An introduction to algebraic topology

Course at Paris VI University, 2005/2006 [1]

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[1] **To the students:** the material covered by these Notes goes beyond the contents of the actual course. All along the semester, the students will be informed of what is required for the exam.

Contents

Introduction

This course is an first introduction to Algebraic Topology from the point of view of Sheaf Theory. An expanded version of these Notes may be found in [23], [24].

Algebraic Topology is usually approached via the study of of homology defined using chain complexes and the fundamental group, whereas, here, the accent is put on the language of categories and sheaves, with particular attention to locally constant sheaves.

Sheaves on topological spaces were invented by Jean Leray as a tool to deduce global properties from local ones. This tool turned out to be extremely powerful, and applies to many areas of Mathematics, from Algebraic Geometry to Quantum Field Theory.

The functor associating to a sheaf F on a topological space X the space $F(X)$ of its global sections is left exact, but not right exact in general. The derived functors $H^j(X; F)$ encode the ‘ ‘obstructions” to pass from local to global. Given a ring k, the cohomology groups $H^j(X; k_X)$ of the sheaf k_X of k-valued locally constant functions is therefore a topological invariant of the space X. Indeed, it is a homotopy invariant, and we shall explain how to calculate $H^j(X; k_X)$ in various situations.

We also introduce the fundamental group $\pi_1(X)$ of a topological space (with suitable assumptions on the space) and prove an equivalence of categories between that of finite dimensional representations of this group and that of local systems on X. As a byproduct, we deduce the Van Kampen theorem from the theorem on the glueing of sheaves defined on a covering.

Lectures will be organized as follows.

Chapter 1 is a brief survey of linear algebra over a ring. It serves as a guide for the theory of additive and abelian categories which is exposed in the subsequent chapters.

In **Chapter 2** we expose the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category $\mathcal{C}$ may be embedded in the category $\mathcal{C}^\wedge$ of contravariant functors on $\mathcal{C}$ with values in the category **Set** of sets. This naturally leads to the concept of representable functor. Next, we study inductive and projective limits in some detail and with many examples.

Chapters **3 and 4** are devoted to additive and abelian categories. The aim is the construction and the study of the derived functors of a left (or right) exact functor F of abelian categories. Hence, we start by studying complexes (and double complexes) in additive and abelian categories. Then we briefly explain the construction of the right derived functor by using injective resolutions and later, by using F-injective resolutions. We apply these results to the case of the functors Ext and Tor.

In Chapter **5**, we study abelian sheaves on topological spaces (with a brief look at Grothendieck topologies). We construct the sheaf associated with a presheaf and the usual internal operations ($\mathcal{H}om$ and $\otimes$) and external operations (direct and inverse images). We also explain how to obtain locally constant or locally free sheaves when glueing sheaves.

In Chapter **6** we prove that the category of abelian sheaves has enough injectives and we define the cohomology of sheaves. We construct resolutions of sheaves using open or closed Čech coverings and, using the fact that the cohomology of locally constant sheaves is a homotopy invariant, we show how to compute the cohomology of spaces by using cellular decomposition. We apply this technique to deduce the cohomology of some classical manifolds.

In Chapter **7**, we define the fundamental groupoid $\pi_1(X)$ of a locally arcwise connected space X as well as the monodromy of a locally constant sheaf and prove that under suitable assumptions, the monodromy functor is an equivalence. We also show that the Van Kampen theorem may be deduced from the theorem on the glueing of sheaves and apply it in some particular sitations.

Conventions. In these Notes, all rings are unital and associative but not necessarily commutative. The operations, the zero element, and the unit are denoted by $+, \cdot, 0, 1$, respectively. However, we shall often write for short ab instead of $a \cdot b$.

All along these Notes, k will denote a *commutative* ring. (Sometimes, k will be a field.)

We denote by $\emptyset$ the empty set and by $\{\mathrm{pt}\}$ a set with one element.

We denote by $\mathbb{N}$ the set of non-negative integers, $\mathbb{N} = \{0, 1, \dots\}$.

Chapter 1

Linear algebra over a ring

This chapter is a short review of basic and classical notions of commutative algebra.

Many notions introduced in this chapter will be repeated later in a more general setting.

Some references: [1], [4].

1.1 Modules and linear maps

All along these Notes, k is a commutative ring.

