An introduction to the geometry of Lubin-Tate spaces

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1 The period morphism to projective space : first definition

1.1 Motivation

Consider the functor F on smooth analytic spaces that associates to S the isomorphism classes of couples (E,ρ) where $E \xrightarrow{p} S$ is an elliptic curve over S and $\rho : \underline{\mathbb{Z}}^2 \xrightarrow{\sim} R^1 p_* \underline{\mathbb{Z}}$ is a rigidification of its Betti cohomology. Then F is representable by Poincaré upper/lower half plane $\mathbb{C}\setminus\mathbb{R} = \mathcal{H}^{\pm}$. The universal elliptic curve is described as $\mathcal{O}_{\mathbb{P}^1}(1)_{|\mathcal{H}^{\pm}}/\underline{\mathbb{Z}}^2$ where $\underline{\mathbb{Z}}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$ is induced by $\underline{\mathbb{Z}}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$.

Now consider the period morphism $\mathcal{H}^{\pm} \longrightarrow \mathbb{P}^1(\mathbb{C})$ defined by $(E,p) \longmapsto \mathrm{Fil}\, \mathcal{O}_S^2$ where $\mathrm{Fil}\, \mathcal{O}_S^2$ is defined by the Hodge filtration of E coupled with the rigidification ρ :

$$\omega_{E/S} = \mathrm{Fil}\mathcal{H}^1_{dR}(E/S) \subset \mathcal{H}^1_{dR}(E/S) = R^1 p_* \underline{\mathbb{Z}} \otimes \mathcal{O}_S \simeq \mathcal{O}_S^2$$

This morphism is a local ismorphism and even an embeding.

We are going to do the same for Lubin-Tate spaces : here the couple (E,ρ) is replaced by (H,ρ) where ρ is a rigidification of the p-adic Betti cohomology : the isogeny class of the reduction. The associated period morphism will be étale thanks to Grothendieck-Messing deformation theory, but not a local isomorphism.

1.2 Notations

Let $\mathbb{H}$ be a height n one dimensional formal p-divisible group over $\overline{\mathbb{F}}_p$. Let

$$\mathfrak{X} \simeq \mathrm{Spf}(W(k)[[x_1,\dots,x_{n-1}]])$$

be the associated Lubin-Tate space clasiffying deformations of $\mathbb{H}$. We note (H,ρ) for the universal deformation, H being a p-divisible group over $\mathfrak{X}$ and

$$\rho : \mathbb{H} \xrightarrow{\sim} H \bmod (p, x_1, \dots, x_{n-1})$$

We have allready described the crystal $(\mathbb{D}(\mathbb{H}), V)$ with its action of the maximal order $\mathcal{O}_D$ of the division algebra D in section ? ? ?.

Set $L = W(k)_{\mathbb{Q}}$. We note $\mathfrak{X}^{rig}$ for the generic fiber of $\mathfrak{X}$ as a rigid space

$$\mathfrak{X}^{rig} \simeq \overset{\circ}{\mathbb{B}}^{n-1} = \{(x_1,\dots,x_{n-1}) \in \mathbb{A}^n \mid \forall i \; v(x_i) > 0 \}$$

As a rigid space over L $\mathfrak{X}^{rig}$ is a growing-up union of closed balls

$$\mathfrak{X}^{rig} = \bigcup_{a\in\mathbb{N}^*} \mathbb{B}^{n-1}(0, p^{-1/a})$$

Where

$$\begin{array}{rcl}\mathbb{B}^{n-1}(0,p^{-1/a}) & = & \{(x_1,\dots,x_{n-1}) \mid \forall i\ v(x_i) \geq \frac{1}{a}\ \} \\ & = & Sp(\mathcal{A}_a)\end{array}$$

where $\mathcal{A}_a$ is the affinoïde alegbra

$$\mathcal{A}_a = L < X_1,\dots,X_{n-1},T_1,\dots,T_{n-1} > /(X_i^a - pT_i)_{1\leq i\leq n-1}$$

and the morphism $Sp(\mathcal{A}_a) \hookrightarrow Sp(\mathcal{A}_b)$ for $a \leq b$ is given by

$$\begin{array}{rcl}\mathcal{A}_b & \longrightarrow & \mathcal{A}_a \\ X_i & \longmapsto & X_i \\ T_i & \longmapsto & X_i^{b-a}T_i\end{array}$$

Recall there is a left action of $\mathcal{O}_D^\times$ on $\mathfrak{X}$ over $\mathrm{Spf}(W(k))$ via

$$\forall g \in \mathcal{O}_D^\times = \mathrm{Aut}(\mathbb{H})\ \ \forall (H_0,\rho_0) \in \mathfrak{X}\quad g.(H_0,\rho_0) = (H_0,\rho_0 \circ g^{-1})$$

One can prove (we won't use this lemma)

Lemme 1. *The action of $\mathcal{O}_D^\times$ is continous in the sens that if $\mathfrak{m} = (p,x_1,\dots,x_{n-1})$ for all $k \in \mathbb{N}^*$ there exits an open compact subgroup $K \subset \mathcal{O}_D^\times$ such that K acts trivialy on Spec$(W(k)[[x_1,\dots,x_{n-1}]]/\mathfrak{m}^k)$.*

