An introduction to the geometry of Lubin-Tate spaces

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1 One dimensional formal group laws

1.1 Definitions

Let R be a ring.

Définition 1. *A one dimensional formal group law over R is a $F(X,Y) \in R[[X,Y]]$ s.t.*
- $F(X,Y) = F(Y,X)$ *(commutativity)*
- $F(X,0) = X$ *and* $F(0,Y) = Y$ *(the section $T = 0$ is the unit section) (in particular* $F(X,Y) \equiv X + Y$ *mod deg* 2*)*
- $F(F(X,Y),Z) = F(X,F(Y,Z))$ *(associativity)*

Lemme 1. *For any one dimensional formal group law F over R there exists a unique series $f \in R[[T]]$ such that $f(0) = 0$ and $F(f(T),T) = 0$.*

Démonstration. This is an easy induction/approximation argument by constructing and verifying its unicity modulo T^k for all k starting with $k = 1$. □

This series f will be denote $[-1]_F$, it is the "formal inversion" on F.

We will often denote $F(X,Y) = X \underset{F}{+} Y$. More generally if A is an R-algebra complete with respect to the ideal $\mathfrak{m}$ and $a, b \in \mathfrak{m}$ we can define $a \underset{F}{+} b$ and thus consider $(\mathfrak{m}, \underset{F}{+})$ that is an abelian group.

Définition 2. *Let F and G be two formal group laws. We define*

$$Hom(F,G) = \{\ f \in TR[[T]] \mid f(X \underset{F}{+} Y) = f(X) \underset{G}{+} f(Y)\ \}$$

And thus we have a category structure on formal group laws over R.

If $R \longrightarrow R'$ is a ring morphism there is an evident base change functor from the category of one dimensional formal groups laws over R to the one over R'.

Exemple 1.
- *The additive groupe law $\widehat{\mathbb{G}}_a$ associated to $F(X,Y) = X + Y$*
- *The multiplicative one $\widehat{\mathbb{G}}_m$ associated to $F(X,Y) = XY + X + Y$ (this is $(X+1)(Y+1) - 1$, for formal group laws 0 is the neutral element thus on has to translate by 1)*
- *If R is a $\mathbb{Q}$-algebra there is an isomorphism of formal group laws* $\log(1+T) : \widehat{\mathbb{G}}_m \xrightarrow{\sim} \widehat{\mathbb{G}}_a$ *where* $\log(1+T) = \sum_{k \geq 1} (-1)^{k-1} \frac{T^k}{k}$

Set $\mathrm{Lie}\, F = R$. We have a linear map :

$$\mathrm{Hom}(F,G) \longrightarrow \mathrm{Hom}(\mathrm{Lie}\, F, \mathrm{Lie}\, G)$$

that associates to f the multiplication by $f'(0)$. Moreover $f \in \mathrm{Hom}(F,G)$ is an isomorphism iff $f'(0) \in R^\times$.

For $n \in \mathbb{Z}$ set $[n]_F \in \mathrm{End}(F)$ be the multiplication by n via $\mathbb{Z} \to \mathrm{End}(F)$. Thus if $n > 0$

$$[n]_F = \underbrace{T \underset{F}{+} \ldots \underset{F}{+} T}_{n-\text{times}}$$

and if $n < 0$

$$[n]_F = \underbrace{[-1]_F(T) \underset{F}{+} \ldots \underset{F}{+} [-1]_F(T)}_{(-n)\,-\text{times}}$$

Then

$$[n]'_F(0) = n$$

and thus if R is a $\mathbb{Z}_{(p)}$-algebra then $\mathrm{End}(F)$ is a $\mathbb{Z}_{(p)}$-algebra. Moreover one verifies that if R is p-adicaly complete then $\mathrm{End}(R)$ is naturaly a $\mathbb{Z}_p$-algebra. The series $[p]_F$ will play an important role in the sequel when R will be p-adically complete.

Définition 3. *An invariant differential form on F is an expression $\omega = f(T)dT$ such that*

$$F_*\omega = f(X)dX + f(Y)dY$$

where

$$F_*\omega = f(F(X,Y))\frac{\partial F}{\partial X}(X,Y)dX + f(F(X,Y))\frac{\partial F}{\partial Y}(X,Y)dY$$

We set ω_F to be the R-module of invariant differential forms on F.

Définition 4. *A continous translation invariant derivation on F is a T-adicaly continuous R-derivation $\partial : R[[T]] \longrightarrow R[[T]]$ such that for any $g \in R[[T]]$*

$$\partial_X(g(X \underset{F}{+} Y)) = (\partial f)(X \underset{F}{+} Y)$$

Being continuous such a derivation ∂ is completly determined by ∂T and is of the form $\partial = f(T)\frac{d}{dT}$ for some $f \in R[[T]]$.

Proposition 1. *The morphism*

$$\begin{array}{ccc} \omega_F & \longrightarrow & R \\ f(T)dT & \longmapsto & f(0) \end{array}$$

is an isomorphism and thus ω_F is free of rank 1. Idem for ω_F^* = translation invariant T-adically continuous derivations on $R[[T]]$ which is identified with $Lie(F) = R$ via

$$\begin{array}{ccc} \omega_F^* & \longrightarrow & R = LieF \\ \partial & \longmapsto & (\partial T)_{|T=0} \end{array}$$

Démonstration. One has to solve the system of equations

$$\begin{cases} f(X) = f(F(X,Y))\frac{\partial F}{\partial X}(X,Y) \\ f(Y) = f(F(X,Y))\frac{\partial F}{\partial Y}(X,Y) \end{cases}$$

which by $f(X,Y) = f(Y,X)$ is reduced to the first equation. Putting $Y = 0$ one finds that necessarily

$$f(Y)\frac{\partial F}{\partial X}(0,Y) = f(0)$$

But the series $g(Y) = \frac{\partial F}{\partial X}(0,Y)$ verifies $g(0) = 1$ and is thus invertible. Thus any solution of the system is a multiple of

$$\frac{\partial F}{\partial X}(0,T)^{-1}$$

Reciprocaly, derivating the equation $F(Z, F(X,Y)) = F(F(Z,X),Y)$ with respect to Z and putting $Z = 0$ one verifies the preceding is a solution of the system. □

Of course $F \longmapsto \omega_F$ is functorial in F and the morphism induced on this modules by a morphism f is multiplication by $f'(0)$. From now on we will consider $\mathrm{Lie}\, F$ as the set of continuous translation invariant derivations on F.