Let A be a k-algebra, that is, a ring endowed with a morphism of rings $\varphi\colon k \to A$ such that the image of k is contained in the center of A. Notice that a ring A is always a $\mathbb{Z}$-algebra. If A is commutative, then A is an A-algebra.

Since we do not assume A is commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module.

Recall that an A-module M is an additive group (whose operations and zero element are denoted $+, 0$) endowed with an external law $A \times M \to M$ satisfying:

$$\begin{cases} (ab)m = a(bm) \\ (a+b)m = am + bm \\ a(m+m') = am + am' \\ 1 \cdot m = m \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that M inherits a structure of a k-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by A^{op} the ring A with the opposite structure. Hence the product ab in A^{op} is the product ba in A and an A^{op}-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Examples 1.1.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \dots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \dots, x_n]]$, the ring of formal powers series with coefficients in A.
(ii) Let k be a field. Then for $n > 1$, the ring $M_n(k)$ of square matrices of rank n with entries in k is non commutative.
(iii) Let k be a field. The *Weyl algebra* in n variables, denoted $W_n(k)$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \leq i, j \leq n)$ with coefficients in k, and relations :

$$[x_i, x_j] = 0,\ [\partial_i, \partial_j] = 0,\ [\partial_j, x_i] = \delta_j^i$$

where $[p, q] = pq - qp$ and δ_j^i is the Kronecker symbol.

The Weyl algebra $W_n(k)$ may be regarded as the ring of differential operators with coefficients in $k[x_1, \dots, x_n]$, and $k[x_1, \dots, x_n]$ becomes a left $W_n(k)$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i.

A morphism $f\colon M \to N$ of A-modules is an A-linear map, *i.e.,* f satisfies:

$$\begin{cases} f(m + m') = f(m) + f(m') & m, m' \in M \\ f(am) = af(m) & m \in M, a \in A. \end{cases}$$

A morphism f is an isomorphism if there exists a morphism $g : N \to M$ with $f \circ g = \mathrm{id}_N, g \circ f = \mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1} : N \to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

A submodule N of M is a subset N of M such that $n, n' \in N$ implies $n + n' \in N$ and $n \in N, a \in A$ implies $an \in N$. A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non trivial ideal, *i.e.,* its only ideals are $\{0\}$ and A. If $A = \mathbb{C}[x]$, then $I = \{P \in \mathbb{C}[x]; P(0) = 0\}$ is a non trivial ideal.

If N is a submodule of M, the quotient module M/N is characterized by the following "universal property": for any module L, any morphism

$h\colon M \to L$ which induces 0 on N factorizes uniquely through M/N. This is visualized by the diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[dr, "0"'] & M \arrow[r, "g"] \arrow[d, "h"] & M/N \arrow[dl, dotted, "h''"] \\
& L &
\end{tikzcd}$$

Let I be a set, and let $\{M_i\}_{i\in I}$ be a family of A-modules indexed by I. The product $\prod_i M_i$ is the set of families $\{(x_i)_{i\in I}\}$ with $x_i \in M_i$, and this set naturally inherits a structure of an A-module. There are natural surjective morphisms:

$$\pi_k\colon \prod_i M_i \to M_k.$$

Note that given a module L and a family of morphisms $f_i\colon L \to M_i$, this family factorizes uniquely through $\prod_i M_i$. This is visualized by the diagram

$$\begin{tikzcd}
& & M_i \\
L \arrow[urr, "f_i"] \arrow[r, dotted] \arrow[drr, "f_j"'] & \prod_k M_k \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
& & M_j.
\end{tikzcd}$$

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ consisting of families $\{(x_i)_{i\in I}\}$ with $x_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, the natural injection $\bigoplus_i M_i \to \prod_i M_i$ is an isomorphism. There are natural injective morphisms:

$$\varepsilon_k : M_k \to \bigoplus_i M_i.$$

We shall sometimes identify M_k to its image in $\bigoplus_i M_i$ by ε_k. Note that given a module L and a family of morphisms $f_i\colon M_i \to L$, this family factorizes uniquely through $\bigoplus_i M_i$. This is visualized by the diagram

$$\begin{tikzcd}
M_i \arrow[dr, "\varepsilon_i"'] \arrow[drr, "f_i"] & & \\
& \bigoplus_k M_k \arrow[r, dotted] & L. \\
M_j \arrow[ur, "\varepsilon_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

If $M_i = M$ for all $i \in I$, one writes:

$$M^{(I)} := \bigoplus_i M_i, \qquad M^I := \prod_i M_i.$$

An A-module M is free of rank one if it is isomorphic to A, and M is free if it is isomorphic to a direct sum $\bigoplus_{i \in I} L_i$, each L_i being free of rank one. If card (I) is finite, say r, then r is uniquely determined and one says M is free of rank r.

