Vector bundles and $p$-adic Galois representations

Laurent Fargues and Jean-Marc Fontaine

Abstract. Let $F$ be a perfect field of characteristic $p > 0$ complete with respect to a non trivial absolute value. Let $E$ be a non archimedean locally compact field whose residue field is contained in $F$. To these data, we associate a “complete regular curve” $X = X_{F,E}$ defined over $E$. If $F$ is an algebraic closure of $F$ and $H = \text{Gal}(F/F)$, there is an equivalence of categories between continuous finite dimensional $E$-linear representations of $H$ and semistable vector bundles over $X$ of slope 0. To construct $X$ we first construct the ring $B$ of “rigid analytic functions of the variable $\pi$ on the punctured unit disk $\{z \in F \mid 0 < |z| < 1\}$.”

Let $C$ be the $p$-adic completion of an algebraic closure $\overline{K}$ of a $p$-adic field $K$. A classical construction from $p$-adic Hodge theory associates to $C$ a field $F = F(C)$ as above and the group $G_K$ acts on the curve $X = X_{F(C),q}$. We study $G_K$-equivariant vector bundles over $X$ and classify those which are “de Rham”. The two main theorems about $p$-adic de Rham representations are recovered by considering the special case of semistable vector bundles of slope 0. This paper is a survey. Details and proofs will appear elsewhere.

1. Curves and vector bundles

1.1. General conventions and notations. If $R$ is a commutative ring and $M_1, M_2$ are $R$-modules, we denote by $\mathcal{L}_R(M_1, M_2)$ the $R$-module of $R$-linear maps $f : M_1 \to M_2$.

If $L$ is a field equipped with a non archimedean absolute value $| \cdot |$ (or a valuation $v$), we denote $\mathcal{O}_L = \{x \in L \mid |x| \leq 1\}$ (or $v(x) \geq 0$) the corresponding valuation ring, $m_L$ the maximal ideal of $\mathcal{O}_L$, and $k_L = \mathcal{O}_L/m_L$ the residue field.

As usual, if $X$ is a noetherian scheme, we view a vector bundle over $X$ as a locally free coherent $\mathcal{O}_X$-module.

If a group $G$ acts on the left on a noetherian scheme $X$, an $\mathcal{O}_X$-representation of $G$ (resp. a $G$-equivariant vector bundle over $X$) is a coherent $\mathcal{O}_X$-module (resp. a vector bundle) $\mathcal{F}$ equipped with a semi-linear action of $G$ in the following sense:

• for all $g \in G$, if $g : X \xrightarrow{\sim} X$ is the action of $g$ on $X$, one is given an isomorphism $c_g : g^*\mathcal{F} \xrightarrow{\sim} \mathcal{F},$
the following cocycle condition is satisfied
\[ c_{g_2} \circ g_2^* c_{g_1} = c_{g_1 g_2}, \quad g_1, g_2 \in G \]

via the identification \( g_2^*(g_1^* F) = (g_1 g_2)^* F \).

If \( X = \text{Spec}(B) \) is affine, an \( \mathcal{O}_X \)-representation of \( G \) is nothing else than a finite type \( B \)-module equipped with a semi-linear left action of \( G \).

In this paper, we use freely the formalism of tensor categories (for which we refer to [DM82]). For instance, if \( G \) is a group acting on a noetherian scheme \( X \), equipped with the tensor product of the underlying \( \mathcal{O}_X \)-modules, the category \( \text{Rep}_{\mathcal{O}_X}(G) \) of \( \mathcal{O}_X \)-representations of \( G \) is an abelian tensor category, though the full subcategory \( \text{Bund}_X(G) \) of \( G \)-equivariant vector bundles is a rigid additive tensor category. If \( X \) is a smooth geometrically connected projective curve over a perfect field \( E \), the full subcategory \( \text{Bund}^0_X(G) \) of \( G \)-equivariant vector bundles which are semistable of slope 0 is a tannakian \( E \)-linear category.

1.2. Complete regular curves. A regular curve \( X \) is a separated integral noetherian regular scheme of dimension 1. In other words, \( X \) is a separated connected scheme obtained by gluing a finite number of spectra of Dedekind rings.

Let \( X \) be a regular curve, \( K = \mathcal{O}_{X, \eta} \) its function field (i.e. the local ring at the generic point \( \eta \)), \( |X| \) the set of closed points of \( X \). For any \( x \in |X| \), let \( v_x \) be the unique discrete valuation of \( K \) such that \( v_x(K^*) = \mathbb{Z} \) and \( \mathcal{O}_{X,x} = \{ f \in K \mid v_x(f) \geq 0 \} \).

The field \( K \), the set of closed points \( |X| \) and the collection of valuations \( (v_x)_{x \in |X|} \) on \( K \) determine completely the curve \( X \):

i) As a set, the underlying topological space is the disjoint union of \( |X| \) and of a set consisting of a single element \( \eta \).

ii) The non empty open subsets are the complements of the finite subsets of \( |X| \). If \( U \) is one of them,
\[ \Gamma(U, \mathcal{O}_X) = \{ f \in K \mid v_x(f) \geq 0 \ \text{for all} \ x \in U \cap |X| \} . \]

If \( X \) is a regular curve, the group \( \text{Div}(X) \) of Weil divisors of \( X \) is the free abelian group generated by the \( [x] \)'s with \( x \in |X| \). If \( f \in K^* \), the divisor of \( f \) is
\[ \text{div}(f) = \sum_{x \in |X|} v_x(f)[x] . \]

If \( X \) is a regular curve, a coherent \( \mathcal{O}_X \)-module is a vector bundle if and only if it is torsion free.

A complete regular curve is a pair \( (X, \text{deg}) \) consisting of a regular curve \( X \) and a degree map
\[ \text{deg} : |X| \to \mathbb{N}_{>0} \]
such that, for any \( f \in K^* \),
\[ \text{deg}(\text{div}(f)) = \sum_{x \in |X|} v_x(f) \text{deg}(x) = 0 . \]

If \( X \) is a complete regular curve, then \( H^0(X, \mathcal{O}_X) \) is a field. We call it the field of definition of \( X \).
Remark. Equipped with the usual definition of the degree, a smooth projective curve over a field is a complete regular curve. Its function field is finitely generated over its field of definition. It won’t be the case for the curves we are going to construct.

Let \( X \) be a complete regular curve. Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module. The rank of \( \mathcal{F} \) is the dimension of its generic fiber \( \mathcal{F}_\eta \) over the function field. If \( r \) is the rank of \( \mathcal{F} \), choose a vector bundle \( \mathcal{E} \) isomorphic to \( \mathcal{O}_X^r \) whose generic fiber \( \mathcal{E}_\eta \) is equal to \( \mathcal{F}_\eta \). For each closed point \( x \in |X| \), let \( \mathcal{F}'_x \) (resp. \( \mathcal{F}''_x \)) the kernel (resp. the image) of the natural map \( \mathcal{F}_x \to \mathcal{F}_\eta \). We set

\[
\text{lg}_x(\mathcal{F}/\mathcal{E}) = \text{lg}_x(\mathcal{F}'_x) + \text{lg}_x(\mathcal{F}''_x/\mathcal{E}_x)
\]

where, if \( M \) is any \( \mathcal{O}_{X,x} \)-module of finite length, \( \text{lg}_x(M) \) is its length and

\[
\text{lg}_x(\mathcal{F}'_x/\mathcal{E}_x) = \text{lg}_x((\mathcal{E}_x + \mathcal{F}'_x)/\mathcal{E}_x) - \text{lg}_x((\mathcal{E}_x + \mathcal{F}'_x)/\mathcal{F}''_x).
\]

We have \( \text{lg}_x(\mathcal{F}/\mathcal{E}) = 0 \) for almost all \( x \). We define the degree of \( \mathcal{F} \)

\[
\text{deg}(\mathcal{F}) = \sum_{x \in |X|} \text{lg}_x(\mathcal{F}/\mathcal{E}). \deg(x).
\]

Granting to (1), it is independent of the choice of \( \mathcal{E} \). The degree may also be defined by:

\[
\text{deg}(\mathcal{F}) = \text{deg}(\mathcal{F}_{\text{tor}}) + \text{deg}(\text{det}(\mathcal{F}/\mathcal{F}_{\text{tor}}))
\]

where

- \( \mathcal{F}_{\text{tor}} \) is the torsion part of \( \mathcal{F} \), a finite direct sum of skyscrapers sheaves of finite length \( \mathcal{O}_{X,x} \)-modules, \( x \in |X| \),
- \( \text{deg}(\mathcal{F}_{\text{tor}}) = \sum_{x \in |X|} \text{lg}_x(\mathcal{F}_x). \deg(x) \),
- if \( \mathcal{L} \) is a line bundle set \( \text{deg}(\mathcal{L}) = \text{deg}(\text{div}(s)) \) where \( s \) is any non-zero meromorphic section of \( \mathcal{L} \), \( \text{div}(s) \) being the Weil divisor associated to \( s \),
- \( \text{det}(\mathcal{F}/\mathcal{F}_{\text{tor}}) \) is the line bundle \( \Lambda^\text{rank}(\mathcal{F}/\mathcal{F}_{\text{tor}}) \).

The point is that, since \( X \) is complete, the degree function on line bundles

\[
\deg : \text{Div}(X) \to \mathbb{Z}
\]

factorizes through the group of principal divisors to give a degree function

\[
\deg : \text{Div}(X)/\sim = \text{Pic}(X) \to \mathbb{Z}.
\]

If \( \mathcal{F} \) is a non-zero coherent \( \mathcal{O}_X \)-module we define the slope of \( \mathcal{F} \) as

\[
\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})/\text{rank}(\mathcal{F})}{\in \mathbb{Q} \cup \{+\infty\}}
\]

(we have \( \mu(\mathcal{F}) = +\infty \) if and only if \( \mathcal{F} \) is torsion).

An \( \mathcal{O}_X \)-module \( \mathcal{F} \) is semistable (resp. stable) if \( \mu(\mathcal{F}') \leq \mu(\mathcal{F}) \) (resp. if \( \mathcal{F} \) is non-zero and if \( \mu(\mathcal{F}') < \mu(\mathcal{F}) \)) for any proper \( \mathcal{O}_X \)-submodule \( \mathcal{F}' \). A non-zero \( \mathcal{O}_X \)-module is semistable of slope \( +\infty \) if and only if it is a torsion module.

The Harder-Narasimhan theorem holds:

**Theorem 1.1.** Let \( \mathcal{F} \) be a non-zero coherent \( \mathcal{O}_X \)-module. There is a unique filtration

\[
0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r \subset \ldots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \mathcal{F}
\]
by $\mathcal{O}_X$-submodules with $\mathcal{F}_i/\mathcal{F}_{i-1} \neq 0$, semistable, and
\[ \mu(\mathcal{F}_1/\mathcal{F}_0) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \ldots > \mu(\mathcal{F}_m/\mathcal{F}_{m-1}) . \]

Moreover, for each $\lambda \in \mathbb{Q} \cup \{+\infty\}$, the full sub-category $\text{Bund}_X^\lambda$ of the category of coherent $\mathcal{O}_X$-modules whose objects are those which are semistable of slope $\lambda$ is an abelian $E$-linear category.

We see that, $\mathcal{F}$ is a vector bundle if and only if $\mu(\mathcal{F}_1/\mathcal{F}_0) \neq +\infty$. In this case, the $\mathcal{F}_i$’s are strict vector subbundles, i.e. the quotients $\mathcal{F}/\mathcal{F}_i$’s are torsion free, hence also vector bundles. If, instead, the torsion sub-module $\mathcal{F}_{tor}$ is not 0, then $\mathcal{F}_{tor} = \mathcal{F}_1$.

2. Bounded analytic functions

2.1. The field $\mathcal{E}_{F,E}$. We fix a non archimedean locally compact field $E$. We denote by $p$ the characteristic of $k_E$ and $q$ the number of elements of $k_E$. We denote by $v_E$ the valuation of $E$ normalized by $v_E(E^*) = \mathbb{Z}$.

Let $F$ be any perfect field containing $k_E$. We denote by $E_{F,E}$ the unique (up to a unique isomorphism) field extension of $E$, complete with respect to a discrete valuation $v$ extending $v_E$ such that
\begin{enumerate}
  \item $v(E_{F,E}^*) = v_E(E^*) = \mathbb{Z}$,
  \item $F$ is the residue field of $E_{F,E}$.
\end{enumerate}

There is a unique section of the projection $\mathcal{O}_{E_{F,E}} \rightarrow F$ which is multiplicative. We denote it $a \mapsto [a]$.

If we choose a uniformizing parameter $\pi$ of $E$, any element $f \in \mathcal{E}_{F,E}$ may be written uniquely
\[ f = \sum_{n \gg -\infty} [a_n] \pi^n \text{ with the } a_n \in F , \]
and $f \in E$ if and only if all the $a_n$’s are in $k_E$.

We see that, if $E$ is of characteristic $p$, the map $a \mapsto [a]$ is an homomorphism of rings. If we use it to identify $F$ with a subfield of $\mathcal{E}$, i.e. if we set $[a] = a$ for all $a \in F$, we get
\[ E = k_E((\pi)) \text{ and } \mathcal{E}_{F,E} = F((\pi)) . \]
Otherwise, $E$ is a finite extension of $\mathbb{Q}_p$. If $W(F)$ (resp. $W(k_E)$) is the ring of Witt vectors with coefficients in $F$ (resp. $k_E$), we see that $\mathcal{E}_{F,E}$ can be identified with $E \otimes_{W(k_E)} W(F)$ and that, for all $a \in F$,
\[ [a] = 1 \otimes (a, 0, 0, \ldots, 0, \ldots) . \]

2.2. Three sub-rings of $\mathcal{E}_{F,E}$. We now fix the perfect field $F$ containing $k_E$ and we assume $F$ to be complete for a given non trivial absolute value $| \cdot |$. Observe that, as $F$ is perfect, the valuation group is $p$-divisible, hence the valuation is not discrete.

If there is no risk of confusion, we set $\mathcal{E} = \mathcal{E}_{F,E}$. We still choose a uniformizing parameter $\pi$ of $E$. The following subsets of $\mathcal{E}$
\[ B^b = B^b_{F,E} = \left\{ \sum_{n \gg -\infty} [a_n] \pi^n \mid \text{there exists } C \text{ such that } |a_n| \leq C, \forall n \right\} , \]
\[ B^{b,+} = B_{F,E}^{b,+} = \left\{ \sum_{n \gg -\infty} [a_n] \pi^n \mid a_n \in \mathcal{O}_F, \forall n \right\} \]

and \[ A = A_{F,E} = \left\{ \sum_{n=0}^{+\infty} [a_n] \pi^n \mid a_n \in \mathcal{O}_F, \forall n \right\} \]

are \( \mathcal{O}_E \)-subalgebras of \( \mathcal{E} \) and are independent of \( \pi \). If \( a \) is any non-zero element of the maximal ideal \( \mathfrak{m}_F \) of \( \mathcal{O}_F \), we have

\[ B^{b,+} = A\left[ \frac{1}{\pi} \right] \quad \text{and} \quad B^b = B^{b,+}\left[ \frac{1}{\pi} \right]. \]

When \( \text{char}(E) = p \), the ring \( B^b \) may be viewed as the ring of rigid analytic functions

\[ f : \Delta = \{ z \in F \mid 0 < |z| < 1 \} \to F \]

which are such that \( \pi^n f \) is analytic and bounded on \( \{ z \in F \mid 0 \leq |z| < 1 \} \), for \( n > 0 \).

### 2.3. Prime ideals of finite degree.

We set \( \mathcal{E}_0 = \mathcal{E}_{k_F} \).

The projection \( \mathcal{O}_F \to k_F \), which we denote as \( a \mapsto \bar{a} \), induces an augmentation map

\[ \varepsilon : B^{b,+} \to \mathcal{E}_0 \text{ sending } \sum_{n \gg -\infty} [a_n] \pi^n \text{ to } \sum_{n \gg -\infty} \bar{a}_n \pi^n. \]

We have \( \varepsilon(A) = \mathcal{O}_{E_0} \). We say that \( \xi \in A \) is primitive if \( \xi \not\equiv \pi A \) and \( \varepsilon(\xi) \neq 0 \). The degree of a primitive element \( \xi \) is

\[ \deg(\xi) = v_{\pi}(\varepsilon(\xi)) \in \mathbb{N}. \]

We see that \( A \) is a local ring whose invertible elements are exactly the primitive elements of degree 0. A primitive element \( \xi \in A \) is irreducible if \( \deg(\xi) > 0 \) and \( \xi \) can’t be written as the product of two primitive elements of degree \( > 0 \). In particular, any primitive element of degree 1 is irreducible.