Démonstration. By rigidity of quasi-isogenies $\forall g \in \mathcal{O}_D\ \exists N \in \mathbb{N}^*\ p^N g$ lifts to an endomorphism of H mod $\mathfrak{m}^k$. But $\mathcal{O}_D$ being a $\mathbb{Z}_p$-module of finite type

$$\exists N \in \mathbb{N}^*\ \forall g \in \mathcal{O}_D\ \ p^N g \in \mathrm{End}(H \bmod \mathfrak{m}^k) \subset \mathrm{End}(\mathbb{H})$$

Thus $\mathrm{End}(H \bmod \mathfrak{m}^k)$ being p-adicaly complete ring

$$\forall g \in \mathcal{O}_D\ \ Id + p^N g \in \mathrm{Aut}(H \bmod \mathfrak{m}^k)$$

and $Id + p^N\mathcal{O}_D$ is a compact open subroup of $\mathcal{O}_D^\times$. □

1.3 Statement of the main theorem

Let $\mathrm{Lie}\,E(H)$ be the universal vector extension of H on $\mathfrak{X}$. It is a free $\mathcal{O}_{\mathfrak{X}}$-module of rank n. It has a flitration

$$0 \longrightarrow \omega_{H^D} \longrightarrow \mathrm{Lie}\,E(H) \longrightarrow \omega_H^* \longrightarrow 0$$

where ω_{H^D} is free of rank $n-1$ and ω_H^* free of rank 1. All those free modules are equivariant $\mathcal{O}_D^\times$-vector bundles on $\mathfrak{X}$. As $\mathcal{O}_D^\times$-vector bundles they are non-trivial at all as they are if we forget the action of $\mathcal{O}_D^\times$. Let

$$\mathrm{Lie}\,E(H)^{rig}$$

be the $\mathcal{O}_D^\times$-equivariant rigid analytic vector bundle over $\mathfrak{X}^{rig}$.

Théorème 1. *$Lie\,E(H)^{rig}$ is a flat $\mathcal{O}_D^\times$-vector bundle : there is an isomorphism*

$$Lie\,E(H)^{rig} \simeq \mathbb{D}(\mathbb{H})_{\mathbb{Q}} \otimes_L \mathcal{O}_{\mathfrak{X}^{rig}}$$

1.4 First proof via the rigidity of quasi-isogenies

For each $a \in \mathbb{N}^*$ consider the following p-adic integral model of $\mathcal{A}_a$

$$A_a = \mathcal{O}_L < X_1, \dots, X_{n-1}, T_1, \dots, T_{n-1} > /(X_i^a - pT_i)_{1 \le i \le n-1}$$

thus $\mathcal{A}_a = A_a[\frac{1}{p}]$. Equip A_a with the p-adic topology. Then there is a morphism

$$\mathrm{Spf}(A_a) \longrightarrow \mathfrak{X}$$

associated to the continous $\mathcal{O}_L$-algebras morphism

$$\begin{array}{rcl} W(k)[[x_1, \dots, x_{n-1}]] & \longrightarrow & A_a \\ x_i & \longmapsto & X_i \end{array}$$

that induces the inclusion $\mathrm{Sp}(\mathcal{A}_a) \hookrightarrow \mathfrak{X}^{rig}$ on the generic fiber. Let (H_a, ρ_a) be the pullback of (H, ρ) to $\mathrm{Spf}(A_a)$ where

$$\rho_a : \mathbb{H} \times_{\mathrm{Spec}(\overline{\mathbb{F}}_p)} \mathrm{Spec}(\overline{\mathbb{F}}_p[T_1, \dots, T_{n-1}]) \xrightarrow{\sim} H_a \bmod (p, X_1, \dots, X_{n-1})$$

Now observe the ideal $(X_1, \dots, X_{n-1})$ is nilpotent in A_a/pA_a. Thus by rigidity of quasi-isogenies there exists two morphisms

$$\mathbb{H} \times_{\mathrm{Spec}(\overline{\mathbb{F}}_p)} \mathrm{Spec}(A_a/pA_a) \underset{f_2}{\overset{f_1}{\rightleftarrows}} H_a \times_{\mathrm{Spec}(A_a)} \mathrm{Spec}(A_a/pA_a)$$

and an integer $N \in \mathbb{N}^*$ such that

$$f_1 \circ f_2 = p^N Id \quad f_2 \circ f_1 = p^N Id$$

Now suppose $p \neq 2$. Then the ideal pA_a of A_a is equiped with nilpotent divided powers (if $p \neq 2$ replace $2A_a$ by $4A_a$). Consider the crystal of

$$\mathbb{H} \times_{\mathrm{Spec}(\overline{\mathbb{F}}_p)} \mathrm{Spec}(A_a/pA_a)$$

evaluated on the P.D. thickening

$$\mathrm{Spec}(A_a/pA_a) \hookrightarrow \mathrm{Spec}(A_a)$$

and noted

$$\mathbb{D}(\mathbb{H})_{(\mathrm{Spec}(A_a/pA_a) \hookrightarrow \mathrm{Spec}(A_a))}$$

