THE HARDER-NARASIMHAN FILTRATION OF FINITE FLAT GROUP SCHEMES

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Motivation

Study the p-adic geometry of Shimura varieties and Rapoport-Zink spaces (both being linked via p-adic uniformization).

– Well known geometry for modular curves (Katz, Lubin)
– For unitary PEL type Shimura varieties of signature $(1, n-1) \times (0, n) \times \cdots \times (0, n)$ at a split prime and Lubin-Tate spaces : linked to the geometry of the Bruhat-Tits building of the linear group (see [4])
– For more general moduli spaces : stratify those p-adic moduli spaces, study the strata, study the action of the Hecke correspondences on those stratifications, study the link with stratifications of the special fiber

References :
- [2] : definition and basic properties of the Harder-Narasimhan filtration of finite flat group schemes
- [3] : application to p-divisible groups, construction of Hecke invariant stratifications of the p-adic moduli spaces, fundamental domains for the actions of Hecke correspondences in each stratum and comparison of those p-adic stratifications with the Newton stratification in the special fiber
- [1] : study of the p-adic geometry near the ordinary locus using the Harder-Narasimhan filtration.

1. An analogy

$K|\mathbb{Q}_p$ complete for $v : K \to \mathbb{R} \cup \{+\infty\}$, $v(p) = 1$ (no hypothesis on v, that may no be discrete, or on the residue field, that may not be perfect).

$\mathcal{C}$ = Category of finite flat group schemes/$\mathcal{O}_K$ commutative of order a power of p.

$\mathcal{D}$ = Category of vector bundles on a smooth projective curve X.

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$\mathcal{D}$	$\mathcal{C}$
• exact category $\hookrightarrow \underbrace{\text{coherent sheaves on } X}_{\text{abelian category}}$	• exact category $\hookrightarrow \underbrace{\text{fppf sheaves of ab. groups on Spec}(\mathcal{O}_K)}_{\text{abelian category}}$ abelian if $e_{K/\mathbb{Q}_p} < p-1$ (Raynaud) but not in general : $u : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mu_{p/\text{Spec}(\mathbb{Z}_p[\zeta_p])}$ $\bar{1} \longmapsto \zeta_p$ u is an iso. in generic fiber $\Rightarrow \ker u = \text{coker} u = 0$ in $\mathcal{C}$ but u is not an isomorphism
$\mathcal{E}' \to \mathcal{E}$ can be inserted in an exact sequence iff $\mathcal{E}' \subset \mathcal{E}$ locally direct factor	$G' \to G$ can be inserted in an exact sequence iff $G' \hookrightarrow G$ is a closed immersion
• additive functions (on exact sequences) : $\text{rk} : \mathcal{D} \to \mathbb{N}$ $\deg : \mathcal{D} \to \mathbb{Z}$	• additive functions : $\text{ht} : \mathcal{C} \to \mathbb{N}$ $\deg : \mathcal{C} \to \mathbb{R}$ $\text{ht}(G) = \log_p \|G\|$ $\deg(G) =$ "valuation of the discriminant of G" $= \sum_i v(a_i)$ if $\omega_G \simeq \oplus_i \mathcal{O}_K/a_i\mathcal{O}_K$
• $\mathcal{D}_\eta =$ Cat. of germs of vector bundles on $\neq \emptyset$ open subsets of X $\simeq k(\eta)$-v.s. of finite dimension $=$ abelian category	• $\mathcal{C}_\eta =$ Cat. of finite K-group schemes $\simeq \text{Rep}_{torsion}(\text{Gal}(\overline{K}\|K))$ $=$ abelian category
$F : \mathcal{D} \to \mathcal{D}_\eta$ generic fiber functor (not fully faithfull)	$F : \mathcal{C} \to \mathcal{C}_\eta$ generic fiber functor, $F(G) = G \otimes K$ F not fully faithfull in general (but, if $e_{K/\mathbb{Q}_p} < p-1$, $F : \mathcal{C} \hookrightarrow \text{Rep}_{tor}(\text{Gal}(\overline{K}\|K))$ fully faith. + description of the essential image of F in terms of the weights of the moderate inertia (Raynaud))
$\forall \mathcal{E} \in \mathcal{D}$, $F : \{\text{subojects of } \mathcal{E}\} \xrightarrow{\sim} \{\text{subobjects of } F(\mathcal{E})\}$	$\forall G \in \mathcal{C}$, $F : \{\text{subojects of } G\} \xrightarrow{\sim} \{\text{subobjects of } G \otimes K\}$ (inverse: schematical closure)
where subobjects=those that can be inserted in exact sequences=locally direct factor subbundles	where subobjects=those that can be inserted in exact sequences= finite flat closed subgroups
• $u : \mathcal{E} \to \mathcal{E}'$ s.t. $F(u)$ is an isomorphism $\Rightarrow \deg(\mathcal{E}) \leq \underbrace{\deg(\mathcal{E}')}_{\deg \mathcal{E} + \text{lenght}(\mathcal{E}'/\mathcal{E})}$ with equality iff u is an iso.	• $u : G \to G'$ s.t. $F(u)$ isomorphisme $\Rightarrow \deg(G) \leq \deg(G')$ with equality iff u is an iso. see following for the explaination of this...