Let $f \colon M \to N$ be a morphism of A-modules. One sets :

$$\begin{aligned} \operatorname{Ker} f &= \{m \in M; \quad f(m) = 0\} \\ \operatorname{Im} f &= \{n \in N; \quad \text{there exists } m \in M, \quad f(m) = n\}. \end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel and the coimage of f:

$$\operatorname{Coker}\ f = N/\operatorname{Im}\ f, \quad \operatorname{Coim} f = M/\operatorname{Ker} f.$$

Since the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism, one shall not use Coim when dealing with A-modules.

If $(M_i)_{i \in I}$ is a family of submodules of an A-module M, one denotes by $\sum_i M_i$ the submodule of M obtained as the image of the natural morphism $\bigoplus_i M_i \to M$. This is also the module generated in M by the set $\bigcup_i M_i$. One calls this module the sum of the M_i's in M.

Example 1.1.2. Let $W_n(k)$ denote as above the Weyl algebra. Consider the left $W_n(k)$-linear map $W_n(k) \to k[x_1, \dots, x_n]$, $W_n(k) \ni P \mapsto P(1) \in k[x_1, \dots, x_n]$. This map is clearly surjective and its kernel is the left ideal generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(k)$-modules:

$$W_n(k)/\sum_j W_n(k)\partial_j \xrightarrow{\sim} k[x_1, \dots, x_n]. \tag{1.1}$$

1.2 Complexes

Definition 1.2.1. A complex $M^\bullet$ of A-modules is a sequence of modules $M^j, j \in \mathbb{Z}$ and A-linear maps $d_M^j : M^j \to M^{j+1}$ such that $d_M^j \circ d_M^{j-1} = 0$ for all j.

One writes a complex as:

$$M^\bullet : \cdots \to M^j \xrightarrow{d_M^j} M^{j+1} \to \cdots$$

If there is no risk of confusion, one writes M instead of $M^\bullet$. One also often write d^j instead of d_M^j.

A morphism of complexes $f : M \to N$ is a commutative diagram:

$$\begin{tikzcd}
\arrow[r] & M^{k-1} \arrow[r, "d_M^{k-1}"] \arrow[d, "f^{k-1}"] & M^k \arrow[r] \arrow[d, "f^k"] & {} \\
\arrow[r] & N^{k-1} \arrow[r, "d_N^{k-1}"'] & N^k \arrow[r] & {}
\end{tikzcd}$$

Remark 1.2.2. One also encounters finite sequences of morphisms

$$M^j \xrightarrow{d^j} M^{j+1} \to \cdots \to M^{j+k}$$

such that $d^n \circ d^{n-1} = 0$ when it is defined. In such a case we also call such a sequence a complex by identifying it to the complex

$$\cdots \to 0 \to M^j \xrightarrow{d^j} M^{j+1} \to \cdots \to M^{j+k} \to 0 \to \cdots .$$

In particular, $M' \xrightarrow{f} M \xrightarrow{g} M''$ is a complex if $g \circ f = 0$.

Consider a sequence

(1.2) $M' \xrightarrow{f} M \xrightarrow{g} M''$, with $g \circ f = 0$. (Hence, this sequence is a complex.)

Definition 1.2.3. (i) The sequence (1.2) is exact if $\operatorname{Im} f \xrightarrow{\sim} \operatorname{Ker} g$.

(ii) More generally, a complex $M^j \to \cdots \to M^{j+k}$ is exact if any sequence $M^{n-1} \to M^n \to M^{n+1}$ extracted from this complex is exact.

(iii) An exact complex $0 \to M' \to M \to M'' \to 0$ is called a short exact sequence.

Example 1.2.4. Let $A = k[x_1, x_2]$ and consider the sequence:

$$0 \to A \xrightarrow{d^0} A^2 \xrightarrow{d^1} A \to 0$$

where $d^0(P) = (x_1P, x_2P)$ and $d^1(Q, R) = x_2Q - x_1R$. One checks immediately that $d^1 \circ d^0 = 0$: the sequence above is a complex.

