$.

On this open ball the Tate module of the universal deformation is an \mathcal{O}_E -étale local system of rank n . The moduli of its trivializations defines a tower of rigid analytic spaces with finite étale transition morphisms

$$(\mathcal{LT}_{\eta,K})_{K \subset \text{GL}_n(\mathcal{O}_E)} \longrightarrow \mathcal{LT}_\eta$$

equipped with an action of $\text{GL}_n(E)^1$ at the limit. There is another group that shows up : the group of automorphisms by quasi-isogenies of \mathbb{H} , $\text{End}(\mathbb{H})_{\mathbb{Q}}^\times$, that is identified with

$$D^\times$$

where D is a division algebra with invariant $\frac{1}{n}$ over E . At the end the tower $(\mathcal{LT}_{\eta,K})_K$ has a commuting action of $(D^\times \times \text{GL}_n(E))^1$, the subgroup of $D \times \text{GL}_n(E)$ formed by elements (d, g) such that $v(\text{Nrd}(d)) + v(\det(g)) = 0$ where Nrd is the reduced norm.

In fact we prefer to work with

$$\mathcal{M}_K = \mathcal{LT}_{\eta,K} \overset{\mathcal{O}_D^\times}{\times} D^\times$$

that is a $\coprod_{\mathbb{Z}}$ of copies of the Lubin-Tate space. The tower $(\mathcal{M}_K)_K$ has an action of $D^\times \times \mathrm{GL}_n(E)$ and a (non-effective since this shifts everything by +1 in the components $\coprod_{\mathbb{Z}}$) descent datum $\mathcal{M}_K^{(\sigma)} \xrightarrow{\sim} \mathcal{M}_K$ from \check{E} to E . We now define

$$R\Gamma(\mathcal{M}_K \hat{\otimes}_{\check{E}} \widehat{E}, \overline{\mathbb{Q}}_\ell) := \bigoplus_{\alpha \in \pi_0(\mathcal{M}_K)} R\Gamma(\mathcal{M}_K^\alpha \hat{\otimes}_{\check{E}} \widehat{E}, \overline{\mathbb{Q}}_\ell).$$

This has an action of $D^\times \times W_E$ where the action of D^\times is smooth and a commuting action of $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(K \backslash \mathrm{GL}_n(E)/K)$.

REMARK 3.2. *As for Harris-Taylor Shimura varieties, the notion of Drinfeld level structure allows us to define some regular integral models of $\mathcal{LT}_{\eta, K}$ when $K = \mathrm{Id} + \pi^m M_n(\mathcal{O}_E)$, a principal congruence subgroup. Those are formal spectrum of complete regular Noetherian rings that are finite free over $\mathcal{O}_{\check{E}}[[x_1, \dots, x_{n-1}]]$.*

3.1. The basic locus as a zero dimensional locally symmetric space. Let I be the algebraic reductive group over \mathbb{Q} that is the endomorphism by quasi-isogenies of an abelian variety over $\overline{\mathbb{F}}_q$ equipped with its additional structures defining an $\overline{\mathbb{F}}_q$ -point of $\overline{S}_{K^p}^{(0)}$. This satisfies

- (1) $I(\mathbb{R})$ is compact modulo its center,
- (2) $I(\mathbb{Q}_p) = D^\times \times \mathbb{Z}_p^\times$ via the action of an automorphism on the Dieudonné module,
- (3) $I(\mathbf{A}_f^p) = G(\mathbf{A}_f^p)$ via the action of an automorphism on the étale cohomology outside p .

In fact I is an inner form of G that is isomorphic to G outside $p\infty$.

The fact is, like for modular curves, that all basic points are in an unique isogeny class. From this we deduce that, after fixing a base point,

$$I(\mathbb{Q}) \backslash (I(\mathbb{Q}_p) / \mathcal{O}_D^\times \times I(\mathbf{A}_f^p) / K^p) \xrightarrow{\sim} \overline{S}_{K^p}^{(0)}(\overline{\mathbb{F}}_q).$$

3.2. Harris-Taylor theorem. From the preceding we obtain that

$$\underbrace{\lim_{\overrightarrow{K}} R\Gamma(\mathrm{Sh}_K \otimes_L \overline{L}, \overline{\mathbb{Q}}_\ell)|_{W_E, \mathrm{cusp}} \text{ at } p}_{\text{expressed in terms of aut. rep. of } G} \xrightarrow{\sim} \underbrace{\mathcal{A}(I)}_{\substack{\text{expressed in} \\ \text{terms of} \\ \text{aut.rep. of } I}} \otimes_{\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(D^\times)}^{\mathbb{L}} \lim_{\overrightarrow{K}} R\Gamma(\mathcal{M}_K \hat{\otimes}_{\check{E}} \widehat{E}, \overline{\mathbb{Q}}_\ell)_{\mathrm{cusp}} \text{ at } p$$

Via a comparison between automorphic representations on the two inner forms I and G (global Jacquet-Langlands) obtained via a comparison of Arthur trace formulas Harris and Taylor prove the following result. This result is obtained via global methods using the fact that any supercuspidal representation globalizes to an automorphic representation that is a discrete series at ∞ .