Exemple 2. *For $\widehat{\mathbb{G}}_m$ the one has $\omega_F = R.\frac{dT}{T+1}$ and $\mathrm{Lie}\, F = R.(1+T)\frac{d}{dT}$. For $\widehat{\mathbb{G}}_a$ one has $\omega_F = R.dT$ and $\mathrm{Lie}\, F = R.\frac{d}{dT}$.*

1.2 Lazard's key lemma

The following is the key lemma to the study of one-dimensional formal group laws.

Définition 5. *Let $\mathfrak{m} = (X,Y) \subset R[[X,Y]]$. Call a truncated at the order n one dimensional formal group law over R a $F \in R[[X,Y]]/\mathfrak{m}^n$ satisfying all the axioms of a formal group law in $R[[X,Y]]/\mathfrak{m}^n$ that is to say modulo degre $\geq n$ polynomials.*

Lemme 2. *Set for all $n \in \mathbb{N}^*$*

$$C_n(X,Y) = \begin{cases} (X+Y)^n - X^n - Y^n & \text{if } n \text{ is not a power of a prime} \\ \dfrac{(X+Y)^n - X^n - Y^n}{p} & \text{if } n = p^\alpha \text{ with } p \text{ prime} \end{cases}$$

Let $n \in \mathbb{N}^*$. Let $F(X,Y)$ be a truncated at the order n formal group law over R that can be extended to a truncated at the order $n+1$ formal group law. Then the set of such extensions G to a truncated at the order $n+1$ formal group law is is a principal homogenous space under R via

$$\forall a \in R\ \forall G \quad a.G = G + aC_n(X,Y)$$

We refer to [**?**] ? ? ? for the details of the proof. Let $G \in R[[X,Y]]/\mathfrak{m}^{n+1}$ be an extension as in the theorem. We look for a polynomial $\Gamma(X,Y) = \sum_{\substack{i+j=n \\ 0<i,j<n}} a_{ij} X^i Y^{n-i}$ such that $F = G + \Gamma$ is a truncated formal group law at the order $n+1$. The first evident condition is $a_{ij} = a_{ji}$. Now writting $F(F(X,Y),Z) = F(X, F(Y,Z))$ modulo degre $n+1$ we find the equivalent cocyle condition

$$\Gamma(X,Y) + \Gamma(X+Y,Z) = \Gamma(Y,Z) + \Gamma(X, Y+Z)$$

The $C_n(X,Y)$ are solutions to this linear system in the unknown (a_{ij}). Reciprocaly one has to see that those are the unique solutions up to a scalar which is a linear algebra problem solved in ...

1.3 The caracteristic zero case

Proposition 2. *Let R be a $\mathbb{Q}$-algebra. Then for a one dimensional formal group law F over R there exists a unique series $\log_F \in TR[[T]]$ called the logarithm of F s.t.*

$$\log_F : F \xrightarrow{\sim} \widehat{\mathbb{G}}_a \quad \text{and } \log_F'(0) = 1$$

Démonstration. The uniqueness assertion is clear since if $f_1, f_2 : F \xrightarrow{\sim} \widehat{\mathbb{G}}_a$ then $f_1^{-1} \circ f_2 \in \mathrm{End}(\widehat{\mathbb{G}}_a)$. And since R is a $\mathbb{Q}$-algebra $\mathrm{End}(\widehat{\mathbb{G}}_a) = R$.

Let now F be such that $F(X,Y) \equiv X + Y \bmod \mathfrak{m}^n$ for some $n \in \mathbb{N}^*$. Then from lemma 2 and since R is a $\mathbb{Q}$-algebra, for a $u \in R$

$$F(X,Y) \equiv X + Y + u((X+Y)^n - X^n - Y^n) \bmod \mathfrak{m}^{n+1}$$

where $\mathfrak{m} = (X,Y)$. Now setting

$$h(T) = T - uT^n$$

we find

$$h(h^{-1}(X) \underset{F}{+} h^{-1}(Y)) \equiv X + Y \mod \mathfrak{m}^{n+1}$$

which expresses the fact that h is a truncated isomorphism

$$h : F \xrightarrow{\sim} \widehat{\mathbb{G}}_a \mod \mathfrak{m}^{n+1}$$

By induction we thus construct a sequence $(h_n)_{n\geq 1}$ such that $h_n \equiv T\,[T^n]$ and $h_n \circ \cdots \circ h_1$ is a truncated isomorphisme modulo $\mathfrak{m}^{n+1}$ between F and $\widehat{\mathbb{G}}_a$. One concludes easily that $\lim_{n\to+\infty}(h_n \circ \cdots \circ h_1)$ converges to the desired series $\log_F$. □

Remarque 1. *This proposition is a very particular case of the following theorem : over a $\mathbb{Q}$-algebra R the Lie algebra functor induces a category equivalence between formal group laws (not necessarily one dimensional or commutative) and Lie algebras over R that are finite free R-modules.*