One defines the k-th cohomology object of a complex $M^\bullet$ as:

$$H^k(M^\bullet) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}.$$

Hence, a complex $M^\bullet$ is exact if all its cohomology objects are zero, that is, $\operatorname{Im} d^{k-1} = \operatorname{Ker} d^k$ for all k.

If $f^\bullet : M^\bullet \to N^\bullet$ is a morphism of complexes, then for each j, f^j sends $\operatorname{Ker} d^j_{M^\bullet}$ to $\operatorname{Ker} d^j_{N^\bullet}$ and sends $\operatorname{Im} d^{j-1}_{M^\bullet}$ to $\operatorname{Im} d^{j-1}_{N^\bullet}$. Hence it defines the morphism

$$H^j(f^\bullet) : H^j(M^\bullet) \to H^j(N^\bullet).$$

One says that f is a *quasi-isomorphism* (a qis, for short) if $H^j(f)$ is an isomorphism for all j.

As a particular case, consider a complex $M^\bullet$ of the type:

$$0 \to M^0 \xrightarrow{f} M^1 \to 0.$$

Then $H^0(M^\bullet) = \operatorname{Ker} f$ and $H^1(M^\bullet) = \operatorname{Coker} f$.

To a morphism $f : M \to N$ one then associates the two short exact sequences :

$$\begin{aligned}
&0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0,\\
&0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0,
\end{aligned}$$

and f is an isomorphism if and only if $\operatorname{Ker} f = \operatorname{Coker} f = 0$. In this case one writes :

$$f : M \xrightarrow{\sim} N.$$

One says f is a monomorphism (resp. epimorphism) if $\operatorname{Ker} f$ (resp. $\operatorname{Coker} f$) $= 0$.

Proposition 1.2.5. *Consider an exact sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0. \tag{1.3}$$

Then the following conditions are equivalent:

(a) *there exists* $h : M'' \to M$ *such that* $g \circ h = \mathrm{id}_{M''}$,

(b) *there exists* $k : M \to M'$ *such that* $k \circ f = \mathrm{id}_{M'}$

(c) *there exists* $h : M'' \to M$ *and* $k : M \to M'$ *such that such that* $\mathrm{id}_M = f \circ k + h \circ g$,

(d) *there exists* $\varphi = (k, g) : M \to M' \oplus M''$ *and* $\psi = (f + h) : M' \oplus M'' \to M$, *such that* φ *and* ψ *are isomorphisms inverse to each other. In other words, the exact sequence (1.3) is isomorphic to the exact sequence* $0 \to M' \to M' \oplus M'' \to M'' \to 0$.

Proof. (a) $\Rightarrow$ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_M - h \circ g) = 0$, which implies that $\mathrm{id}_M - h \circ g$ factors through $\operatorname{Ker} g$, that is, through M'. Hence, there exists $k : M \to M'$ such that $\mathrm{id}_M - h \circ g = f \circ k$.
(b) $\Rightarrow$ (c). The proof is similar and left to the reader.
(c) $\Rightarrow$ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{M''}) \circ g = 0$. Since g is onto, this implies $g \circ h - \mathrm{id}_{M''} = 0$.
(c) $\Rightarrow$ (b). The proof is similar and left to the reader.
(d) $\Leftrightarrow$ (a)&(b)&(c) is obvious. q.e.d.

Definition 1.2.6. In the above situation, one says that the exact sequence (1.3) splits.

If A is a field, all exact sequences split, but this is not the case in general. For example, the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

1.3 The functor Hom

In this section, A denotes a k-algebra and the notation $M \in \mathrm{Mod}(A)$ means that M is an A-module. One calls $\mathrm{Mod}(A)$ the category of A-modules. (A precise definition will be given in Chapter 2.)

Let M and N be two A-modules. One denotes by $\mathrm{Hom}_A(M, N)$ the set of A-linear maps $f \colon M \to N$. This is clearly a k-module. In fact one defines the action of k on $\mathrm{Hom}_A(M, N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a \lambda f(m) = a(\lambda f(m))$, and $\lambda f \in \mathrm{Hom}_A(M, N)$.

We shall often set for short

$$\mathrm{Hom}\,(M, N) = \mathrm{Hom}_k(M, N).$$

Notice that if K is a k-module, then $\mathrm{Hom}\,(K, M)$ is an A-module.