Explaination of the last property of finite flat group schemes. $u : G \to G'$ inducing an isomorphism $G \otimes K \xrightarrow{\sim} G' \otimes K$.

Algebras of G and G' are flat locally complete intersections over $\mathcal{O}_K \Rightarrow 2$ different definitions for their discriminant :

- algebraic one via ω_G and $\omega_{G'}$ (the preceding one)
- arithmetic one via the discriminant of the quadratic form trace

More precisely, if $G = \mathrm{Spec}(A)$, $A =$ finite free $\mathcal{O}_K$-algebra of rank $|G|$. Quadratic form :

$$\begin{array}{rcl} \mathrm{tr} : A \times A & \longrightarrow & \mathcal{O}_K \\ (a, a') & \longmapsto & \mathrm{tr}_{A/\mathcal{O}_K}(aa'). \end{array}$$

$G \otimes K$ étale $\Rightarrow$ perfect after inverting p.

$$\mathrm{disc}(\mathrm{tr}) \in (\mathcal{O}_K \setminus \{0\})/(\mathcal{O}_K^\times)^2$$

and one can define the valuation of this discriminant

$$v(\mathrm{disc}(\mathrm{tr})) \in \mathbb{R}_+.$$

Then, A locally complete intersection implies

$$\boxed{\deg(G) = \frac{1}{|G|} v(\mathrm{disc}(\mathrm{tr}))}.$$

If A' is the algebra of G' and $u^* : A' \hookrightarrow A$, one deduces from this that

$$\boxed{\deg(G') = \deg(G) + \frac{2}{|G|} \sum_j v(b_j)}$$

where $A/u^*A' \simeq \oplus_j \mathcal{O}_K/b_j\mathcal{O}_K$.

2. The Harder-Narasimhan filtration

Theorem 1. *Objects of $\mathcal{C}$ admit Harder-Narasimhan filtrations for the slope function $\mu = \frac{\deg}{\mathrm{ht}}$. Moreover, μ takes values in $[0, 1]$.*

Means the following :

Definition 1. *$G \in \mathcal{C}$ is semi-stable if $G = 0$ or $\forall G' \subset G$, $G' \neq 0$, $\mu(G') \leq \mu(G)$.*

- Then, $\forall G$, $\exists!$ filtration $0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G$ s.t. :
– $\forall i, G_i/G_{i-1}$ is semi-stable
– $\mu(G_1/G_0) > \cdots > \mu(G_r/G_{r-1})$.