THEOREM 3.3. *The cuspidal part of the middle degree cohomology*

$$\varinjlim_K H^{n-1}(\mathcal{M}_K \hat{\otimes}_{\check{E}} \widehat{E}, \overline{\mathbb{Q}}_\ell)$$

is, up to a Tate twist, of the form

$$\bigoplus_{\substack{\pi \\ \text{supercuspidal}}} \text{JL}^{-1}(\pi) \otimes \pi \otimes \varphi_\pi$$

where φ_π is an n -dimensional $\overline{\mathbb{Q}}_\ell$ -representation of W_E . The correspondence $\pi \mapsto \varphi_\pi$ defines a local Langlands correspondence for $\text{GL}_{n/E}$.

Third lecture - Sept 26

We are now dealing with **period morphisms for p -divisible groups**.

1. Some general thoughts on period morphisms

For $p = \infty$ there is only one period morphism and this is a $G(\mathbb{R})$ -equivariant embedding

$$G(\mathbb{R}) \hookrightarrow X \xrightarrow{\text{open}} \mathcal{F}_{\mu_h} \hookrightarrow G(\mathbb{C})$$

where G is a reductive group over \mathbb{R} , X is a hermitian symmetric space defined by the $G(\mathbb{R})$ -conjugacy class of $h : \mathbb{S} \rightarrow G$, and \mathcal{F}_{μ_h} is the complex flag manifold defined by μ_h . This embedding is nothing else than the map that sends a Hodge structure to the Hodge filtration.

Moreover, the image of this embedding is easy to describe. In fact, the complex conjugate of μ_h is $\mu_h^c = w_h \cdot \mu_h^{-1}$ with $w_h : \mathbb{G}_m \rightarrow G$ central and thus complex conjugation defines $\overline{(-)} : \mathcal{F}_{\mu_h} \xrightarrow{\sim} \mathcal{F}_{\mu_h^{-1}}$, and

$$\begin{aligned} X &\underset{\text{open/closed}}{\subset} \{z \in \mathcal{F}_{\mu_h} \mid z \text{ and } \bar{z} \text{ are opposite parabolic subgroups}\} \\ &= \{z \in \mathcal{F}_{\mu_h} \mid \text{inv}(z, \bar{z}) = 1\} \end{aligned}$$

where here $P_{\mu_h^{-1}}$ is opposite to P_{μ_h} and

$$\text{inv} : G_{\mathbb{C}}/P_{\mu_h} \times G_{\mathbb{C}}/P_{\mu_h^{-1}} \longrightarrow P_{\mu_h^{-1}} \backslash G_{\mathbb{C}}/P_{\mu_h}.$$

Here the open/closed condition defining X is that for z satisfying $\text{inv}(z, \bar{z}) = 1$, one has an associated $h_z : \mathbb{G}_m \rightarrow G$ and we ask this is $G(\mathbb{R})$ -conjugated to h .

EXAMPLE 1.1. (1) Consider $G = \text{Gsp}_{2n}$. Then, \mathcal{F}_{μ_h} is the variety of Lagrangians in \mathbb{C}^{2n} equipped with the standard symplectic structure. Moreover, for a Lagrangian subspace $L \subset \mathbb{C}^{2n}$, the condition defining our open subset is that $L \cap \bar{L} = (0)$. It is clear that if $L \cap (\mathbb{C}^n \oplus (0)) \neq (0)$ then L is not in our open subset. The subset of \mathcal{F}_{μ_h} formed by Lagrangian subspace L satisfying $L \cap (\mathbb{C}^n \oplus (0)) = \emptyset$ is identified with the affine space of symmetric matrices $A \in M_n(\mathbb{C})$, ${}^t A = A$. To such a matrix A one associated the image of $\mathbb{C}^n \oplus (0)$ by $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$. Now, the associated Lagrangian subspace L satisfies $L \cap \bar{L} = (0)$ iff $\text{Im}(A)$ (imaginary part) is invertible. Our open subset has thus n connected components given by the signature of the symmetric non-singular matrix $\text{Im}(A)$.

The open/closed subspace X is the union of the two connected components that correspond to the signatures $(n, 0)$ and $(0, n)$ that is to say $\text{Im}(A)$ or $-\text{Im}(A)$ is positive definite. This is $\pm\mathcal{H}_n$ where \mathcal{H}_n is Siegel upper half space.

- (2) Let $G = GU(1, n - 1)$ with $h(z) = \text{diag}(z, \bar{z}, \dots, \bar{z})$. One has $\mathcal{F}_{\mu_h} = \mathbb{P}^{n-1}(\mathbb{C})$ and our open subset is $\{[z_1 : \dots : z_n] \mid |z_1|^2 - \sum_{i=2}^n |z_i|^2 \neq 0\}$. This has two connected components : the first one is an open ball $\{[1 : z_2 : \dots : z_n] \mid \sum_{i=2}^n |z_i|^2 < 1\} \subset \mathbb{C}^{n-1}$ and the other one is $\{[1 : z_2 : \dots : z_n] \mid \sum_{i=2}^n |z_i|^2 > 1\} \cup \{[0 : z_2 : \dots : z_n] \in \mathbb{P}^{n-2}(\mathbb{C})\}$. The space X is the first connected component identified with an open ball.

For $p \neq \infty$ the story is different :

- (1) There are two period maps and two groups acting
- (2) Those are linked to the two cohomology theories : crystalline cohomology and p -adic étale cohomology. For $p = \infty$ we only have Betti cohomology.
- (3) Those two period maps correspond to the two spectral sequences : the Hodge to de Rham spectral sequence and the Hodge-Tate spectral sequence
- (4) The period maps aren't embeddings in general.