There is a natural isomorphism $\mathrm{Hom}_A(A, M) \simeq M$: to $u \in \mathrm{Hom}_A(A, M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\mathrm{Hom}_A(A/I, M) \simeq \{m \in M; Im = 0\}$.

Functors

Although the general definition of a functor will be given in Chapter 2, we give it in the particular case of categories of modules.

Let A and B be two k-algebras. A functor $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ associates a B-module $F(M)$ to each A-module M and associates a B-linear map $F(f)\colon F(M) \to F(N)$ to each A-linear map $f\colon M \to N$ such that:

$$\begin{aligned} &F(\mathrm{id}_M) = \mathrm{id}_{F(M)} \text{ for any } A\text{-module } M, \\ &F(g \circ f) = F(f) \circ F(g) \text{ for any morphisms } M \xrightarrow{f} N \xrightarrow{g} L. \end{aligned}$$

A functor $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is k-additive if it commutes to finite direct sums, *i.e.*, $F(M \oplus N) \simeq F(M) \oplus F(N)$ and the map

$$F\colon \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(F(M), F(N)$$

is k-linear.

A contravariant functor is almost the same as a functor, with the difference that it reverses the direction of the arrows. Hence, to a morphism $f\colon M \to N$, a contravariant functor $G\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ associates a morphism $G(f)\colon G(N) \to G(M)$ and satisfies

$$G(g \circ f) = G(f) \circ G(g).$$

The functors $\mathrm{Hom}_A(M, \bullet)$ and $\mathrm{Hom}_A(\bullet, N)$

Let $M \in \mathrm{Mod}(A)$. The functor

$$\mathrm{Hom}_A(M, \bullet)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(k)$$

associates $\mathrm{Hom}_A(M, K)$ to the A-module K and to an A-linear map $g : K \to L$ it associates

$$\begin{aligned} \mathrm{Hom}_A(M, g) : \mathrm{Hom}_A(M, K) &\xrightarrow{g\circ} \mathrm{Hom}_A(M, L) \\ (M \xrightarrow{h} K) &\mapsto (M \xrightarrow{h} K \xrightarrow{g} L). \end{aligned}$$

Clearly, $\mathrm{Hom}_A(M, \bullet)$ is a functor from the category $\mathrm{Mod}(A)$ of A-modules to the category $\mathrm{Mod}(k)$ of k-modules.

Similarly, for $N \in \mathrm{Mod}(A)$, the contravariant functor

$$\mathrm{Hom}_A(\bullet, N)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(k)$$

associates $\mathrm{Hom}_A(K, N)$ to the A-module K and to an A-linear map $g : K \to L$ it associates

$$\begin{array}{rcl} \mathrm{Hom}_A(g, N) : \mathrm{Hom}_A(L, N) & \xrightarrow{\circ g} & \mathrm{Hom}_A(K, N) \\ (L \xrightarrow{h} N) & \mapsto & (K \xrightarrow{g} L \xrightarrow{h} N). \end{array}$$

Clearly, the two functors $\mathrm{Hom}_A(M, \bullet)$ and $\mathrm{Hom}_A(\bullet, N)$ commute to finite direct sums or finite products, *i.e.*,

$$\begin{array}{rcl} \mathrm{Hom}_A(K \oplus L, N) & \simeq & \mathrm{Hom}_A(K, N) \times \mathrm{Hom}_A(L, N) \\ \mathrm{Hom}_A(M, K \times L) & \simeq & \mathrm{Hom}_A(M, K) \times \mathrm{Hom}_A(M, L). \end{array}$$

Hence, these functors are additive.

Exactness

Proposition 1.3.1. (a) *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ be a complex of A-modules. The assertions below are equivalent.*

(i) *the sequence is exact,*

(ii) *M' is isomorphic by f to* $\mathrm{Ker}\, g$*,*

(iii) *any morphism $h : L \to M$ such that $g \circ h = 0$, factorizes uniquely through M'* (i.e., *$h = f \circ h'$, with $h' : L \to M'$*). *This is visualized by*

$$\begin{tikzcd} & & L \arrow[dl, dotted, "h'"'] \arrow[d, "h"] \arrow[dr, "0"] & \\ 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \end{tikzcd}$$

(iv) *for any module L, the sequence of k-modules*

$$(1.4) \qquad 0 \to \mathrm{Hom}_A(L, M') \to \mathrm{Hom}_A(L, M) \to \mathrm{Hom}_A(L, M'')$$

is exact.