We say that two primitive irreducible elements \( \xi \) and \( \xi' \) are associated (we write \( \xi \sim \xi' \)) if there exists \( \eta \) primitive of degree 0 such that \( \xi' = \xi \eta \). This is an equivalence relation and we set

\[ \{ Y_{F,E} \} = \{ Y \} = \{ \text{primitive irreducible elements} \} / \sim. \]

If \( y \in \{ Y \} \) is the class of \( \xi \), we set \( \deg(y) = \deg(\xi) \).

We say that an ideal \( \mathfrak{a} \) of \( A \), \( B^{b,+} \) or \( B^b \) is of finite degree if it is a principal ideal which is generated by a primitive element \( \xi \) of \( A \). The degree of such an \( \mathfrak{a} \) is the degree of \( \xi \).

**Proposition 2.3.1.** Let \( y \in \{ Y \} \) be the class of a primitive irreducible element \( \xi \). The ideal \( p_y \) (resp. \( p_y^{b,+} \), resp. \( p_y^b \)) of \( A \) (resp. \( B^{b,+} \), resp. \( B^b \)) generated by \( \xi \) is prime and depends only on \( y \). The map

\[ y \mapsto p_y \ ( \text{resp. } y \mapsto p_y^{b,+}, \text{ resp. } y \mapsto p_y^b) \]

induces a bijection between \( \{ Y \} \) and the set of prime ideals of finite degree of \( A \) (resp. \( B^{b,+} \), resp. \( B^b \)).

To describe what are the quotients of these rings by a prime ideal of finite degree, it is convenient to introduce the notion of \( p \)-perfect field.
2.4. p-perfect fields. A p-perfect field is a field \( L \) complete with respect to a non trivial non archimedean absolute value \( | | \) whose residue field \( k_L \) is of characteristic \( p \) and which is such that the endomorphism \( x \mapsto x^p \) of \( \mathcal{O}_L/p\mathcal{O}_L \) is surjective.

If \( L \) is the fraction field of a complete discrete valuation ring, we see that \( L \) is a p-perfect field if and only if \( k_L \) is perfect of characteristic \( p \) and \( \mathfrak{m}_L \) is generated by \( p \).

A strictly p-perfect field is a p-perfect field \( L \) such that \( \mathcal{O}_L \) is not a discrete valuation ring.

Let \( L \) be a field complete with respect to a non trivial non archimedean absolute value, with char(\( k(p) \)) = \( p \) and \( \mathcal{O}_L \) not a discrete valuation ring. It is easy to see that

- if \( a \) is any element of the maximal ideal \( \mathfrak{m}_L \) of \( \mathcal{O}_L \) such that \( p \in (a) \), then \( L \) is strictly p-perfect if and only if the map
  \[
  \mathcal{O}_L/(a) \hookrightarrow \mathcal{O}_L/(a) \quad \text{sending } x \text{ to } x^p
  \]
is onto,

- if \( L \) is of characteristic \( p \), \( L \) is strictly p-perfect if and only if \( L \) is perfect.

Let \( L \) be a p-perfect field. We consider the set

\[
F(L) = \{ x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in L \text{ and } (x^{(n+1)})^p = x^{(n)} \}.
\]

If \( x, y \in F(L) \), we set

\[
(x + y)^{(n)} = \lim_{m \to +\infty} (x^{(n+m)} + y^{(n+m)})^p, \quad (xy)^{(n)} = x^{(n)}y^{(n)}
\]

(it is easy to see that the limit above exists).

Proposition 2.4.1. Let \( L \) be a p-perfect field. Then \( F(L) \) is a perfect field of characteristic \( p \), complete with respect to the absolute value \( | | \) defined by \( |x| = |x^{(0)}| \).

Moreover

i) If \( a \subset \mathfrak{m}_L \) is a finite type (i.e. principal) ideal of \( \mathcal{O}_L \) containing \( p \) and if \( u \mapsto \bar{u} \) denote the projection \( \mathcal{O}_L \to \mathcal{O}_L/a \), the map

\[
\mathcal{O}_{F(L)} \to \lim_{n \to \infty} \mathcal{O}_L/a^n
\]

(with transition maps \( v \mapsto v^p \)) sending \( (x^{(n)})_{n \in \mathbb{N}} \) to \( (\bar{x}^{(n)})_{n \in \mathbb{N}} \) is an isomorphism of topological rings.

ii) If \( L \) contains \( E \) as a closed subfield, the map

\[
\theta_{L,E} : B^{b+}_{F(L),E} \to L
\]

sending \( \sum_{n \geq 0} a_n \pi^n \) to \( \sum_{n \geq 0} a_0^{(n)} \pi^n \) is a surjective homomorphism of \( E \)-algebras (independent of the choice of \( \pi \)). Moreover,

1. If \( \mathcal{O}_L \) is a discrete valuation ring, \( F(L) \) is the residue field of \( L \) equipped with the trivial valuation and \( \theta_{L,E} \) is an isomorphism.
2. If \( L \) is strictly p-perfect, we have \( |F(L)| = |L| \) and the kernel of \( \theta_{L,E} \) is a prime ideal of \( B^{b+}_{F(L),E} \) of degree 1. We have

\[
\theta_{L,E}(B^{b+}_{F(L),E}) = L \quad \text{and} \quad \theta_{L,E}(A_{F(L),E}) = \mathcal{O}_L.
\]
Remarks. 
(1) If $L$ is of characteristic $p$, the map $x \mapsto x^{(0)}$ is a canonical isomorphism of the field $F(L)$ onto the residue field of $L$ if $L$ is not strictly $p$-perfect and onto $L$ otherwise. Then, all the results are obvious. If $L$ is strictly $p$-perfect and if $\lambda$ is the unique element of $F(L)$ such that $\lambda^{(0)} = \pi$, then $\pi - [\lambda]$ is a generator of $\ker \theta_{L,E}$.

(2) If $L$ is strictly perfect of characteristic 0, it's not always true that there exists $\lambda \in F(L)$ such that $\pi - [\lambda]$ is a generator of $\ker \theta_{L,E}$ (which is equivalent to saying that $\lambda^{(0)} = \pi$). This is true if $F$ is algebraically closed, but such a $\lambda$ is not unique!

All the ideals of degree 1 are obtained by this construction: Let $L$ be the set of isomorphism classes of pairs $(L, \iota)$ where $L$ is a $p$-perfect field containing $E$ as a closed subfield and $\iota : F(L) \to F$ is an isomorphism of topological fields.

If $(L, \iota)$ is such a pair, let $\theta_L : B^0 \to L$ be the homomorphism deduced from $\theta_{L,E} : B^0_{F(L),E} \to L$ by transport de structure.

Proposition 2.4.2. The map $L \to \{ \text{ideals of degree 1} \}$ sending the class of $(L, \iota)$ to the kernel of $\theta_L$ is bijective.

2.5. Algebraic extensions of strictly $p$-perfect fields.

Proposition 2.5.1. Let $L_0$ be a strictly $p$-perfect field containing $E$ as a closed subfield, $F_0 = F(L_0)$ and $\mathfrak{m}$ the kernel of the map $\theta_{L_0,E} : B^0_{F_0,E} \to L_0$.

i) If $L$ is a finite extension of $L_0$, then $L$ is strictly $p$-perfect and $F(L)$ is a finite extension of $F(L_0)$ of the same degree.

ii) If $F$ is a finite extension of $F_0$, the ideal $B^0_{F,E} \mathfrak{m}$ of $B^0_{F,E}$ is maximal and the quotient of $B^0_{F,E}$ by this ideal is a finite extension of $L_0$ of the same degree.

The functor $L \to F(L)$ is an equivalence of categories between finite extensions of $L_0$ and finite extensions of $F_0$. The functor $F \mapsto B^0_{F,E}/B^0_{F,E} \mathfrak{m}$ is a quasi-inverse.

Remark. This equivalence extends in an obvious way to étale algebras. Hence, we see that the small étale site of $L_0$ can be identified with the small étale site of $F_0$.

2.6. Finite divisors. We can now give a complete description of the prime ideals of finite degree.

Proposition 2.6.1. If $F$ is algebraically closed, a primitive element is irreducible if and only if it is of degree 1.

Proposition 2.6.2. Let $y \in |Y|$, $d = \deg(y)$, $\xi = \sum_{n=0}^{\infty} [c_n] \pi^n$ a primitive element lifting $y$, $L_y = B^0/p^0_y$ and $\theta_y : B^0 \to L_y$ the projection. We set $||y|| = |c_0|^{1/d}$.

i) The ideals $p^0_y$ and $p^{0,+}_y$ are maximal and

$B^{0,+}/p^{0,+}_y = L_y$.

ii) There is a unique absolute value $|\cdot|$ on the field $L_y$ such that $|\theta_y([a])|_y = |a|$ for all $a \in F$. Equipped with this absolute value, $L_y$ is a $p$-perfect field containing $E$ as a closed subfield. Moreover $|\pi|_y = ||y||$.

iii) The map $F \to F(L_y)$ sending $a$ to $(\theta_y([a] \pi^n))_{n \in \mathbb{N}}$ is a continuous homomorphism of topological fields identifying $F(L_y)$ with a finite extension of $F$ of degree $d$.

iv) The ring $A/p_y$ is a $O_{L_y}$-subalgebra of $O_{L_y}$ whose fraction field is $L_y$. 


We define the group \( \text{Div}_f(Y) \) of \textit{finite divisors} of \( Y \) as the free abelian group with basis the \( [y] \)'s for \( y \in |Y| \). Hence any finite divisor may be written uniquely
\[
D = \sum_{y \in |Y|} n_y[y] \quad \text{with} \quad n_y \in \mathbb{Z}, \text{almost all 0}
\]
The degree of such a \( D \) is \( \sum_{y \in |Y|} n_y \deg(y) \).

We denote \( \text{Div}_f^+(Y) \) the monoid of \textit{finite effective divisors}, i.e. of divisors \( D = \sum n_y[y] \) with \( n_y \geq 0 \) for all \( y \). From the previous proposition, one deduces:

**Corollary 2.6.1.** The map from \( \text{Div}_f^+(Y) \) to the multiplicative monoid of ideals of finite degree of \( A \) (resp. \( B^{b,+} \), resp. \( B^b \)) sending \( \sum_{y \in |Y|} n_y[y] \) onto \( \prod_{y \in |Y|} [p_y]^{n_y} \) (resp. \( \prod_{y \in |Y|} (p_y^{b,+})^{n_y} \), resp. \( \prod_{y \in |Y|} (p_y^{b})^{n_y} \) ) is an isomorphism of monoids.

### 3. The rings of rigid analytic functions

#### 3.1. Norms and completions.

For \( f = \sum_{n \gg -\infty} a_n \pi^n \in B^b \), and \( 0 < \rho < 1 \), we define
\[
|f|_\rho = \max_{n \in \mathbb{Z}} |a_n| \rho^n.
\]

We also set
\[
|f|_0 = q^{-r} \quad \text{if } r \text{ is the smallest integer such that } a_r \neq 0, \text{ and } |f|_1 = \sup_{n \in \mathbb{Z}} |a_n|.
\]

For any \( \rho \in [0,1] \), the map \( f \mapsto |f|_\rho \) is a multiplicative norm on \( B^b \), i.e. we have
\[
|f + g|_\rho \leq \max\{|f|_\rho, |g|_\rho\}, \quad |fg|_\rho = |f|_\rho |g|_\rho \quad \text{and} \quad |f|_\rho = 0 \iff f = 0.
\]

For any non empty interval \( I \subset [0,1] \), we denote
\[
B_I = B_{F,E,I}
\]
the completion of \( B^b \) for the family of the \( |.|_\rho \)'s for \( \rho \in I \).

**Proposition 3.1.1.** Let \( I \subset [0,1] \) be a non empty interval. For any \( \rho \in I \), \( |.|_\rho \) is a norm on \( B_I \) (i.e., if \( b \in B_I \) is \( \neq 0 \), then \( |b|_\rho \neq 0 \)). Moreover:

i) If \( J \subset I \) is an interval, the induced map
\[
B_I \to B_J
\]
is a continuous injective map.

ii) If \( I = [\rho_1, \rho_2] \) is a non empty closed interval contained in \([0,1]\), then \( B_I \) is a Banach \( E \)-algebra: if we set
\[
A_{F,E,I}^b = A_I^b = \{ f \in B^{b,+} \mid |f|_{\rho_1} \leq 1 \text{ and } |f|_{\rho_2} \leq 1 \},
\]
then \( B_I = A_I[1/\pi] \) where \( A_I = A_{F,E,I} \) is the \( \pi \)-adic completion of \( A_I^b \).

\footnote{Say that a sequence \( (f_n)_{n \in \mathbb{N}} \) is a Cauchy sequence over the interval \( I \) if for any \( \rho \in I \) and any \( \epsilon > 0 \), there exists \( N \) such that \( |f_m - f_n|_\rho < \epsilon \) if \( m \) and \( n \) are \( \geq N \). Say that two Cauchy sequences \((f_n)_{n \in \mathbb{N}} \) and \((g_n)_{n \in \mathbb{N}} \) are equivalent if, for any \( \rho \in I \) and any \( \epsilon > 0 \), there exists \( N \) such that \( |f_n - g_n|_\rho < \epsilon \) if \( n \geq N \). An element of \( B_{F,E,I} \) may be viewed as an equivalence class of Cauchy sequences over \( I \).}
iii) If \( I \subset [0,1[ \) is not restricted to \( I = \{0\} \), then \( B_I \) is a Fréchet-\( E \)-algebra (inverse limit of Banach \( E \)-algebras): If \( \mathcal{I}_I \) is the set of closed intervals contained in \( I \), the map

\[
B_I \to \varprojlim_{J \in \mathcal{I}_I} B_J
\]

is a homeomorphism of topological rings.

iv) We have \( B_{[0,1]} = B^b \) and \( B_{[0]} = E \).

In what follow, if \( J \subset I \), we use the injective map \( B_I \to B_J \) to identify \( B_I \) with a subring of \( B_J \).

If \( I \subset [0,1[ \) contains 0 then \( B_I \) can be identified with a subring of \( E \):

\[
B_I = \left\{ \sum_{n, n \to -\infty} [a_n] \pi^n \in E \mid \forall \rho \in I, |a_n| \rho^n \to 0 \text{ for } n \to +\infty \right\}.
\]

If \( I \subset [0,1[ \) contains 0, we set

\[
B_I^+ = \{ b \in B_I \mid |b|_0 \leq 1 \} = B_I \cap O_E.
\]

Similarly if \( I \subset [0,1[ \) contains 1, we set

\[
B_I^+ = \{ b \in B_I \mid |b|_1 \leq 1 \}.
\]

We have

\[
B_{[0,1]}^+ = B^{b,+} \text{ and } A = B^{b,+} \cap O_E = \{ b \in B^b = B_{[0,1]} \mid |b|_0 \leq 1 \text{ and } |b|_1 \leq 1 \}.
\]

We also write

\[
B_{[0,1]}^{E,E,I} = B^+ = B_{[0,1]}^{b,+} \text{ and } B_{F, E} = B = B_{[0,1]}.
\]

If \( \text{char}(E) = p \) and if \( I \subset [0,1[ \) the ring \( B_I \) can be identified with the ring of rigid analytic functions

\[
f : \{ z \in F \text{ with } |z| \in I \} \to F.
\]

In particular \( B := B_{[0,1]} \) is the ring of rigid analytic functions on the punctured open unit disk.

Similarly, if \( \text{char}(E) = p \) and if \( 0 \in I \subset [0,1[ \), then \( B_I^+ \) may be identified with the ring of analytic functions

\[
f : \{ z \in F \text{ with } |z| \in I \} \to F,
\]

though \( B_I \) is the ring of meromorphic rigid analytic functions in the same range, with no pole away from 0.