2. The case of Lubin-Tate spaces

2.1. The Lubin-Tate tower. Take $E = \mathbb{Q}_p$ to simplify. Let

$$\mathbb{H}$$

be a one dimensional 1-dimensional formal p -divisible group over $\overline{\mathbb{F}}_p$ (such an \mathbb{H} is unique up to a non-unique isomorphism). This can be seen, after fixing a coordinate $\text{Spf}(\overline{\mathbb{F}}_p[[T]]) \xrightarrow{\sim} \mathbb{H}$ as a one dimensional formal group law $\mathfrak{F} \in \overline{\mathbb{F}}_p[[X, Y]]$ that gives the addition : $X + Y = \mathfrak{F}(X, Y)$.

Let n be the height of \mathbb{H} that is to say $[p]_{\mathfrak{F}} = aT^{p^n} + \dots$ with $a \neq 0$.

DEFINITION 2.1. *The moduli space of deformations of \mathbb{H} over complete local $W(\overline{\mathbb{F}}_p)$ -algebras is the Lubin-Tate space*

$$\begin{array}{c} \mathcal{LT} \\ \downarrow \\ \text{Spf}(W(\overline{\mathbb{F}}_p)). \end{array}$$

This is non-canonically isomorphic to

$$\text{Spf}(W(\overline{\mathbb{F}}_p)[[x_1, \dots, x_{n-1}]]).$$

Let D be a division algebra with invariant $\frac{1}{n}$ over \mathbb{Q}_p , $D = \mathbb{Q}_{p^n}[\Pi]$ where \mathbb{Q}_{p^n} is the degree n unramified extension of \mathbb{Q}_p , $\Pi^n = p$ and if σ is the Frobenius of $\mathbb{Q}_{p^n}|\mathbb{Q}_p$ then $\Pi\sigma\Pi^{-1} = x^\sigma$. One has an identification

$$\mathcal{O}_D = \text{End}(\mathbb{H})$$

where \mathcal{O}_D is the maximal order in D , $\mathcal{O}_D = \mathbb{Z}_p^n[\Pi]$.

There is an evident action of \mathcal{O}_D^\times on \mathcal{LT}

$$\begin{array}{c} \mathcal{LT} \\ \curvearrowright \\ \mathcal{O}_D^\times \end{array}$$

DEFINITION 2.2. Let \mathcal{LT}_η be the generic fiber of \mathcal{LT} as a locally of finite type adic space over $\mathrm{Spa}(\mathbb{Q}_p)$.

After fixing some formal coordinates

$$\mathcal{LT}_\eta \simeq \mathring{\mathbb{B}}_{\mathbb{Q}_p}^{n-1}$$

that is again equipped with an action of \mathcal{O}_D^\times . The Tate module of the universal deformation defines an étale \mathbb{Z}_p -local system T of rank n on \mathcal{LT}_η .

DEFINITION 2.3. For $K \subset \mathrm{GL}_n(\mathbb{Z}_p)$ we note

$$\mathcal{LT}_{\eta,K}$$

the moduli space of trivializations mod K of the \mathbb{Z}_p -local system T .

→ we force the monodromy of our local system to leave in K . This means

$$\mathcal{LT}_{\eta,K} = \underline{(K/\mathrm{Id} + p^m M_n(\mathbb{Z}_p))} \backslash \underline{\mathrm{Isom}}((\underline{Z/p^m \mathbb{Z}})^n, T/p^m T)$$

for $m \gg 0$.

We obtain a tower

$$\begin{array}{c} (\mathcal{LT}_{\eta,K})_K \begin{array}{c} \leftarrow \curvearrowright \rightarrow \\ \mathrm{GL}_n(\mathbb{Q}_p)^1 \end{array} \\ \curvearrowright \\ \mathcal{O}_D^\times \end{array}$$

where

- the action of \mathcal{O}_D^\times is horizontal,
- the action of $\mathrm{GL}_n(\mathbb{Q}_p)^1$ is vertical : for $g \in \mathrm{GL}_n(\mathbb{Q}_p)^1$, $g : \mathcal{LT}_{\eta,K} \xrightarrow{\sim} \mathcal{LT}_{\eta,gKg^{-1}}$,
- both actions commute.

Here the action of $\mathrm{GL}_n(\mathbb{Z}_p)$ is the evident one. To extend it to an action of $\mathrm{GL}_n(E)^1$ we have to go back to some integral models of $\mathcal{LT}_{\eta,K}$ for K a principal congruence subgroup, $K = \mathrm{Id} + p^m M_n(\mathbb{Z}_p)$, $m \geq 1$. This is given by this notions of Drinfeld level structure that defines a an integral model $\mathrm{Spf}(R_m)$ where R_m is a complete regular $W(\overline{\mathbb{F}}_p)$ -algebra. We now use the following two elementary results ;

- (1) if S is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ and H a one dimensional height n formal p -divisible group over S equipped with a level m Drinfel structure

$$\eta : (\mathbb{Z}/p^m)^n \longrightarrow H[p^m]$$

then any subgroup M of $(\mathbb{Z}/p^m\mathbb{Z})^n$ defines a finite flat closed subgroup scheme $G \subset H[p^m]$ such that $\eta|_M : \underline{M} \rightarrow G$.