(b) *Let $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a complex of A-modules. The assertions below are equivalent.*

(i) *the sequence is exact,*

(ii) *M'' is isomorphic by g to* $\mathrm{Coker}\, f$*,*

(iii) *any morphism* $h : M \to L$ *such that* $h \circ f = 0$*, factorizes uniquely through* M'' (i.e., $h = h'' \circ g$*, with* $h'' : M'' \to L$)*. This is visualized by*

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[dr, "0"'] & M \arrow[r, "g"] \arrow[d, "h"] & M'' \arrow[r] \arrow[dl, dotted, "h''"] & 0 \\ & L & & \end{tikzcd}$$

(iv) *for any module* L*, the sequence of* k*-modules*

$$0 \to \mathrm{Hom}_A(M'', L) \to \mathrm{Hom}_A(M, L) \to \mathrm{Hom}_A(M', L) \tag{1.5}$$

is exact.

Proof. (a) (i) $\Rightarrow$ (ii) is obvious, as well as (ii) $\Rightarrow$ (iii), since any linear map $h : L \to M$ such that $g \circ h = 0$ factorizes uniquely through $\mathrm{Ker}\, g$, and this characterizes $\mathrm{Ker}\, g$. Finally, (iii) $\Leftrightarrow$ (iv) is tautological.
(b) The proof is similar. q.e.d.

Definition 1.3.2. (i) An additive functor $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is left exact if for any exact sequence $0 \to M' \to M \to M''$ in $\mathrm{Mod}(A)$, the sequence $0 \to F(M') \to F(M) \to F(M'')$ in $\mathrm{Mod}(B)$ is exact.

(ii) An additive functor $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is right exact if for any exact sequence $M' \to M \to M'' \to 0$ in $\mathrm{Mod}(A)$, the sequence $F(M') \to F(M) \to F(M'') \to 0$ in $\mathrm{Mod}(B)$ is exact.

(iii) An additive contravariant functor $G\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is left exact if for any exact sequence $M' \to M \to M'' \to 0$ in $\mathrm{Mod}(A)$, the sequence $0 \to G(M'') \to G(M) \to G(M')$ in $\mathrm{Mod}(B)$ is exact.

(iv) An additive contravariant functor $G\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is right exact if for any exact sequence $0 \to M' \to M \to M''$ in $\mathrm{Mod}(A)$, the sequence $G(M'') \to G(M) \to G(M') \to 0$ in $\mathrm{Mod}(B)$ is exact.

(v) An additive functor is exact if it is both right and left exact.

Hence, the fact that (a)–(i) $\Leftrightarrow$ (a)–(iv) and (b)–(i) $\Leftrightarrow$ (b)–(iv)) is formulated by saying that $\mathrm{Hom}_A(\,\bullet\,, L)$ and $\mathrm{Hom}_A(L, \bullet\,))$ are left exact functors.

Note that if $A = k$ is a field, then $\mathrm{Hom}_k(M, k)$ is the algebraic dual of M, the vector space of linear functional on M, usually denoted by M^*. If M is finite dimensional, then $M \simeq M^{**}$. If $u : L \to M$ is a linear map, the map $\mathrm{Hom}_k(u, k) : M^* \to L^*$ is usually denoted by ${}^t u$ and called the transpose of u.

Lemma 1.3.3. *Consider an additive functor* $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ *and assume that for each exact sequence* $0 \to M' \to M \to M'' \to 0$ *in* $\mathrm{Mod}(A)$*, the sequence* $0 \to F(M') \to F(M) \to F(M'')$ *is exact in* $\mathrm{Mod}(B)$*. Then* F *is left exact.*

Proof. Consider an exact sequence $0 \to M' \xrightarrow{f} M \to L$ and denote by M'' the cokernel of f. We get an exact sequence $0 \to M' \to M \to M'' \to 0$, hence, by the hypothesis, an exact sequence $0 \to F(M') \to F(M) \to F(M'')$. On the other hand, the map $M \to L$ factorizes through a map $h\colon M'' \to L$ which is clearly injective. Consider the exact sequence $0 \to M'' \to L \to \mathrm{Coker}\, h \to 0$. Applying the functor F, we obtain that $F(M'') \to F(L)$ is injective. Since the map $F(M) \to F(L)$ factorizes through $F(M) \to F(M'')$, the kernel of $F(M) \to F(L)$ is isomorphic to the kernel of $F(M) \to F(M'')$. It follows that the sequence $0 \to F(M') \to F(M) \to F(L)$ is exact. q.e.d.

