An introduction to the geometry of Lubin-Tate spaces

Laurent Fargues
CNRS-IHES-université Paris-Sud Orsay

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A Finite flat group schemes

All schemes will be separated. If S is a scheme we will consider the associated fppf topology. It is the topology associated to all schemes over S and whose covering morphisms are generated by faithfully flat of finite presentation morphisms of S-schemes (by topology associated to we mean the topology associated to the preceding pretopology). We refer to [SGA 3, tome I, exposé IV] for generalities on Grothendieck topologies and quotients of sheaves.

In the following notes we will work with general hypothesis like "of finite presentation" or "finite localy free". For a first approach the reader can suppose he is working over a noetherian base and all its schemes are of finite type. This is what the author of this note used to do until the day he really had to use the theory over non-noetherian bases.

A.1 General results of algebraic geometry

A.1.1 Finite flat/finite localy free

Recall the following very basic lemma from commutative algebra, since we use it without quoting.

Lemme 1. *Let $f : X \longrightarrow S$. Then X is finite localy free over S iff f is finite flat of finite presentation.*

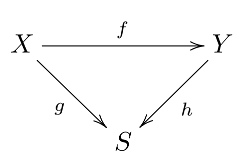
In particular if S is localy noetherian finite flat over S is equivalent to finite localy free over S.

Recall moreover that a finite morphism $f : X \longrightarrow S$ is of finite presentation iff $f_*\mathcal{O}_X$ is localy of finite presentation as an $\mathcal{O}_S$-module. That is to say a finite A-algebra B is of finite presentation as an A-algebra iff it is of finite presentation as an A-module.

A.1.2 Flatness fiber by fiber

Recall :

Théorème 1 (Grothendieck, EGA IV Coro. 11.3.11)**.** *Let*



be a diagram of localy of finite presentation morphisms of schemes. Then the following are equivalent
- *g is flat and $\forall s \in S$ $f_s : X_s \longrightarrow Y_s$ is flat*
- *f is flat and h is flat at all points of $f(X)$*

A.1.3 Monomorphisms

Proposition 1 (EGA IV, proposition 8.1.5)**.** *A proper morphism of schemes is a monomorphism iff it is a closed immersion.*

Corollaire 1. *Let S be a scheme. A morphism from a finite S-group-scheme to a finite type S-groups-scheme is a monomorphism iff it is a closed immersion iff the kernel is trivial.*

Remarque 1. *We refer to [SGA 3 exposé XVI, chapter 1] for examples of monomorphisms of group schemes that are not immersions.*

A.1.4 Connected components of a finite flat morphism

Recall for a quasi-finite morphism $f : X \longrightarrow S$ and $s \in S$ the separable rank of the fiber X_s is $|X_s(k(s)^{sep})|$ where $k(s)^{sep}$ is a separable closure of $k(s)$. One has $X_s \simeq \text{Spec}(A)$ where A is a finite $k(s)$-algebra. It is thus artinian isomorphic to a finite product

$$A \simeq \prod_{i \in I} B_i$$

where each B_i is artinian local with residue field k_i that is a finite extension of $k(s)$. Then the separable rank of X_s is

$$\sum_{i \in I} [k_i : k(s)]^{sep}$$

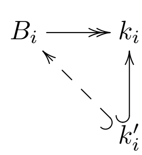
where $[. : .]^{sep}$ means separable degre.

Lemme 2 ([1], lemma 4.8 p.63)**.** *Let $f : X \longrightarrow S$ be finite and localy free. Then there exists a factorization*

$$f : X \xrightarrow{g} Y \xrightarrow{h} S$$

with h and g finite localy free, h étale and g radicial surjective iff the separable degre of the fibers of f is localy constant. If this is the case then this factorization is unique up to a unique isomorphism and functorial in X/S.