If $\log_F$ is the logarithm of F then $\log_F^* dT$ is an invariant by translations differential form on F since dT is one on $\widehat{\mathbb{G}}_a$. Reciprocaly if $\omega = f(T)dT$ is an invariant by translations differential form on F such that $f(0) = 1$ then $\log_F = \int_0^T \omega$ and thus

$$\log_F = \int_0^T \frac{\partial F}{\partial X}(0,T)^{-1} dT$$

If R is p-adically complete and without p-torsion one has the following formula for the logarithm of F after inverting p still denoted $\log_F$ allthough F is defined over R

$$\log_F = \lim_{k\to+\infty} \frac{1}{p^k}[p^k]_F(T)$$

where the limit is for the p-adic topology on each of the coefficients of the power series and this limit lies in $R[\frac{1}{p}][[T]]$.

1.4 Lazard's ring

Définition 6. *Let Λ be the ring representing the functor*

$$\begin{array}{ccl} \textit{Rings} & \longrightarrow & \textit{Sets} \\ R & \longmapsto & \textit{the set of one dim. formal group laws over } R \end{array}$$

In fact this ring has the following presentation :

$$\Lambda = \mathbb{Z}[a_{ij}]_{i,j\geq 1}/I$$

where the a_{ij} are interderterminates, the universal group law is $F(X,Y) = X + Y + \sum_{i,j\geq 1} a_{ij} X^i Y^j$ and the ideal I is generated by the relations $a_{ij} = a_{ji}$ and the one given by writting $F(F(X,Y),Z) = F(X, F(Y,Z))$.

Lemme 3. *With the preceding presentation for Λ put $\deg a_{ij} = i + j - 1$. Then the ideal I is homogenous.*

Démonstration. Put $\deg X = \deg Y = -1$. Then $F(X,Y)$ is homogenous of degre -1 in the variables $(a_{ij})_{i,j}$, X and Y. Thus if we put $\deg Z = -1$ the series $F(F(X,Y),Z)$ and $F(X,F(Y,Z))$ are homogenous of degre -1 in the variables $(a_{ij})_{i,j}, X, Y$ and Z. Thus the equation $F(F(X,Y),Z) = F(X,F(Y,Z))$ gives degre 0 homogenous equations in the $(a_{ij})_{i,j}$. □

Thus the ring Λ is graded by

$$\Lambda = \bigoplus_{k\geq 0} \Lambda^k$$

with $\Lambda^0 = \mathbb{Z}$. Now put

$$\forall k \in \mathbb{N} \;\; \widetilde{\Lambda}^k = \Lambda^k / < xy \mid x \in \Lambda^i, y \in \Lambda^j \;\; i+j = k \text{ and } i,j > 0 >$$

wher $< . >$ means the R-submodule generated. Then a translation of lemma 2 is

Corollaire 1. *For all $k \in \mathbb{N}$ the $\mathbb{Z}$-module $\widetilde{\Lambda}^k$ if free of rank 1.*

Démonstration. For all $n \in \mathbb{N}$ consider the ideal $\bigoplus_{k \geq n} \Lambda^k$ of Λ and the quotient ring

$$\Lambda / \bigoplus_{k \geq n} \Lambda^k \simeq \bigoplus_{0 \leq k \leq n-1} \Lambda^k$$

This quotient ring represents the functor

$$\begin{array}{rcl} \text{Rings} & \longrightarrow & \text{Sets} \\ R & \longmapsto & \text{truncated at the order } n+1 \text{ one dim. formal group laws} \end{array}$$

Now for a truncated at the order $n+1$ formal group law F corresponding to a morphism

$$f : \bigoplus_{0 \leq k \leq n-1} \Lambda^k \longrightarrow R$$

we look for the truncated at the order $n+1$ formal group laws that are congruent to F mod degre $\geq n$. This corresponds to $\mathbb{Z}$-linear maps $\psi : \Lambda^{n-1} \longrightarrow R$ such that if we set

$$\forall k \leq n-1 \;\; \forall x \in \Lambda^k \;\; f'(x) = \begin{cases} f(x) \text{ if } k < n-1 \\ f(x) + \psi(x) \text{ if } k = n-1 \end{cases}$$

then f' is a ring morphism. One sees easily that this is equivalent to says that ψ factorises as a morphism of abelian groups as $\psi : \widetilde{\Lambda}^{n-1} \longrightarrow R$. □

From this one deduces :

Théorème 1 (Lazard)**.** *The graded ring Λ is a polynomial algebra*

$$\Lambda \simeq \mathbb{Z}[t_k]_{k \geq 1}$$

with $\deg t_k = k$.

Démonstration. Choose for all k a lifting to Λ^k of a generator of $\widetilde{\Lambda}^k$ and define a map from $\mathbb{Z}[t_k]_{k \geq 1}$ to Λ by sending t_k to this generator. On has to see this is an isomorphism. But this is a morphism of graded rings $A \longrightarrow B$ inducing an isomorphism a the level of the $\mathbb{Z}$-modules $\widetilde{A}^k \longrightarrow \widetilde{B}^k$ for all k where $\widetilde{A}^k$ and $\widetilde{B}^k$ are defined as $\widetilde{\Lambda}^k$. From this one deduces that this morphism is surjective. Now to prove injectivity it suffices to prove it after tensoring with $\mathbb{Q}$, for the map

$$\mathbb{Q}[t_k]_{k \geq 1} \longrightarrow \Lambda \otimes \mathbb{Q}$$

But now define the following “universal logarithm” series

$$h(T) = T + \sum_{k \geq 1} u_k T^{k+1} \in \mathbb{Q}[u_k]_{k \geq 1}[[T]]$$

and set $G(X,Y) = h^{-1}(h(X)+h(Y)) \in \mathbb{Q}[u_k]_{k \geq 1}[[X,Y]]$. By the universal property of Λ associated to G there is a graded morphism

