An introduction to the geometry of Lubin-Tate spaces

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1 The Lubin-Tate tower

1.1 Motivation

As allready remarked, the upper/lower Poincaré half plane $\mathcal{H}^{\pm} = \mathbb{C} \setminus \mathbb{R}$ represents the functor from smooth analytic spaces S to sets that associates to S the isomorphism classes of couples (E, ρ) where $E \xrightarrow{p} S$ is an elliptic curve and $\rho : \underline{\mathbb{Z}}^2 \xrightarrow{\sim} R^1 p_* \underline{\mathbb{Z}}$. Instead of trivialising the Betti cohomology one can trivialse it partialy : for $N \geq 3$ one can consider the functor $X(N)$ that associates to S the isomorphism classes of couples (E, ρ) where E is as before and

$$\rho : (\underline{Z}/N\underline{\mathbb{Z}})^2 \xrightarrow{\sim} R^1 p_* \underline{\mathbb{Z}}/N\underline{\mathbb{Z}}$$

This is the level N modular curve.

On Lubin-Tate spaces the situation is more complicated : there are two fiber functors on p-divisible groups in unequal caracteristic, the Dieudonné functor (crystalline cohomology of the special fiber) and the Tate module functor (the p-adic étale cohomology of the generic fiber). We have allready trivialized the Dieudonné functor on Lubin-Tate spaces by specifying the isomorphism class of our p-divisible group on the special fiber. Now, our space is not simply connected as is $\mathcal{H}^{\pm}$. We still can trivialize the Tate-module, at least partialy like in the case of $X(N)$.

1.2 Level structures on the generic fiber

Let $\mathfrak{X} = \mathrm{Spf}(R)$ be the Lubin-Tate space of deformations of the p-divisible group $\mathbb{H}$ of height n. Let (H, ρ) be the universal deformation. We still note H for the associated p-divisible group over $\mathrm{Spec}(R)$.

Let's consider the Tate module $T_p(H)$ of $H \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[\frac{1}{p}])$. We can see it as an étale p-adic local system on $\mathfrak{X}^{rig}$: the system $(H[p^k]^{rig})_{k \geq 1}$.

Définition 1. *Let $X = \mathfrak{X}^{rig}$. We define forall $k \geq 1$ X_k to be the functor on rigid analytic spaces over $\mathfrak{X}^{rig}$ that associates to $S \longrightarrow X$ the sets of isomorphism of étale sheaves*

$$\eta : (p^{-k}\mathbb{Z}/\mathbb{Z})^n \xrightarrow{\sim} H[p^k]^{rig} \times_X S$$

called a level k structure.

Lemme 1. *X is represented by a finite étale Galois covering of X with Galois group $GL_n(\mathbb{Z}/p^k\mathbb{Z})$.*

Démonstration. There is a right action of $\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ on X by $\eta \mapsto \eta \circ g$. After a finite étale covering of X trivializing $H[p^k]$ that is to say $X' \longrightarrow X$ such that $H[p^k] \times_S X' \simeq (p^{-k}\mathbb{Z}/\mathbb{Z})^n$, $X_k \times_X X'$ becomes a $\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$-Galois cover over X'. Thus by finite étale descent this proves the assertion. □

We thus have a tower of étale coverings

$$\ldots \longrightarrow X_{k+1} \longrightarrow X_k \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X$$

with an action of $\mathrm{GL}_n(\mathbb{Z}_p)$ wich make it "pro-Galois covering of X with Galois group $\mathrm{GL}_n(\mathbb{Z}_p)$".