We refer to [1] for the details of the proof, but let's point out that when $S = \text{Spec}(k)$ is the spectrum of a field the proof is the following. Let $A \simeq \prod_{i \in I} B_i$ be a finite k-algebra as before with B_i local artinian and residue field k_i. Then if for each i k_i' is the separable closure of k in k_i thanks to the fact that $k_i'|k$ is separable (and thus étale) there exists a unique lift



of k-algebra morphisms from k_i' into B_i. Then $Y = \text{Spec}(\prod_{i \in I} k_i')$ and at the level of algebras the factorization is

$$k \longrightarrow \prod_{i \in I} k_i' \longrightarrow \prod_{i \in I} B_i = A$$

Here is a particular case of [EGA IV, proposition 15.5.1] :

Proposition 2. *Suppose S is localy noetherian. Let $X \longrightarrow S$ be finite flat. Then the function $S \ni s \longmapsto$ (separable rank of X_s) is lower semi-continuous.*

This means for each $n \in \mathbb{N}$ the sets of $s \in S$ such that $|X_s(k(s)^{sep})| \geq n$ is open in S. The proof of [EGA IV, proposition 15.5.1] is, using the fact one konw this function is allready constructible, by reduction to the case when S is the spectrum of a discrete valuation ring $S = \mathrm{Spec}(\mathcal{O})$. Then one has to prove the separable rank of the special fiber of $X/\mathcal{O}$ is greater than the one of its generic fiber.

Corollaire 2. *Let S be noetherian. Let $X \longrightarrow S$ be finite flat. Then there exists a "découpage" of X into dijsoints localy closed subsets*

$$|X| = \coprod_{1 \leq i \leq N} Z_i$$

such that $\forall k \ \bigcup_{i \leq k} Z_i$ is closed and if Z_i stands for the associated reduced localy closed subscheme of X then over Z_i there is a factorization

$$X \times_S Z_i \longrightarrow Y_i \longrightarrow Z_i$$

as in lemma 2.

A.2 Quotients

This is one of the main theorem of [SGA 3, tome I, exposé V].

Théorème 2 (Grothendieck)**.** *Let S be a scheme. Let G be a finite localy free S-group scheme acting on an S-scheme X via $q : G \times_S X \longrightarrow X$. Suppose each orbit of G is contained in an affine open subset of X that is to say if $p : G \times_S X \longrightarrow X$ is the projection $\forall x \in X$ the set $q(p^{-1}(x))$ is contained in an affne open subset. Then the fppf quotient sheaf $G\backslash X$ is representable by an S-scheme. Moreover :*

- *The quotient scheme $G\backslash X$ coincide with the quotient in the category of ringed spaces*
- *The morphism $X \longrightarrow G\backslash X$ is integral*
- *If X is affine so is $G\backslash X$*
- *If G acts freely on X then $X \longrightarrow G\backslash X$ is a finite localy free morphism and the morphism*

$$\begin{array}{ccc} G \times_S X & \longrightarrow & X \times_{G\backslash X} X \\ (g, x) & \longmapsto & (x, g.x) \end{array}$$

  is an isomorphism. Thus $X \longrightarrow G\backslash X$ is an fppf G-torsor.

In the preceding theorem by quotient in the category of ringed spaces we mean : consider the diagram of ringed spaces

$$G \times_S X \overset{q}{\underset{p}{\rightrightarrows}} X$$

then the quotient ringed spaces has as underlying topological space the quotient Y of the diagram

$$|G \times_S X| \overset{q}{\underset{p}{\rightrightarrows}} |X|$$

in the category of topological spaces and has as a structural sheaf the subsheaf of $\delta_* \mathcal{O}_X$, where $\delta : |X| \longrightarrow Y$ is the projection, consisting of elements x such that $q^* x = p^* x$.

If $S = \mathrm{Spec}(R), G = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ then in the diagram

$$B \overset{p^*}{\underset{q^*}{\rightrightarrows}} A \otimes_R B$$

(where $p^*b = 1 \otimes b$) if

$$C = \{b \in B \mid q^*b = p^*b\}$$

we have

$$G \backslash X = \mathrm{Spec}(C)$$

In the preceding theorem when we say G acts freely on X we mean : for all S-scheme U $G(U)$ acts freely on $X(U)$, and it suffices to verify it for $U = X$ and $Id : X \longrightarrow X$ that is to say the square

$$\begin{array}{ccc} X & \xrightarrow{Id} & X \\ \downarrow{\scriptstyle (e,Id)} & & \downarrow{\scriptstyle \Delta_{X/S}} \\ G \times_S X & \xrightarrow{(p,q)} & X \times_S X \end{array}$$

is cartesian (where e is the unit section and $\Delta_{X/S}$ the diagonal).