There is a morphism of P.D. thickenings

$$\begin{array}{ccc} \mathrm{Spec}(A_a/pA_a) & \hookrightarrow & \mathrm{Spec}(A_a) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{\mathbb{F}}_p) & \hookrightarrow & \mathrm{Spec}(W(\overline{\mathbb{F}}_p)) \end{array}$$

Thus by the crystal property there is a canonical isomorphism

$$\mathbb{D}(\mathbb{H})_{(\mathrm{Spec}(A_a/pA_a) \hookrightarrow \mathrm{Spec}(A_a))} \simeq \mathbb{D}(\mathbb{H}) \otimes_{\mathcal{O}_L} A_a$$

where $\mathbb{D}(\mathbb{H})$ without any subscript means the Dieudonné module of $\mathbb{H} : \mathbb{D}(\mathbb{H}) = \mathbb{D}(\mathbb{H})_{(\mathrm{Spec}(\overline{\mathbb{F}}_p) \hookrightarrow \mathrm{Spec}(W(\overline{\mathbb{F}}_p)))}$. Now the crystal of $H_a \times_{\mathrm{Spec}(A_a)} \mathrm{Spec}(A_a/pA_a)$ evaluated on the P.D. thickening $\mathrm{Spec}(A_a/pA_a) \hookrightarrow \mathrm{Spec}(A_a)$ is

$$\mathrm{Lie}\, E(H_a)$$

By functoriality of the crystal functor f_1 and f_2 induce

$$\mathbb{D}(\mathbb{H}) \otimes_{\mathcal{O}_L} A_a \underset{\mathbb{D}(f_2)}{\overset{\mathbb{D}(f_1)}{\rightleftarrows}} \operatorname{Lie} E(H_a)$$

such that $\mathbb{D}(f_1) \circ \mathbb{D}(f_2) = \mathbb{D}(f_1 \circ f_2) = \mathbb{D}(p^N Id) = p^N Id$ and $\mathbb{D}(f_2) \circ \mathbb{D}(f_1) = p^N Id$. Thus there is a canonical isomorphism

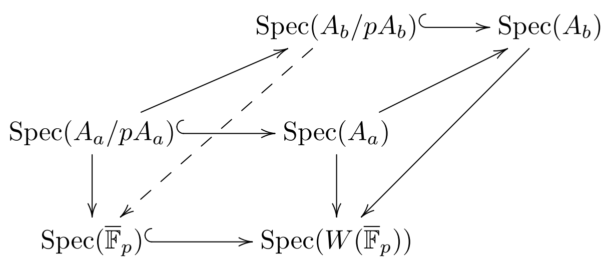
$$\operatorname{Lie} E(H_a)[\frac{1}{p}] \simeq \mathbb{D}(\mathbb{H}) \otimes_{\mathcal{O}_L} \mathcal{A}_a$$

induced by $p^k \mathbb{D}(f_1)$ where $k \in \mathbb{Z}$ and $p^k f_1 \equiv \rho_a$ modulo $(X_1, \ldots, X_{n-1})$.

Now those morphisms are compatible for varying a : for $a \leq b$ there is a restriction morhpism $\mathcal{A}_a \longrightarrow \mathcal{A}_b$ and the following diagramm commutes

$$\begin{array}{ccc} \operatorname{Lie} E(H_b)[\frac{1}{p}] & \xrightarrow{\sim} & \mathbb{D}(\mathbb{H}) \otimes \mathcal{A}_b \\ \downarrow & & \downarrow \\ \operatorname{Lie} E(H_a)[\frac{1}{p}] & \xrightarrow{\sim} & \mathbb{D}(\mathbb{H}) \otimes \mathcal{A}_a \end{array}$$

This follows from the commutativity of the following diagramm of P.D. thickenings



Now let's verify this isomorphism is $\mathcal{O}_D^\times$-equivariant. For each $g \in \mathcal{O}_D^\times$ for each $a \in \mathbb{N}^*$ $\exists b \in \mathbb{N}^*$ such that

$$g(\mathbb{B}(0, p^{-1/a})) \subset \mathbb{B}(0, p^{-1/b})$$

Thus the associated co-morphism induces

$$g^* : \mathcal{A}_b \longrightarrow \mathcal{A}_a$$

But one can verify $\forall c$ $A_c = \{f \in \mathcal{A}_c \mid \|f\|_{\mathcal{A}_c,\infty} \leq 1\}$ where $\|f\|_{\mathcal{A}_c,\infty} = \sup\{|f(x)| \mid x \in \operatorname{Sp}(\mathcal{A}_c)\}$. Thus g^* induces a morphism at the level of the integral models

$$g^* : A_b \longrightarrow A_a$$

since $\forall f \in \mathcal{A}_b$ $\|g^* f\|_{\mathcal{A}_a,\infty} \leq \|f\|_{\mathcal{A}_b,\infty}$. But now for the quasi-isogeny

$$\mathbb{H} \times_{\operatorname{Spec}(\overline{\mathbb{F}}_p)} \operatorname{Spec}(A_b/pA_b) \xrightarrow{\rho_b} H_b \times_{\operatorname{Spec}(A_b)} \operatorname{Spec}(A_b/pA_b)$$