1.3 Drinfeld level structures

Définition 2. *Let $(A, \mathfrak{m})$ be a complete local ring with caratetristic p residue field. Let $F \in A[[X,Y]]$ be a height n formal group law on R. A level k Drinfeld structure on F is a group morphism*

$$\eta : (p^{-k}\mathbb{Z}/\mathbb{Z})^n \longrightarrow F[p^k](R)$$

that is to say a group morphism $\eta : (p^{-k}\mathbb{Z}/\mathbb{Z})^n \longrightarrow (\mathfrak{m}, \underset{F}{+})$ such that

$$\prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x)) \mid [p^k]_F(T)$$

Remarque 1. *The preceding definition depends only on the formal group and not on the group law since for any $f \in A[[T]]$, $f(0) = 0$ and $f'(0) \in A^\times$*

$$\prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x)) \mid [p^k]_F(T) \Longleftrightarrow \prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T - f(\eta(x))) \mid f \circ [p^k]_F \circ f^{-1}$$

Hence the preceding definition makes sens for a formal p-divisible group H without any particular choice of an isomorphism of pointed formal schemes $Spf(A[[T]]) \xrightarrow{\sim} H$.

We will use repeteadly the following form of Weierstrass division

Lemme 2. *Let A be an I-adicaly compete ring. Let $g = \sum_{i\geq 0} a_i T^i \in A[[T]]$. Suppose $\exists n\ a_n \in A^\times$ and $\forall i < n\ a_i \in I$. Then for all $f \in A[[T]]$ $\exists Q, R$ such that $Q \in A[[T]]$, $R \in A[T]$, $\deg R < n$ and*

$$f = Qg + R$$

Lemme 3. *Let $F(X,Y)$ be a formal group law over a ring B. Then $\exists w \in B[[X,Y]]^\times$ such that*

$$X \underset{F}{-} Y = (X - Y) \times w$$

Démonstration. We apply Weierstrass division theorem in the ring $C = B[[Y]]$, complete for the Y-adic topology. The series $X - Y \in C[[X]]$ verifies that the coefficient of X is 1, a unit, and $-Y$ is topologicaly nilpotent. Thus we can apply Weierstrass division

$$\exists f \in C[[X]]\ \exists g \in C\ \ X \underset{F}{-} Y = (X - Y)f + g$$

But puting $X = Y$ one sees $g = 0$. Thus $X \underset{F}{-} Y = (X - Y)f$. Now since $X \underset{F}{-} Y \equiv X - Y \mod \deg 2$ one sees $f(0) = 1$ and thus f is a unit. □

Remarque 2. *Last lemma is false for $X \underset{F}{+} Y$ and $X + Y$ as one sees in the case of $\widehat{\mathbb{G}}_m$.*

Corollaire 1. *In the definition of a Drinfeld level structure one can replace*

$$\prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x))$$

by

$$\prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x))$$

Proposition 1. *With the hypothesis of the preceding definition for a morphism*

$$\eta : (p^{-k}\mathbb{Z}/\mathbb{Z})^n \longrightarrow (\mathfrak{m}, \underset{F}{+})$$

the following are equivalent

— *η is a Drinfeld level structure*

— $[p^k]_F(T) = u \times \prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x))$ *where* $u \in A[[T]]^\times$.

— $\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x)) \mid [p]_F(T)$

— $[p]_F(T) = v \times \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x))$ *with* $v \in A[[T]]^\times$.

Démonstration. The second assertion clearly implies the first. Let's see the first assertion implies the second one. If

$$[p^k]_F(T) = f \times \prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T - \eta(x))$$

for somme $f \in A[[T]]$ then by reducing modulo the maximal ideal of A, using the fact that $[p^k]_F$ is congruent to unit $\times T^{p^n}$ modulo $\mathfrak{m}$ and degre $\geq p^n+1$, one sees $f(0)$ is a unit and thus $f \in A[[T]]^\times$. Thus the two first assertiion are equivalent.

In the same way one proves assertions three and four are equivalent.