There is a similar result for right exact functors and for contravariant functors. Moreover:

Lemma 1.3.4. *Consider an additive functor* $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$*. The conditions below are equivalent:*

(i) F *is exact,*

(ii) *for any exact sequence* $M' \to M \to M''$ *in* $\mathrm{Mod}(A)$*, the sequence* $F(M') \to F(M) \to F(M'')$ *is exact in* $\mathrm{Mod}(B)$*,*

(iii) *for any exact sequence* $0 \to M' \to M \to M'' \to 0$ *in* $\mathrm{Mod}(A)$*, the sequence* $0 \to F(M') \to F(M) \to F(M'') \to 0$ *is exact in* $\mathrm{Mod}(B)$*.*

The proof is left as an exercise.

Example 1.3.5. The functors $\mathrm{Hom}_A(\,\bullet\,, L)$ and $\mathrm{Hom}_A(M, \bullet\,)$ are not right exact in general. In fact choose $A = k[x]$, with k a field, and consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0 \tag{1.6}$$

(where $\cdot x$ means multiplication by x). Apply $\mathrm{Hom}_A(\cdot, A)$ to this sequence. We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\,\bullet\,, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.

Similarly, apply $\mathrm{Hom}_A(A/Ax, \bullet)$ to the exact sequence (1.6). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.

Notice moreover that the functor $\mathrm{Hom}_A(\bullet, \bullet)$ being additive, it sends split exact sequences to split exact sequences. This shows again that (1.6) does not split.

The next result is of constant use.

Proposition 1.3.6. *Let $f : M \to N$ be a morphism of A-modules. The conditions below are equivalent:*

(i) *f is an isomorphism,*

(ii) *for any A-module L, the map $\mathrm{Hom}_A(L, M) \xrightarrow{f\circ} \mathrm{Hom}_A(L, N)$ is an isomorphism,*

(iii) *for any A-module L, the map $\mathrm{Hom}_A(N, L) \xrightarrow{\circ f} \mathrm{Hom}_A(M, L)$ is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious.
(ii) $\Rightarrow$ (i). Choose $L = A$.
(iii) $\Rightarrow$ (i). By choosing $L = M$ and $\mathrm{id}_M \in \mathrm{Hom}_A(M, M)$ we find that there exists $g : N \to M$ such that $g \circ f = \mathrm{id}_M$. Hence, f is injective and moreover, by Proposition 1.2.5 there exists an isomorphism $N \simeq M \oplus P$. Therefore, $\mathrm{Hom}_A(P, L) \simeq 0$ for all module L, hence $\mathrm{Hom}_A(P, P) \simeq 0$, and this implies $P \simeq 0$. q.e.d.

Injective and projective modules

Definition 1.3.7. (i) An A-module I is injective if for any exact sequence $0 \to M' \to M \to M''$ in $\mathrm{Mod}(A)$, the sequence $\mathrm{Hom}_A(M'', I) \to \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(M', I) \to 0$ is exact in $\mathrm{Mod}(k)$ or, equivalently, if the functor $\mathrm{Hom}_A(\bullet, I)$ is exact.

(ii) An A-module P is projective if for any exact sequence $M' \to M \to M'' \to 0$ in $\mathrm{Mod}(A)$, the sequence $\mathrm{Hom}_A(P, M') \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, M'') \to 0$ is exact in $\mathrm{Mod}(k)$ or, equivalently, if the functor $\mathrm{Hom}_A(P, \bullet)$ is exact.

Proposition 1.3.8. *An A-module I is injective if and only if for any solid diagram in which the row is exact:*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "k"'] & M \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd}$$

the dotted arrow may be completed, making the diagram commutative.

Proof. (i) Assume that I is injective and let M'' denote the cokernel of the map $M' \to M$. Applying $\operatorname{Hom}_A(\,\bullet\,, I)$ to the sequence $0 \to M' \to M \to M''$, one gets the exact sequence:

$$\operatorname{Hom}_A(M'', I) \to \operatorname{Hom}_A(M, I) \xrightarrow{\circ f} \operatorname{Hom}_A(M', I) \to 0.$$

Thus there exists $h : M \to I$ such that $h \circ f = k$.
(ii) Conversely, consider an exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$. Then the sequence $0 \to \operatorname{Hom}_A(M'', I) \xrightarrow{\circ h} \operatorname{Hom}_A(M, I) \xrightarrow{\circ f} \operatorname{Hom}_A(M', I) \to 0$ is exact by Proposition 1.3.1 and the hypothesis.