- (2) if S is a reduced \mathbb{F}_p -scheme and $f : H \rightarrow H'$ is a height 0 quasi-isogeny between one dimensional formal p -divisible groups then f is an isomorphism.

At the end we obtain an action of $(\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times)^1$ on our tower.

2.2. The de Rham period morphism. Let

$$D = \mathbb{D}(\mathbb{H})$$

be the covariant rational Dieudonné module of \mathbb{H} . This is an n -dimensional \check{Q}_p -vector space equipped with a crystalline Frobenius φ ,

$$D \curvearrowright \varphi.$$

The matrix of the associated Verschiebung $p\varphi^{-1}$ is given in a suitable basis by

$$p\varphi^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma^{-1}$$

We now use the following property. Recall that a quasi-isogeny between p -divisible groups H and H' over a quasi-compact scheme S is an element of $f \in \mathrm{Hom}(H, H')[\frac{1}{p}]$ such that there exists $g \in \mathrm{Hom}(H', H)[\frac{1}{p}]$ satisfying $g \circ f = \mathrm{Id}$ and $f \circ g = \mathrm{Id}$.

LEMMA 2.4 (rigidity of quasi-isogenies). *Let $S_0 \hookrightarrow S$ be a nilpotent closed immersion of schemes and H, H' be p -divisible groups over S . Then, reduction to S_0 induces an isomorphism*

$$\mathrm{Qisog}(H, H') \xrightarrow{\sim} \mathrm{Qisog}(H \times_S S_0, H' \times_S S_0).$$

We now use **the crystalline nature of the Dieudonné crystal of a p -divisible group**. Let R be a p -adic ring, H a p -divisible group over $\mathrm{Spf}(R)$ and H_0 be a p -divisible group over $\mathrm{Spec}(R/pR)$. Suppose given a quasi-isogeny

$$\rho : H_0 \rightarrow H \otimes_R R/pR.$$

Let \mathcal{E} be the covariant Dieudonné crystal of H on $(\mathrm{Spec}(R)/\mathrm{Spec}(\mathbb{Z}_p))_{\mathrm{crys}}$ and \mathcal{E}_0 be the one of H_0 on $(\mathrm{Spec}(R/pR)/\mathrm{Spec}(\mathbb{Z}_p))_{\mathrm{crys}}$. This gives rise to an isomorphism

$$\rho_* : \mathcal{E}_{0, R \rightarrow R/pR}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{E}_{R \rightarrow R}[\frac{1}{p}].$$

From this and the rigidity of quasi-isogenies we deduce the following result.

PROPOSITION 2.5. *Let (\mathcal{E}, ∇) be the convergent isocrystal on \mathcal{LT}_η associated to the universal deformation as an \mathcal{O}_D^\times -equivariant vector bundle equipped with an integrable connexion. There is a canonical \mathcal{O}_D^\times -equivariant isomorphism*

$$(\mathbf{D} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{LT}_\eta}, \text{Id} \otimes d) \xrightarrow{\sim} (\mathcal{E}, \nabla)$$

and thus (\mathcal{E}, ∇) is generated by its horizontal sections that are identified with D ,

$$D \xrightarrow{\sim} \mathcal{E}^{\nabla=0}.$$

The rank n vector bundle \mathcal{E} can be thought of as being the $(\mathcal{H}_{dR}^1)^\vee$ of the universal deformation. There is an **Hodge filtration**

$$\text{Fil } \mathcal{E} \subset \mathcal{E}$$

that is identified with $\omega_{HD}[\frac{1}{p}]$ where H is the universal deformation and fits into the Hodge exact sequence

$$0 \longrightarrow \underbrace{\omega_{HD}[\frac{1}{p}]}_{\text{rk. } n-1} \longrightarrow \mathcal{E} \longrightarrow \underbrace{\omega_H[\frac{1}{p}]}_{\text{rk. } 1} \longrightarrow 0$$

DEFINITION 2.6 (de Rham period morphism for Lubin-Tate spaces). *We note*

$$\pi_{dR} : \mathcal{LT}_\eta \longrightarrow \mathbb{P}(D) \simeq \mathbb{P}^{n-1}$$

for the \mathcal{O}_D^\times -equivariant morphism defined by the Hodge filtration and Proposition 2.5.

Grothendieck-Messing theory says that to deform a p -divisible group is the same as to deform its Hodge filtration. From this the following basic result is elementary.

PROPOSITION 2.7. *The de Rham period morphism π_{dR} satisfies the following :*

- (1) *It is (partially proper) étale,*
- (2) *Its geometric fibers are the Hecke orbits*

The following result is quite deep and will be later reinterpreted in terms of the curve.

THEOREM 2.8 (Gross-Hopkins). *The de Rham period morphism*

$$\pi_{dR} : \mathcal{LT}_\eta \longrightarrow \mathbb{P}_{\mathbb{Q}_p}^{n-1}$$

is surjective.

At the end we thus have an étale cover

$$\mathring{\mathbb{B}}_{\mathbb{Q}_p}^{n-1} \longrightarrow \mathbb{P}_{\mathbb{Q}_p}^{n-1}$$

with infinite discrete fibers.