**Remark.** Let \( I \subset [0,1[ \). Let \( (a_n)_{n \in \mathbb{Z}} \) be elements in \( F \) such that, for all \( \rho \in I \), we have \( |a_n| \rho^n \to 0 \) whenever \( n \to +\infty \) and also when \( n \to -\infty \). Then the series

\[
\sum_{n \in \mathbb{Z}} [a_n] \pi^n
\]

converges (in both directions) to an element of \( B_I \). If \( \text{char}(E) = p \), any element of \( B_I \) may be written uniquely like that. If \( \text{char}(E) = 0 \), we don’t know if it is always possible and, when it is possible, we don’t know if this writing is unique (but it seems unlikely in general).
3.2. Newton polygons. Let \( v \) the valuation of \( F \) normalized by \( |a| = q^{-v(a)} \) for all \( a \in F \). Let \( I \subset [0,1] \) be an interval containing 0. The map
\[
(a_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} [a_n] \pi^n
\]
is a bijection between the set of sequences \((a_n)_{n \in \mathbb{Z}}\) of elements of \( F \) such that
i) \( a_n = 0 \) for \( n \ll 0 \),
ii) for all \( \rho \in I \), \( a_n \rho^n \to 0 \) for \( n \to +\infty \)
and \( B_I \). If \( f = \sum_{n > -\infty} [a_n] \pi^n \in B_I \) is non-zero, the Newton polygon of \( f \) is the convex hull \( \text{Newt}(f) \) of the points of the real plane of coordinates \((n,v(a_n))\) for \( n \in \mathbb{Z} \). If \( J \subset I \) is an interval, \( \text{Newt}_J(f) \) is the sub-polygon of \( \text{Newt}(f) \) obtained by deleting all segments whose slopes \( s \) are such that \( q^s \notin I \).

Proposition 3.2.1. Let \( I \subset [0,1] \) be an interval and let \( T \) be the smallest interval containing \( I \) and 0. Then \( B_T \) is a dense subring of \( B_I \). If \( f \in B_I \) and \( (f_n)_{n \in \mathbb{N}} \) is a sequence of elements of \( B_T \) converging to \( f \), then the sequence \((\text{Newt}_I(f_n))_{n \in \mathbb{N}}\) has a limit, i.e., for any closed interval \( J \subset I \), the sequence of the \( \text{Newt}_J(f_n) \) is stationary. This limit is independent of the choice of the sequence \((f_n)_{n \in \mathbb{N}}\).

We call this limit \( \text{Newt}_I(f) \).

3.3. Divisors. For any interval \( I \subset [0,1] \) different from \( \emptyset, \{1\} \), we set
\[
|Y_I| = \{ y \in |Y| \mid ||y|| \in I \}
\]
and we define the group \( \text{Div}(Y_I) \) of divisors of \( Y_I \):

i) If \( I \) is closed and \( I \subset [0,1[ \), we set
\[
\text{Div}(Y_I) = \left\{ \sum_{y \in |Y_I|} n_y[y] \mid n_y = 0 \text{ for almost all } y \right\}.
\]

ii) If \( I \subset [0,1[ \) is not closed and if \( J_I \) denote the set of closed ideals \( J \subset I \), we set
\[
\text{Div}(Y_I) = \left\{ \sum_{y \in |Y_I|} n_y[y] \mid \forall J \in J_I, n_y = 0 \text{ for almost all } y \text{ with } ||y|| \in J \right\}.
\]

iii) If \( 1 \in I \), we define \( I' \) as the complement of 1 in \( I \), we choose \( \rho_0 \in I' \) and we set
\[
\text{Div}(Y_I) = \left\{ \sum_{y \in |Y_I|} n_y[y] \in \text{Div}(Y_I) \mid \sum_{||y|| \geq \rho_0} n_y \log(||y||) > -\infty \right\}
\]
(independent of the choice of \( \rho_0 \)).

For any \( I \), we denote by \( \text{Div}^+(Y_I) \) the monoid of effective divisors i.e. of divisors \( D = \sum n_y[y] \in \text{Div}(Y_I) \) such that \( n_y \geq 0 \) for all \( y \).

---

\(^2\)See the remark 3.4.1 below for a geometric interpretation of these constructions.
3.4. Closed ideals. For any \( y \in |Y| \), we choose a primitive element \( \xi_y \) representing \( y \).

**Proposition 3.4.1.** Let \( I \subset [0, 1] \) be a non empty interval and \( y \in |Y| \). If \( ||y|| \notin I \), then \( \xi_y \) is invertible in \( B_I \). If \( ||y|| \in I \) and if \( L_y = B_I^{\#}(\xi_y) \), the projection of \( B_I^{\#} \) to \( L_y \) extends by continuity to a surjective homomorphism of \( E \)-algebras

\[
\theta_y : B_I \to L_y
\]

whose kernel is the maximal ideal generated by \( \xi_y \).

The map

\[
y \mapsto m_{I, y} = \text{ideal of } B_I \text{ generated by } \xi_y
\]

is an injective map from \( |Y| \) to the set of maximal ideals of \( B_I \).

**Theorem 3.1.** Let \( I \subset [0, 1] \) an interval different from \( \emptyset, \{1\} \). For any \( y \in |Y| \), we have \( \cap_{n \in \mathbb{N}} (m_{I, y})^n = \emptyset \). Let \( f \in B_I \) a non-zero element. For any \( y \in |Y| \), let \( v_y(f) \) be the biggest integer \( n \) such that \( f \in (m_y)^n \). Then

\[
\text{div}(f) = \sum_{y \in |Y|} v_y(f)[y] \in \text{Div}^+(Y_I)
\]

Moreover, for any \( \rho = q^{-r} \in I \) with \( r > 0 \), the length \( \mu_{\rho}(f) \) of the projection on the horizontal axis of the segment of Newt.(f) of slope \( -r \) is finite and

\[
\sum_{|y| = \rho} v_y(f) \deg(y) = \mu_{\rho}(f).
\]

**Corollary 3.4.1.** Let \( I \subset [0, 1] \) an interval different from \( \emptyset, \{1\} \). Then:

i) Any non-zero closed prime ideal of \( B_I \) is maximal and principal.

ii) The map \( |Y| \to \{ \text{closed maximal ideals of } B_I \} \) sending \( y \) to \( m_{I, y} \) is a bijection.

iii) If \( I \subset [0, 1] \) and is closed, any ideal of \( B_I \) is closed and \( B_I \) is a principal domain.

**Proposition 3.4.2.** Let \( I \subset [0, 1] \) a non empty interval. For any non-zero closed ideal \( a \) of \( B_I \) and any \( y \in |Y| \), let \( v_y(a) \) the biggest integer \( n \leq 0 \) such that \( a \subset (m_{I, y})^n \). Then

\[
\text{div}(a) = \sum_{y \in |Y|} v_y(a)[y] \in \text{Div}^+(Y_I)
\]

The map

\[
\{ \text{non-zero closed ideals of } B_I \} \to \text{Div}^+(Y_I)
\]

so defined, is an isomorphism of monoids.

**Remark 3.4.1.** Let \( I \subset [0, 1] \) an interval different from \( \emptyset, \{1\} \).

- If \( I \) is closed, we see that \( \text{Div}(Y_I) \) is nothing but the group of divisors of the regular curve \( Y_I = \text{Spec}(B_I) \) and that \( |Y_I| \) may be identified to the set of closed points of \( Y_I \).

- Otherwise, we may consider the inductive system of regular curves

\[
Y_I = (Y_J = \text{Spec}(B_J))_{J \subseteq I_I}.
\]

If \( J_1 \subset J_2 \) belong to \( I_I \), we have morphisms of abelian groups

\[
\text{Div}(Y_{J_1}) \to \text{Div}(Y_{J_2}) \text{ and } \text{Div}(Y_{J_2}) \to \text{Div}(Y_{J_1})
\]
induced by the fact that, if \( a \) is a non-zero ideal of \( B_J \) then \( a \cap B_J \) is a non-zero ideal of \( B_J \), though, if \( b \) is a non zero ideal of \( B_J \), then \( B_J b \) is a non zero ideal of \( B_J \). We see that \( \text{Div}(Y_J) \) is the inverse limit of the \( \text{Div}(Y_J) \) for \( J \in I \). The direct limit of these groups consists of the subgroup

\[
\text{Div}_f(Y_I) = \left\{ \sum_{y \in |Y_I|} n_y[y] \in \text{Div}(Y_I) \mid n_y = 0 \text{ for almost all } y \in |Y_I| \right\}.
\]

3.5. Factorization. From the above proposition, we see that the analogue in this context of the classical question “does there exist an analytic function which has a given set of zeros with fixed multiplicities ” becomes the question:

“Let \( D \in \text{Div}^+(Y_I) \). Does there exist \( f \in B_I \) such that \( \text{div}(f) = D \) ?”

The answer to this question is “yes for any \( D \)” if and only if any closed ideal is principal.

The answer to this question is obviously “yes” if \( I \subset [0, 1] \) is closed. This is also “yes” if \( I = [0, \rho] \) for some \( \rho \in ]0, 1[ \) (see cor. 3.5.1 below). But it is “no” in general.

Recall that one says that the field \( F \) is spherically complete if the intersection of any decreasing sequence of non empty balls contained in \( F \) is non empty.

For instance, if \( k \) is an algebraically closed field of characteristic \( p \),

i) the completion of an algebraic closure of the field \( k((u)) \) is not spherically complete,

ii) If \( G \) is a divisible totally ordered abelian group (e.g. \( G = \mathbb{Q} \) or \( \mathbb{R} \)), we may consider the subset \( F \) of all formal series of the form

\[
f = \sum_{g \in G} a_g g \quad \text{with } a_g \in k,
\]

such that the support of \( f \)

\[
\text{supp}(f) = \{ g \in G \mid a_g \neq 0 \}
\]

is a well ordered subset of \( G \). Then, with the obvious addition, multiplication and absolute value, \( F \) is an algebraically closed field which is spherically complete [Poo93].

**Proposition 3.5.1.** Let \( I \subset [0, 1[ \) be a non closed interval. Then:

i) If \( F \) is not spherically complete, there are closed ideals of \( B_I \) which are not principal.

ii) If \( F \) is spherically complete and \( \text{char}(E) = p \), any closed ideal of \( B_I \) is principal.

It is likely that (ii) remains true whenever \( \text{char}(E) = 0 \).

Without any assumption on \( F \), if \( I \) is an interval whose closure contains 0, any divisor

\[
\sum_{y \in |Y_I|} n_y[y]
\]

such that \( n_y = 0 \) if \( ||y|| \geq \rho \) for \( \rho \in I \) big enough, is the divisor of a function.

More precisely, for any \( y \in |Y_I| \) we denote by \( d_y \) the degree of \( y \) and we choose a \( \pi \)-primitive element \( \xi \) (i.e. an element \( \xi \in A \) such that \( |\xi - \pi^d_y|_1 < 1 \)) representing \( y \) (one can show that such an element always exists). Then:
Proposition 3.5.2. Let \( I \subset [0,1] \) an interval containing 0, not reduced to \( \{0\} \), and \( \bar{I} \) the complement of \( \{0\} \) in \( I \). Let
\[
D = \sum_{y \in \bar{I}} n_y[y] \in \text{Div}^+(Y_I) .
\]
i) For any \( \rho \in \bar{I} \), the infinite product
\[
f_{\leq \rho} = \prod_{||y|| \leq \rho} \xi_y \pi_y^{-n_y}
\]
converges in \( B^+_{[0,1]} \subset B_I \) and \( \text{div}(f_{\leq \rho}) = \sum_{||y|| \leq \rho} n_y[y] \).

ii) If there exists \( f \in B_I \) such that \( \text{div}(f) = D \) then \( f = f_{\leq \rho}f_{> \rho} \) for some \( f_{> \rho} \in B_\mathfrak{p} \) and \( \text{div}(f_{> \rho}) = \sum_{||y|| > \rho} n_y[y] \).

In particular, if \( I = \{0,1\} \), \( f_{\leq \rho} \in B^h_{[0,1]} \). In this case, \( f \in B_{[0,1]}^+ \) (resp \( B_{[0,1]}^+ \)) if and only if \( f_{> \rho} \in B^h \) (resp. \( B^h^+ \)).

Corollary 3.5.1. i) If \( I = [0,\rho] \) for some \( \rho \in [0,1[ \), any closed ideal of \( B_I \) is principal.

ii) An ideal of \( B_{[0,1]} \) or of \( B_{[0,1]} \) is closed if and only if it is an intersection of principal ideals.

3.6. Units. The ring \( A \) is a local ring. Therefore, if \( \mathfrak{m}_A \) is its maximal ideal, the multiplicative group \( A^* \) of invertible elements of \( A \) is the complement of \( \mathfrak{m}_A \) in \( A \). With obvious notations, we have also
\[
A^* = [\mathcal{O}_F^*] \times \mathcal{U}_F \quad \text{with} \quad \mathcal{U}_F = \{1 + \sum_{n=1}^{\infty} [a_n] \pi^n \mid a_n \in \mathcal{O}_F \} .
\]
We have also
\[
(B^h^+)^* = \pi^2 \times A^* = \pi^2 \times [\mathcal{O}_F^*] \times \mathcal{U}_F \quad \text{and} \quad (B^h)^* = \pi^2 \times [F^*] \times \mathcal{U}_F .
\]
If \( f \) is an invertible element of \( B_{[0,1]} \) we must have \( \text{div}(f) = 0 \), which implies that \( f \in B^h \). Therefore,
\[
(B_{[0,1],1})^* = (B_{[0,1],1})^* = (B^h)^* \quad \text{and} \quad (B^h)^* = (B^h^+)^* .
\]

4. The curve \( X \) in the case where \( F \) is algebraically closed

4.1. Construction of the curve. The Frobenius automorphism \( \varphi \) on \( B^h \) is the unique \( E \)-automorphism which is continuous for \( \| \cdot \|_0 \) and induces \( x \mapsto x^q \) on \( F \). It satisfies
\[
\varphi \left( \sum_{n \geq 0} [a_n] \pi^n \right) = \sum_{n \geq 0} [a_n^q] \pi^n .
\]
For any \( f \in B^h \) and any \( \rho \in [0,1], \) we have \( \| \varphi(f) \|_\rho = (|f|_\rho)^q \). This implies that \( \varphi \) extends by continuity to an automorphism (still denoted \( \varphi \)) of \( B = B_{[0,1]} \).

We consider the graded \( E \)-algebra
\[
P_\pi = P_{\pi,E,\pi} = \bigoplus_{d \in \mathbb{N}} P_{\pi,d} \quad \text{with} \quad P_{\pi,d} = P_{\pi,E,\pi,d} = \{ b \in B \mid \varphi(b) = \pi^d b \} .
\]
The natural map \( P_\pi \to B \) is injective and we use it to identify \( P_\pi \) with a subring of \( B \). We have \( P_\pi \subset B^+ \).
We define the scheme
\[ X = X_{F,E} = \text{Proj} \ P_{\pi}. \]
One can show that \( X \) is independent of the choice of \( \pi \). If \( \pi' \) is another uniformizing parameter of \( E \) and if \( X' = \text{Proj} \ P_{\pi'} \), the function field of \( X' \) (viewed as a subfield of the fraction field of \( B \)) is the function field \( K \) of \( X \) and the set of closed points of \( X' \) (viewed as a subset of the set of normalized discrete valuations on \( K \)) is the set of closed points of \( X \).

On the other hand, the line bundles
\[ \mathcal{O}_X(d)_{\pi} = \bigoplus_{n \in \mathbb{Z}} P_{\pi,n+d} \]
(with the convention that \( P_{\pi,m} = 0 \) for \( m < 0 \)) depend on the choice of \( \pi \).

We have
\[ P_{\pi,0} = \{ u \in B \mid \varphi(u) = u \} = E. \]

4.2. The Lubin-Tate formal group. Set
\[ \ell_{x}(X) = \sum_{n=0}^{+\infty} X^{q_{n}} \pi^{n} \in E[[X]] \]
and \( \Phi_{x}(X,Y) \in E[[X,Y]] \) the unique formal power series \( \equiv X + Y \) (mod \( (X,Y)^2 \)) such that
\[ \ell_{x}((\Phi_{x}(X,Y))) = \ell_{x}(X) + \ell_{x}(Y). \]
Then, \( \Phi_{x}(X,Y) \in \mathcal{O}_{E}[[X,Y]] \) and defines a one parameter formal group law over \( \mathcal{O}_{E} \) which is a Lubin-Tate formal group over \( \mathcal{O}_{E} \) associated to the uniformizing parameter \( \pi \) ([LT65], [Ser67], §3).

For any linearly topologized complete \( \mathcal{O}_{E} \)-algebra \( \Lambda \), we may consider the topological \( \mathcal{O}_{E} \)-module \( \Phi_{x}(\Lambda) \): The underlying topological space is the topological space underlying the ideal of elements of \( \Lambda \) which are topologically nilpotent, with the addition \( (x,y) \mapsto \Phi_{x}(x,y) \) and the multiplication by \( \alpha \in \mathcal{O}_{E} \) given by \( x \mapsto \Phi_{x}(\alpha x) \).