Let's prove assertion three implies the first one. First by the preceding lemma we have to prove

$$\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \mid [p]_F(T) \Longrightarrow \prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \mid [p^k]_F(T)$$

Thus let's suppose

$$[p]_F(T) = f(T) \times \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x))$$

for some $f(T)$. Let's prove by induction on i, starting at $i=1$ and finishing at $i=k$, the following assertion

$$\prod_{x\in(p^{-i}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \mid [p^i]_F(T)$$

Suppose it is verified for $i-1$. Then

$$[p]_F(T) = f(T)\times \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \Longrightarrow [p^i]_F(T) = f([p^{i-1}]_F(T))\times \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} ([p^{i-1}]_F(T) \underset{F}{-} \eta(x))$$

If we choose for each $x \in (p^{-1}\mathbb{Z}/\mathbb{Z})^n$ an element $\widetilde{x} \in (p^{-i}\mathbb{Z}/\mathbb{Z})^n$ such that $p^{i-1}\widetilde{x} = x$ then

$$\begin{aligned} [p^i]_F(T) &= f([p^{i-1}]_F(T)) \times \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} ([p^{i-1}]_F(T) \underset{F}{-} [p^{i-1}]_F(\eta(\widetilde{x}))) \\ &= f([p^{i-1}]_F(T)) \times \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} [p^{i-1}]_F(T \underset{F}{-} \eta(\widetilde{x})) \end{aligned}$$

By the induction hypothesis

$$\exists g(T) \;\; [p^{i-1}]_F = g(T) \times \prod_{y\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(y))$$

And thus

$$[p^i]_F(T) = f([p^{i-1}]_F(T)) \times \prod_{\substack{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n \\ y\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n}} g(T \underset{F}{-} \eta(\widetilde{x}))\, \underbrace{(T \underset{F}{-} \eta(\widetilde{x}) \underset{F}{-} \eta(y))}_{T \underset{F}{-} \eta(\widetilde{x}+y)}$$

and thus

$$\prod_{\substack{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n\\ y\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n}} (T \underset{F}{-} \eta(\widetilde{x}+y)) \mid [p^i]_F(T)$$

But the family of $\widetilde{x}+y$, with x,y as in the preceding formula, is exactly $(p^{-i}\mathbb{Z}/\mathbb{Z})^n$.

Now let's prove assertion two implies assertion three. Thus suppose

$$[p^k]_F(T) = w \times \prod_{x\in(p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x))$$

with $w \in A[[T]]^\times$. Writte

$$[p]_F(T) = \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \times Q(T) + R(T)$$

for the Weierstrass division of $[p]_F$ by the polynomial $\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n}(T \underset{F}{-} \eta(x))$ that is to say $Q \in A[[T]]$ and $R \in A[T]$ $\deg R < p^n$. As before one verifies $Q \in A[[T]]^\times$. Moreover by reducing modulo $\mathfrak{m}$, since $[p]_T \equiv \text{unit} \times T^{p^n} \mod \deg\ > p^n$ we have $R \in \mathfrak{m}[T]$. Putting $T=0$ one finds

$$R \in T\mathfrak{m}[T]$$

Now suppose $R \in \mathfrak{m}^k[T]$ for some $k \geq 1$. We will now prove $R \in \mathfrak{m}^{k+1}[T]$. This will allow us to conclude since then $R \in (\cap_{k\geq 1}\mathfrak{m}^k)[T] = (0)$.

Thus, let's compute in $A/\mathfrak{m}^{k+1}[[T]]$. Then we prove by induction on i for $1 \leq i \leq k$

$$[p^i]_F \equiv \prod_{x\in(p^{-i}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \times Q_i(T) + R(a_i T^{p^{(i-1)n}}) \mod (\mathfrak{m}^{k+1}[[T]], T^{p^{in}}\mathfrak{m}^k[[T]])$$

where $Q_i \in A[[T]]^\times$ and $a_i \in A^\times$. In fact if it is verified for $i-1$ that is to say

$$[p^{i-1}]_F \equiv \prod_{x\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \times Q_{i-1}(T) + R(a_{i-1} T^{p^{(i-2)n}}) \mod (\mathfrak{m}^{k+1}[[T]], T^{p^{(i-1)n}}\mathfrak{m}^k[[T]])$$