if one takes the pullback $g^*\rho_b$ of ρ_b to $\operatorname{Spec}(A_a/pA_a)$ via $g : \operatorname{Spec}(A_a/pA_a) \longrightarrow \operatorname{Spec}(A_b/pA_b)$ one has a commutative diagramm

$$\begin{array}{ccc} \mathbb{H} \times_{\operatorname{Spec}(\overline{\mathbb{F}}_p)} \operatorname{Spec}(A_a/pA_a) & \xrightarrow{g^*\rho_b} & H_a \times_{\operatorname{Spec}(A_a)} \operatorname{Spec}(A_a/pA_a) \\ \downarrow{\scriptstyle g^{-1}} & & \| \\ \mathbb{H} \times_{\operatorname{Spec}(\overline{\mathbb{F}}_p)} \operatorname{Spec}(A_a/pA_a) & \xrightarrow{\rho_a} & H_a \times_{\operatorname{Spec}(A_a)} \operatorname{Spec}(A_a/pA_a) \end{array}$$

where $g^{-1} \in \mathrm{Aut}(\mathbb{H})$. From this it follows the isomorphism

$$h_a : \mathbb{D}(\mathbb{H}) \otimes_{\mathcal{O}_L} \mathcal{A}_a \xrightarrow{\sim} \mathrm{Lie}\, E(H_a)[\frac{1}{p}]$$

we onstructed before verifies for $g^* : \mathcal{A}_b \longrightarrow \mathcal{A}_a$ the following diagram commutes

$$\begin{array}{ccc} \mathbb{D}(\mathbb{H}) \otimes \mathcal{A}_a & \xrightarrow{g^* h_b} & \mathrm{Lie}\, E(H_a)[\frac{1}{p}] \\ \downarrow{\scriptstyle \mathbb{D}(g)^{-1} \otimes Id} & & \| \\ \mathbb{D}(\mathbb{H}) \otimes \mathcal{A}_a & \xrightarrow{h_a} & \mathrm{Lie}\, E(H_a)[\frac{1}{p}] \end{array}$$

□

1.5 Second proof via Dwork's trick

Let

$$\nabla : \mathrm{Lie}\, E(H) \longrightarrow \mathrm{Lie}\, E(H) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/W(\overline{\mathbb{F}}_p)}$$

be the Gauss Manin connection. Let's look at the system of differential equations

$$\left(\mathrm{Lie}\, E(H)^{rig}\right)^{\nabla=0}$$

as a sheaf on the grothendieck topology of admissible open subsets where $\nabla v = 0$ means $\forall 1 \leq i \leq n_1 \ \nabla_{\frac{\partial}{\partial x_i}} v = 0$. One wants to prove $\mathrm{Lie}\, E(H)^{rig}$ is generated by the solutions of this system as in the Riemann Hilbert correspondence bewteen local systems of $\mathbb{C}$-vector spaces and vector bundles equiped with an integrable connection on a complexe manifold S

$$\mathcal{F} \longmapsto (\mathcal{F} \otimes_{\mathbb{C}} \mathcal{O}_S, Id \otimes d)$$

$$\mathcal{E}^{\nabla=0} \longleftarrow\!\shortmid (\mathcal{E}, \nabla)$$

and if S is simply connected $\mathcal{E}^{\nabla=0}$ is trivial and thus $\mathcal{E}$ is generated by it global solutions.

Let's first see on

$$\mathbb{B}(0, p^{-1/(p-1)})$$

$\mathrm{Lie}\, E(H)^{rig}$ is generated by its flat section. In fact if

$$B = \mathcal{O}_L < X_1, \ldots, X_{n-1}, T_1, \ldots, T_{n-1} > /(X_i^{p-1} - pT_i)_{1 \leq i \leq n-1}$$

then $\mathrm{Spf}(B)$ is an integral model of $\mathbb{B}(0, p^{-1/(p-1)})$ such that the ideal

$$(p, X_1, \ldots, X_{n-1})$$

in B is equiped with topologicaly nilpotent divided powers ($p \neq 2$). Thus there is an isomorphism

$$\mathrm{Lie}(E(H)) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathrm{Spf}(B)} \simeq \mathbb{D}(\mathbb{H}) \otimes_{\mathcal{O}_L} \mathcal{O}_{\mathrm{Spf}(B)}$$

Now let $\mathcal{E}$ be the crystal of H mod p and $\mathcal{E}^{(p)}$ its pullback by the Frobenius. There are two morphisms

$$\mathcal{E} \underset{F}{\overset{V}{\rightleftarrows}} \mathcal{E}^{(p)}$$

such that $FV = p$ and $VF = p$ induced by

$$G \underset{V}{\overset{F}{\rightleftarrows}} G^{(p)}$$

Let $\varphi : \mathfrak{X} \longrightarrow \mathfrak{X}$ be the lifting of the Frobenius morphism defined by

$$\forall i \;\; \varphi^* x_i = x_i^p$$

and $\varphi = \sigma$ on $W(\overline{\mathbb{F}}_p)$. Then the main constatation is

$$\forall 0 < \epsilon < 1 \;\; \varphi^{rig}(\mathbb{B}(0,\epsilon)) \subset \mathbb{B}(0,\epsilon^p)$$

