Geometrization of the local Langlands correspondence

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Local Langlands parameters

- E local field residue field $\mathbb{F}_q$, $[E:\mathbb{Q}_p] < +\infty$ or $E = \mathbb{F}_q((\pi))$
- G reductive group over E
- $\ell \neq p$, $\Lambda =$ any $\mathbb{Z}_\ell[\sqrt{q}]$-algebra ($\ell \gg 0$ if ℓ not invertible)
- ${}^L G = \hat{G} \rtimes W_E$ Langlands dual over Λ
- π smooth representation of $G(E)$ with coefficients in Λ, Schur irreducible i.e. $\mathrm{End}(\pi) = \Lambda$
- We construct

$$\varphi_\pi : W_E \to {}^L G$$

  its semi-simple Langlands parameter.
- Compatible with parabolic induction and usual class field theory for tori. Usual local Langlands for GL_n (Harris-Taylor, Henniart)
- semi-simple : $N = 0$ when $\Lambda = \overline{\mathbb{Q}}_\ell$. For example : $\varphi_{\mathrm{triv}} = \varphi_{\mathrm{Steinberg}}$ for GL_n

Morphisms between centers

In fact we do much more.

- For Λ a $\mathbb{Z}_\ell$-algebra make it a condensed ring via $\Lambda := \Lambda^{disc} \otimes_{\mathbb{Z}_\ell^{disc}} \mathbb{Z}_\ell$
- There is a scheme$/\mathbb{Z}_\ell$, $\coprod_{\text{infinite}}$ affine schemes,

$$Z^1(W_E, \hat{G})$$

  Value on Λ is condensed 1-cocycles $W_E \to \hat{G}(\Lambda)$
- Studied in details by Dat-Helm-Kurinczuk-Moss
- Then

$$\mathrm{LocSys}_{\hat{G}} := [Z^1(W_E, \hat{G})/\hat{G}]$$

  is a zero dimensional locally complete intersection algebraic stack$/\mathbb{Z}_\ell$. Moduli of Langlands parameters.

Morphisms between centers

- Coarse moduli space

$$Z^1(W_E, \hat{G}) /\!\!/ \hat{G}$$

$\coprod_{\text{infinite}}$ affine schemes finite type$/\mathbb{Z}_\ell$.

- Functions on it

$$\mathfrak{Z}^{\text{spec}}(G, \mathbb{Z}_\ell) = \mathcal{O}(Z^1(W_E, \hat{G}))^{\hat{G}}$$

- Example : $G = \mathsf{GL}_n$, $\mathfrak{Z}^{\text{spec}}(G, \mathbb{Z}_\ell) \to \Lambda =$ pseudo-representations $W_E \to \mathsf{GL}_n(\Lambda)$.
- $\mathfrak{Z}(G(E), \mathbb{Z}_\ell) =$ Bernstein center $=$ center of the category of smooth representations of $G(E)$ with coefficients in $\mathbb{Z}_\ell$
- We construct a morphism

$$\mathfrak{Z}^{\text{spec}}(G, \mathbb{Z}_\ell[\sqrt{q}]) \longrightarrow \mathfrak{Z}(G(E), \mathbb{Z}_\ell[\sqrt{q}])$$

Morphism between centers

Conjecture

The morphism

$$\mathfrak{Z}^{\mathrm{spec}}(G, \mathbb{Z}_\ell[\sqrt{q}]) \longrightarrow \mathfrak{Z}(G(E), \mathbb{Z}_\ell[\sqrt{q}])$$

is "independant of $\ell \gg 0$" in the sens that it is induced by a morphism

$$\mathfrak{Z}^{\mathrm{spec}}(G, \mathbb{Z}[\tfrac{1}{N}, \sqrt{q}]) \longrightarrow \mathfrak{Z}(G(E), \mathbb{Z}[\tfrac{1}{N}, \sqrt{q}])$$

with $p \mid N$ (both centers are defined over $\mathbb{Z}[\frac{1}{p}]$).

The real deal : Bun_G

In fact we do much much more.

- S an $\overline{\mathbb{F}}_q$-perfectoid space $\rightsquigarrow$ $X_S = E$-adic space "the relative curve parametrized by S"
- i.e there is a way to put in family the collection of curves

$$(X_{k(s),k(s)^+})_{s\in S}$$

where $X_{k(s),k(s)^+}$ is the curve defined and studied with Fontaine attached to the perfectoid field $k(s)$

We will consider the v-topology on $\overline{\mathbb{F}}_q$-perfectoid spaces = some kind of analog of fpqc topology for schemes

$$* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$$

final object of the v-topos (not representable)

Bun_G

Theorem

The correspondence $S \mapsto \{$principal G-bundles on $X_S\}$ defines a v-stack

$$\mathrm{Bun}_G \longrightarrow *$$

that is an "Artin v-stack" (ℓ-cohomologically) smooth of dimension 0.