- Set $\mathrm{HN}(G) =$ concave polygon with slopes $\big(\mu(G_i/G_{i-1})\big)_{1 \leq i \leq r}$ and multiplicities $\big(\mathrm{ht}(G_i/G_{i-1})\big)_{1 \leq i \leq r}$.
Then, if $G' \subset G$, $\deg(G') \leq \mathrm{HN}(G)\big(\mathrm{ht}(G')\big)$ i.e.

$$\boxed{\mathrm{HN}(G) = \text{concave hull of } \big(\mathrm{ht}(G'), \deg(G')\big)_{G' \subset G}}.$$

- For $\lambda \in [0, 1]$, if $\mathcal{C}_{ss}^\lambda =$ semi-stable objects of slope λ, then

$$\boxed{\mathcal{C}_{ss}^\lambda = \text{abelian category stable under extensions}}$$

In particular, if G is semi-stable then, $\forall n$, $G[p^n]$ is flat semi-stable of slope $\mu(G)$.

Exemple 1. – *G étale $\Leftrightarrow \mu(G) = 0$.*
– *G of multiplicative type $\Leftrightarrow \mu(G) = 1$.*

And thus the family of abelian categories $\big(\mathcal{C}_{ss}^\lambda\big)_{\lambda \in [0,1]}$ interpolates between the abelian category of étale and multiplicative group schemes over $\mathcal{O}_K$.

Exemple 2. *If H is a p-divisible group over $\mathcal{O}_K$ then, $\forall n \geq 1$, $\mu(H[p^n]) = \dfrac{\dim H}{\mathrm{ht}\, H}$.*

H.N. filtration and duality. If G^D is the Cartier dual of G then

$$\mu(G^D) = 1 - \mu(G).$$

This implies the H.N. filtration of G^D = orthogonal of the H.N. filtration of G via the perfect pairing

$$G(\mathcal{O}_{\overline{K}}) \times G^D(\mathcal{O}_{\overline{K}}) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(1).$$

3. Example : the monogenous case

"Naïve" lower ramification filtration. Let $\overline{K}$ be an algebraic closure of K.

$G =$ any finite flat group scheme over $\mathcal{O}_K$. For $\lambda \in v(\overline{K})_{>0}$ let $\mathfrak{m}_{\overline{K},\lambda} = \{x \in \mathcal{O}_{\overline{K}} \mid v(x) \geq \lambda\}$. Set

$$\underbrace{G_\lambda(\mathcal{O}_{\overline{K}})}_{\text{subgalois-module of } G(\mathcal{O}_{\overline{K}})} = \ker\big(G(\mathcal{O}_{\overline{K}}) \longrightarrow G(\mathcal{O}_{\overline{K}}/\mathfrak{m}_{\overline{K},\lambda})\big).$$

Subgalois-module $G_\lambda(\mathcal{O}_{\overline{K}}) \longmapsto$ subgroup scheme of the étale group scheme $G \otimes K$
$\underset{\text{schematical closure}}{\longmapsto}$ $G_\lambda \subset G$ closed sub finite flat group scheme.

$$\big(G_\lambda\big)_{\lambda \in v(\overline{K})_{>0}} = \text{decreasing filtration of } G.$$

The monogenous case. Suppose $G = \mathrm{Spec}\big(\mathcal{O}_K[T]/(f)\big)$ with

$$\begin{cases} f \in \mathcal{O}_K[T] \text{ unitary} \\ f(0) = 0 \\ \text{neutral section of } G \leftrightarrow T = 0 \end{cases}$$

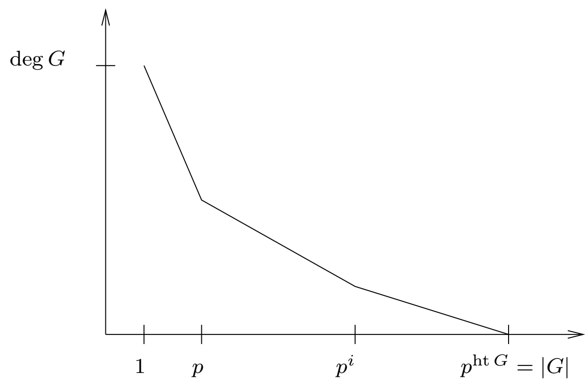

Fig. 1. The Newton polygon of f. $\deg(G) = v(f'(0))$. breakpoints$\leftrightarrow$ jumps of $(G_\lambda)_\lambda$. slopes$\leftrightarrow$slopes λ at jumps of $(G_\lambda)_\lambda$

$$G_\lambda = \mathrm{Spec}\Big(\mathcal{O}_K[T]/\big(\prod_{\substack{\alpha \text{ root of } f \\ v(\alpha) \geq \lambda}} (T - \alpha)\big)\Big)$$