The following result can be verified in an elementary way. We note $\mathbb{Q}_p^{cyc} := \widehat{\cup_{n \geq 1} \mathbb{Q}_p(\zeta_n)}$.

PROPOSITION 2.9. *The projective limit*

$$\mathcal{LT}_{\eta,\infty} := \varprojlim_K \mathcal{LT}_{\eta,K}$$

makes sense as a $\mathbb{Q}_p^{\text{cyc}}$ -perfectoid space.

At the end we obtain the following picture.

$$\begin{array}{c} \mathcal{LT}_{\eta,\infty} \\ \left(\begin{array}{c} \downarrow \text{GL}_n(\mathbb{Z}_p) \\ \mathcal{LT}_{\eta} \\ \downarrow \pi_{dR} \\ \mathbb{P}^{n-1}_{\mathbb{Q}_p} \end{array} \right) \\ \text{GL}_n(\mathbb{Q}_p)^1 \end{array}$$

where the torsors are pro-étale torsors.

2.3. The Hodge-Tate period morphism. We now come to the other period morphism in the game.

Recall that if G is a (commutative) finite locally free group scheme over a scheme there is a morphism of fppf sheaves

$$\begin{aligned} G = \mathcal{H}om(G^D, \mathbb{G}_m) &\longrightarrow \omega_{G^D} \\ f &\longmapsto f^* \frac{dT}{T}. \end{aligned}$$

from G toward the fppf sheaf associated to the coherent sheaf ω_{G^D} .

Let now H be a p -divisible group over $\text{Spec}(R)$ where R is a p -torsion free p -adic ring. Suppose moreover that R is integrally closed in $R[\frac{1}{p}]$. The preceding construction applied to the collection $(H[p^n])_{n \geq 1}$ defines a \mathbb{Z}_p -linear morphism

$$\alpha_H : \underbrace{\text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H_\eta)}_{\mathbb{Z}_p\text{-module}} \xrightarrow[\substack{R \text{ int. closed} \\ \text{in } R[\frac{1}{p}]}]{\cong} \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H) \longrightarrow \underbrace{\omega_{H^D}}_{\substack{\text{projective of} \\ \text{finite type} \\ R\text{-module} \\ \text{of rk. ht}(H) - \dim(H)}}$$

where H_η is the étale p -divisible group $H \otimes_R R[\frac{1}{p}]$. We note $\alpha_H \otimes 1$ for its linearization

$$\alpha_H \otimes 1 : \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H_\eta) \otimes_{\mathbb{Z}_p} R \longrightarrow \omega_{H^D}.$$

The key result is now the following.

PROPOSITION 2.10 (Faltings, F.). *If $R = \mathcal{O}_C$ with $C|\mathbb{Q}_p$ a complete algebraically closed extension of \mathbb{Q}_p then the preceding induces a complex*

$$0 \longrightarrow \omega_H^\vee(1) \xrightarrow{(\alpha_{HD} \otimes 1)^\vee(1)} T_p(H) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{\alpha_H \otimes 1} \omega_{HD} \longrightarrow 0$$

whose cohomology is killed by $p^{\frac{1}{p-1}}$ if $p \neq 2$ and 4 if $p = 2$. In particular one has an Hodge-Tate exact sequence

$$0 \longrightarrow \omega_H^\vee(1) \left[\frac{1}{p} \right] \xrightarrow{(\alpha_{HD} \otimes 1)^\vee(1)} V_p(H) \otimes_{\mathbb{Q}_p} C \xrightarrow{\alpha_H \otimes 1} \omega_{HD} \left[\frac{1}{p} \right] \longrightarrow 0$$

Let us remark that $\frac{1}{p-1} = v_p(2i\pi)$ in the preceding proposition. Using this result we can construct a morphism

$$\pi_{HT} : \mathcal{LT}_{\eta, \infty} \longrightarrow \check{\mathbb{P}}_{\mathbb{Q}_p}^{n-1}$$

that is $\mathrm{GL}_n(\mathbb{Q}_p)^1$ -equivariant and \mathcal{O}_D^\times -invariant. Here $\check{\mathbb{P}}_{\mathbb{Q}_p}^{n-1}$ is the dual projective space classifying rank $n-1$ quotients of \mathcal{O}^n . Let us fix the isomorphism $\check{\mathbb{P}}_{\mathbb{Q}_p}^{n-1} \xrightarrow{\sim} \mathbb{P}_{\mathbb{Q}_p}^{n-1}$ given by the identification of $(\mathcal{O}^n)^\vee$ and \mathcal{O}^n deduced from the dual of the canonical basis. This commutes with the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ twisted by $g \mapsto {}^t g^{-1}$.

THEOREM 2.11 (Faltings, F.). *The image of*

$$\pi_{HT} : \mathcal{LT}_{\eta, \infty} \longrightarrow \check{\mathbb{P}}_{\mathbb{Q}_p}^{n-1} \xrightarrow{\sim} \mathbb{P}_{\mathbb{Q}_p}^{n-1}$$

is Drinfeld's space Ω . Moreover, $\mathcal{LT}_{\eta, \infty} \rightarrow \Omega$ is a pro-étale \mathcal{O}_D^\times -torsor that is identified with Drinfeld-tower.