Let \( C \) be an algebraically closed field containing \( E \), complete for an absolute value extending the given absolute value on \( E \). We may consider the Tate module
\[ T_{C}(\Phi_{x}) = \mathcal{L}_{\mathcal{O}_{E}}(E/\mathcal{O}_{E}, \Phi_{x}(\mathcal{O}_{C})). \]
This is a free-\( \mathcal{O}_{E} \)-module of rank one. If we denote by \( \Phi_{x}(\mathcal{O}_{E}) \) the inductive limit (or the union) of the \( \Phi_{x}(\mathcal{O}_{E}) \), for \( E' \) varying through the finite extensions of \( E \) contained in \( C \), we have also \( T_{C}(\Phi) = \mathcal{L}_{\mathcal{O}_{E}}(E/\mathcal{O}_{E}, \Phi_{x}(\mathcal{O}_{E})). \)

If \( V_{C}(\Phi_{x}) \) is the one dimensional \( E \)-vector space \( E \otimes_{\mathcal{O}_{E}} T_{C}(\Phi_{x}) \), we have a short exact sequence
\[ 0 \to V_{C}(\Phi_{x}) \to \mathcal{L}_{\mathcal{O}_{E}}(E, \Phi_{x}(\mathcal{O}_{C})) \to C \to 0 \]
where the map \( \mathcal{L}_{\mathcal{O}_{E}}(E, \Phi_{x}(\mathcal{O}_{C})) \to C \) is \( f \mapsto \ell_{x}(f(1)) \).

The perfectness of \( \mathcal{O}_{F} \) implies that multiplication by \( \pi \) on the \( \mathcal{O}_{E} \)-module \( \Phi_{x}(\mathcal{O}_{F}) \) is bijective, so \( \Phi_{x}(\mathcal{O}_{F}) \) is an \( E \)-vector space. We see that \( \Phi_{x}(\mathcal{O}_{F}) \) depends
only on the special fiber $\Phi_{\pi,k_E}$ of $\Phi_{\pi}$ (a formal $\mathcal{O}_E$-module over the residue field $k_E$ of $\mathcal{O}_E$).

**Proposition 4.2.1.** For any $x$ in the maximal ideal $\mathfrak{m}_F$ of $\mathcal{O}_F$, the series
\[ \sum_{n \in \mathbb{Z}} \pi^{-n} [x^n] \]
converges in $B$ and its sum $L_\pi(x)$ belongs to $P_{\pi,1}$. The map
\[ L_\pi : \Phi_{\pi}(\mathcal{O}_F) \to P_{\pi,1} \]
so defined is an isomorphism of topological $E$-vector spaces.

**Remark.** This construction can be generalized: For $d \in \mathbb{N}$, one may interpret $P_d$ as being “the sections over $\mathcal{O}_F$ of an $E$-sheaf $S_{d,E}$ for the syntomic topology over $k_E$”.

In the rest of the section 4, we assume $F$ algebraically closed.

The automorphism $\varphi$ generates a torsion free cyclic group $\varphi^\mathbb{Z}$ of automorphisms of $B$. This group acts also on $|Y|$ and on $\text{Div}(Y) = \text{Div}(Y_{[0,1]})$. If $\lambda, \lambda'$ are non-zero elements of $m_F$ such that $\pi - [\lambda]$ and $\pi - [\lambda']$ have the same image in $|Y|$, this implies that $[\lambda] = [\lambda']$. If $\pi - [\lambda]$ is a lifting of $y \in |Y|$ and $n \in \mathbb{Z}$ then $\pi - [\lambda^n]$ is a lifting of $\varphi^n(y)$, so if $y \in |Y|$ then the $\varphi^n(y)$’s for $n \in \mathbb{Z}$ are all distinct.

This implies that it is possible to choose for each $y \in |Y|$ an element $\lambda_y \in m_F$ such that $\pi - [\lambda_y]$ is a lifting of $y$ and, for all $y$,
\[ \lambda_{\varphi(y)} = (\lambda_y)^n. \]

We make such a choice once and for all. If $y \in |Y|$, the field
\[ L_y = B^+/([\pi - [\lambda_y]]) = B^+/([\pi - [\lambda_y]]) = B/([\pi - [\lambda_y]]) \]
is algebraically closed. The multiplicative map $\mathcal{O}_F \to \mathcal{O}_{L_y}$ sending $a$ to $\theta_y([a])$ induces, by passing to the quotients, an isomorphism of rings
\[ \mathcal{O}_F/\mathcal{O}_{L_y} \to \mathcal{O}_{L_y}/\mathcal{O}_{L_y}. \]
Moreover, $\varphi$ induces a canonical isomorphism $L_y \to L_{\varphi(y)}$.

For any linearly topologized complete $\mathcal{O}_E$-algebra $\Lambda$, we denote $\mathcal{V}_{E,\pi}(\Lambda)$ the $E$-vector space $\mathcal{L}_{\mathcal{O}_E(E,\Phi_{\pi}(\Lambda))}$.

**Proposition 4.2.2.** Let $y \in |Y|$. The natural maps
\[ \mathcal{V}_{E,\pi}(\mathcal{O}_C) \to \mathcal{V}_{E,\pi}(\mathcal{O}_{L_y}/\mathcal{O}_{L_y}) \to \mathcal{V}_{E,\pi}(\mathcal{O}_{L_y}/\mathcal{O}_{L_y}) \to \mathcal{V}_{E,\pi}(\mathcal{O}_F) \to \Phi_{\pi}(\mathcal{O}_F) \to P_{\pi,1} \]
are all isomorphisms.

**ii)** We have a commutative diagram
\[
\begin{array}{cccccc}
0 & \to & V_C(\Phi_{\pi}) & \to & V_{E,\pi}(\mathcal{O}_C) & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
0 & \to & P_{\pi,1} \cap \ker \theta_y & \to & P_{\pi,1} & \to & C & \to & 0
\end{array}
\]
where the lines are exact and the vertical arrows are isomorphisms.

**Remark.** There is an explicit way to construct a generator $t$ of $P_{\pi,1} \cap \ker \theta_y$:
From the fact that $F$ is algebraically closed, one deduces easily that one can find $t_+ \in A$ not divisible by $\pi$ such that
\[ \varphi(t_+) = (\pi - [\lambda_y])t_+. \]
On the other hand the infinite product

\[ t_\omega = \prod_{n=0}^{+\infty} \left( 1 - \frac{|\lambda^n_y|}{\pi} \right) \]

converges in \( B^\times \). We may take \( t = t_- t_+ \).

### 4.3. Divisors of \( X \).

Let \( \text{Div}(Y)_{\varphi=1} \) the subgroup of \( \text{Div}(Y) \) consisting of the divisors \( D \) such that \( \varphi(D) = D \) and \( \text{Div}^+(Y)_{\varphi=1} \) the submonoid of \( \text{Div}^+(Y) \) consisting of effective divisors such that \( \varphi(D) = D \).

If \( D = \sum_{y \in Y} n_y y \in \text{Div}(Y) \) we have \( \varphi(D) = \sum_{y \in Y} n_y |\varphi(y)| \), therefore \( D \in \text{Div}(X) \) if and only if \( n_y = n_{\varphi(y)} \) for all \( y \).

Choose \( \rho \in [0,1[. \) As \( \rho^i, \rho^j \subset [\rho^j, \rho] \), we have \( n_y = 0 \) for almost all \( y \) such that \( \rho^i < ||y|| \leq \rho \). On the other hand, for any \( y \in Y \), there is a unique \( n \in \mathbb{Z} \) such that \( \rho^i < ||\varphi^n(y)|| \leq \rho \). Therefore:

**Proposition 4.3.1.** For any \( y \in Y \), set \( \delta(y) = \sum_{n \in \mathbb{Z}} |\varphi^n(y)| \) \( \in \text{Div}(Y)_{\varphi=1} \) and

\[ \Delta = \{ D \in \text{Div}(Y) \mid \text{there exists } y \in Y \text{ such that } D = \delta(y) \} \]

Then \( \text{Div}(Y)_{\varphi=1} \) \( \text{(resp. Div}^+(Y)_{\varphi=1} \) is a free abelian group \( \text{(resp. monoid) and the elements of } \Delta \text{ form a basis.} \)

**Proposition 4.3.2.**

i) Let \( y \in Y \) and \( t \) a generator of \( E_y = P_{x,1} \cap m_y \). Then

\[ \text{div}(t) = \delta(y) \]

ii) Let \( d \in \mathbb{N}_{>0} \) and \( u \in P_{x,d} \) non-zero. There exists \( t_1, t_2, \ldots, t_d \in P_{x,1} \) such that

\[ u = t_1 t_2 \ldots t_d \]

Moreover, if \( t'_1, t'_2, \ldots, t'_d \in P_{x,1} \) are such that \( u = t'_1 t'_2 \ldots t'_d \), there exists \( \sigma \in \mathcal{G}_d \) and \( \lambda_1, \lambda_2, \ldots, \lambda_d \in E^* \) such that \( t'_i = \lambda_i t_{\sigma(i)} \) for all \( i \).

This proposition is an easy consequence of what we already know: (i) is formal. To prove (ii), we observe that the ideal generated by \( u \) is fixed by \( \varphi^n \) for all \( n \in \mathbb{Z} \), hence \( \text{div}(u) \in \text{Div}^+(Y)_{\varphi=1} \). Therefore we can write

\[ \text{div}(u) = D_1 + D_2 + \ldots + D_r \]

with \( D_i \in \Delta \). If \( D_i = \delta(y_i) \), if \( m_y \) is the maximal ideal of \( B \) corresponding to \( y_i \), and if \( t_i \) is a generator of \( P_{x,1} \cap m_y \), then we must have

\[ u = \lambda t_1 t_2 \ldots t_r \]

with \( \lambda \in B^* \). Therefore, we must have \( r = d \) and \( \varphi(\lambda) = \lambda \), hence \( \lambda \in E^* \). The assertion follows.

An easy consequence of this proposition is the following result:

**Theorem 4.1.** Let \( [X] \) the set of closed points of \( X \) and set \( \text{deg}(x) = 1 \) for all \( x \in [X] \). Then \( X \) is a complete curve whose field of definition is \( E \). Moreover:

i) Let \( D \in \Delta, t \in P_{x,1} \) non-zero such that \( \text{div}(t) = D, y \in [Y] \) such that \( D = \delta(y) \) and \( L_D = L_y \). Then

a) the homogeneous ideal of \( P_x \) generated by \( t \) defines a closed point \( x_D \) of \( X \) whose local ring is a discrete valuation ring and residue field is \( L_D \),
b) the complement of $x_D$ in $X$ is an affine scheme which is the spectrum of a principal domain.

ii) The map $D \mapsto x_D$ is a bijection $\Delta \to |X|$ inducing canonical isomorphisms
$$\text{Div}(Y)_{\varphi=1} \to \text{Div}(X) \text{ and } \text{Div}^+(Y)_{\varphi=1} \to \text{Div}^+(X).$$

4.4. Vector bundles. For each $d \in \mathbb{Z}$, $\mathcal{O}_X(d)$ is a line bundle of degree $d$.

Proposition 4.3.2 implies trivially:

Proposition 4.4.1. We have

$$\text{Pic}^0(X) = 0,$$

i.e., for any $d \in \mathbb{Z}$, a line bundle $\mathcal{L}$ is of degree $d$ if and only if $\mathcal{L} \simeq \mathcal{O}_X(d)_\varphi$.

In particular, if $\pi'$ is any other uniformizing parameter, $\mathcal{O}_X(1)_{\pi'}$ is isomorphic (not canonically) to $\mathcal{O}_X(1)_{\pi_3}$.

Let $h$ be a positive integer. We may consider

$$X_h = \text{Proj} \bigoplus_{d \in \mathbb{N}} P_{h,\pi,d}$$

with $P_{h,\pi,d} = \{ \varphi^h(u) = \pi^d u \}$.

If $E_h$ denotes the unramified extension of $E$ whose residue field is the unique extension of degree $h$ of the residue field $k_E$ of $E$ which is contained in $F$, we see that $X_h = X_{F,E,h}$. It is a complete regular curve whose field of definition is $E_h$.

If $x \in P_{\pi,d}$ then $x \in P_{h,\pi,dh}$. It is easy to see that the induced map

$$\oplus P_{\pi,d} \to \oplus P_{h,\pi,d}$$

induces a morphism

$$\nu_h : X_h \to X$$

which is a cyclic cover of degree $h$ identifying $X_{F,h}$ with $X \times_{\text{Spec } E} \text{ Spec } E_h$.

For each $\lambda \in \mathbb{Q}$, if $\lambda = d/h$, with $d,h \in \mathbb{Z}$ relatively prime and $h > 0$, we set

$$\mathcal{O}_X(\lambda)_\pi = (\nu_h)_*(\mathcal{O}_{X_{F,h}}(d)_\pi).$$

This is a vector bundle over $X$ of rank $h$ and degree $d$, hence of slope $\lambda$.

Theorem 4.2. For any non-zero coherent $\mathcal{O}_X$-module $\mathcal{F}$, the Harder-Narasimhan filtration on $\mathcal{F}$ splits (non canonically). Moreover, if $\lambda \in \mathbb{Q}$, then $\mathcal{F}$ is stable (resp. semistable) of slope $\lambda$ if and only if $\mathcal{F} \simeq \mathcal{O}_X(\lambda)_\pi$ (resp. there is an integer $n > 0$ such that $\mathcal{F} \simeq \mathcal{O}_X(\lambda)_\pi^{\oplus n}$).

Corollary 4.4.1. The functor

$$\text{Pic}^0(X) = 0,$$

is a quasi-inverse.

The proof of the theorem is easily reduced to the proof of the corollary. By dèvissage, one sees that it is enough to prove the following statement:

---

When $F$ is not algebraically closed, this result remains true if and only if the residue field $k_F$ of $F$ is algebraically closed.
LEMME 4.2.1. Let $h$ be a positive integer and $\mathcal{F}$ be a vector bundle extension of $\mathcal{O}_X(1)$ by $\mathcal{O}_X(-1/h)$. Then

$$H^0(X, \mathcal{F}) \neq 0.$$ 

This lemma can be deduced by elementary manipulations on modifications of vector bundles from:

PROPPOSITION 4.4.2. Let $h$ be a positive integer and

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{E} \to 0$$

a short exact sequence of coherent $\mathcal{O}_X$-modules, with $\mathcal{E}$ torsion of length 1. Then:

i) If $\mathcal{F} \simeq \mathcal{O}_X(1/h)$, then $\mathcal{F}' \simeq \mathcal{O}_X^h$.

ii) If $\mathcal{F}' \simeq \mathcal{O}_X^h$, then $\mathcal{F} \simeq \mathcal{O}_X(1/r) \oplus \mathcal{O}_X^{h-r}$ for some $r$ with $1 \leq r \leq h$.

Let $C$ be the residue field of $X$ at the closed point which is the support of $\mathcal{E}$. This is an algebraically closed extension of $\mathcal{E}$, complete with respect to an absolute value extending the given absolute value on $\mathcal{E}$. This proposition can be translated:

i) in terms of Banach-Colmez spaces over $C$, i.e. the “Espaces de Banach de dimension finie” introduced by Colmez [Col02],

ii) or in terms of free $B$-modules equipped with a $\varphi$-semi-linear automorphism,

iii) or in terms of Barsotti-Tate groups over $O_C$.

This leads to three different proofs of the proposition which becomes a consequence of the work of Colmez (loc. cit.) or of Kedlaya ([Ke05], [Ke08]) or of a result of Laflaille ([Laf79], also proved in [GH94]) for the first part and of Drinfel’d ([Dr76], also proved in [Laf85]) for the second part.

A consequence of the previous theorem is that the geometric étale $\pi_1$ of the curve $X$ is trivial. More precisely:

PROPPOSITION 4.4.3. Let $X' \to X$ be a finite étale morphism and $E' = H^0(X', \mathcal{O}_{X'})$. The natural morphism

$$X' \to X \times_{\text{Spec} E} \text{Spec} E'$$

is an isomorphism.

4.5. The topology on $\mathcal{O}_X$. The multiplicative norms $| \cdot |_\rho$ for $0 < \rho < 1$ extend to the fraction field of $B$. For each open subset $U$ of $X$, we endow the ring $\Gamma(U, \mathcal{O}_X) \subset \text{Frac}(B)$ with the topology defined by the restriction of this family of norms. The transition maps

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$$

for $V \subset U$ open is obviously continuous. This endows $\mathcal{O}_X$ with a natural structure of sheaf of locally convex $E$-algebras which plays an important role in the study of $O_X$-representations of certain topological groups.