Thus

$$\forall 0 < \epsilon < 1 \; \exists i \;\; (\varphi^{rig})^i \left(\mathbb{B}(0,\epsilon)\right) \subset \mathbb{B}(0,p^{-1/(p-1)})$$

or

$$\boxed{\mathfrak{X}^{rig} = \bigcup_{i\geq 1} (\varphi^{rig})^{-i}\left(\mathbb{B}(0,p^{-1/(p-1)})\right)}$$

Let's set

$$\forall k \geq 1 \; B_k = \mathcal{O}_L < X_1,\dots,X_{n-1},T_1,\dots,T_{n-1} > /(X_i^{k(p-1)} - pT_i)_{1\leq i\leq n-1}$$

thus $\mathrm{Spf}(B_k)^{rig} = \mathbb{B}(0,p^{-\frac{1}{k(p-1)}})$ and the morphism $\mathrm{Spf}(B_k) \longrightarrow \mathrm{Spf}(B_{k+1})$ given by $X_i \mapsto X_i$ and $T_i \mapsto X_i^{p-1}T_i$ induces the inclusion $\mathbb{B}(0,p^{-\frac{1}{k(p-1)}}) \hookrightarrow \mathbb{B}(0,p^{-\frac{1}{(k+1)(p-1)}})$ on the generic fiber

Moreover φ induces

$$\forall k\geq 1 \;\; \varphi : \mathrm{Spf}(B_{pk}) \longrightarrow \mathrm{Spf}(B_k)$$

Now for all $i \in \mathbb{N}^*$

$$\left(\mathcal{E}^{(p^i)}\right)_{(\mathrm{Spec}(B_{p^i}/pB_{p^i})\hookrightarrow \mathrm{Spec}(B_{p^i}))} \simeq (\varphi^i)^*\mathcal{E}_{(\mathrm{Spec}(B_1/pB_1)\hookrightarrow\mathrm{Spec}(B_1))}$$

since there is a morphism of P.D. thickenings

$$\begin{array}{ccc} \mathrm{Spec}(B_{p^i}/pB_{p^i}) & \hookrightarrow & \mathrm{Spec}(B_{p^i}) \\ \downarrow{\scriptstyle Frob} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Spec}(B_1/pB_1) & \hookrightarrow & \mathrm{Spec}(B_1) \end{array}$$

Now consider the following sequence of morphisms

$$\begin{array}{ccccc} \mathrm{Lie}\, E(H)\otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathrm{Spf}(B_{p^i})} & = & \mathcal{E}_{(\mathrm{Spec}(B_{p^i}/pB_{p^i})\hookrightarrow\mathrm{Spec}(B_{p^i}))} & \xrightarrow{V^i} & \left(\mathcal{E}^{(p^i)}\right)_{(\mathrm{Spec}(B_{p^i}/pB_{p^i})\hookrightarrow\mathrm{Spec}(B_{p^i}))} \\ & & & & \| \\ \mathbb{D}(\mathbb{H})\otimes_{\mathcal{O}_L,\sigma^i}\mathcal{O}_{\mathrm{Spf}(B_{p^i})} & = & \left(\mathbb{D}(\mathbb{H})\otimes_{\mathcal{O}_L}\mathcal{O}_{\mathrm{Spf}(B_1)}\right)\otimes_{\mathcal{O}_{\mathrm{Spf}(B_1)},(\varphi^i)^*}\mathcal{O}_{\mathrm{Spf}(B_{p^i})} & \xleftarrow{\simeq} & (\varphi^i)^*\mathcal{E}_{(\mathrm{Spec}(B_1/pB_1)\hookrightarrow\mathrm{Spec}(B_1))} \\ \downarrow{\scriptstyle F^i\otimes Id} & & & & \\ \mathbb{D}(\mathbb{H})\otimes_{\mathcal{O}_L}\mathcal{O}_{\mathrm{Spf}(B_{p^i})} & & & & \end{array}$$

Note the composite

$$\Delta_i : \mathrm{Lie}\, E(H)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathrm{Spf}(B_{p^i})} \longrightarrow \mathbb{D}(\mathbb{H})\otimes_{\mathcal{O}_L}\mathcal{O}_{\mathrm{Spf}(B_{p^i})}$$

Then consider

$$\begin{array}{ccc} \frac{1}{p^i}\Delta_i : \mathrm{Lie}\, E(H)^{rig}\otimes_{\mathcal{O}_{\mathfrak{X}^{rig}}}\mathcal{O}_{Sp(B_{p^i}[\frac{1}{p}])} & \longrightarrow & \mathbb{D}(\mathbb{H})\otimes_{\mathcal{O}_L}\mathcal{O}_{Sp(B_{p^i}[\frac{1}{p}])} \\ \| & & \| \\ \left(\mathrm{Lie}\, E(H)^{rig}\right)_{|\mathbb{B}(0,p^{-\frac{1}{p^i(p-1)}})} & & \mathbb{D}(\mathbb{H})\otimes\mathcal{O}_{\mathbb{B}(0,p^{-\frac{1}{p^i(p-1)}})} \end{array}$$

This is an isomorphism since we have defines Δ_i as $\Delta_i = F^i \otimes Id \circ \alpha_i \circ V^i$ for some isomorphism α_i and then $\frac{1}{p^i} F^i \circ \alpha_i^{-1} \circ V^i \otimes Id$ is a reciproc isomorphism to $\frac{1}{p^i}\Delta_i$.