$$\Lambda \otimes \mathbb{Q} \longrightarrow \mathbb{Q}[u_k]_{k \geq 1}$$

and this is an isomorphism since by proposition 2 this morphism defines an isomorphism of functors between $\text{Hom}(\mathbb{Q}[u_k]_{k \geq 1}, -)$ and $\text{Hom}(\Lambda \otimes \mathbb{Q}, -)$. More precisely, the inverse of this morphism is constructed in the following way : by proposition **??** there exists a unique $g \in \Lambda \otimes \mathbb{Q}[[T]]$ such that $g(0) = 0$, $g'(0) = 1$ and if F is the universal group law on $\Lambda \otimes \mathbb{Q}$ then

$$F(X, Y) = g^{-1}(g(X) + g(Y))$$

Then the inverse of $\Lambda \otimes \mathbb{Q} \longrightarrow \mathbb{Q}[u_k]_{k \geq 1}$ is given by sending u_k to the $k+1$-th coefficient of g.

Now this isomorphism

$$\Lambda \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}[u_k]_{k \geq 1}$$

is a graded isomorphism if we set $\deg u_k = k$. In fact, putting $\deg T = -1$ we see the universal logarithm $h(T) \in \mathbb{Q}[u_k]_{k \geq 1}[[T]]$ is homogenous with degree -1. Thus, with $\deg X = \deg Y = -1$, $G(X, Y) = h^{-1}(h(X) + h(Y))$ is homogenous with degre -1 which implies our isomorphism respects the grading.

Now look at the composition

$$\mathbb{Q}[t_k]_{k \geq 1} \twoheadrightarrow \Lambda \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}[u_k]_{k \geq 1}$$

it is a surjective graded morphism

$$\mathbb{Q}[t_k]_{k \geq 1} \twoheadrightarrow \mathbb{Q}[u_k]_{k \geq 1}$$

and thus an isomorphism since it induces a surjective morphism of finite dimensional $\mathbb{Q}$-vector spaces of the same dimension between elements of degre less than a given integer. □

1.5 The theory of one-dim. formal groups laws is unobstructed

Corollaire 2. *Let R be a ring.*

- *Any truncated at the order k one dimensional formal group law over R extends to a formal group law over R.*
- *Let I be an ideal in R. Any one dimensional formal group law over R/I lifts to a formal group law over R.*

Démonstration. This is a consequence of Lazard's theorem, since "morphisms of polynomials algebras can be lifted". □

1.6 Classification over car. p algebraically closed fields

1.6.1 Frobenius morphism

Let R be a ring such that $pR = 0$. Let G be a one dim. formal group law over R.

Définition 7. *We let $G^{(p)}$ be the formal group law obtained by applying the p-power to all the coefficients of G.*

Définition 8. *We denote*

$$F : G \longrightarrow G^{(p)}$$

the Frobenius morphism defined by the power series T^p.

1.6.2 Height

Let k be a caracteristic p field and G defined over k.

Lemme 4. *Let $f \in End(F)$ be s.t. $f'(0) = 0$ and $f \neq 0$. Then $\exists! h \in \mathbb{N}^* \ \exists g \in k[[T]] \quad g(0) = 0, g'(0) \neq 0$ and*

$$f = g(T^{p^h})$$

Démonstration. Differentiationg the equality $f(F(X,Y)) = F(f(X), f(Y))$ with respect to X one finds

$$f'(F(X,Y))\frac{\partial F}{\partial X}(X,Y) = \frac{\partial F}{\partial X}(f(X), f(Y))f'(X)$$

with $X = 0$ one finds

$$f'(Y)\frac{\partial F}{\partial X}(0,Y) = 0 \Longrightarrow f' = 0$$

since $\frac{\partial F}{\partial X}(0,Y)$ is invertible.

Thus f can be written as a composite

$$\underbrace{G \xrightarrow{F} G^{(p)} \xrightarrow{g} G}_{f}$$

where $g \in \mathrm{End}(F)$, and one can apply by induction the preceding process to g. □

Définition 9. *Let $[p]_G \in End(G)$. If $[p]_G = 0$ we say G has infinite height. If not let $h \geq 1$ be the integer defined in the previous lemma which is the biggest integer k such that $[p]_F$ can be factorized by $F^{(p^k)}$. We say G has height h.*

Thus if G has height $h < +\infty$ then $[p]_F$ can be factorized as

$$\underbrace{G \xrightarrow{F^h} G^{(p^h)} \xrightarrow[\simeq]{g} G}_{[p]_G}$$

where g is an isomorphism. From this it is clear that the height of G is an invariant of the isomorphism class of G.

1.7 The infinite height case

We keep the hypothesis of the last section.

Proposition 3. *G has infinite height $\Longleftrightarrow G \simeq \widehat{\mathbb{G}}_a$.*

Démonstration. Supppose $G \equiv X + Y \bmod \deg\ \geq n$. Then $\exists a \in k$

$$G \equiv X + Y + aC_n(X,Y) \bmod \deg\ \geq n+1$$

If n is not a power of p then puting $h(T) = t - aT^n$ one finds

$$h(h^{-1}(X) \underset{G}{+} h^{-1}(Y)) \equiv X + Y \bmod \deg\ \geq n+1$$

If n is a power of p then one finds $[p]_G \equiv -aT^n \bmod \deg\ \geq n+1$ (cf. lemma 5) and thus from the hypothesis $[p]_G = 0$ we find $a = 0$. By induction we thus can construct for all n truncated isomorphisms $G \xrightarrow{\sim} \widehat{\mathbb{G}}_a \bmod \deg \geq n$ which in the limit gives us an isomorphism as in the proof of proposition 2. □

1.8 The finite height case

The notations are the same as in the preceding section.