Remarque 2. *In the preceding theorem if G acts freely on X and X is of finite presentation over S then $G\backslash X$ is of finite presentation over S. This results from the fact that $X \longrightarrow G\backslash X$ being finite localy free it is of finite presentation and EGA IV proposition 1.4.3 (v).*

Corollaire 3. *Let G be an S-group scheme of finite presentation and $H \subset G$ a sub-group-scheme finite localy free over S such that H is distinguished in G. Then G/H is representable by a finite presentation S-group scheme and $G \longrightarrow G/H$ is finite localy free.*

A.3 epimorphism/faithfully flat

Any faithfully flat of finite presentation morphism of schemes is an fppf epimorphism but in general the converse is false : take for example X a scheme, Y an X-scheme that is not flat over X and the morphism $X \coprod Y \longrightarrow X$ that is the identity on X and the structural morphism of Y on Y. Since it has a section it is an epimorphism but is not flat.

Proposition 3. *Let G be a finite localy free S-group scheme and H an S-group scheme of finite presentation. Let $f : G \longrightarrow H$ be a morphism of S-group schemes. Then the following are equivalent*

- *f is an fppf epimorphism*
- *f is faithfully flat*
- *$\forall s \in S$ $f_s : G_s \longrightarrow H_s$ is an épimorphism*
- *$\forall s \in S$ $f_s : G_s \longrightarrow H_s$ is faithfully flat*

And if one of those conditions is verified then H is finite localy free over S.

Démonstration. We use the fact that if $s \in S$ and X is an S-scheme then if $f : \mathrm{Spec}(k(s)) \longrightarrow S$ and $\mathcal{F}$ is the fppf sheaf associated to X $f^*\mathcal{F}$ is represented by the $\mathrm{Spec}(k(s))$-scheme X_s. This is due to the fact we took all schemes in the definition of the fppf topology (for example this fact is false in general for the small étale site). Thus in particular if $X \longrightarrow Y$ is an epimorphism then $\forall s \in S$ $X_s \longrightarrow Y_s$ is an epimorphism.

By Grothendieck's theorem on flatness fiber by fiber everything is now reduced to prove that if S is the spectrum of a field then f is an epimorphism iff it is faithfully flat. Of course if it is faithully flat is an epimorphism. Now suppose it is an epimorphism. The kernel of f is a finite flat group scheme since S is the spectrum of a field. The morphism of sheaves $G/\ker f \longrightarrow H$ is an isomorphism. But by grothendieck's theorem on quotients by finite flat group schemes the morphism $G \longrightarrow G/\ker f$ is faithully flat, which proves the result. □

A.4 Exact sequences

Proposition 4. *Consider an exact sequence*

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

of finite S-group schemes that are of finite presentation over S. Then

— *If G_2 is localy free over S so are G_1 and G_3*
— *If G_1 and G_3 are localy free over S so is G_2*

Démonstration. If G_2 is localy free over S then by the preceding proposition so is G_3 and moreover the morphism $G_2 \longrightarrow G_3$ is faithfully flat which implies G_1 is finite localy free.

If G_1 and G_3 are localy free then G_2 being a G_1-torsor over G_3 thus if flat over G_3 thus over S. □

Lemme 3. *Consider an exact sequence*

$$0 \longrightarrow G_1 \xrightarrow{u} G_2 \xrightarrow{v} G_3$$

of finite localy free S-group schemes. Then the sequence

$$0 \longrightarrow G_1 \xrightarrow{u} G_2 \xrightarrow{v} G_3 \longrightarrow 0$$

is exact iff

$$rank(G_2) = rank(G_1) \times rank(G_3)$$

Démonstration. If it is exact then G_2 being a G_1-torsor over G_3 we have $G_1 \times_S G_2 \xrightarrow{\sim} G_2 \times_{G_3} G_2$. Thus $\text{rank}_S(G_1) \times \text{rank}_S(G_2) = \text{rank}_{G_3}(G_2)^2$. But $\text{rank}_S(G_2) = \text{rank}_S(G_3) \times \text{rank}_{G_3}(G_2)$. Thus combining both equality $\text{rank}(G_2) = \text{rang}(G_1) \times \text{rank}(G_3)$.