Furthermore one can verify easily the isomorphisms $\frac{1}{p^i}\Delta_i$ are compatible when i varies (use once more $FV = p$). □

1.6 The link between both proofs

Let $S_0 \hookrightarrow S$ be a nilpotent immersion of quasi-compact $\mathbb{F}_p$-schemes. Let G be a p-divisible group over S and set $G_0 = G \times_S S_0$. Suppose there exists a section

$$S_0 \underset{\epsilon}{\overset{\curvearrowleft}{\hookrightarrow}} S$$

Lemme 2. *For $i >> 0$ $\quad G^{(p^i)} \simeq G_0^{(p^i)} \times_{S_0,\epsilon,} S$*

Démonstration. easy □

Thus for $i >> 0$ there is an isogeny

$$\delta_i : G \xrightarrow{F^i} G^{(p^i)} \simeq G_0^{(p^i)} \times_{S_0,\epsilon} S \xrightarrow{V^i \times Id} G_0 \times_{S_0,\epsilon} S$$

and thus a degre 0 quasi-isogeny : $\frac{1}{p^i}\delta_i$.

Now apply this to $S = \mathrm{Spec}(A_k/pA_k)$ and $S_0 = V(X_1, \ldots, X_{n-1})$.

1.7 The period morphism

We have a filtration by localy free sheaves

$$0 \longrightarrow V(H)^{rig} \longrightarrow \mathrm{Lie}\, E(H)^{rig} \longrightarrow \underbrace{\mathrm{Lie}\,(H)^{rig}}_{\text{rank } 1} \longrightarrow 0$$

that is $\mathcal{O}_D^\times$-equivariant. Moreover we have an $\mathcal{O}_D^\times$-equivariant isomorphism

$$\mathrm{Lie}\, E(H)^{rig} \simeq \mathbb{D}(\mathbb{H}) \otimes \mathcal{O}_{\mathfrak{X}^{rig}}$$

Définition 1. *We note $\breve{\pi} : \mathfrak{X}^{rig} \longrightarrow \mathbb{P}(\mathbb{D}(\mathbb{H}))$ for the $\mathcal{O}_D^\times$-equivariant rigid analytic morphism induced by*

$$\mathbb{D}(\mathbb{H}) \otimes \mathcal{O}_{\mathfrak{X}^{rig}} \simeq Lie\, E(H)^{rig} \twoheadrightarrow Lie\,(H)^{rig}$$

2 First properties of the period morphism

2.1 Étalness

Proposition 1. *The period morphism $\breve{\pi}$ is étale.*

Démonstration. This is a consequence of Grothendieck-Messing deformation theory. One has to check for any affinoïd algebra $\mathcal{A}$ equiped with a squared zero ideal I the diagram

$$\begin{array}{ccc} \mathfrak{X}^{rig}(\mathcal{A}) & \longrightarrow & \mathbb{P}(\mathbb{D}(\mathbb{H}))^{rig}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathfrak{X}^{rig}(\mathcal{A}/I) & \longrightarrow & \mathbb{P}(\mathbb{D}(\mathbb{H}))^{rig}(\mathcal{A}/I) \end{array}$$

is cartesian. This is a bit technical because one has to chose models for Spf($\mathcal{A}$)...this is done in ? ? ?. But at the end this is a consequence of Grothendieck-Messing deformation theory. □

Remarque 1. *There are different notions of étalness for morphisms between rigid spaces. For example the inclusion of the unit ball $\mathbb{B}(0,1) \hookrightarrow \mathbb{A}^1$ is not étale in Berkovich's theory but is in the classical theory of Tate (since it is an open immersion). In terms of classical rigid spaces the difference is between étale morphisms and overconvergent étale one (the overconvergent one being the étale morphisms in Berkovich's theory). But for morphisms between $\mathfrak{X}^{rig}$ and $\mathbb{P}^n$ both notions coïncide (in terms of Berkovich's langage : $\partial\mathfrak{X}^{an} = \emptyset$ and $\partial\mathbb{P}^n = \emptyset$ thus for any $f : \mathfrak{X}^{an} \longrightarrow \mathbb{P}^n$ $\partial(\mathfrak{X}^{an}/\mathbb{P}^n) = \emptyset$).*

2.2 Fibers

Let $K|L$ be a valued extension for a valuation $v : K \longrightarrow \mathbb{R} \cup \{+\infty\}$ and such that K is complete for this valuation.