1.8.1 Normalized group law

Définition 10. *Let $h \in \mathbb{N}^*$. The height h group law G over k is normalized if $[p]_G = T^{p^h}$.*

Remarque 2. *If G is normalized then $G \in \mathbb{F}_{p^h}[[X,Y]]$.*

As we have seen we can writte

$$[p]_G = g \circ F^h$$

where g is an isomorphism between $G^{(p^h)}$ and G. Now the problem to know wether or not G is isomorphic to a normalized group law is equivalent to know wether or not g can be written as

$$g = g_1 g_1^{-p^h}$$

for some $g_1 \in \mathrm{Aut}(G)$, that is to say if g is "a coboundary".

Remarque 3. *This problem is related to the problem to know if in a height h one dimensional crystal over k one can find an non-zero element e s.t. $F^h.e = p.e$.*

Proposition 4. *If k is separably closed then any finite height one dimensional group law over k is isomorphic to a normalized one.*

Démonstration. If

$$[p]_G \equiv aT^{p^h} \; [T^{p^h+1}]$$

with $a \neq 0$ then if $b \in k$ is such that $a = b^{1-p^h}$ and $g(T) = bT$ then $g \circ [p]_G \circ g^{-1} \equiv T^{p^h} \; [T^{p^h+1}]$. And thus we can suppose the coefficient of T^{p^h} is 1. Now if for some $k \geq 2$

$$[p]_F \equiv T^{p^h} + aT^{kp^h} \; [T^{(k+1)p^h}]$$

putting $g(T) = T - bT^k$ we find

$$g \circ [p]_G \circ g^{-1}(T) \equiv T^{p^h} + (a + b^{p^h} - b)T^{kp^j} \; [T^{(k+1)p^h}]$$

and thus using a b solution of the Artin-Schreier equation $a = b - b^{p^h}$ we can suppose

$$[p]_G \equiv T^{p^h} \; [T^{(k+1)p^h}]$$

We conclude as usual by induction and passing to the limit. □

Lemme 5. *Let G_1, G_2 be two formal group laws over R s.t.*

$$G_1 \equiv G_2 + aC_{p^i}(X,Y) \; mod \; \deg p^i + 1$$

then

$$[p]_{G_1} \equiv [p]_{G_2} + a(-1 + p^{p^i - 1})T^{p^i} \; mod \; \deg p^i + 1$$

In particular if $pR = 0$ then

$$[p]_{G_1} \equiv [p]_{G_2} - aT^{p^i} \; mod \; \deg p^i + 1$$

Démonstration. By an easy induction argument one finds

$$\forall n \in \mathbb{N}^* \;\; [n]_{G_1} \equiv [n]_{G_2} + a \sum_{k=1}^{n-1} C_{p^i}(T, kT) \bmod \deg p^i + 1$$

Moreover

$$\begin{aligned} \sum_{k=1}^{n-1} C_{p^i}(T, kT) &= \sum_{k=1}^{n-1} \frac{(k+1)^{p^i} T^{p^i} - k^{p^i} T^{p^i} - T^{p^i}}{p} \\ &= (-1 + p^{p^i - 1}) T^{p^i} \end{aligned}$$

□

Proposition 5. *Let G_1, G_2 be two formal group laws over k separably closed. Then $G_1 \simeq G_2$ iff $ht(G_1) = ht(G_2)$.*

Démonstration. Suppose $\mathrm{ht}(G_1) = \mathrm{ht}(G_2) = h$, G_1 and G_2 are normalized, and we have constructed $g \in \mathbb{F}_{p^h}[[T]]$ s.t. $g(0) = 0$, $g'(0) \neq 0$ and

$$g(X \underset{G_1}{+} Y) \equiv g(X) \underset{G_2}{+} g(Y) \text{ mod deg } \geq n$$

Then replacing G_1 by $g(G_1(g^{-1}(X) + g^{-1}(Y)))$ we can suppose

$$G_1 \equiv G_2 \text{ mod } \text{ deg } \geq n$$

and thus

$$\exists a \in \mathbb{F}_{p^h} \;\; G_1 \equiv G_2 + aC_n(X,Y) \text{ mod deg } \geq n+1$$

If n is not a power of p, using $g(T) = T - aT^n$ we have $g(G_1(g^{-1}(X), g^{-1}(Y))) \equiv G_2$ mod deg $\geq n+1$. If n is a power of p then by the preceding lemma

$$[p]_{G_1} \equiv [p]_{G_2} - aT^n \; [T^{n+1}]$$

$$\Longrightarrow a = 0 \text{ since } [p]_{G_1} = [p]_{G_2} = T^{p^h}$$

Thus starting from a truncated at the order n isomorphism between G_1 and G_2 we have constructed a truncated isomorphism at the order $n+1$. Moreover since $g \in \mathbb{F}_{p^h}[[T]]$ the new formal group laws are still normalized. By induction and a limit process one then construct an isomorphism $G_1 \xrightarrow{\sim} G_2$. □

Corollaire 3. *Any finite height h one dim. formal group law is isomorphic to a normalized one F s.t.*

$$F(X,Y) \equiv X + Y + C_{p^h}(X,Y) \textit{ mod deg} \geq p^h + 1$$

Démonstration. This can be deduced from the proof of the last proposition. □

Théorème 2. *The isomorphism classes of one dimensional formal group laws over a separably closed field are in bijection with $\mathbb{N}^* \cup \{\infty\}$, the bijection being given by the height.*