- diagonal of Bun_G representable in locally spatial diamonds
- there is a surjection $U \to \mathrm{Bun}_G$ that is (ℓ-coho.) smooth with U a locally spatial diamond s.t. $U \to *$ is (ℓ-coho.) smooth

Bun_G : points

- Set $\breve{E} = \widehat{E^{un}}$ with its Frobenius σ. One has

$$X_S = Y_S/\varphi^{\mathbb{Z}}$$

  with $Y_S \to \mathrm{Spa}(\breve{E})$, $\varphi =$some Frobenius that extends σ on $\breve{E}$.
- Functor

$$\text{Isocrystals} \longrightarrow \text{vector bundles on } X_S$$

$$(D, \varphi) \longmapsto Y_S \overset{\varphi^{\mathbb{Z}}}{\times} D$$

- $B(G) = G(\breve{E})/\sigma$-conjugation, $b \sim gbg^{-\sigma}$, Kottwitz set of G-isocrystals
- $b \in G(\breve{E}) \rightsquigarrow \mathcal{E}_b$ principal G-bundle on X_S

Bun_G : points

Theorem (Fargues-Fontaine (GL_n), Fargues)

F alg. closed

$$\begin{array}{ccc} B(G) & \xrightarrow{\sim} & H^1_{ét}(X_F, G) \\ [b] & \mapsto & [\mathcal{E}_b] \end{array}$$

- Dictionary : reduction theory (Atiyah-Bott) for G-bundles / Kottwitz description of $B(G)$.
- *Example* : $\mathcal{E}_b$ semi-stable $\Leftrightarrow$ b is basic (isoclinic)

Thus, identification

$$B(G) = |\mathrm{Bun}_G|$$

Bun_G : geometry

- $c_1 : \pi_0(\mathrm{Bun}_G) \xrightarrow{\sim} \pi_1(G)_\Gamma$
- Nice Harder-Narasimhan stratification, in particular

$$\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G \text{ is open}$$

  Each connected component has a unique ss point and

$$\mathrm{Bun}_G^{ss} = \coprod_{[b] \text{ basic}} \underbrace{[*/\underline{G_b(E)}]}_{\text{classifying stack of pro-etale torsors}}$$

  with $G_b =$ inner form of G ($G_1 = G$ for example)
- More generally for any $[b] \in B(G)$ the associated HN strata is a classifying stack

$$[*/\widetilde{G}_b]$$

  with $\widetilde{G}_b = \widetilde{G}_b^0 \rtimes \underline{G_b(E)}$, $\widetilde{G}_b^0 =$ unipotent diamond $G_b =$ inner form of a Levi

The real deal : $D_{lis}(\mathrm{Bun}_G, \Lambda)$

- Λ any $\mathbb{Z}_\ell$-algebra
- We define a triangulated category

$$D_{lis}(\mathrm{Bun}_G, \Lambda)$$

that is $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ when Λ is torsion and a sub-category of $D_{\text{proét}}(\mathrm{Bun}_G, \Lambda_{\blacksquare})$ in general

- For $[b] \in B(G)$ inclusion of HN stratum

$$i^b : [*/\widetilde{G}_b] \hookrightarrow \mathrm{Bun}_G$$

induces

$$(i^b)^* : D_{lis}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{lis}([*/\widetilde{G}_b], \Lambda) \underbrace{=}_{\ell \neq p} D(G_b(E), \Lambda)$$

(derived category of smooth representations of $G_b(E)$)