Reciprocaly, there is a monomorphism $\text{coker}(u) \hookrightarrow G_3$ that is a closed immersion since $\text{coker}(u)$ is finite localy free. Moreover by the preceding case and the hypothesis $\text{rank}(\text{coker}(u))$ is equal to $\text{rank}(G_3)$. Thus, a closed immersion of finite localy free S-schemes of the same rank being an isomorphism we conclude. □

A.5 Étale/connected finite flat group-schemes

Let G be a finite localy free S-group scheme.

We say G is étale, resp. radicial, if the morphism $G \longrightarrow S$ is étale, resp. radicial.

We remark if ω_G is the conormal sheaf of the unit section of G then $\Omega^1_{G/S} \simeq p^*\omega_G$ where $p : G \longrightarrow S$.

Lemme 4. *Write $G = Spec(\mathcal{O}_S \oplus \mathcal{I})$ where $\mathcal{I}$ is the augmentation ideal of G.*

The group G is étale iff $\omega_G = \mathcal{I}/\mathcal{I}^2 = 0$.

The group G is radicial iff $\mathcal{I}$ is (localy on S) a nil-ideal.

A.6 Finite flat group-schemes over fields

Let k be a field.

Lemme 5. *Let G be finite group-scheme over k. Then*

Lemme 6. *Let G be a finite group scheme over k. Then there is an exact sequence*

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{ét} \longrightarrow 0$$

where G^0 is connected and $G^{ét}$ étale. If k is perfect then $G^0 = G_{red}$. If k is separably closed this sequence is split.

Démonstration.

A.7 The order of a finite flat group-scheme

In this section all group-schemes are commutative.

Définition 1. *For G a finite localy free group-scheme over S we note $|G|$ the order of G that is to say the rank of $\mathcal{O}_G$ as a localy free $\mathcal{O}_S$-module.*

Proposition 5 (Deligne)**.** *Any finite localy free G over S is killed by its order.*

The proof is given in ? ? ?. When S is the spectrum of a field there is a more elementary proof. In fact if $S = \mathrm{Spec}(k)$, if k has caracteristic 0 then G is étale and this is the usual statement for abstract finite groups (after going to the algebraic closure of k). If k has caracteristic p, one can suppose k is algebraically closed. Then $G \simeq G^0 \times G^{\text{ét}}$ and one is reduced to the statement for G connected : $G = G^0$. Then if $G = \mathrm{Spec}(k \oplus I)$ were I is the augmentation ideal of the unit section then multiplication by p on G induces multiplication by $p \in k$ on I/I^2 and is thus zero. Thus if $[p] \in End(G)$ is multiplication by p then $[p]^*I \subset I^2$. With this one easily concludes.

Remarque 3. *The preceding proof over a field implies at least that for any base S any finite localy free G over S is torsion, which is weaker than Deligne's statement but sufficient for most applications.*

Lemme 7. *Let G finite localy free over S and $N \in \mathbb{N}$ its annihilator that is to say $N\mathbb{Z} = \ker(\mathbb{Z} \longrightarrow End(G))$. Then the prime divisors of N and $|G|$ are the same.*

Démonstration. By the preceding proposition $N| \, |G|$. Let now p be a prime factor of $|G|$. Suppose $p \nmid N$. Then $p : G \longrightarrow G$ is an isomorphism. We want to see this is impossible. By specialization we can suppose S is the spectrum of an algebraicaly closed field. Then $G \simeq G^0 \times G^{\text{ét}}$. If ℓ is the caracteristic of this field G^0 is killed by a power of ℓ. Thus $G^0 = (0)$ or $\ell \neq p$. If $G^0 = (0)$ we are reduced to the case of $G = G^{\text{ét}}$ an abstract finite group and it is clear $p : G^{\text{ét}} \longrightarrow G^{\text{ét}}$ canot be an isomorphism. Suppose now $\ell \neq p$. We know the order of G^0 is a power of ℓ (? ? ?). Thus $p||G^{\text{ét}}|$. Thus as before this is impossible by the case of usual abstract finite groups. □