Démonstration. It remains to prove that for any $h \in \mathbb{N}^*$ there exists a height h formal group law over k. But if $\Lambda \simeq \mathbb{Z}[t_i]_{i \geq 1}$ is Lazard's ring then if we define the ring morphism $\Lambda \longrightarrow k$ by sending $t_i \mapsto 0$ if $i < p^h - 1$, $t_{p^h-1} \mapsto 1$ and $t_i \mapsto$ any element of k for other i then if $G(X,Y)$ is the corresponding formal group law over k we have

$$G(X,Y) \equiv X + Y + C_{p^h}(X,Y) \text{ mod deg } \geq p^h + 1$$

which implies as allready seen

$$[p]_G \equiv -T^{p^h} \; [T^{p^h+1}]$$

□

1.9 Endomorphism algebra

One can show that for a G of height $h < +\infty$ over k separably closed

$$\mathrm{End}(G) = \mathcal{O}_D$$

where D is a division algebra with invariant $\frac{1}{n}$ and $\mathcal{O}_D$ is its maximal order. This is a consequence of the classification of p-divisible groups via crystals. Concretely

$$\mathcal{O}_D = \mathbb{Z}_{p^n}[\Pi]$$

where $\mathbb{Z}_{p^n} = W(\mathbb{F}_{p^n})$ is the ring of integers in the degre n unramified extension of $\mathbb{Q}_p$ and Π is the uniformizing element in D verifying

$$\forall x \in \mathbb{Z}_{p^n} \quad \Pi x = x^\sigma \Pi$$

where σ stands for the Frobenius of $\mathbb{Q}_{p^n}$.

It results from the proof of theorem 2 that there exists one dim. formal group law of any height defined over $\mathbb{F}_p$. Then such a formal group law G admits the Frobenius as an endomorphism. In the identification $\mathrm{End}(G) = \mathcal{O}_D$ the Frobenius is sent to a uniformizing element of $\mathcal{O}_D$ and thus can be identified with Π.

2 Deformation theory in the unobstructed case

In this section we present a particular case of Schlessinger's theorem.

2.0.1 Definitions

Let k be a field and $\mathcal{C}$ be the category of local Artin rings with residue field k, morphisms being morphisms inducing the identity on k. Let

$$F : \mathcal{C} \longrightarrow \{ \text{ Sets } \}$$

be a covariant functor. We will make the assumption $F(k) = \{\emptyset\}$ that is to say F is "connected".

Définition 11. *The functor F is formally smooth if for all $A, B \in \mathcal{C}$ and $A \twoheadrightarrow B$ a surjective morphism then $F(A) \longrightarrow F(B)$ is surjective.*

Of course, as usual, in the preceding definition one can assume the kernel I of $A \twoheadrightarrow B$ verifies $I^2 = (0)$ and even $\mathfrak{m}.I = (0)$ where $\mathfrak{m}$ is the maximal ideal of A.

Définition 12. *The functor F satisfies the Mayer-Vietoris property if for all diagramms in $\mathcal{C}$*

$$\begin{array}{ccc} A & & B \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

The natural morphism $F(A \times_C B) \longrightarrow F(A) \times_{F(C)} F(B)$ is an isomorphism.

Exemple 3. *If $F \simeq \mathfrak{X}$ where $\mathfrak{X}$ is a formal scheme then F satisfies the Mayer Vietoris property.*

2.0.2 Tangent functor

Let's suppose F satisfies the Mayer-Vietoris property. Let $\mathcal{M}$ be the category of finite dimensional k-vector spaces. There is a functor

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{C} \\ V & \longmapsto & k \oplus V \end{array}$$

where $k \oplus V$ is the trivial extension of k by the squared zero ideal V. For example if $V = k$, $k \oplus V = k[\epsilon]$ is the dual number algebra.

Définition 13. *We denote $TF : \mathcal{M} \longrightarrow$ Sets to be the composite $\mathcal{M} \longrightarrow \mathcal{C} \longrightarrow$ Sets.*

There is a canonical factorisation

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{TF} & \text{Sets} \\ & \searrow \quad \nearrow & \\ & k\text{-vector spaces} & \end{array}$$

defined in the following way : $\forall V \in Ob\,\mathcal{M}$ there is a morphism in $\mathcal{M}$

$$\begin{array}{ccc} V \times V & \longrightarrow & V \\ (x, y) & \longmapsto & x + y \end{array}$$

inducing by functoriality a morphism in $\mathcal{C}$

$$k \oplus V \times k \oplus V \xrightarrow{+} k \oplus V$$

In the same way forall $x \in k$ there is a morphism in $\mathcal{M}$

$$\begin{array}{ccc} V & \longrightarrow & V \\ v & \longmapsto & x.v \end{array}$$

inducing

$$m_x : k \oplus V \longrightarrow k \oplus V$$

They induce

$$TF(V) \times TF(V) \xrightarrow{\simeq} F(k \oplus V \times k \oplus V) \xrightarrow{F(+)} TF(V)$$

and

$$\forall x \in k \;\; TF(V) \xrightarrow{F(m_x)} TF(V)$$

And one verifies those two operations induce a k-vector space structure on $TF(V)$ (in fact $k \oplus V$ has a structure of k-vector space in the category $\mathcal{C}$). One verifies immediatly that this vector space structure is functorial in V. Thus we have our factorization.