4A locally convex $E$-vector space is a topological $E$-vector space whose topology can be defined by a family of semi-norms.
4.6. $\mathcal{O}_X$-representations. We denote by $\mathcal{G}_F$ the group of continuous automorphisms of the field $F$ (an automorphism of the field $F$ is continuous if and only if it sends the valuation of $F$ to a strictly positive multiple of it). We equip $\mathcal{G}_F$ and its subgroups with the pointwise convergence topology, that is to say the weakest topology making the applications

$$\mathcal{G}_F \rightarrow F$$

$$g \mapsto g(x)$$

continuous when $x$ goes through $F$. If $F = \widehat{\tilde{F}}_0$ where $\tilde{F}_0$ is complete valued then $\text{Gal}(\tilde{F}_0/F_0) \subset \mathcal{G}_F$ is a closed subgroup and the induced topology on $\text{Gal}(\tilde{F}_0/F_0)$ is the usual profinite topology. By functoriality, $\mathcal{G}_F$ acts on $X$. We'll need slightly more. The action of $\mathcal{G}_F$ on $\mathcal{O}_X$ is continuous, i.e., for any open subset $U$ of $X$, the subgroup

$$\mathcal{G}_{F,U} = \{ g \in \mathcal{G}_F \mid g(U) = U \}$$

is a closed subgroup of $\mathcal{G}_F$ and the natural map

$$\mathcal{G}_{F,U} \times \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$$

is continuous.

Let $H$ be any closed subgroup of $\mathcal{G}_F$. We explained in §1.1 what is a $\mathcal{O}_X$-representations of $H$. We now use the topology on the sheaf $\mathcal{O}_X$ to put a continuity condition on these representations. More precisely if $E$ is an $\mathcal{O}_X$-representation of $H$ we require, for any open subset $U$ of $X$, the subgroup

$$\mathcal{G}_{F,U} = \{ h \in H \mid h(U) = U \}$$

is a closed subgroup of $\mathcal{G}_F$ and the natural map

$$H_U \times \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$$

(where $H_U = \{ h \in H \mid h(U) = U \}$) to be continuous.

From now on an $\mathcal{O}_X$-representation of $H$ will mean a continuous one.

5. Galois descent

5.1. The curve $X$ when $F$ may not be algebraically closed. We don’t assume anymore $F$ algebraically closed and we consider the curve

$$X = X_{F,E} = \text{Proj} \ P_\pi$$

We choose an algebraic closure $\overline{F}$ of $F$ and we set $H = \text{Gal}(\overline{F}/F)$. The absolute value $||$ of $F$ extends uniquely to $\overline{F}$ and to its completion $\overline{\tilde{F}}$ (which is algebraically closed). We set

$$\overline{B} = B_{\overline{F},E}, \quad \overline{P}_\pi = P_{\overline{F},E,\pi}, \quad \text{and} \quad \overline{X} = X_{\overline{F},E} = \text{Proj} \ \overline{P}_\pi$$

The action of $H$ on $\overline{F}$ extends uniquely to a continuous action on $\overline{\tilde{F}}$ and by functoriality to a continuous action on $\overline{B}$ and $\overline{P}_\pi$. As we may identify $H$ with a closed subgroup of the group $\mathcal{G}_{\overline{F}}$ of continuous automorphisms of the field $\overline{F}$, $H$ also acts on the curve $\overline{X}$.

Theorem 5.1. i) The natural maps

$$B \rightarrow \overline{B}^H \text{ and } P_\pi \rightarrow \overline{P}_\pi^H$$

are isomorphisms.

ii) The map $P_\pi \rightarrow \overline{P}_\pi$ induces a morphism of schemes

$$\nu : \overline{X} \rightarrow X$$
independent of the choice of \( \pi \).

iii) Define the degree of any closed point \( x \in X \) by
\[
\deg(x) = \text{cardinality of } \nu^{-1}(x) .
\]

Then \( X \) is a complete regular curve defined over \( E \).

iv) The morphism \( \nu \) induces an isomorphism
\[
\text{Div}(X) \rightarrow (\text{Div}(\tilde{X}))^H .
\]

Let \( H^* \) be the group of characters of \( H \), i.e. the group of continuous homomorphisms from \( H \) to the multiplicative group \( E^* \) of \( E \). If \( D \in \text{Div}^+(X) = (\text{Div}^+(\tilde{X}))^H \) is an effective divisor of degree \( d \in \mathbb{N} \) and if \( u \in \tilde{P}_{\pi,d} \) is a generator of the homogeneous ideal of \( \tilde{P} \) corresponding to \( D \), there is \( \xi_D \in H^* \) such that, for all \( h \in H \),
\[
h(u) = \xi_D(h)u
\]
and \( \xi_D \) is independent of the choice of \( u \). The map \( D \mapsto \xi_D \) extends uniquely to an homomorphism of groups
\[
\text{Div}(X) \rightarrow H^* .
\]

This map induces an isomorphism
\[
\text{Pic}^0(X) \rightarrow H^* .
\]

More precisely,

**Proposition 5.1.1.** Let \( K = \mathcal{O}_{X,q} \) the function field of \( X \). The sequence
\[
0 \rightarrow E^* \rightarrow K^* \rightarrow \text{Div}(X) \rightarrow \mathbb{Z} \times H^* \rightarrow 0 ,
\]
where \( \text{Div}(X) \rightarrow \mathbb{Z} \times H^* \) is the map sending \( D \) to \( (\deg(D), \xi_D) \), is exact.

Moreover, for all \( \xi_0 \in H^* \), there exists an infinite set of effective divisors \( D \) of degree 1 such that \( \xi_D = \xi_0 \).

If \( \mathcal{F} \) is a coherent \( \mathcal{O}_X \)-module (resp. a vector bundle over \( X \)), then \( \nu^* \mathcal{F} \) may be viewed as an \( \mathcal{O}_{\tilde{X}} \)-representation of \( H \) (resp. an \( H \) equivariant vector bundle over \( \tilde{X} \)).

Conversely, if \( \mathcal{E} \) is an \( \mathcal{O}_{\tilde{X}} \)-representation of \( H \), we define the \( \mathcal{O}_X \)-module \( \mathcal{E}^H \) by setting, for all open subset \( U \) of \( X \)
\[
\Gamma(U, \mathcal{E}^H) = \Gamma(\nu^{-1}(U), \mathcal{E})^H
\]
(and obvious restriction maps).

**Theorem 5.2.** The functor
\[
\nu^* : \{ \text{coherent } \mathcal{O}_X \text{-modules} \} \rightarrow \{ \mathcal{O}_{\tilde{X}} \text{-representations of } H \}
\]
is an equivalence of tensor categories, respecting the rank, the degree and the Harder-Narasimhan filtration.

For any \( \mathcal{O}_{\tilde{X}} \)-representation \( \mathcal{E} \) of \( H \), the \( \mathcal{O}_X \)-module \( \mathcal{E}^H \) is coherent. The functor
\[
\mathcal{E} \mapsto \mathcal{E}^H
\]
is a quasi-inverse of the functor \( \mathcal{F} \mapsto \nu^* \mathcal{F} \).
5.2. The étale fundamental group. Let $F'$ be a finite extension of $F$ and $E'$ be a finite extension of $E$.

- When, the residue field $k_{E'}$ is embedded in $k_F$ we have defined the curve $X_{F',E'}$ and the natural morphism

$$X_{F,E} \longrightarrow X_{F',E} \otimes E'$$

is an isomorphism.

- Therefore, we may define in general the curve $X_{F',E'}$ by

$$X_{F',E'} = X_{F',E} \otimes E'.$$

We have

$$X_{F',E} = \text{Proj } P_{F',E}$$

and the obvious map $P_{F',E} \rightarrow P_{F',E'}$ induces a morphism

$$X_{F',E} \rightarrow X$$

which is a finite étale cover of $X$ of degree $[F' : F]$, independent of the choice of $\pi$.

Therefore

$$X_{F',E} \rightarrow X$$

is a finite étale cover of $X$ of degree $[F' : F][E' : E]$.

Choose a closed point $\bar{x} = \text{Spec } C$ of $\bar{X}$. Then $C$ is algebraically closed and we denote by $\pi$ the geometric point of $X$

$$\text{Spec } C \rightarrow \bar{X} \rightarrow X.$$  

Let $I$ the set of pairs $(F',E')$ with $F'$ be a finite Galois extension of $F$ contained in the field $F(C)$ introduced in §2.4 and $E'$ a finite Galois extension of $E$ contained in $C$.

The inclusion $F' \rightarrow F(C)$ induces an extension of the morphism

$$\pi : \text{Spec } C \rightarrow X$$

to a morphism of $X$-schemes

$$\text{Spec } C \rightarrow X_{F',E},$$

which, using the inclusion $E' \rightarrow C$, extends also to a morphism of $X$-schemes

$$\text{Spec } C \rightarrow X_{F',E'}.$$

**Proposition 5.2.1.** For each $(F',E') \in I$, the morphism $X_{F',E} \rightarrow X$ is a finite étale Galois cover whose Galois group is $\text{Gal}(F'/F) \times \text{Gal}(E'/E)$.

Moreover the projective system

$$(X_{F',E'} \rightarrow X)_{(F',E') \in I}$$

(with obvious transition maps) induces an isomorphism

$$\pi_1^{\text{et}}(X,\pi) \rightarrow \text{Gal}(E'/E) \times \text{Gal}(\overline{F}/F),$$

where $E^s$ (resp. $\overline{F}$) denote the separable closure of $E$ in $C$ (resp. of $F$ in $F(C)$).

In particular, the geometric étale $\pi_1$ of $X$ may be identified with $\text{Gal}(\overline{F}/F)$. 
6. de Rham $G_K$-equivariant vector bundles

In this section, $K$ is a field of characteristic 0 which is the fraction field of a complete discrete valuation ring $\mathcal{O}_K$ whose residue field $k$ is perfect of characteristic $p > 0$. We choose an algebraic closure $\overline{K}$ of $K$ and we set $G_K = \text{Gal}(\overline{K}/K)$. We denote by $C$ the completion of $\overline{K}$. This is an algebraically closed field, therefore it is a strictly $p$-perfect field and the field $F = F(C)$ is algebraically closed.

6.1. The curve $X = X_{F(C), \mathbb{Q}_p}$. We consider the curve $X = X_{F, \mathbb{Q}_p}$.

We set $B = B_{F, \mathbb{Q}_p}$ and $B^+ = B^+_{F, \mathbb{Q}_p}$.

We have $X = \text{Proj} \, P_p$ with $P_p = \bigoplus_{d \in \mathbb{N}} P_{p,d}$ and $P_{p,d} = \{ u \in B \mid \varphi(u) = p^d u \}$. The natural map $P_p \to B$ is injective, with image contained in $B^+$, and we identify $P_p$ with its image.

As $F = F(C)$, we have a canonical continuous surjective homomorphism of $\mathbb{Q}_p$-algebras $\theta : B \to C$ (the restriction of $\theta$ to $B^b$ is the map $\sum_{n > -\infty} [a_n]p^n \mapsto \sum_{n > -\infty} a_n^{(0)} p^n$).

We fix $\varpi \in F$ such that $\varpi^{(0)} = p$. Then the kernel of $\theta$ is the principal ideal generated by $p - [\varpi]$. As usual in $p$-adic Hodge theory [Fon94a], we denote $B^b_{dR}$ the completion of $B^b$ for the $(p - [\varpi])$-adic topology. This is also the completion of $B$ (or of $B^+$) for the ker $\theta$-adic topology. As $\theta$ is $G_K$-equivariant, the action of $G_K$ on $B$ extends to $B^b_{dR}$.

As usual (loc. cit.), we fix $\varepsilon \in F$ such that $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$. We set $t = \log([\varpi]) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \in B^+$. Then $t$ is a generator of the $\mathbb{Q}_p$-line $P_{p,1} \cap \ker \theta$. The homogeneous ideal of $P_p$ generated by $t$ defines a closed point $\infty$ of $X$ which is the image in $|X|$ of the maximal ideal $\ker \theta$ of $B$.

Therefore $\infty$ is fixed under $G_K$, its residue field is $C$ and the completion of the discrete valuation ring $\mathcal{O}_{X,\infty}$ is $B^+_{dR}$. We set $X_\infty = X \setminus \{ \infty \}$.

This is an affine open subset, stable under $G_K$. We see that $B_\infty := \Gamma(X_\infty, \mathcal{O}_X) = \{ \text{homogeneous elements of degree } 0 \text{ of } P_p([\frac{1}{2}] \}$ is a principal ideal domain. We set $B_{cr} = B^+([\frac{1}{2}])$.

The Frobenius $\varphi$ on $B^+$ extends uniquely to an automorphism of $B_{cr}$ and we have $B_\infty = \{ b \in B_{cr} \mid \varphi(b) = b \}$.
Remark. The ring $B^+$ is sometimes denoted $B_{crys}^+$ (e.g. [Ber02], §1 where $F = F(C)$ is denoted $\mathfrak{E}$, though $A$ is denoted $\mathfrak{A}$ and $B^h+$ is denoted $B^+$). Traditionally [FP94], one defines the ring $A_{crys}$ as the $p$-adic completion of the divided power envelop of the ring $A$ with respect to the ideal generated by $p - [\pi]$ and $B_{crys}^+ = A_{crys}[1/p]$. The inclusion of $A[1/p] = B^h+$ into $B^+$ extends by continuity to a canonical injective map from $B^+$ into $B_{crys}^+$. Hence, we may identify $B^+$ with a subring of $B_{crys}^+$ and $B^+[1/t]$ with a subring of $B_{crys} = B_{crys}^+[1/t]$. We then have
\[ B^+ = \cap_{n\in\mathbb{N}}\varphi^n(B_{crys}^+) \text{ and } B^+[\frac{1}{t}] = \cap_{n\in\mathbb{N}}\varphi^n(B_{crys}) , \]
so, we have also
\[ B_c = \{ b \in B_{crys} \mid \varphi(b) = b \} \]
and the definition of $B_c$ given here agrees with the definition of [FP94], chap.I, §3.3.

6.2. $B_c$-representations of $G_K$. Recall that a $B_c$-representation of $G_K$ is a $B_c$-module of finite type equipped with a semi-linear and continuous action of $G_K$. Those are the (continuous) $\mathcal{O}_X$-representations of $G_K$. They form an abelian category. A $G_K$-equivariant vector bundle over Spec $B_c$ is a $B_c$-representation of $G_K$ such that the underlying $B_c$-module is locally free, hence free as $B_c$ is a principal domain. It turns out that this condition is automatic:

Proposition 6.2.1. The $B_c$-module underlying any $B_c$-representation of $G_K$ is torsion free. The category of $B_c$-representations of $G_K$ is an abelian category.

Granted what we already know, the proof of this proposition is easy: The second assertion results from the first. To show the first assertion, it is enough to show, that if $V$ is a $B_c$-representation of $G_K$ such that the underlying $B_c$-module is a torsion module, then $V = 0$. We observe that the annihilator of $V$ is a non-zero ideal $\mathfrak{a}$ stable under $G_K$. Then $\mathfrak{a}$ is the product of finitely many maximal ideals. If $\mathfrak{m}$ is one of them, for all $g \in G_K$, $g(\mathfrak{m})$ must contain $\mathfrak{a}$. But the maximal ideals corresponds to the closed points of $X_c = X\setminus\{\infty\}$ and one can show that $\infty$, which is fixed under $G_K$, is the unique closed point of $X$ whose orbit under $G_K$ is finite. Therefore $\mathfrak{a} = B_c$ and $V = 0$.

Remarks. (1) This result implies that the tensor category of $B_c$-representations is a tannakian $\mathbb{Q}_p^\times$-linear category.

(2) It is easy to see that $B_c^* = \mathbb{Q}_p^*$. This implies that any continuous 1-cocycle $\alpha : G_K \to (B_c)^*$
takes its values in $\mathbb{Q}_p^*$. It means that, if $V$ is a one dimensional $B_c$-representation, the $\mathbb{Q}_p$-line generated by a basis of $V$ over $B_c$ is stable under $G_K$. In other words, any one dimensional $B_c$-representation of $G_K$ comes by scalar extension from a one dimensional $p$-adic representation of $G_K$.