then

$$[p^i]_F \equiv \prod_{x\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n} [p]_F(T \underset{F}{-} \eta(\widetilde{x})) \times Q_{i-1}\circ[p]_F + R(a_{i-1}[p]_F^{p^{(i-2)n}}) \mod (\mathfrak{m}^{k+1}[[T]], [p]_F^{p^{(i-1)n}}\mathfrak{m}^k[[T]])$$

where $\widetilde{x} \in (p^{-i}\mathbb{Z}/\mathbb{Z})^n$ is such that $p\widetilde{x} = x$. But the ideal $(\mathfrak{m}^{k+1}[[T]], [p]_F^{p^{(i-1)n}}\mathfrak{m}^k[[T]])$ is equal to $(\mathfrak{m}^{k+1}[[T]], T^{p^{in}})$ since $[p]_F \equiv \text{unit} \times T^{p^n} \mod (\mathfrak{m}[[T]], T^{p^n+1}\mathfrak{A}[[T]])$. Now in the same way one has

$$R(a_{i-1}[p]_F^{p^{(i-2)n}}) \equiv R(\text{unit} \times T^{p^{(i-1)n}})$$

thus, we have modulo $(\mathfrak{m}^{k+1}[[T]], T^{p^{in}}\mathfrak{m}^k[[T]])$

$$[p^i]_F \equiv \prod_{x\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n} \left(\prod_{y\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(\widetilde{x}+y)).Q(T \underset{F}{-} \eta(\widetilde{x})) + R(T \underset{F}{-} \eta(\widetilde{x})) \right) .Q([p]_F) + R(a_i T^{p^{(i-1)n}})$$

Now $R(T \underset{F}{-} \eta(\widetilde{x})) \equiv R(T)$, $R(T)^2 \equiv 0$ and $Q(T \underset{F}{-} \eta(\widetilde{x}))R(T) \equiv Q(T)R(T)$. Thus

$$[p^i]_F \equiv Q([p]_F) \prod_{\substack{x\in(p^{-i+1}\mathbb{Z}/\mathbb{Z})^n\\ y\in(p^{-1}\mathbb{Z}/\mathbb{Z})^n}} (T \underset{F}{-} \eta(\widetilde{x}+y))Q(T \underset{F}{-} \eta(\widetilde{x})) + p^{i-1}Q([p]_F)R(T) + R(a_i T^{p^{(i-1)n}})$$

But $p^{i-1} \in \mathfrak{m}$ and thus $p^{i-1}R(T) \equiv 0$. We thus obtain the induction hypothesis for i easily.

Thus we now have

$$[p^k]_F \equiv \prod_{x \in (p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x)) \times Q_k(T) + R(a_k(T^{p^{(k-1)n}})) \bmod (\mathfrak{m}^{k+1}[[T]], T^{p^{kn}}\mathfrak{m}^k[[T]])$$

where $Q_k \in A[[T]]^\times$ and $a_x k \in A^\times$. And by hypothesis

$$[p^k]_F(T) = w \times \prod_{x \in (p^{-k}\mathbb{Z}/\mathbb{Z})^n} (T \underset{F}{-} \eta(x))$$

with $w \in A[[T]]^\times$. Using Q_k and w are units plus the fact that $\deg R < p^n$ one concludes $R \equiv 0 \bmod \mathfrak{m}^{k+1}[T]$. $\square$

1.4 Drinfeld integral models

Proposition 2. *Consider the functor on artin local ring with residue field $\overline{\mathbb{F}}_p$ that associates to A the set of isomorphism classes of triples (H, ρ, η) where H is a p-divisible group over A,*

$$\rho : \mathbb{H} \xrightarrow{\sim} H \otimes_A \overline{\mathbb{F}}_p$$

and

$$\eta : (p^{-k}\mathbb{Z}/\mathbb{Z})^n \longrightarrow H[p^k](A)$$

is a Drinfeld level structure. This functor is pro-representable by a formal scheme $\mathfrak{X}_k = Spf(R_k)$ where R_k is a complete local ring.