Lemme 6. *There is a natural isomorphism of functors*

$$TF(-) \simeq TF(k) \otimes_k (-)$$

Démonstration. Fix an equivalence bewteen $\mathcal{M}$ and the subcategory of vector spaces $(k^n)_{n \geq 0}$. It suffices to construct natural isomorphisms

$$TF(k^n) \simeq TF(k) \otimes_k k^n$$

They are deduced from the Mayer -Vietoris property : $TF(k \times \cdots \times k) \simeq TF(k)^n$. The naturality is easily checked. □

Proposition 6. *Let $A \in \mathcal{C}$ with maximal ideal $\mathfrak{m}$ and let I be an ideal of A s.t. $\mathfrak{m}.I = 0$. Let $f : F(A) \longrightarrow F(A/I)$. Then for each $x \in F(A/I)$ either $f^{-1}(\{x\}) = \emptyset$ either $f^{-1}(\{x\})$ is a principal homogenous space under $TF(I)$.*

Démonstration. There is an isomorphism in $\mathcal{C}$

$$\begin{array}{ccc} A \times_{A/I} A & \xrightarrow{\sim} & A \times (k \oplus I) \\ (a_1, a_2) & \mapsto & (a_1, \bar{a}_1 \oplus (a_1 - a_2)) \end{array}$$

where $\bar{a}_1$ is the reduction of a_1 modulo the maximal ideal of A. Applying the Mayer-Vietoris property we find

$$F(A) \times TF(I) \xrightarrow{\sim} F(A) \times_{F(A/I)} F(A)$$

and one verifies easily this defines an action of $TF(I)$ on $TF(A)$. □

2.1 Formaly étale morphisms

Définition 14. *A morphism $f : F \longrightarrow G$ of covariant functors on $\mathcal{C}$ is formay étale if for all $A \in \mathcal{C}$ and ideal I of A the following diagram is cartesian*

$$\begin{array}{ccc} F(A) & \longrightarrow & G(A) \\ \downarrow & & \downarrow \\ F(A/I) & \longrightarrow & G(A/I) \end{array}$$

Of course as usual, by devissage, it suffices to consider the case when $I^2 = 0$ or even when $\mathfrak{m}.I = (0)$ where $\mathfrak{m}$ is the maximal ideal of A.

Lemme 7. *A morphism $f : F \longrightarrow G$ is an isomorphism iff it is formaly étale and induces a bijection $F(k) \longrightarrow G(k)$.*

Démonstration. This is clear since in a cartesian square if the bottom horizontal line is a bijection then so is the top horizontal one. □

Proposition 7. *Suppose F is formaly smooth. Suppose F and G satisfy the Mayer Vietoris property. A morphism $f : F \longrightarrow G$ is formaly étale iff it induces an isomorphism $TF(k) \xrightarrow{\sim} TG(k)$.*

Démonstration. If f induces an isomorphism between $TF(k)$ and $TG(k)$ then by lemma 6 is induces an isomorphism of the tangent functors $TF(-) \xrightarrow{\sim} TG(-)$. Then you conclude by proposition 6. □

2.1.1 Main theorem

If k is a perfect field any complete local ring with residue field k is naturaly a $W(k)$-algebra. Thus the category $\mathcal{C}$ is the category of local artinian $W(k)$-algebras with residue field k and F can be considered as a functor over $\mathrm{Spf}(W(k))$.

Théorème 3. *Suppose k is perfect. Suppose F satisfies the Mayer-Vietoris property, is formally smooth verifies $F(k) = \{\emptyset\}$ and satisfies*

$$\dim_k TF(k) = n < +\infty$$

Then

$$F \simeq Spf(W(k)[[T_1, \ldots, T_n]])$$

Démonstration. Let $(e_1, \ldots, e_n)$ be a basis of $TF(k)$. It gives an element of

$$F(k[T_1, \ldots, T_n]/(T_1^2, \ldots, T_n^2)) \simeq \prod_{i=1}^{n} TF(k)$$

By the formal smoothness property this can be lifted to an element of

$$F(W(k)[[T_1, \ldots, T_n]]) := \varprojlim_{k} \; F(W(k)[[T_1, \ldots, T_n]]/I^k)$$

where $I = (p, T_1, \ldots, T_n)$. This gives a morphism

$$f : G = \mathrm{Spf}(W(k)[[T_1, \ldots, T_n]]) \longrightarrow F$$

Now because of our choices

$$TG(k) \xrightarrow{\sim} TF(k)$$

and thus by proposition 7 f is an isomorphism. □

3 Lubin-Tate spaces : first approach via formal group laws

3.1 Definition

Let F be a one dimensional formal group law of finite height n over $\overline{\mathbb{F}}_p$.

Définition 15. *Let A be a local artinian algebra with maximal ideal $\mathfrak{m}$ and residue field $\overline{\mathbb{F}}_p$.*
- *A formal group law G over A is a deformation of G if $G \equiv F$ modulo $\mathfrak{m}$*
- *Two deformations G_1, G_2 are isomorphic if there is an isomorphism*

$$f : G_1 \xrightarrow{\sim} G_2$$

such that $f \equiv Id$ mod $\mathfrak{m}$ (that is ot say $f(T) \equiv T$).

Let $\mathcal{D}ef$ be the associated functor of isomorphism classes of deformations of F on artin local rings with residue field $\overline{\mathbb{F}}_p$.

3.2 Computation of the tangent space

We note $k = \overline{\mathbb{F}}_p$.