Proposition 2. *Let $x = (H, \rho) \in \mathfrak{X}^{rig}(K) = \mathfrak{X}(\mathcal{O}_K)$. Then the fiber*

$$\breve{\pi}^{-1}(\breve{\pi}(x)) \subset \mathfrak{X}^{rig}(K)$$

is the set of (H', ρ') such that there exists an isogeny $f : H \longrightarrow H'$ over Spec$(\mathcal{O}_K)$ and an integer $N \in \mathbb{N}$ s.t. the following diagram commutes

$$\begin{array}{ccc} \mathbb{H} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_K/p\mathcal{O}_K & \xrightarrow{\rho} & H \ mod\ p \\ \big\downarrow p^N & & \big\downarrow f\ mod\ p \\ \mathbb{H} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_K/p\mathcal{O}_K & \xrightarrow{\rho'} & H' \ mod\ p \end{array}$$

or equivalently such that the quasi-isogeny $\rho' \circ \rho^{-1}$ between H mod p and H' mod p lifts to a quasi-isogeny between H and H' over Spec$(\mathcal{O}_K)$.

Démonstration. If the quasi-isogeny $\rho' \circ \rho^{-1}$ over Spf$(\mathcal{O}_K)$ lifts to a quasi-isogeny f over Spec$(\mathcal{O}_K)$ then it induces an isomorphism

$$\begin{array}{ccccc} \mathrm{Fil}\,\mathrm{Lie}\,E(H)[\frac{1}{p}] & \hookrightarrow & \mathrm{Lie}\,E(H)[\frac{1}{p}] & & \\ \big\downarrow f_* \simeq & & \big\downarrow f_* \simeq & \searrow^{\simeq} & \mathbb{D}(\mathbb{H})[\frac{1}{p}] \otimes K \\ \mathrm{Fil}\,\mathrm{Lie}\,E(H')[\frac{1}{p}] & \hookrightarrow & \mathrm{Lie}\,E(H')[\frac{1}{p}] & \nearrow_{\simeq} & \end{array}$$

by functoriality of the universal vector extension. Thus they have the same image by the period morphism. Reciprocaly, if (H', ρ') has the same image by the period morphism as (H, ρ) then the induced morphism at the level of the evaluation of the crystals of H mod p and H' mod p on $\mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/p\mathcal{O}_K$ by the quasi-isogeny $\rho' \circ \rho^{-1}$ induces an isomorphism like in the preceding diagram. Thus for $N >> 0$ $p^N \rho' \circ \rho^{-1}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}\,\mathrm{Lie}\,E(H) & \hookrightarrow & \mathrm{Lie}\,E(H) \\ \big\downarrow & & \big\downarrow \mathbb{D}(p^N\rho' \circ \rho^{-1}) = p^N \mathbb{D}(\rho' \circ \rho^{-1}) \\ \mathrm{Fil}\,\mathrm{Lie}\,E(H') & \hookrightarrow & \mathrm{Lie}\,E(H') \end{array}$$

and thus by Messing's theorem $p^N \rho' \circ \rho^{-1}$ lifts to $\mathcal{O}_K$. In the same way for $N >> 0$ $p^N \rho \circ \rho'^{-1}$ lifts to $\mathcal{O}_K$. □

Lemme 3. *Let H be a p-divisible group over $\mathcal{O}_K$. Then the isomorphism classes of couples (H', f) where H' is a p-divsible group over $\mathcal{O}_K$ and $f : H \longrightarrow H'$ is a quasi-isogeny over Spec$(\mathcal{O}_K)$ are in bijection with Gal$(\overline{K}|K)$-stable lattices inside $V_p(H)$.*

Démonstration. Such isomorphism classes are in bijection with equivalence classes of couples (N, H) where $N \in \mathbb{Z}$ and H is a finite localy free sub-group scheme of H over $\mathcal{O}_K$ and where the equivalence relation is the one generated by (N, H) is equivalent to $(N-1, p^{-1}(H))$. To (N, H) is associated $p^N\times$ the isogeny $H \longrightarrow H/N$.

But since $\mathcal{O}_K$ is a valuation ring, for any $k \in \mathbb{N}^*$ the map $H \longmapsto H \otimes_{\mathcal{O}_K} K$ from finite flat sub-group schemes of $H[p^k]$ to the one of $H[p^k] \otimes_{\mathcal{O}_K} K$ is a bijection. The reverse bijection is given by schematical closure.

Thus couples (N, H) are in bijection with couples (N, W) where $N \in \mathbb{Z}$ and $W \subset T_p(H) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is a Gal$(\overline{K}|K)$-stable finite sub-group. Now equivalence classes of such couples are in bijection with Galois-stable lattices in $V_p(H)$. To Λ is associated $(N, p^{-N}\Lambda/T_p(H))$ for $N >> 0$. □