6.3. Vector bundles and their cohomology. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then
- the $B_c$-module $\mathcal{F}_c = \Gamma(X_c, \mathcal{F})$
is of finite type,
- the completion $\mathcal{F}_{dR}^+$ of the fiber at $\infty$ is a $B_{dR}^+$-module of finite type,
we have a canonical isomorphism

\[ \iota_\mathcal{F} : B_{dR} \otimes B_e \mathcal{F}_e \to B_{dR} \otimes B_{dR}^+ \mathcal{F}_{dR}^+ \]

With an obvious definition for the morphisms, the triples

\[ (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F}) \]

with \( \mathcal{F}_e \) a \( B_e \)-module of finite type, \( \mathcal{F}_{dR}^+ \) a \( B_{dR}^+ \)-module of finite type and

\[ \iota_\mathcal{F} : B_{dR} \otimes B_e \mathcal{F}_e \to B_{dR} \otimes B_{dR}^+ \mathcal{F}_{dR}^+ \]

an isomorphism of \( B_{dR}^+ \)-modules form a tensor abelian category. The correspondence

\[ \mathcal{F} \mapsto (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F}) \]

just defined induces a tensor equivalence of categories. We use it to identify these two categories.

Then \( \mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F}) \) is a vector bundle if and only if \( \mathcal{F}_e \) is free over \( B_e \) and \( \mathcal{F}_{dR}^+ \) is free over \( B_{dR}^+ \). In this case, to give \( \iota_\mathcal{F} \) is the same as giving an isomorphism from \( \mathcal{F}_{dR}^+ \) onto a \( B_{dR}^+ \)-lattice of finite type generating \( B_{dR} \otimes B_e \mathcal{F}_e \) as a \( B_{dR} \) vector space.

Therefore, we may as well see a vector bundle over \( X \) as a pair

\[ (\mathcal{F}_e, \mathcal{F}_{dR}^+) \]

where \( \mathcal{F}_e \) is a free \( B_e \)-module of finite rank and \( \mathcal{F}_{dR}^+ \) is a \( B_{dR}^+ \)-lattice in \( \mathcal{F}_{dR} = B_{dR} \otimes B_e \mathcal{F}_e \).

The cohomology of \( \mathcal{F} \) is easy to compute: we have an exact sequence

\[ 0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_e \oplus \mathcal{F}_{dR}^+ \to \mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to 0 \]

where the middle map is \( (b, b') \mapsto b - b' \). In the special case of \( \mathcal{O}_X \), we have \( H^0(X, \mathcal{O}_X) = \mathbb{Q}_p \) and \( H^1(X, \mathcal{O}_X) = 0 \), giving rise to the “fundamental exact sequence of \( p \)-adic Hodge theory”

\[ 0 \to \mathbb{Q}_p \to B_e \oplus B_{dR}^+ \to B_{dR} \to 0 \].

6.4. \( G_K \)-equivariant vector bundles. As \( \infty \) is stable under \( G_K \), we see that:

- We may identify the abelian tensor category of \( \mathcal{O}_X \)-representations of \( G_K \) with the category of triples

\[ (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F}) \]

where

i) \( \mathcal{F}_e \) is a \( B_e \)-representation of \( G_K \),

ii) \( \mathcal{F}_{dR}^+ \) is a \( B_{dR}^+ \)-representation of \( G_K \),

iii) \( \iota_\mathcal{F} : B_{dR} \otimes B_e \mathcal{F}_e \to B_{dR} \otimes B_{dR}^+ \mathcal{F}_{dR}^+ \)

is a \( G_K \)-equivariant isomorphism of \( B_{dR} \) vector spaces.

- We may identify the category of \( G_K \)-equivariant vector bundles over \( X \) to the category of pairs

\[ (\mathcal{F}_e, \mathcal{F}_{dR}^+) \]

where

i) \( \mathcal{F}_e \) is a \( B_e \)-representation of \( G_K \),

ii) \( \mathcal{F}_{dR}^+ \) is a \( G_K \)-stable \( B_{dR}^+ \)-lattice in \( \mathcal{F}_{dR} = B_{dR} \otimes B_e \mathcal{F}_e \).

The category of such pairs has already been considered by Berger [Ber08].
Remark. Let $\mathcal{F}$ be an $O_X$-representation of $G_K$. The fact that $\infty$ is the only closed point of $X$ whose orbit under $G_K$ is finite implies that the torsion of $\mathcal{F}$, if any, is concentrated at $\infty$. If $\mathcal{F}$ is a vector bundle, i.e. is torsion free and if $\mathcal{G}$ is a $G_K$-equivariant modification of $\mathcal{F}$ (i.e. $\mathcal{F}$ and $\mathcal{G}$ have the same generic fiber), we have $\mathcal{G}_e = \mathcal{F}_e$ though $G_{dR}^e$ may be any $G_K$-stable $B_{dR}^+$-lattice of $\mathcal{F}_{dR}$.

6.5. The hierarchy of $O_X$-representations. Let $B^?$ be any topological ring equipped with a continuous action of $G_K$. We say that a $B^?$-representation $V$ of $G_K$ is trivial if the natural map

$$B^? \otimes_{(B^?)^{G_K}} V^{G_K} \rightarrow V$$

is an isomorphism.

We introduce the ring

$$B_{lcr} = B_{cr}[\log([\varpi])]$$

of polynomials in the indeterminate $\log([\varpi])$ with coefficients in $B_{cr}$.

Consider the continuous maps

$$\chi : G_K \rightarrow \mathbb{Z}_p^* \quad \text{and} \quad \eta : G_K \rightarrow \mathbb{Z}_p$$

such that, for all $g \in G_K$,

$$g(t) = \chi(g)t \quad \text{and} \quad g(\varpi) = \varpi^{\eta(g)}.$$

The action of $G_K$ on $B^+$ extends to $B_{lcr}$ by setting, for all $g \in G_K$,

$$g(\frac{1}{t}) = \frac{1}{\chi(g)t} \quad \text{and} \quad g(\log([\varpi])) = \log([\varpi]) + \eta(g)t.$$

We say that a $B_e$-representation $V$ is de Rham (resp. log-crystalline, resp. crystalline) if the representation $B_{dR} \otimes_{B_e} V$ (resp. $B_{lcr} \otimes_{B_e} V$, resp. $B_{cr} \otimes_{B_e} V$) is trivial. We say that $V$ is potentially log-crystalline if there is a finite extension $L$ of $K$ contained in $\overline{K}$ such that $V$, viewed as a $B_e$-representation of $G_L = \text{Gal}(\overline{K}/L)$ is log-crystalline.

For any property which makes sense for a $B_e$-representation, we say that a $G_K$-equivariant vector bundle $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+)$ over $X_E$ satisfies this property if $\mathcal{F}_e$ does.

The following result is easy to prove:

**Proposition 6.5.1.** Let

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

a short exact sequence of $B_e$-representations or of $G_K$-equivariant vector bundles. If $\mathcal{F}$ is de Rham (resp. potentially log-crystalline, resp. log-crystalline, resp. crystalline), so are $\mathcal{F}'$ and $\mathcal{F}''$.

Therefore we may say that an $O_X$-representation of $G_K$ is de Rham (resp. potentially log-crystalline, resp. log-crystalline, resp. crystalline) if it is isomorphic to a quotient of a $G_K$-equivariant vector bundle which has this property.

It is easy to show (see more details in §6.7 below) that:

– if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two $O_X$-representations of $G_K$ having one of those four properties, then any sub-$O_X$-representation of $\mathcal{F}_1$, any quotient of $\mathcal{F}_1$, the representation $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $L_{O_X}(\mathcal{F}_1, \mathcal{F}_2)$ have the same properties,
we have the implications
\[
\text{crystalline} \implies \text{log-crystalline} \implies \text{potentially log-crystalline} \implies \text{de Rham}.
\]

It is a deep result (see §7 below) that, conversely, any de Rham $O_X$-representation is potentially log-crystalline.

6.6. Log-crystalline $B_e$-representations and $(\varphi, N)$-modules. Let $K_0 = \text{Frac} \ W(k)$. One can show that
\[
(B_{lcr})^{G_K} = K_0.
\]

If $V$ is any $B_e$-representation of $G_K$, we set
\[
D_{lcr}(V) = (B_{lcr} \otimes_{B_e} V)^{G_K}.
\]

This is a $K_0$-vector space and we denote
\[
\alpha_V : B_{lcr} \otimes_{K_0} D_{lcr}(V) \to B_{lcr} \otimes_{B_e} V
\]

the $B_{lcr}$-linear map deduced by scalar extension from the inclusion $D_{lcr}(V) \subset B_{lcr} \otimes_{B_e} V$.

By definition $V$ is log-crystalline if and only if $\alpha_V$ is bijective. It is not hard to see that $\alpha_V$ is injective, that the dimension over $K_0$ of $D_{lcr}(V)$ is $\leq$ the rank of $V$ over $B_e$ and that equality holds if and only if $\alpha_V$ is bijective (this last statement comes from the fact that any $B_e$-representation of $G_K$ of rank one comes, by scalar extension, from a one dimensional $p$-adic representation of $G_K$ and that any non-zero element $b \in B_{lcr}$ such that the $\mathbb{Q}_p$-vector space generated by $b$ is stable under $G_K$ is invertible).

The Frobenius $\varphi$ on $B_e$ extends to $B_{lcr}$ by setting
\[
\varphi(\frac{1}{x}) = \frac{1}{x^p} \quad \text{and} \quad \varphi(\log([\varpi])) = p \log([\varpi]).
\]

One denotes $N : B_{lcr} \to B_{lcr}$ the unique $B^+$-derivation such that $N(\log([\varpi])) = -1$. We get
\[
N \varphi = p \varphi N.
\]

The action of $\varphi$ and of $N$ commute with the action of $G_K$. On $K_0$ we have $N = 0$ and the Frobenius $\varphi$ is the absolute Frobenius, i.e. the unique continuous automorphism inducing $x \mapsto x^p$ on the residue field.

A $(\varphi, N)$-module over $k$ is a finite dimensional $K_0$-vector space $D$ equipped with two operators
\[
\varphi, N : D \ni \varphi, N
\]

with $\varphi$ semi-linear with respect to the action of $\varphi$ on $K_0$ and bijective, $N K_0$-linear and $N \varphi = p \varphi N$.

With an obvious definition of the morphisms, the $(\varphi, N)$-modules over $k$ form an abelian category $\text{Mod}(\varphi, N)_k$. It has an obvious structure of a tannakian $\mathbb{Q}_p$-linear category.

Let $V$ be a $B_e$-representation of $G_K$. The free $B_{lcr}$-module $B_{lcr} \otimes_{B_e} V$ is equipped with operators $\varphi$ and $N$ by setting
\[
\varphi(b \otimes v) = \varphi(b) \otimes v \quad \text{and} \quad N(b \otimes v) = Nb \otimes v \quad \text{if} \quad b \in B_{lcr} \quad \text{and} \quad v \in V.
\]
commuting with the action of $G_K$. Therefore

$$D_{lcr}(V) = (B_{lcr} \otimes_{B_e} V)^{G_K}$$

is stable under $\varphi$ and $N$ and becomes a $(\varphi, N)$-module over $k$.

If $D$ is a $(\varphi, N)$-module over $k$, then $G_K$, $\varphi$ and $N$ act on $B_{lcr} \otimes_{K_0} D$ via

$$g(b \otimes x) = g(b) \otimes x, \varphi(b \otimes x) = \varphi(b) \otimes \varphi(x) \ N(b \otimes x)$$

$$= Nb \otimes x + b \otimes Nx$$

for $g \in G_K, b \in B_{lcr}, x \in D$.

It is easy to see that the $B_e$-module

$$V_{lcr}(D) = \{ v \in B_{lcr} \otimes_{K_0} D \mid \varphi_E(v) = v \text{ and } Nv = 0 \}$$

is free of rank equal to the dimension of $D$ over $K_0$, hence is a $B_e$-representation of $G_K$.

Let $\text{Rep}_{B_{lcr}}(G_K)$ be the full sub-category of the category $\text{Rep}_{B_e}(G_K)$ of $B_e$-representations of $G_K$ whose objects are the representations which are log-crystalline. The proof of the following statement is straightforward and formal:

**Theorem 6.1.** For any $(\varphi, N)$-module $D$ over $k$, the $B_e$-representation $V_{lcr}(D)$ of $G_K$ is log-crystalline. The functor

$$V_{lcr} : \text{Mod}(\varphi, N)_k \rightarrow \text{Rep}_{B_{lcr}}(G_K)$$

is an equivalence of categories and the functor

$$V \mapsto D_{lcr}(V)$$

is a quasi-inverse.

**Remarks.** (1) It is easy to see that a $B_e$-representation $V$ of $G_K$ is crystalline if and only if it is log-crystalline and $N = 0$ on $D_{lcr}(V)$.

(2) The relation $N\varphi = p\varphi N$ implies that $N$ is nilpotent on any object of $\text{Mod}(\varphi, N)_k$ and that the kernel of $N$ is a sub-object.

In particular, the semi-simplification of a log-crystalline $B_e$-representation of $G_K$ is a crystalline $B_e$-representation of $G_K$. If $k$ is algebraically closed, the full sub-category $\text{Mod}(\varphi)_k$ of $\text{Mod}(\varphi, N)_k$ whose objects are those on which $N = 0$ is semi-simple [Man63], [62]. Therefore a $B_e$-representation of $G_K$ is crystalline if and only if it is log-crystalline and semi-simple.

(3) The category $\text{Rep}_{B_{lcr}}(G_K)$ is a tannakian subcategory of $\text{Rep}_{B_e}(G_K)$, i.e., it is stable under taking sub-objects, quotients, direct sums, tensor products, internal hom and contains the unit representation $B_e$. The functor $V_{lcr}$ is an equivalence of tannakian categories.

Let $I_K \subset G_K$ the inertia subgroup. We have $C^{I_K} = \hat{K}_{nr}$, the $p$-adic completion of the maximal unramified extension of $K$ contained in $\mathbb{K}$. The algebraic closure of $\hat{K}_{nr}$ in $C$ is a dense subfield of $C$ and $I_K$ can be identified with the Galois group of this algebraic closure over $\hat{K}_{nr}$.

If $V$ is any $B_e$-representation of $G_K$, denote by $\text{Res}_{I_K}(V)$ the $B_e$-representation of $I_K$ which is $V$ with the action of $I_K$ deduced from the inclusion of $I_K$ into $G_K$.

If $\mathbb{K}$ is the residue field of $\hat{K}_{nr}$, and $G_k = \text{Gal}(\mathbb{K}/k) = G_K/I_K$, we have

$$D_{lcr}(V) = (D_{lcr}(\text{Res}_{I_K}(V)))^{G_k}.$$
From the fact that, if \( \tilde{K}_{0, nr} \) is the fraction field of \( W(\mathfrak{k}) \) and \( D \) is a finite dimensional \( \tilde{K}_{0, nr} \)-vector space equipped with a semi-linear and continuous action of \( \mathcal{G}_\ell \), the natural map

\[
\tilde{K}_{0, nr} \otimes_{K_0} D_{\mathcal{G}_\ell} \to D
\]

is an isomorphism, we deduce:

**Proposition 6.6.1.** Let \( V \) be a \( B_\ell \)-representation of \( G_K \). Then \( V \) is log-crystalline if and only if \( \text{Res}_{I_K}(V) \) is log-crystalline.

6.7. Log-crystalline vector bundles and filtered \((\varphi, N)\)-modules. As \( B^+ \) is separated for the \( \ker \theta \)-adic topology, we may view \( B^+ \) as a subring of \( B^{+}_{dR} \) and \( B_{cr} = B^{+[1/t]} \) as a sub \( B_\ell \)-algebra of \( B_{dR} = B^{+[1/t]}_{dR} \).

Extending the \( p \)-adic logarithm by deciding that \( \log(p) = 0 \), one can identify \( B_{cr} \) with a sub-\( B_\ell \)-algebra of \( B_{dR} \) by setting

\[
\log(\varpi) = \log(\varpi)/p = -\sum_{n=1}^{\infty} \left( \frac{p - [\varpi]}{np^n} \right).
\]

If \( \mathcal{F} = (\mathcal{F}_e, \mathcal{F}^e_{dR}) \) is a \( G_K \)-equivariant vector bundle over \( X \), and if \( \mathcal{F}_{dR} = B_{dR} \otimes_{B_\ell} \mathcal{F}_e = B_{dR} \otimes_{B^{+}_{dR}} \mathcal{F}^e_{dR} \), we set

\[
\mathcal{D}_{cfr}(\mathcal{F}) = \mathcal{D}_{cfr}(\mathcal{F}^e) = (B_{cr} \otimes_{B_\ell} \mathcal{F}^e)^{G_K} \quad \text{and} \quad \mathcal{D}_{dR}(\mathcal{F}) = (\mathcal{F}_{dR})^{G_K}
\]

If \( \mathcal{F} \) is of rank \( r \), then:

i) \( \mathcal{D}_{cfr}(\mathcal{F}) \) is a \((\varphi, N)\)-module over \( K_0 \) whose dimension over \( K_0 \) is \( \leq r \) with equality if and only if \( \mathcal{F} \) is log-crystalline.

ii) The natural map

\[
B_{dR} \otimes_K \mathcal{D}_{dR}(\mathcal{F}) \to \mathcal{F}_{dR}
\]

is always injective, therefore the \( K \)-vector space \( \mathcal{D}_{dR}(\mathcal{F}) \) is of dimension \( \leq r \) with equality if and only if \( \mathcal{F} \) is de Rham.