$D_{lis}(\mathrm{Bun}_G, \Lambda)$ when Λ is not torsion

Λ not torsion

- $$(i^1)^* : D_{\text{proét}}(\mathrm{Bun}_G, \Lambda_\blacksquare) \longrightarrow D_{\text{proét}}([*/\underline{G(E)}], \Lambda_\blacksquare) = D(G(E), \Lambda_\blacksquare)$$
  derived category of representations of $G(E)$ as a condensed group in condensed solid Λ-modules $\to$ too big.
- Example : $V = \mathbb{Q}_\ell$-vector space defines a solid $\mathbb{Q}_\ell$-vector space $V^{disc} \otimes_{\mathbb{Q}_\ell^{disc}} \mathbb{Q}_\ell$ whose value on A profinite is
  $$\{f : A \to V \mid \dim \mathrm{Vect} f(A) < +\infty \text{ and } A \xrightarrow{f} \mathrm{Vect} f(A) \text{ continuous}\}$$
  $$\underbrace{\mathrm{Vect}_{\mathbb{Q}_\ell}}_{\text{usual discrete v.s.}} \subsetneq \underbrace{\mathrm{Vect}_{\mathbb{Q}_\ell,\blacksquare}}_{\text{condensed solid } \mathbb{Q}_\ell\text{-v.s.}} \supset \mathbb{Q}_\ell\text{-Banach spaces}$$
- Rep. of $G(E)$ as a condensed group in $\mathrm{Vect}_{\mathbb{Q}_\ell}$ =smooth rep. of $G(E)$ in $\mathbb{Q}_\ell$-v.s. (use $\ell \neq p$)

$D_{lis}(\mathrm{Bun}_G, \Lambda)$ when Λ is not torsion

Define subcategory $D_{lis}(\mathrm{Bun}_G, \mathbb{Q}_\ell) \subset D_{\text{proét}}(\mathrm{Bun}_G, \mathbb{Q}_{\ell,\blacksquare})$ such that via $i^1 : * \to \mathrm{Bun}_G$,

$$\begin{array}{ccc} D_{lis}(\mathrm{Bun}_G, \mathbb{Q}_\ell) & \hookrightarrow & D_{\text{proét}}(\mathrm{Bun}_G, \mathbb{Q}_{\ell,\blacksquare}) \\ \downarrow {\scriptstyle (i^1)^*} & & \downarrow {\scriptstyle (i^1)^*} \\ D(\mathrm{Vect}_{\mathbb{Q}_\ell}) & \hookrightarrow & D(\mathrm{Vect}_{\mathbb{Q}_{\ell,\blacksquare}}) \end{array}$$

Définition

$D_{lis}(\mathrm{Bun}_G, \Lambda)$ = *smallest triangulated category stable under all direct sums that contains the* $f_\natural \Lambda$ *for all* $f : U \to \mathrm{Bun}_G$ *representable in locally spatial diamonds cohomologically smooth.*

Here $f_\natural$ = relative homology (5 functors ($f_\natural, Rf_*, f^*, R\mathcal{H}om, \overset{\mathbb{L}}{\otimes}$) for solid pro-étale sheaves on locally spatial diamonds)

$D_{lis}(\mathrm{Bun}_G, \Lambda)$

- Via $(i^1)_!$ and $(i^1)^*$

$$D(G(E), \Lambda) \subset D_{lis}(\mathrm{Bun}_G, \Lambda)$$

  is a direct factor.

- Good object for the local Langlands program is not a smooth representation π or a complex in $D(G(E), \Lambda)$ but an object of $D_{lis}(\mathrm{Bun}_G, \Lambda)$! ! ! Have to think the local Langlands program from this point of view !
- Usual notions of admissible, finite representations or Bernstein-Zelevinsky duality extend to $D_{lis}(\mathrm{Bun}_G, \Lambda)$

$D_{lis}(\mathrm{Bun}_G, \Lambda)$

More precisely for $\Lambda = \overline{\mathbb{Q}}_\ell$ (to simplify) :

Theorem

For $A \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$

1. *A is compact iff it has finite support and for all $[b] \in B(G)$, $(i^b)^* A \in D(G(E), \Lambda)$ is bounded with finite type cohomology*
2. *A is ULA iff for all $[b] \in B(G)$, $(i^b)^* A$ is a bounded complex with admissible cohomology*
3. *There is a duality functor $\mathbb{D}_{BZ} : D_{lis}(\mathrm{Bun}_G, \Lambda)^\omega \longrightarrow D_{lis}(\mathrm{Bun}_G, \Lambda)^\omega$ that extends Bernstein-Zelevinsky duality on $D^b(G(E), \Lambda)$*

Let's be serious now : the spectral action

Theorem

There is a monoidal action of $\mathrm{Perf}(\mathrm{LocSys}_{\hat{G}})$ *on* $D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$.