Corollaire 1. *Suppose $K = \overline{K}$. Let $x = (H, \rho) \in \mathfrak{X}^{rig}(K) = \mathfrak{X}(\mathcal{O}_K)$. Then the fiber*

$$\breve{\pi}^{-1}(\breve{\pi}(x)) \subset \mathfrak{X}^{rig}(K)$$

is a homogenous space under $GL(V_p(H))^1$ the stabilizer of x being $GL(T_p(H))$. This fiber is thus in bijection with

$$GL_n(\mathbb{Q}_p)^1/GL_n(\mathbb{Z}_p)$$

3 An explicit formula for the period morphism

For any ring R we note $W(R)$ fo its witt vectors. There is an augmentation $W(R) \twoheadrightarrow R$ that sends $[x_0, \dots]$ to x_0. The associated augmentation ideal is equiped with divided powers (functorialy in R, as usual is suficies to verify it in a universal situation when R is torsion free, $R = \mathbb{Z}[X_i]_{i\geq 0}$). This divided powers are compatible with the one on the ideal (p) of $\mathbb{Z}_{(p)}$. If R is a p-adic ring then $W(R)$ is p-adic and if $p \neq 2$ the divided powers on the augmentation ideal of $W(R)$ are topologicaly nilpotent. Moerover $W(R)$ is canonicaly equiped with a litfting of Frobenius $F : W(R) \longrightarrow W(R)$ relative to $W(R) \longrightarrow R/pR$:

$$\begin{array}{ccc} W(R) & \longrightarrow & R/pR \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle Frob} \\ W(R) & \longrightarrow & R/pR \end{array}$$

and if $I_R = \ker(W(R) \longrightarrow R)$ then (I_R, p) is equiped with topologicaly nilpotent divided powers $(p \neq 2)$ since the divided powers on I_R are compatible with the one on (p).

Théorème 2 (Zink)**.** *There exists a system of formal coordinates $(x_1, \dots, x_{n-1})$ on $\mathfrak{X}$ s.t. if $R = W(\overline{\mathbb{F}}_p)[[x_1, \dots, x_{n-1}]]$ and H is the universal deformation over $\mathfrak{X}$ then the evaluation of the crystal of H over $W(R)$ relative to the P.D. thickening $W(R) \twoheadrightarrow R$ is*

$$\mathbb{D}(H)_{W(R)\twoheadrightarrow R} = W(R)e_1 \oplus \cdots \oplus W(R)e_n$$

the associated morphism $\mathbb{D}(H)_{W(R)\twoheadrightarrow R} \longrightarrow \mathbb{D}(H)_{R\xrightarrow{Id}R} = Lie\, E(H) \longrightarrow Lie\, H$ giving the filtration of the filtered module is identified with the projection

$$W(R)e_1 \oplus \dots W(R) \oplus e_n \twoheadrightarrow Re_1$$

Moerover the associated Frobenius morphism associated to the Verschiebung $V : G^{(p)} \longrightarrow G$, the P.D. thickening $W(R) \twoheadrightarrow R/pR$ and the lifting of Frobenius F on $W(R)$

$$\begin{array}{ccccc} \mathbb{D}(H)_{W(R)\twoheadrightarrow R} \otimes_{W(R),F} W(R) & = & \mathbb{D}\left((H \otimes_R R/pR)^{(p)}\right)_{W(R)\twoheadrightarrow R/pR} & \xrightarrow{\mathbb{D}(V)} & \mathbb{D}(H \otimes_R R/pR)_{W(R)\twoheadrightarrow R/pR} \\ & \searrow^{\text{Frobenius}} & & & \| \\ & & & & \mathbb{D}(H)_{W(R)\twoheadrightarrow R} \end{array}$$

has as a matrix in the bases $(e_i \otimes 1)_i$ and $(e_i)_i$

$$\begin{pmatrix} [x_1] & p[x_2] & \dots & p[x_{n-1}] & p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & p & & 0 & 0 \\ & 0 & \ddots & \vdots & \vdots \\ & & & p & 0 \end{pmatrix}$$

where $[x_i] \in W(R)$ is the Teichmüller lift of x_i.

In this coordinates if

$$\begin{pmatrix} x_1 & px_2 & \dots & px_{n-1} & p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & p & & 0 & 0 \\ & 0 & \ddots & \vdots & \vdots \\ & & & p & 0 \end{pmatrix}$$

is the reduction of the preceding matrix through $W(R) \twoheadrightarrow R$ and

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & p & & 0 & 0 \\ & 0 & \ddots & \vdots & \vdots \\ & & & p & 0 \end{pmatrix}$$

is the matrix of the Frobenius of the crystal of $\mathbb{H}$ then one has the following formula for the period morphism

$$\boxed{\breve{\pi} = \lim_{k\to+\infty} [1 : 0 : \dots : 0].AA^{(\sigma)} \dots A^{(\sigma^{k-1})}.B^{-k}}$$

where $A^{(\sigma^k)}$ is for the matrix A where one replaces all x_i by $x_i^{p^k}$. This limit has to be taken in the sens of the Frechet topology of $\Gamma(\mathfrak{X}^{rig}, \mathcal{O}_{\mathfrak{X}^{rig}})$ defined by uniform convergence on each closed ball with radius < 1.

3.1 The GL$_2$-case

Théorème 3. *Let $\mathfrak{X} = Spf(W(\overline{\mathbb{F}}_p)[[x]])$ be the Lubin-Tate space for $n = 2$. The period morphism $\breve{\pi} = [f_0 : f_1]$ induces an isomorphism from the open ball*

$$\{x \in \mathfrak{X}^{rig} \mid v(x) > \frac{1}{p+1}\}$$

to the open ball

$$\{[y_0 : y_1] \in \mathbb{P}^1 \mid v(y_1/y_0) > \frac{1}{p+1}\}$$

moreover on this ball we have

$$v(f_1(x)/f_0(x)) = v(x)$$