At the end we obtain the following diagram.

$$\begin{array}{ccc}
 & \underbrace{[\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times]^1} & \\
 & \curvearrowright & \\
 & \mathcal{LT}_{\eta, \infty} & \\
 \underbrace{\mathrm{GL}_n(\mathbb{Q}_p)^1} \swarrow & & \searrow \underbrace{\mathcal{O}_D^\times} \\
 \pi_{dR} \swarrow & & \searrow \pi_{HT} \\
 \check{\mathbb{P}}_{\mathbb{Q}_p}^{n-1} & & \Omega_{\check{\mathbb{Q}}_p} \\
 \curvearrowright & & \curvearrowright \\
 \mathcal{O}_D^\times & & \underbrace{\mathrm{GL}_n(\mathbb{Q}_p)^1}
 \end{array}$$

3. Rapoport-Zink spaces

3.1. Integral models in hyperspecial level. Rapoport-Zink spaces are generalizations of Lubin-Tate and Drinfeld spaces. We only explain the $G = \mathrm{GL}_n$ -case.

Let \mathbb{H} be a p -divisible group over $\overline{\mathbb{F}}_p$ of dimension n and dimension d . Let (\mathbf{D}, φ) be its covariant rational Dieudonné isocrystal. We note :

- (1) $G = \mathrm{GL}_n$,
- (2) G_b the reductive algebraic group over \mathbb{Q}_p whose R -points are $\mathrm{Aut}(D \otimes_{\mathbb{Q}_p} R, \varphi \otimes \mathrm{Id})$.

Here the $b \in G(\check{\mathbb{Q}}_p)$ refers to the matrix of Frobenius in a basis of \mathbf{D} , in which case φ can be identified with $b\sigma \in G(\check{\mathbb{Q}}_p) \rtimes \sigma$. Then, G_b is identified with the twisted centralizer of b ,

$$G_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} \check{\mathbb{Q}}_p) \mid gb\sigma = b\sigma g\}$$

that is to say

$$gbg^{-\sigma} = b.$$

If $(\lambda_1, \dots, \lambda_r)$ are the slopes of (\mathbf{D}, φ) with respective multiplicities (m_1, \dots, m_r) , then

$$G_b \simeq \prod_{i=1}^r \mathrm{GL}_{m_i}(D_{-\lambda_i})$$

where D_λ is the division algebra with invariant λ over \mathbb{Q}_p .

DEFINITION 3.1. We note \mathcal{M} for the functor on $W(\overline{\mathbb{F}}_p)$ -schemes on which p is locally nilpotent such that

$$\mathcal{M}(S) = \{(H, \rho)\} / \sim$$

where

- H is a p -divisible group over $\overline{\mathbb{F}}_p$,
- $\rho : \mathbb{H} \times_{\overline{\mathbb{F}}_p} (S \bmod p) \rightarrow H \times_S (S \bmod p)$ is quasi-isogeny.

The $\overline{\mathbb{F}}_p$ -points of this moduli are identified via Dieudonné theory with

$$\mathcal{M}(\overline{\mathbb{F}}_p) = \{M \subset \mathbf{D} \text{ a lattice s.t. } pM \subset \varphi(M) \subset M\}.$$

This can be rewritten in the following way. Let $\mu : \mathbb{G}_m \rightarrow G$ be the Hodge cocharacter

$$\mu(z) = (\underbrace{z, \dots, z}_{d \text{ times}}, \underbrace{1, \dots, 1}_{n-d \text{ times}}).$$

Then we have

$$\mathcal{M}(\overline{\mathbb{F}}_p) = \left\{ g \in G(W(\overline{\mathbb{F}}_p)[\frac{1}{p}]) / G(W(\overline{\mathbb{F}}_p)) \mid \mathrm{inv}(bg^\sigma, g) = \{\mu\} \right\}$$

where

$$\begin{aligned} \mathrm{inv} : G(W(\overline{\mathbb{F}}_p)[\frac{1}{p}]) / G(W(\overline{\mathbb{F}}_p)) \times G(W(\overline{\mathbb{F}}_p)[\frac{1}{p}]) / G(W(\overline{\mathbb{F}}_p)) &\longrightarrow G(W(\overline{\mathbb{F}}_p)) \backslash G(W(\overline{\mathbb{F}}_p)[\frac{1}{p}]) / G(W(\overline{\mathbb{F}}_p)) \\ ([g_1], [g_2]) &\longmapsto [g_1^{-1}g_2] \end{aligned}$$

that is identified with $\text{Hom}(\mathbb{G}_m, G)/G$ -conjugacy via $\mu \mapsto [\mu(p)]$.

Thus, **the $\overline{\mathbb{F}}_p$ -points of \mathcal{M} can be identified with an affine Deligne-Lusztig set** (we say “set” instead of “variety” because à priori we don’t know how to put a geometric structure on this now).

Further more, for any $x \in \mathcal{M}(\overline{\mathbb{F}}_p)$ if H_x is the associated p -divisible group, there is an identification

$$\widehat{\mathcal{M}}_{/x} = \text{Def}(H_x)$$

that is representable by a formal scheme isomorphic to

$$\text{Spf}(W(\overline{\mathbb{F}}_p)[[x_1, \dots, x_{d(n-d)}]]).$$

The moduli space \mathcal{M} is a much subtler version on the naive formal scheme

$$\coprod_{x \in \mathcal{M}(\overline{\mathbb{F}}_p)} \text{Def}(H_x).$$

We have in fact the following theorem.