Then,

$$\begin{aligned} \mathrm{Newt}(f) &= \text{convex hull of } \big(\underbrace{p^{\mathrm{ht} G'}}_{|G'|}, \deg G'\big)_{G' \subset G} \\ \mathrm{HN}(G) &= \text{convex hull of } \big(\mathrm{ht}(G'), \deg(G')\big)_{G' \subset G} \end{aligned}$$

$\mathrm{Newt}(f) \longmapsto \mathrm{HN}(G)$: apply $\log_p$ on x-coordinate + convexify + reverse to make it concave. May loose information when you convexify.

$\Rightarrow$ the HN filtration is extracted from $(G_\lambda)_\lambda$.

Exemple 3. *$G = E[p]$, E supersingular elliptic curve. $w = \mathrm{Newt}(f)(p)$. Then*

$$\begin{aligned} (G_\lambda)_\lambda \textit{ has a break of height } 1 &\Leftrightarrow w < \frac{p}{p+1} \\ \textit{HN filtration has a break of height } 1 &\Leftrightarrow w < \frac{1}{2} \end{aligned}$$

4. Families

$F|\mathbb{Q}_p$ discrete

$\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_F)$-formal scheme topologically of finite type without p-torsion (admissible in the sens sof Raynaud)

$\mathfrak{X}^{an}$ = Berkovich generic fiber

$\mathfrak{X}^{rig}$ = rigid generic fiber

$$\underbrace{|\mathfrak{X}^{rig}|}_{\text{tot. discontinuous}} \underset{\text{dense}}{\subset} \overbrace{|\mathfrak{X}^{an}|}^{\text{compact loc.arcwise connected}}$$

$$\mathfrak{X}^{rig} = \{x \in \mathfrak{X}^{an} \mid [\mathcal{K}(x) : F] < +\infty\}.$$

$G/\mathfrak{X}$ finite locally free group scheme of order a power of p

$$x \in \mathfrak{X}^{an} \longmapsto G_x/\mathcal{O}_{\mathcal{K}(x)} \text{ finite flat group scheme}$$

Theorem 2. (1) *The function*

$$\begin{array}{rcl} |\mathfrak{X}^{an}| & \longrightarrow & \textit{Polygons} \\ x & \longmapsto & \mathrm{HN}(G_x) \end{array}$$

is continuous. Moreover if $\mathcal{P}$ = polygon with rationnal coordinates breakpoints

$$\begin{array}{rcl} \{x \in \mathfrak{X}^{rig} \mid \mathrm{HN}(G_x) \leq \mathcal{P}\} & = & \textit{admissible open} \\ \{x \in \mathfrak{X}^{an} \mid \mathrm{HN}(G_x) \leq \mathcal{P}\} & = & \textit{closed analytic domain} \end{array}$$

(2) *Suppose $\exists i_0 \in \mathbb{N}\cap]0, \mathrm{ht}G[$ s.t. $\forall x \in \mathfrak{X}^{rig}$, $\mathrm{HN}(G_x)$ has a breakpoint at the x-coordinate i_0. Then $\exists\ \widetilde{\mathfrak{X}} \to \mathfrak{X}$ formal admissible blow up and $G' \subset G \times_{\mathfrak{X}} \widetilde{\mathfrak{X}}$ closed sub finite flat group scheme with $\mathrm{ht}(G') = i_0$ and $\forall x \in \mathfrak{X}^{rig}$, G'_x is the memmer of the HN filtration of G_x of height i_0.*

Références

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