We see also that \( \mathcal{D}_{dR}(\mathcal{F}) \) is a *filtered \( K \)-vector space*, i.e. a finite dimensional \( K \)-vector space \( \Delta \) equipped with a decreasing filtration, indexed by \( \mathbb{Z} \), by sub \( K \)-vector spaces

\[
\ldots \supset F^{i+1} \Delta \supset F^i \Delta \supset F^{i-1} \Delta \supset \ldots
\]

such that \( F^i \Delta = 0 \) for \( i \gg 0 \) and \( = \Delta \) for \( i \ll 0 \); The filtration is defined by

\[
F^i \mathcal{D}_{dR}(\mathcal{F}) = (F^i B_{dR} \otimes B^{+}_{dR} \mathcal{F}^e_{dR})^{G_K}
\]

where \( F^i B_{dR} = B^{+}_{dR} t^i \) is the fractional ideal of the discrete valuation ring \( B^{+}_{dR} \) which is the \( i \)-th power of its maximal ideal.

The inclusion \( K \otimes_{K_0} B_{cr} \to B_{dR} \) induces an injective map

\[
K \otimes_{K_0} \mathcal{D}_{cfr}(\mathcal{F}) \to \mathcal{D}_{dR}(\mathcal{F})
\]

For dimension reasons, if \( \mathcal{F} \) is log-crystalline, *this map is an isomorphism*, \( \mathcal{F} \) is de Rham and the pair \( \mathcal{D}_{cfr,K}(\mathcal{F}) \) consisting of \( \mathcal{D}_{cfr}(\mathcal{F}) \) and the filtration on \( K \otimes_{K_0} \mathcal{D}_{cfr}(\mathcal{F}) \) induced by this isomorphism is a *filtered \((\varphi, N)\)-module over \( K \) (cf. [Fon94b]), i.e. it is a finite dimensional \( K_0 \)-vector space \( D \) equipped with operators \( \varphi, N \) giving to \( D \) the structure of a \((\varphi, N)\)-module over \( k \), plus a filtration \( F \) (i.e. a structure of filtered \( K \) vector space) on the \( K \) vector space \( D_K = K \otimes_{K_0} D \).

A morphism of filtered \((\varphi, N)\)-modules over \( K \)

\[
f : (D, F) \to (D', F)
\]
is a $K_0$-linear map commuting with $\varphi$ and $N$ and such that, if $f_K : D_K \to D'_K$ is the $K$-linear map deduced from $f$ by scalar extension, then $f_K(F^iD_K) \subset F^iD'_K$ for all $i \in \mathbb{Z}$.

The category $\text{MF}_K(\varphi, N)$ of filtered $(\varphi, N)$-modules over $K$ is an additive $\mathbb{Q}_p$-linear category.

If there is no risk of confusion on the filtration, we write $D = (D, F)$ for any object $(D, F)$ of $\text{MF}_K(\varphi, N)$. The following result is now obvious:

**Theorem 6.2.** The functor

$$D_{lcr,K} : \{ \text{log-cryst. } G_K\text{-equiv. vector bundles over } X \} \to \text{MF}_K(\varphi, N)$$

is an equivalence of categories. A quasi-inverse is given by the functor $\mathcal{F}_{lcr}$ defined by

$$\mathcal{F}_{lcr}(D) = (\mathcal{V}_{lcr}(D), F^0(B_{dR} \otimes_K D_K))$$

where $\mathcal{V}_{lcr}(D)$ is the $B_e$-representation of $G_K$ associated to the $(\varphi, N)$-module over $k$ underlying $D$ and

$$F^0(B_{dR} \otimes_K D_K)) = \sum_{i \in \mathbb{Z}} F^iB_{dR} \otimes_K F^{-i}D_K \subset B_{dR} \otimes_K D_K = B_{dR} \otimes_{B_e} \mathcal{V}_{lcr}(D).$$

**Remarks.** (1) We say that a sequence of morphisms of log-crystalline $G_K$-equivariant vector bundles over $X$ is exact if the underlying sequence of $\mathcal{O}_X$-modules is exact. Similarly we say that a sequence of morphisms

$$\ldots \to (D', F) \to (D, F) \to (D'', F) \to \ldots$$

of $\text{MF}_K(\varphi, N)$ is exact if, for any $i \in \mathbb{Z}$, the induced sequence of $K$-vector spaces

$$\ldots F^iD'_K \to F^iD_K \to F^iD''_K \ldots$$

is exact.

With these definitions (or rather with the restriction of this definition to short exact sequences) these two categories are exact categories ([Qui73], §2). The functors $D_{lcr,K}$ and $\mathcal{F}_{lcr,K}$ turn exact sequences into exact sequences.

(2) The category of $G_K$-equivariant vector bundles over $X$ and the category $\text{MF}_K(\varphi, N)$ both have a natural structure of a $\mathbb{Q}_p$-linear tensor category ([Font94b], §4.3.4, for the later). The functors $\mathcal{F}_{lcr,K}$ and $\mathcal{V}_{lcr,K}$ are tensor functors.

(3) Let $\mathcal{F}$ be a log-crystalline $G_K$-equivariant vector bundle over $X$ and let $D = D_{lcr}(V)$. If $\mathcal{G}$ is a $G_K$-equivariant modification of $\mathcal{F}$, then $\mathcal{G}$ is still log-crystalline and $D_{lcr}(\mathcal{G}) = D$. Therefore, to give such a modification is the same as changing the filtration on $D_K$.

(4) We have a functor $D \to (D, F_{\text{triv}})$ from the category of $(\varphi, N)$-modules over $k$ to $\text{MF}_K(\varphi, N)$ consisting of adding to a $(\varphi, N)$-module $D$ the trivial filtration on $D_K$ (i.e. $F_{\text{triv}}^iD_K = D_K$ if $i \leq 0$ and if $i > 0$).

(5) Let $D$ be a $(\varphi, N)$-module over $k$, and choose a basis $e_1, e_2, \ldots, e_r$ of $D$ over $K_0$. If we set $\varphi(e_j) = \sum_{i=1}^r a_{ij}e_i$, the $p$-adic valuation of the determinant of the matrix of the $a_{ij}$ is independent of the choice of the basis and is denoted $t_N(D)$. It is easy to see that

$$\text{rank} \mathcal{V}_{lcr,K}(D, F_{\text{triv}}) = \dim_{K_0} D \text{ and } \deg \mathcal{V}_{lcr,K}(D, F_{\text{triv}}) = -t_N(D).$$
If now $F$ is a filtration on $D$, so that $V_{lcr,K}(D,F)$ is a modification of $V_{lcr,K}(D,F_{lcr})$, it’s easy to see that, if $t_H(D,F) = \sum_{i \in \mathbb{Z}} i \dim_K(F^i D_K / F^{i+1} D_K)$, then
\[
\text{rank } V_{lcr,K}(D,F) = \text{rank } V_{lcr,K}(D,F_{lcr})
\]
and
\[
\deg V_{lcr,K}(D,F) = \deg V_{lcr,K}(D,F_{lcr}) + t_H(D,F).
\]
This remark suggests to define the rank, the degree and the slope of a non-zero filtered $(\varphi, N)$-module $(D,F)$ over $K$ by
\[
\text{rank}(D,F) = \dim_K D, \quad \deg(D,F) = t_H(D,F) - t_N(D) \quad \text{and } \mu(D,F) = \frac{\deg(D,F)}{\text{rank}(D,F)}.
\]
Let $f : (D',F) \to (D,F)$ a morphism of $MF_K(\varphi,N)$, with $f_K : D'_K \to D_K$ the underlying $K$-linear map. We say that $f$ is strict if it is strictly compatible to the filtrations, i.e. if $f_K(F^i D'_K) = F^i D_K \cap f_K(D'_K)$ for all $i \in \mathbb{Z}$. If $f_K$ is injective, it is equivalent to saying that $f$ fits into a short exact sequence of $MF_K(\varphi,N)$
\[
0 \to (D',F) \to (D,F) \to (D'',F) \to 0.
\]
A sub-object $(D',F)$ of a filtered $(\varphi,N)$-module $(D,F)$ is a morphism $(D',F) \to (D,F)$ such that the $(\varphi,N)$-module $D'$ is a sub-object of $D$.

The strict sub-objects of an object $(D,F)$ correspond bijectively to the sub-objects of the underlying $(\varphi,N)$-module via the map
\[
D' \mapsto (D',F) \text{ with } F^i D'_K = F^i D_K \cap D'_K \text{ for all } i \in \mathbb{Z}.
\]
If $(D',F)$ is such a sub-object, the quotient $(D,F)/(D',F)$ is the cokernel of $(D',F) \to (D,F)$.

We say that a filtered $(\varphi,N)$-module $(D,F)$ is semistable if, for any non-zero sub-object $(D',F)$ of $(D,F)$, we have $\mu(D',F) \leq \mu(D,F)$. It is enough to check it for strict sub-objects.

The following assertion is entirely formal:

**Proposition 6.7.1.** i) For any non-zero filtered $(\varphi,N)$-module $D$ over $K$, there is a unique filtration (called the Harder-Narasimhan filtration) by strict sub-objects
\[
0 = D_0 \subset D_1 \subset \ldots \subset D_{i-1} \subset D_i \subset \ldots \subset D_{m-1} \subset D_m = D
\]
with each $D_i/D_{i-1}$ non-zero and semistable such that
\[
\mu(D_i/D_{i-1}) > \mu(D_{i+1}/D_i) > \ldots > \mu(D_m/D_{m-1}).
\]

ii) The functors $D_{lcr,K}$ and $V_{lcr,K}$ respect the rank, the degree, the slope and the Harder-Narasimhan filtration.

**6.8. p-adic Hodge theory.** The corollary 4.4.1 implies that we have an equivalence of tannakian categories between $p$-adic representations (i.e. $\mathbb{Q}_p$-representations) of $G_K$ and $G_K$-equivariant vector bundles over $X$ which are semistable of slope $0$:
\[
V \mapsto F(V) = \mathcal{O}_X \otimes_{\mathbb{Q}_p} V = (B_\mathbb{A} \otimes_{\mathbb{Q}_p} V, B_{dR}^+ \otimes_{\mathbb{Q}_p} V)
\]
(with $F \mapsto V(F) = H^0(X,F)$ as a quasi-inverse).

We say that $V$ is de Rham (resp. potentially log-crystalline, resp. log-crystalline, resp. crystalline if $F(V)$ has this property.
Classically one introduces [Fon94a] the ring
\[ B_{st} = B_{cris}[\log(\varpi)] . \]
If \( V \) is a \( p \)-adic representation of \( G_K \), one says that \( V \) is de Rham (resp. crystalline, resp. semistable, resp. potentially semistable) if \( B_{dR} \otimes_{\mathbb{Q}_p} V \) is trivial (resp. \( B_{cris} \otimes_{\mathbb{Q}_p} V \) is trivial, resp. \( B_{st} \otimes_{\mathbb{Q}_p} V \) is trivial, resp. there is a finite extension \( L \) of \( K \) contained in \( \overline{K} \) such that \( V \) is semistable as a \( p \)-adic representation of \( G_L \)).

The origin of this terminology lies in the facts that, if \( Z \) is any proper and smooth variety over \( K \), \( i \in \mathbb{N} \) and \( V = H^i_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_p) \), then ([Fa89], [Ts99], [Ni08])
- the \( p \)-adic representation \( V \) is de Rham and the filtered \( K \)-vector space \( D_{dR}(V) = D_{dR}(\mathcal{F}(V)) \) can be identified with \( H^i_{dR}(Z) = H^i(Z, \Omega^*_Z/K) \) equipped with the Hodge filtration,
- if there exists \( Z \) over \( \mathcal{O}_K \) proper and smooth such that \( \text{Spec } K \times_{\text{Spec } \mathcal{O}_K} Z = Z \),

then \( V \) is crystalline and \( D_{cris}(V) = D_{cr}(\mathcal{F}(V)) \) is the \( i^{th} \)-crystalline cohomology group of the special fiber of \( Z \) (equality respecting the Frobenius and compatible with the filtration via the de Rham comparison isomorphism),
- if there exists \( Z \) over \( \mathcal{O}_K \) proper and semistable such that \( \text{Spec } K \times_{\text{Spec } \mathcal{O}_K} Z = Z \),

then \( V \) is semistable and \( D_{st}(V) = D_{lcr}(\mathcal{F}(V)) \) is the \( i^{th} \)-log-crystalline cohomology group of the log special fiber of \( Z \) (equality respecting \( \varphi \) and \( N \) and compatible with the filtration via the de Rham comparison isomorphism).

It is easy to check that
- the definition given in §6.5 of a de Rham and of a crystalline \( p \)-adic representation agrees with the classical definition,
- a \( p \)-adic representation \( V \) is log-crystalline (resp. potentially log-crystalline) if and only if it is semistable (resp. potentially semistable).

We made this change of terminology to avoid confusion between the two notion of semistability (semistable model of a variety and semistable vector bundle).

As a corollary of the proposition 6.7.1, denoting \( \text{Rep}_{\mathbb{Q}_p,lcr}(G_K) \) the full subcategory of the category \( \text{Rep}_{\mathbb{Q}_p}(G_K) \) of \( p \)-adic representations of \( G_K \) whose objects are the log-crystalline ones and \( \text{MF}_{K}^0(\varphi, N) \) the full sub-category of \( \text{MF}_{K}(\varphi, N) \) whose objects are those which are semistable of slope 0, we get:

**Theorem 6.3.** For any \( p \)-adic log-crystalline representation of \( G_K \),
\[ D_{lcr,K}(V) = D_{lcr,K}(\mathcal{O}_X \otimes V) \]
is a filtered (\( \varphi, N \))-module over \( K \) which is semistable of slope 0.

The category \( \text{Rep}_{\mathbb{Q}_p,lcr}(G_K) \) is a tannakian subcategory of \( \text{Rep}_{\mathbb{Q}_p}(G_K) \) and
\[ D_{lcr,K} : \text{Rep}_{\mathbb{Q}_p,lcr}(G_K) \to \text{MF}_{K}^0(\varphi, N) \]
is an equivalence of tensor categories. The functor
\[ V_{lcr,K} : \text{MF}_{K}^0(\varphi, N) \to \text{Rep}_{\mathbb{Q}_p,lcr}(G_K) , \]
defined by

\[ V_{lcr,K}(D) = \Gamma(X, V_{lcr,K}(D)) \]

is a quasi-inverse.

This important result of p-adic Hodge theory was first proved in [CF00] where a filtered \((\varphi,N)\)-module over \(K\) is said to be weakly admissible whenever it is semistable of slope 0.

7. de Rham = potentially log-crystalline

To finish, we explain the main lines of the proof of:

**Theorem 7.1.** Any p-adic representation of \(G_K\), any \(B\)-representation of \(G_K\) or any \(G_K\)-equivariant vector bundle over \(X\) is de Rham if and only if it is potentially log-crystalline.

The case of p-adic representations is another important result of p-adic Hodge theory. The first proof was given by Berger [Ber02] relying on Crew’s conjecture first proved by André [An02] and Mebkhout [Meb02].

We know that the condition of the theorem is sufficient and it is obviously enough to show that, if \(V\) is a \(B\)-representation of \(G_K\) which is de Rham, then \(V\) is potentially log-crystalline.

We first reduce the proof to the case where \(k\) is algebraically closed: Let \(\hat{K}_{nr} \subset C\) the p-adic closure of the maximal unramified extension \(K_{nr}\) of \(K\) contained in \(\bar{K}\). Let \(\overline{K}_{nr}\) the algebraic closure of \(\hat{K}_{nr}\). Then \(\overline{K}_{nr}\) is stable under the action of the inertia subgroup \(I_K\) of \(G_K\). This gives an identification of \(I_K\) to the Galois group \(\text{Gal}(\overline{K}_{nr}/\hat{K}_{nr})\).

**Proposition 7.2.** Let \(V\) be a \(B\)-representation of \(G_K\). Then \(V\) is log-crystalline if and only if \(V\) is log-crystalline as a representation of \(I_K = \text{Gal}(\overline{K}_{nr}/\hat{K}_{nr})\).