THEOREM 3.2 (Rapoport-Zink). *The functor \mathcal{M} is a representable by a $\text{Spf}(W(\overline{\mathbb{F}}_p))$ -formal scheme locally formally of finite type that is to say locally isomorphic to*

$$\text{Spf}(W(\overline{\mathbb{F}}_p)[[X_1, \dots, X_s]]\langle Y_1, \dots, Y_t \rangle / \text{Ideal}).$$

Moreover the irreducible components of \mathcal{M}_{red} are projective algebraic varieties over $\overline{\mathbb{F}}_p$.

The action of $G_b(\mathbb{Q}_p)$ on the quasi-isogeny ρ defines a continuous action of $G_b(\mathbb{Q}_p)$ on \mathcal{M} ,

$$\mathcal{M} \curvearrowright^{G_b(\mathbb{Q}_p)}$$

EXAMPLE 3.3. *From the fact that any degree 0 quasi-isogeny between 1-dimensional formal p -divisible groups over $\overline{\mathbb{F}}_p$ we deduce that in the Lubin-Tate case*

$$\mathcal{M} = \mathcal{LT} \times_{\mathcal{O}_D^\times} D^\times$$

that is (non-canonically) isomorphic to $\coprod_{\mathbb{Z}} \mathcal{LT}$ where the action of \mathcal{O}_D^\times on the factor \mathcal{LT} associated to $k \in \mathbb{Z}$ is the canonical one twisted by $d \mapsto \Pi^k d \Pi^{-k}$.

REMARK 3.4. *Although \mathcal{M} is formally smooth, in general \mathcal{M}_{red} is not smooth.*

3.2. The tower. Let

$$\mathcal{M}_\eta \curvearrowright G_b(\mathbb{Q}_p)$$

be the generic fiber of \mathcal{M} as a locally of finite type adic space over $\mathrm{Spa}(\check{\mathbb{Q}}_p)$. As before with the Lubin-Tate tower one obtains a tower

$$\begin{array}{ccc} & & G(\mathbb{Q}_p) \\ & \curvearrowright & \\ (\mathcal{M}_{\eta,K})_K & & \\ \uparrow & & \\ G_b(\mathbb{Q}_p) & & \end{array}$$

where K goes through the set of compact open subgroups of $G(\mathbb{Q}_p)$ and both actions commute. The definition of the action of $G(\mathbb{Q}_p)$ is more subtle than in the Lubin-Tate case since there is no “good notion” of integral level structures like this is the case for one dimensional p -divisible groups according to Drinfeld.

This relies on **Raynaud’s flatification by blow-ups** : if S is a quasi-compact quasi-separated scheme, $G \rightarrow S$ is a finite locally free group scheme, $U \subset S$ is an open subset and $H \subset G \times_S U$ is a closed finite locally free sub-group scheme then after a blow-up supported on $S \setminus U$ we can suppose that H extends to a closed subgroup scheme of G finite locally free over S .

EXAMPLE 3.5. For the Lubin-Tate tower, the associated RZ tower is

$$(\mathcal{M}_{\eta,K})_K = (\mathcal{LT}_{\eta,K})_K \begin{array}{c} (\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times)^1 \\ \times \\ \mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \end{array}$$

3.3. Period morphisms. As for Lubin-Tate spaces, if (\mathcal{E}, ∇) is the convergent isocrystal associated to the universal deformation H on \mathcal{M} , the universal quasi-isogeny ρ induces an isomorphism

$$(D \otimes_{\check{\mathbb{Q}}_p} \mathcal{O}_{\mathcal{M}_\eta}, \mathrm{Id} \otimes d) \xrightarrow{\sim} (\mathcal{E}, \nabla).$$

The Hodge filtration then defines a $G_b(\mathbb{Q}_p)$ -equivariant morphism

$$\pi_{dR} : \mathcal{M}_\eta \longrightarrow \mathcal{F}_\mu$$

where \mathcal{F}_μ is the rigid analytic flag manifold associated to μ . This satisfies :

- This is étale and thus in particular its image is open,
- Its geometric fibers are the Hecke orbits.

The image of the étale morphism π_{dR} ,

$$\mathcal{F}_\mu^a := \mathrm{Im}(\pi_{dR}),$$

is the so-called **admissible open subset** of \mathcal{F}_μ . This is a partially proper open subset inside the flag manifold \mathcal{F}_μ . Little is known in general about it outside of the fact that

- there is an inclusion

$$\mathcal{F}_\mu^a \subset \mathcal{F}_\mu^{wa}$$

where \mathcal{F}_μ^{wa} is the so-called **weakly admissible open subset**, a very concrete open subset that is of the form

$$\mathcal{F}_\mu \setminus \bigcup_{\text{profinite}} \text{Schubert varieties.}$$