Lemme 8. *The tangent space to $\mathcal{D}ef$ is the k-vector space $H^1(C^\bullet)$ where*

$$C^\bullet = \Big[\underset{0}{Tk[[T]]} \xrightarrow{\partial} \underset{+1}{XYk[[X,Y]]^{\mathfrak{S}_2}} \xrightarrow{\partial} \underset{+2}{k[[X,Y,Z]]} \Big]$$

where $k[[X,Y]]^{\mathfrak{S}_2}$ stands for the symmetric formal power series and

$$\begin{aligned} \partial\varphi(T) &= \varphi(X \underset{F}{+} Y) - \varphi(X) - \varphi(Y) \\ \partial\psi(X,Y) &= \psi(Y,Z) - \psi(X \underset{F}{+} Y, Z) + \psi(X, Y \underset{F}{+} Z) - \psi(X,Y) \end{aligned}$$

Démonstration. Writte $F'(X,Y) = F(X,Y) + \epsilon\psi(X,Y)$ with $\epsilon^2 = 0$. Then

$$\partial\psi = 0 \Longleftrightarrow F'(F'(X,Y),Z) = F'(X, F'(Y,Z))$$

And if $h(T) = T + \epsilon\varphi(T)$,

$$F' = h(F(h^{-1}(X), h^{-1}(Y))) \Longleftrightarrow \psi = \partial\varphi$$

□

Suppose now F is normalized and satisfies

$$F(X,Y) \equiv X + Y + C_{p^n}(X,Y) \text{ mod deg} \geq p^n + 1$$

(cf. corollary 3). We will suppose more explicitely that F is associated to the morphism from Lazard's ring $\mathbb{Z}[t_i]_{i\geq 1}$ to k defined by $t_i \mapsto 0$ if $i < p^n - 1$, $t_{p^n-1} \mapsto 1$ and for other i t_i maps to anything.

Proposition 8. *The tangent space to $\mathcal{D}ef$ is $n-1$ dimensional generated by cocyles $\psi_1, \dots, \psi_{n-1}$ that verify $\psi_i(X,Y) \equiv C_{p^i}(X,Y)$ mod deg $\geq p^i + 1$.*

Démonstration. We have

$$\begin{aligned} \partial T^k &\equiv (X+Y)^k - X^k - Y^k \text{ mod deg} \geq k+1 \\ \partial T^{p^i} &\equiv C_{p^{i+n}}(X,Y) \text{ mod deg} \geq p^i + 1 \end{aligned}$$

Moereover if a cocyle ψ verifies

$$\psi \equiv 0 \text{ mod deg} \geq k$$

then thanks to lemma 2

$$\psi \equiv aC_k(X,Y) \text{ mod deg} \geq k+1$$

for some $a \in \mathbb{F}_{p^h}$. With this one can prove that if $k \geq p^h$ and ψ is a cocyle that is a coboundary mod deg$\geq k$ then ψ is a couboundary mod deg$\geq k+1$.

Now for each $1 \leq i \leq n-1$ let ψ_i consider the map from Lazard's ring $\mathbb{Z}[t_j]_{j\geq 1}$ to $k[\epsilon]$ défined by $t_j \mapsto 0$ if $j < p^i - 1$, $t_{p^i-1} \mapsto \epsilon$, $t_j \mapsto 0$ if $p^i - 1 < j < p^n - 1$, $t_{p^n-1} \mapsto 1$ and t_j maps to anything for other j (the same anything as the one we chosed for defining F). This defines a deformation of F and thus a cocyle ψ_i. It follows by computing modulo T^{p^n} and the formulas given for ∂T^k with k not a power of p that those cocyles form a basis of the tangent space to $\mathcal{D}ef$. □

3.3 Représentability

Proposition 9. *There is an isomorphism*

$$\mathcal{D}ef \simeq Spf(W(k)[[T_1,\ldots,T_{n-1}]])$$

Démonstration. We apply theorem 3. The functor $\mathcal{D}ef$ clearly satisfies Mayer-Vietoris property. Formal smoothness follows from corollary 2 and the assertion about the tangent space is proposition 8. □

3.4 Good coordinates on Lubin-Tate spaces

Proposition 10. *There is a system of formal coordinates $(x_1,\ldots,x_{n-1})$ on $\mathcal{D}ef$ that is to say an isomorphism $\mathcal{D}ef \simeq Spf((W(k)[[x_1,\ldots,x_{n-1}]])$ such that if G is the associated universal formal group law then*

$$\forall 1 \leq i \leq n-1 \;\; G(X,Y) \equiv x_i C_{p^i}(X,Y) \;\; mod\; (deg\; p^i + 1, x_1, \ldots, x_{i-1})$$

Démonstration. Let G be the group law over $W(k)[[x_1,\ldots,x_{n-1}]]$ associated to the map from Lazard's ring

$$\Lambda = \mathbb{Z}[t_k]_{k\geq 1}$$

to $W(k)[[x_1,\ldots,x_{n-1}]]$ that maps t_{p^i-1} to x_i for $1 \leq i \leq n-1$, t_{p^n-1} to 1 and other t_i to 0. It gives a morphism from Spf($W(k)[[x_1,\ldots,x_{n-1}]]$) to $\mathcal{D}ef$. To see this is an isomorphism we have to prove it is one at the level of the tangent spaces, that is to say look at the reduction of G to $k[x_1,\ldots,x_{n-1}]/(x_1^2,\ldots,x_{n-1}^2)$. But this is clear from the computation of the tangent space of $\mathcal{D}ef$. □

Corollaire 4. *There is a system of formal coordinates $(x_1,\ldots,x_{n-1})$ on $\mathcal{D}ef$ such that if G is the associated universal formal group law then*

$$[p]_G = pu_0T + x_1u_1T^p + \cdots + x_{n-1}u_{n-1}T^{p^{n-1}} + u_nT^{p^n}$$

where $u_0,\ldots,u_n \in W(k)[[x_1,\ldots,x_{n-1}]][[T]]^\times$ are units.

Démonstration. This follows from lemma 5. □

Remarque 4. *In the following sections we will explain the link with the Newton stratification and this type of coordinates.*