Démonstration. By associating (H, ρ) to (H, ρ, η) one sees this functor is over Lubin-Tate space $\mathfrak{X} = \mathrm{Spf}(R)$. Moreover if $F(X, Y) \in R[[X, Y]]$ is a universal formal group law over R this functor is a subfunctor of

$$\mathfrak{Y} = F[p^k] \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} F[p^k]$$

where $F[p^k] = \mathrm{Spf}(R[[T]]/([p^k]_F))$. We have $\mathfrak{Y} = \mathrm{Spf}(B)$ with

$$B = R[[T_1]]/([p^k]_F(T_1))\hat{\otimes}_R \dots \hat{\otimes}_R R[[T_n]]/([p^k]_F(T_n))$$

and over $\mathfrak{Y}$ there ia a universal η :

$$\eta : (p^{-k}\mathbb{Z}/\mathbb{Z})^n \longrightarrow H \times_{\mathfrak{X}} \mathfrak{Y}$$

given by $e_i \mapsto T_i$ where $(e_i)_{1 \leq i \leq n}$ is the canonical base of $(p^{-k}\mathbb{Z}/\mathbb{Z})^n$. Now our functor is the subfunctor of $\mathfrak{Y}$ where this η satisfies Drinfeld divisibility condition. First let's remark B is a complete local ring finite over R. In fact By Weierstrass $R[[T]]/([p^k]_F(T))$ is a finite R-algebra thus B is finite over R (finiteness implies one can replace $\hat{\otimes}$ by $\otimes$) and

$$B \simeq R[[T_1, \dots, T_n]]/([p]_F(T_1), \dots, [p]_F(T_n))$$

with each $[p^k]_F(T_i)$ in the maximal ideal $(\mathfrak{m}, T_1, \dots, T_n)$ of the complete local ring $R[[T_1, \dots, T_n]]$ with $\mathfrak{m}$ being the maximal ideal of R, which implies B is local and complete.

Now we can apply Weierstrass division theorem to the ring B, $[p^k]_F(U) \in B[[U]]$ and $\prod_{x \in (p^{-k}\mathbb{Z}/\mathbb{Z})^n}(U - T_i)$:

$$[p^k]_F(U) = \prod_{x \in (p^{-k}\mathbb{Z}/\mathbb{Z})^n} (U - T_i).Q(U) + R(U)$$

where $R(U) \in B[U]$ and $\deg R < p^{nk}$. Writte

$$R = \sum_{i=0}^{p^{nk}-1} a_i U^i$$

Reducing modulo the maximal ideal of B the preceding division on sees forall i a_i lies in this maximal ideal. Thus

$$R_k = B/(a_1, \ldots, a_{p^{nk}-1})$$

is a complete local ring such that $\mathrm{Spf}(R_k)$ represents our functor. $\square$

Théorème 1. *For all k the complete local rings R_k are regular with a system of parameter given by $(\eta(p^{-k}e_1), \ldots, \eta(p^{-k}e_n))$, η being the universal level structure and $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{Z}^n$.*

Démonstration. We have seen in the preceding proof that R_k is a finite R-algebra. Since $\dim R = n$ we deduce $\dim R_k = n$. Thus it suffices to prove

$$R_k/(\eta(p^{-k}e_1), \ldots, \eta(p^{-k}e_n)) = \overline{\mathbb{F}}_p$$

But by proposition ??? and the preceding proof giving an explicit construction of R_k

$$R_k = R_1[[T_1, \ldots, T_n]]/\left(([p^{k-1}]_F(T_1) - \eta(p^{-1}e_1)), \ldots, ([p^{k-1}]_F(T_n) - \eta(p^{-1}e_n))\right)$$

where F is the universal formal group law and on R_k

$$\forall i \;\; \eta(p^{-k}e_i) = T_i$$

Thus

$$R_k/(\eta(p^{-k}e_1), \ldots, \eta(p^{-k}e_n)) = R_1/(\eta(p^{-1}e_1), \ldots, \eta(p^{-1}e_n))$$