— For $[K : \mathbb{Q}_p] < +\infty$,

$$\mathcal{F}_\mu^a(K) = \mathcal{F}_\mu^{wa}(K)$$

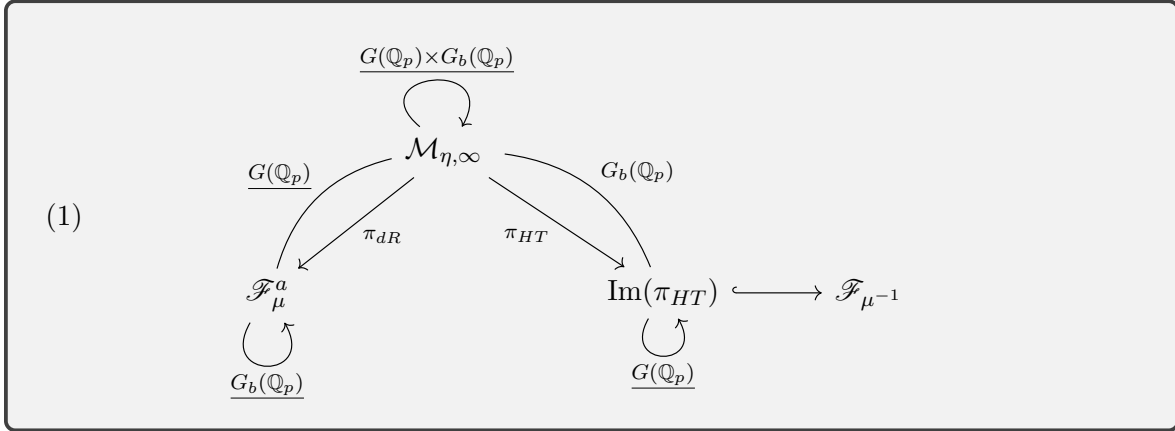
that is to say \mathcal{F}_μ^a and \mathcal{F}_μ^{wa} have the same Tate classical points.

— There is a complete characterization of when $\mathcal{F}_\mu^a = \mathcal{F}_\mu^{wa}$.

The picture at this point for the Hodge-Tate period morphism is more difficult to describe since we first need to give a meaning to

$$\mathcal{M}_{\eta,\infty} := \varprojlim_K \mathcal{M}_{\eta,K}.$$

The fact is that this is a perfectoid space (if \mathbb{H} is not étale) but we can make a sense out of it using integral models and blow-ups as in the Lubin-Tate case. At the end there is picture



where $\text{Im}(\varphi_{HT}) \subset \mathcal{F}_{\mu-1}$ is not an open subset in general and is well defined in general only as a locally spatial diamond. When b is basic i.e. the isocrystal (D, φ) is isoclinic then $\text{Im}(\pi_{HT})$ is open inside the dual flag manifold $\mathcal{F}_{\mu-1}$ and this is a classical rigid analytic open subset.

3.4. Cohomology. As for Lubin-Tate spaces one can use the cohomology spaces

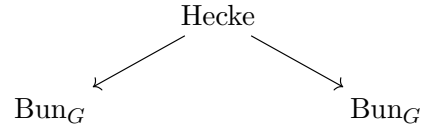
$$H_c^\bullet(\mathcal{M}_K \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$$

as representations of $\mathcal{H}_{\mathbb{Q}_\ell}(K \backslash G(\mathbb{Q}_p)/K)$ and $G_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ to define a **kernel for the local Langlands correspondence**. More precisely, we look at the correspondence

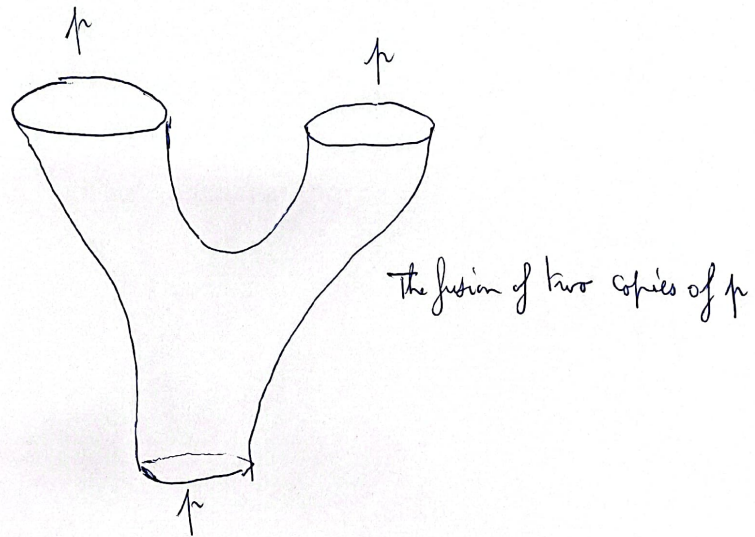
$$(2) \quad \underbrace{\pi}_{\text{smooth rep. of } G_b(\mathbb{Q}_p)} \longmapsto \underbrace{\varprojlim_K \text{Ext}_{G_b(\mathbb{Q}_p)}^\bullet(H_c^\bullet(\mathcal{M}_K \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}), \pi)}_{\text{smooth} \times \text{continuous representation of } G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}} .$$

4. Final thoughts

Diagram (1) has been a great motivation for the geometrization conjecture of the local Langlands correspondence with relation with the correspondence given by the Hecke stack



The “cohomological kernel” of equation (2) given by the cohomology of Rapoport-Zink spaces is even a reminder that the preceding correspondence should be upgraded to a cohomological correspondence.



$$\text{Div}^1 \hookrightarrow \text{Div}^1 \times \text{Div}^1$$

$$\text{Div}^1 = \text{Spa}(\mathbb{Q}_t)^\diamond / \varphi^{\mathbb{Z}}$$

Chapitre 4

Fourth lecture - October 10