- This monoidal action defines the morphism between centers

$$\underbrace{\mathfrak{Z}^{spec}(G, \mathbb{Z}_\ell)}_{\text{spectral stable Bernstein center}} = \mathfrak{Z}(\mathrm{Perf}(\mathrm{LocSys}_{\hat{G}}))$$

$$\to \underbrace{\mathfrak{Z}(D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell))}_{\text{geometric stable Bernstein center}} \to \underbrace{\mathfrak{Z}(G(E), \Lambda)}_{\text{Bernstein center}}$$

- Defined using some geometric Satake correspondence for sheaves of Λ-modules on the B_{dR}-affine Grassmanian + some enhanced version of Beilinson-Drinfeld/Vincent Lafforgue formalism (quantum field theory/factorization sheaves)

The spectral action

More precisely :

- for a finite set I and $V \in \mathrm{Rep}_\Lambda({}^L G)^I$, using the geometric Satake correspondence, we construct a functor

$$T_V : D_{lis}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{lis}(\mathrm{Bun}_G \times [*/\underline{W_E}^I], \Lambda).$$

  Those are compatible when I and V vary.

- The category $D_{lis}(\mathrm{Bun}_G, \Lambda)$ has a natural enhancement as a Λ-linear condensed stable ∞-category, $\mathcal{C} = \mathcal{D}_{lis}(\mathrm{Bun}_G, \Lambda)$. We get compatible functors

$$\mathrm{Rep}_\Lambda({}^L G)^I \longrightarrow \mathrm{End}(\mathcal{C})^{BW_E^I}$$

  when I vary and those define the spectral action.

The wormhole : the geometrization conjecture

G quasisplit. Fix $\psi : U(E) \to \overline{\mathbb{Z}}_\ell$ non-degenerate. Let

$$\mathcal{W}_\psi = (i^1)_! \big(c\text{−}\mathrm{ind}_{U(E)}^{G(E)} \psi \big) \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$$

be the "Whittaker sheaf".

Conjecture

The functor

$$\begin{aligned} \mathrm{Perf}(\mathrm{LocSys}_{\hat{G}}/\overline{\mathbb{Z}}_\ell) &\longrightarrow D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell) \\ \mathcal{F} &\longmapsto \mathcal{F} * \mathcal{W}_\psi \end{aligned}$$

extends to an equivalence compatible with the spectral action

$$\mathrm{Coh}_{Nilp}(\mathrm{LocSys}_{\hat{G}}/\overline{\mathbb{Z}}_\ell) \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega$$

Spectral side *Geometric side*

The wormhole

- Here *Nilp* = Arinkin-Gaitsgory singular support condition (perfect complexes correspond to the condition : the singular support is contained in the zero section).
- This condition disappears overs $\overline{\mathbb{Q}}_\ell$ (automatic).
- Thus, have to think of local Langlands as a "non-abelian Fourier transform" with "kernel given by the Whittaker representation" ! !

Some final thoughts

- Looks like the natural objects are not smooth representations of $G(E)$, or element of $D(G(E), \Lambda)$, but rather obects in $D_{lis}(\mathrm{Bun}_G, \Lambda)$:
  - Extension of the notion of finite type, resp. admissible representation.
  - Extension of Zelevinsly involution.
  - For $A \in D_{lis}(\mathrm{Bun}_G, \Lambda)$ Schur irreducible we can define its semi-simple Langlands parameter $\varphi_A : W_E \to {}^L G$
- Let $f : {}^L G \to {}^L H$ be an L-homomorphism over $\overline{\mathbb{Q}}_\ell$ with G and H quasisplit. Geometrization conjecture implies the existence of a kernel of functoriality

$$A_f \in D_{lis}(\mathrm{Bun}_G \times \mathrm{Bun}_H, \overline{\mathbb{Q}}_\ell)$$

  that induces the classical Langlands functoriality $D(G(E), \overline{\mathbb{Q}}_\ell) \to D(H(E), \overline{\mathbb{Q}}_\ell)$.

Some final thoughts

- A_f is naturally constructed. To obtain the functoriality

$$D(G(E),\overline{\mathbb{Q}}_\ell) \to D(H(E),\overline{\mathbb{Q}}_\ell)$$

  one needs to use the inclusion $D(G(E),\overline{\mathbb{Q}}_\ell) \hookrightarrow D_{lis}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)$ and the projection $D_{lis}(\mathrm{Bun}_H,\overline{\mathbb{Q}}_\ell) \twoheadrightarrow D(H(E),\overline{\mathbb{Q}}_\ell)$.
- Functoriality is more natural from $D_{lis}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)$ to $D_{lis}(\mathrm{Bun}_H,\overline{\mathbb{Q}}_\ell)$.
- In the global case : are really automorphic representations the natural objects to which the Langlands functoriality program applies ?