We thus have to prove the assertion for R_1. But now thanks to ??? we can choose formal coordinates $(x_1, \ldots, x_{n-1})$ on R such that

$$[p]_F = pu_0T + x_1u_1T^p + \cdots + x_{n-1}u_{n-1}T^{p^{n-1}} + u_nT^{p^n}$$

with $\forall i\ u_i \in W(\overline{\mathbb{F}}_p)[[x_1, \ldots, x_{n-1}]][[T]]^\times$. Writting Drinfeld divisibility condition we find

$$\text{unit} \times T^{p^n} = pu_0T + x_1u_1T^p + \cdots + x_{n-1}u_{n-1}T^{p^{n-1}} + u_nT^{p^n} \text{ modulo } (\eta(p^{-1}e_1), \ldots, \eta(p^{-1}e_n))$$

and thus

$$\forall i \;\; x_i \in (\eta(p^{-1}e_1), \ldots, \eta(p^{-1}e_n)) \text{ and } p \in (\eta(p^{-1}e_1), \ldots, \eta(p^{-1}e_n))$$

But, as seen in the explicit given construction of R_1 from R, R_1 as an R-algebra is generated by $(\eta(p^{-1}e_1), \ldots, \eta(p^{-1}e_n))$. Thus

$$R_1/(\eta(p^{-1}e_1), \ldots, \eta(p^{-1}e_n)) = R/(p, x_1, \ldots, x_{n-1}) = \overline{\mathbb{F}}_p$$

$\square$

Remarque 3. *This proof consists just in verifying any Lubin-Tate group with a trivial Drinfeld level structure is trivial that is to say isomorphic to $\mathbb{H}$ since Spf($R_k/((\eta(p^{-k}e_1), \ldots, \eta(p^{-k}e_n)))$) is the moduli space of (H, ρ, η) with trivial η.*

Corollaire 2. *Putting $R = R_0$, forall $k' \geq k \geq 0$ $R_{k'}$ is a finite flat R_k-algebra.*

Démonstration. We have allready seen it is finite. But both ring being regular it is flat since any finite type morphism between noetherian regular rings with fibers of the same dimension is flat. $\square$

Now there is a right action of $\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ on $\mathfrak{X}_k$ over $\mathfrak{X}$ by $\eta \longmapsto \eta \circ g$.

Proposition 3. *Spec($R_k[\frac{1}{p}]$) $\longrightarrow$ Spec($R[\frac{1}{p}]$) with its action of $GL_n(\mathbb{Z}/p^k\mathbb{Z})$ is the moduli space of trivializations of the finite étale group scheme $H[p^k] \otimes_R R[\frac{1}{p}]$.*

Démonstration. Writte $[p^k]_F = u(T) \times f(T)$ with u an invertible series and $f(T)$ a polynomial. After inverting p Drinfeld condition becomes : all $\eta(x)$ form distinct roots of the separable polnyomial $f(T)$. □

Corollaire 3. *Via the preceding action Spec(R_k) is an fppf $GL_n(\mathbb{Z}/p^k\mathbb{Z})$-torsor over Spec$(R)$. Thus R_k is finite flat of degre $|GL_n(\mathbb{Z}/p^k\mathbb{Z})|$.*

Démonstration. The quotient $\mathrm{Spec}(R_k)/\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ is the spectrum of a finite R-algebra equal to R after inverting p by the preceding lemma. But R being regular it is integraly closed in $R[\frac{1}{p}]$. Thus

$$\mathrm{Spec}(R_k)/\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z}) = \mathrm{Spec}(R)$$

But $\mathrm{Spec}(R_k) \longrightarrow \mathrm{Spec}(R)$ being flat this implies $\mathrm{Spec}(R_k)$ is an fppf-torsor over $\mathrm{Spec}(R)$ under $\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$. □