Let \(k\) be the residue field of \(\hat{K}_{nr}\) and \(\hat{K}_{0,nr}\) the fraction field of \(W(\bar{k})\). The group \(\text{Gal}(\bar{k}/k) = G_K/I_K\) acts semi-linearly on the finite dimensional \(\hat{K}_{0,nr}\) vector space

\[ \mathcal{D}_{lcr,nr}(V) = (B_{lcr} \otimes_{B_k} V)^{I_K} \]

and we have

\[ \mathcal{D}_{lcr}(V) = (\mathcal{D}_{lcr,nr}(V))^{G_k} \]

It is well known that, if \(n\) is any positive integer, the pointed set \(H^1_{cont}(G_k, GL_n(\hat{K}_{0,nr}))\) is trivial. This implies that the natural map

\[ \hat{K}_{0,nr} \otimes_{K_0} \mathcal{D}_{lcr}(V) \to \mathcal{D}_{lcr,nr}(V) \]

is an isomorphism. Therefore \(\dim_{K_0} \mathcal{D}_{lcr}(V) = \dim_{\hat{K}_{0,nr}} \mathcal{D}_{lcr,nr}(V)\).

If \(r\) is the rank of \(V\) over \(B_e\), then \(V\) is log-crystalline as a \(B_e\)-representation of \(G_K\) (resp. \(I_K\)) if and only if \(\dim_{K_0} \mathcal{D}_{lcr}(V) = r\) (resp. \(\dim_{\hat{K}_{0,nr}} \mathcal{D}_{lcr,nr}(V) = r\)).

The proposition follows.

From now on, we assume \(k\) algebraically closed.
Let $E$ be a finite extension of $\mathbb{Q}_p$ and $\tau : E \to K$ a $\mathbb{Q}_p$-embedding. We choose a uniformizing parameter $\pi$ of $E$. For $d \in \mathbb{N}$, we consider the 1-dimensional $E$-representations of $G_K$

$$E(d)_{\tau} = \text{Sym}^d V_C(\Phi_{\pi}) \quad \text{and} \quad E(-d)_{\tau} = \text{the $E$-dual of } E(d)_{\tau},$$

where $V_C(\Phi_{\pi})$ is the 1-dimensional representation associated to the Lubin-Tate formal group $\Phi_{\pi}$ (§4.2). If we use $\tau$ to see $E$ as a closed subfield of $C$, then $V_C(\Phi_{\pi}) = E \otimes T_{\tau}(\Phi_{\pi})$ where

$$T_{\tau}(\Phi_{\pi}) = \lim_{\substack{n \to \infty}} \Phi_{\pi}(\mathcal{O}_C)_{\tau^n}$$

is the Tate module of $\Phi_{\pi}$.

We say that a $E$-representation $V$ of $G_K$ is $\tau$-ordinary if there is a decreasing filtration $(F^iV)_{i \in \mathbb{Z}}$ of $V$ by sub-$E$-vector spaces stable under $G_K$ such that $F^dV = V$ for $d \leq 0$, $F^dV = 0$ for $d > 0$, each $F^dV$ is stable under $G_K$ and $G_K$ acts trivially on $(F^dV/F^{d+1}V) \otimes_E E(-d)_{\tau}$.

If $\pi'$ is another uniformizing parameter of $E$, then $V_C(\Phi'_{\pi})$ and $V_C(\Phi_{\pi})$ are isomorphic. Therefore, the condition of being $\tau$-ordinary is independent of the choice of $\pi$.

The theorem follows from these three propositions:

**Proposition 7.3.** Any $B_{c}$-representation $V$ of $G_K$ which is potentially de Rham (i.e. de Rham as a representation of $GL$ for a suitably chosen finite extension $L$ of $K$ contained in $\mathbb{K}$) is de Rham.

**Proposition 7.4.** Let $\tau : E \to K$ be a $\mathbb{Q}_p$-embedding of a finite extension $E$ of $\mathbb{Q}_p$ into $K$. Any $E$-representation of $G_K$ which is $\tau$-ordinary is log-crystalline.

**Proposition 7.5.** Let $V$ be a $B_{c}$-representation of $G_K$ which is de Rham. There exists an integer $h_V \geq 1$ such that, for any finite extension $E$ of $\mathbb{Q}_p$ of degree divisible by $h_V$ and any embedding $\tau : E \to K$, one can find

1) a finite extension $L$ of $K$ contained in $\mathbb{K}$ and containing $\tau(E)$,
2) a $\tau$-ordinary $E$-representation $V$ of $GL = \text{Gal}(\mathbb{K}/L)$,
3) a $GL$-equivariant $B_{c} \otimes_{\mathbb{Q}_p} E$-linear bijection $B_{c} \otimes_{\mathbb{Q}_p} V \simeq E \otimes_{\mathbb{Q}_p} V$.

The field $\mathbb{K}$ is naturally embedded into $B_{dR}$ and the proposition 7.3 becomes a formal consequence of the fact that, for any positive integer $n$, the pointed set $H^1(G_K, GL_n(\mathbb{K}))$ is trivial.

The proof of the proposition 7.4 relies on some hard computation in Galois cohomology which can be done using the techniques of Herr [He98] to compute Galois cohomology by the way of the theory of $(\phi, \Gamma)$-modules [Fon90]. This computation has been done by Berger showing a much more general result : any extension of two semi-stable $E$-representations which is de Rham is semistable (unpublished, see also [Ber02], §6).

The proof of the proposition 7.5 runs as follows:

Say that a $G_K$-equivariant vector bundle $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^{+})$ is trivial at $\infty$ if it is de Rham and $\mathcal{F}_{dR}^{+} = B_{dR}^{+} \otimes_K D_{dR}(\mathcal{F})$. 


To any $B_c$-representation $\mathcal{W}$ of $G_K$ which is de Rham, setting $D_{dR}(\mathcal{W}) = (B_{dR} \otimes_{B_c} \mathcal{W})^{G_K}$, one can associate to $\mathcal{W}$ the $G_K$-equivariant vector bundle $\tilde{\mathcal{W}} = (\mathcal{W}, B_{dR}^+ \otimes_K D_{dR}(\mathcal{W}))$

which is trivial at $\infty$. The correspondence $\mathcal{W} \mapsto \tilde{\mathcal{W}}$ is a functor inducing a tensor equivalence between the category of de Rham $B_c$-representations of $G_K$ and $G_K$-equivariant vector bundles over $X$ which are trivial at $\infty$.

If $\mathcal{F}$ is any de Rham $G_K$-equivariant vector bundle over $X$, then $\tilde{\mathcal{F}}_c$ is a modification of $\mathcal{F}$ and $\mathcal{F}$ is trivial at $\infty$ if and only if $\tilde{\mathcal{F}}_c = \mathcal{F}$.

Let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \cdots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \mathcal{V}$

be the Harder-Narasimhan filtration of $\tilde{\mathcal{V}}$. By unicity of this filtration, each $\mathcal{F}_i$ is stable under $G_K$. Setting $\mathcal{V}_i = (\mathcal{F}_i)_c$, we get a decreasing filtration

$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{i-1} \subset \mathcal{V}_i \subset \cdots \subset \mathcal{V}_{m-1} \subset \mathcal{V}_m = \mathcal{V}$

by sub-$B_c$-representations of $G_K$. For $1 \leq i \leq m$, $\mathcal{F}_i$ and $\tilde{\mathcal{F}}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ are trivial at $\infty$ (we have $\mathcal{F}_i = \mathcal{V}_i$ and $\tilde{\mathcal{F}}_i = \mathcal{V}_i$, where $\mathcal{V}_i = \mathcal{V}_i / \mathcal{V}_{i-1}$).

Let $\mu_i$ be the slope of the semistable vector bundle $\tilde{\mathcal{F}}_i$ and let $h_{\mathcal{V}_i}$ be the smallest positive integer such that

$h_{\mathcal{V}_i} \mu_i \in \mathbb{Z} \quad \text{for} \quad 1 \leq i \leq m$.

Let $E$ be a finite extension of $\mathbb{Q}_p$, of degree $h$ divisible by $h_{\mathcal{V}_i}$, and let $\tau$ be a $\mathbb{Q}_p$-embedding of $E$ into $\overline{K}$ and $K'$ a finite extension of $K$ contained in $\overline{K}$ and containing $\tau(E)$. The curve $X_E = X_{E,E}$ is a cyclic étale cover of $X$ of degree $h$ equipped with an action of $G_{K'}$ and the natural morphism $\nu : X_E \rightarrow X$ is $G_{K'}$-equivariant.

Choose a uniformizing parameter $\pi$ of $E$. For each $d \in \mathbb{Z}$, the line bundle $\mathcal{O}_{X_E}(d)_{\pi}$ is equipped with an action of $G_{K'}$ and

$\mathcal{O}_X(d/h)_{\pi} = \nu_* \mathcal{O}_{X_E}(d)_{\pi}$

is a $G_{K'}$-equivariant vector bundle over $X$ which is semistable of slope $d/h$. For $1 \leq i \leq m$, the $G_{K'}$-equivariant vector bundle $\mathcal{G}_i = \nu \circ \mathcal{F}_{i/c}$ is semistable of slope 0, hence $W_i = H^0(X, \mathcal{G}_i)$ is a $p$-adic representation of $G_{K'}$ and $\mathcal{G}_i = \mathcal{O}_X \otimes_{\mathbb{Q}_p} W_i$.

On the other hand, $\mathcal{G}_i = \tilde{\mathcal{W}}_i$, where $\mathcal{W}_i$ is the de Rham $B_c$-representation of $G_{K'}$.

$\mathcal{W}_i = \mathcal{L}_{B_c}(\Gamma(X, \mathcal{O}_X(\mu_i))_c, \mathcal{V}_i)$,

hence $\mathcal{G}_i$ is trivial at $\infty$. Therefore, the natural map

$B_{dR}^+ \otimes_K (B_{dR} \otimes_{\mathbb{Q}_p} W_i)^{G_{K'}} \rightarrow B_{dR}^+ \otimes_{\mathbb{Q}_p} W_i$

is an isomorphism. A fortiori, the natural map

$C \otimes_K (C \otimes_{\mathbb{Q}_p} W_i)^{G_{K'}} \rightarrow C \otimes_{\mathbb{Q}_p} W_i$

is an isomorphism (i.e. the $p$-adic representation $W_i$ of $G_{K'}$ is Hodge-Tate, with all its Hodge-Tate weights equal to 0). A deep result of Sen [Sen73] implies that $G_{K'}$ acts on $W_i$ through a finite quotient. Therefore, one can find a finite extension $L$ of $K'$ contained in $\overline{K}$ such that $G_L$ acts trivially on each $W_i$. One easily checks
that it implies the existence of a positive integer $r_i$ and of an isomorphism of $G_L$-equivariant vector bundles

$$f_i : (\mathcal{O}_X(\mu_i) \cdot \pi)^{\otimes i} \to E \otimes_{\mathbb{Q}_p} \mathcal{F}_i.$$  

For all $d \in \mathbb{Z}$, there is a canonical isomorphism

$$\mathcal{O}_X(d/h)_\pi \simeq B_e \otimes_{\mathbb{Q}_p} E[d]_\pi$$

and therefore, for $1 \leq i \leq m$, if $\mu_i = d_i/h$, we get a $G_L$-equivariant $B_e \otimes_{\mathbb{Q}_p}$-linear bijection

$$B_e \otimes_{\mathbb{Q}_p} (E[d_i])^{\otimes i} \simeq E \otimes_{\mathbb{Q}_p} \mathcal{V}_i.$$  

In particular, this concludes the proof when $m = 1$. Assume $m \geq 2$. By induction, we may assume there is a $\tau$-ordinary representation $V'$ of $G_L$ and a $G_L$-equivariant $B_e \otimes_{\mathbb{Q}_p}$-linear bijection

$$B_e \otimes_{\mathbb{Q}_p} V' \simeq E \otimes_{\mathbb{Q}_p} \mathcal{V}_{m-1}.$$  

Set $B_{e, E} = B_e \otimes_{\mathbb{Q}_p} E$. We get an exact sequence of $B_{e, E}$-representations of $G_L$

$$0 \to B_{e, E} \otimes_{\mathbb{E}} V' \to E \otimes_{\mathbb{Q}_p} V \to B_{e, E} \otimes_{\mathbb{E}} (E[m])^{\otimes \infty} \to 0.$$  

Twisting by $E(-d_m)$, we are reduced to show, that, if we have a short exact sequence of $B_{e, E}$-representations of $G_L$

$$(*) \quad 0 \to B_{e, E} \otimes_{\mathbb{E}} W' \to W \to B_{e, E} \to 0$$  

with $W'$ a $\tau$-ordinary $\mathbb{E}$-representation of $G_L$, then $W$ comes by scalar extension from an $\mathbb{E}$-representation of $G_L$ which is an extension of $E$ by $W'$. Setting

$$B_{dR, E} = E \otimes_{\mathbb{Q}_p} B_{dR}, \quad B_{dR, E}^+ = E \otimes_{\mathbb{Q}_p} B_{dR}^+ \text{ and } \tilde{B}_{dR, E} = B_{dR, E}/B_{dR, E}^+,$$

we get from the fundamental exact sequence ($\S 6.3$), a short exact sequence

$$0 \to E \to B_{e, E} \to \tilde{B}_{dR, E} \to 0.$$  

Tensoring with $W'$, we get an exact sequence

$$0 \to W' \to B_{e, E} \otimes_{\mathbb{E}} W' \to \tilde{B}_{dR, E} \otimes_{\mathbb{E}} W' \to 0,$$

inducing an exact sequence of continuous $G_L$-cohomology

$$\ldots \to H^1_{cont}(G_L, W') \to H^1_{cont}(G_L, B_{e, E} \otimes_{\mathbb{E}} W') \to H^1_{cont}(G_L, \tilde{B}_{dR, E} \otimes_{\mathbb{E}} W') \to \ldots$$  

The short exact sequence $(*)$ defines an element $c \in H^1_{cont}(G_L, B_{e, E} \otimes_{\mathbb{E}} W')$. What we need to show is that $c$ comes from an element of $H^1_{cont}(G_L, W')$ or equivalently goes to $0$ in $H^1_{cont}(G_L, \tilde{B}_{dR, E} \otimes_{\mathbb{E}} W')$. The map

$$H^1_{cont}(G_L, B_{e, E} \otimes_{\mathbb{E}} W') \to H^1_{cont}(G_L, \tilde{B}_{dR, E} \otimes_{\mathbb{E}} W')$$

factors through $H^1_{cont}(G_L, B_{dR, E} \otimes_{\mathbb{E}} W')$ and this comes from the fact that the extension is de Rham which means that the image of $c$ is already $0$ in $H^1_{cont}(G_L, B_{dR, E} \otimes_{\mathbb{E}} W')$.

Remark. Let $\mathcal{F}$ a de Rham $G_K$-equivariant vector bundle over $X$. Choose a finite Galois extension $L$ of $K$ contained in $\overline{K}$ such that $\mathcal{F}$ is log-crystalline as a $G_L$-vector bundle. Then the $(\varphi, N)$ module over $L$  

$$\mathcal{D}_{cr, L}(\mathcal{F})$$
is equipped with an action of $G_{L/K}$ defined in an obvious way. This gives to $\mathcal{D}_{\text{der},L}(F)$ the structure of what can be called a filtered $(\varphi, N, G_{L/K})$-module over $K$. The inductive limit (in a straightforward way) of the categories of filtered $(\varphi, N, G_{L/K})$-modules over $K$, when $L$ runs through all the finite Galois extensions of $K$ contained in $\overline{K}$, is the category

$$MF_K(\varphi, N, G_K)$$

of filtered $(\varphi, N, G_K)$-modules over $K$. This is, in an obvious way, a $\mathbb{Q}_p$-linear tensor category, with an obvious definition of the rank, the degree and the slope of any non-zero object. The Harder-Narasimhan filtration of any object can be defined.

We see that the $\mathcal{D}_{\text{der},L}$'s induce a tensor equivalence of categories

$$\text{de Rham } G_K\text{-equivariant vector bundles over } X \iff \text{Mod}_K(\varphi, N, G)$$

respecting rank, degree, slopes and the Harder-Narasimhan filtration.

The restriction of this equivalence to semistable vector bundles of slope 0 leads to the "classical" equivalence ([Fon94b], [Ber02]) of categories between de Rham $p$-adic representations of $G_K$ and "weakly admissible" (or semistable of slope 0) filtered $(\varphi, N, G_K)$-modules over $K$.

References