To conclude, apply Lemma 1.3.3. q.e.d.

By reversing the arrows, we get a similar result assuming P is projective. In other words, P is projective if and only if for any solid diagram in which the row is exact:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "k"] & \\ M \arrow[r, "f"] & M'' \arrow[r] & 0 \end{tikzcd}$$

the dotted arrow may be completed, making the diagram commutative.

A free module is projective and if $A = k$ is a field, all modules are both injective and projective.

Generators and relations

Suppose one is interested in studying a system of linear equations

$$\sum_{j=1}^{N_0} p_{ij} u_j = v_i, \quad (i = 1, \ldots, N_1) \tag{1.7}$$

where the p_{ij}'s belong to the ring A and u_j, v_i belong to some left A-module L. Using matrix notations, one can write equations (1.7) as

$$Pu = v \tag{1.8}$$

where P is the matrix (p_{ij}) with N_1 rows and N_0 columns, defining the A-linear map $P\cdot : L^{N_0} \to L^{N_1}$. Now consider the right A-linear map

$$\cdot P : A^{N_1} \to A^{N_0}, \tag{1.9}$$

where $\cdot P$ operates on the right and the elements of A^{N_0} and A^{N_1} are written as rows. Let $(e_1, \dots, e_{N_0})$ and $(f_1, \dots, f_{N_1})$ denote the canonical basis of A^{N_0} and A^{N_1}, respectively. One gets:

$$f_i \cdot P = \sum_{j=1}^{N_0} p_{ij} e_j, \quad (i = 1, \dots, N_1). \tag{1.10}$$

Hence $\operatorname{Im} P$ is generated by the elements $\sum_{j=1}^{N_0} p_{ij} e_j$ for $i = 1, \dots, N_1$. Denote by M the quotient module $A^{N_0}/A^{N_1} \cdot P$ and by $\psi : A^{N_0} \to M$ the natural A-linear map. Let $(u_1, \dots, u_{N_0})$ denote the images by ψ of $(e_1, \dots, e_{N_0})$. Then M is a left A-module with generators $(u_1, \dots, u_{N_0})$ and relations $\sum_{j=1}^{N_0} p_{ij} u_j = 0$ for $i = 1, \dots, N_1$. By construction, we have an exact sequence of left A-modules:

$$A^{N_1} \xrightarrow{\cdot P} A^{N_0} \xrightarrow{\psi} M \to 0. \tag{1.11}$$

Applying the left exact functor $\operatorname{Hom}_A(\bullet, L)$ to this sequence, we find the exact sequence of k-modules:

$$0 \to \operatorname{Hom}_A(M, L) \to L^{N_0} \xrightarrow{P\cdot} L^{N_1}. \tag{1.12}$$

Hence, the k-module of solutions of the homogeneous equations associated to (1.7) is described by $\operatorname{Hom}_A(M, L)$.

1.4 Tensor product

The tensor product, that we shall construct below, solves a "universal problem". Namely, consider a right A-module N, a left A-module M, and a k-module L. Let us say that a map $f : N \times M \to L$ is (A, k)-bilinear if f is additive with respect to each of its arguments and satisfies $f(na, m) = f(n, am)$, $f(n(\lambda), m) = \lambda(f(n, m))$ for all $(n, m) \in N \times M$ and $a \in A, \lambda \in k$.

We shall construct a k-module denoted $N \otimes_A M$ such that f factors uniquely through the bilinear map $N \times M \to N \otimes_A M$ followed by a k-linear map $N \otimes_A M \to L$. This is visualized by:

$$\begin{tikzcd} N \times M \arrow[r] \arrow[dr, "f"'] & N \otimes_A M \arrow[d, dotted] \\ & L \end{tikzcd}$$

First, remark that one may identify a set I to a subset of $k^{(I)}$ as follows: to $i \in I$, we associate $\{l_j\}_{j\in I} \in k^{(I)}$ given by

$$l_j = \begin{cases} 1 \text{ if } & j = i, \\ 0 \text{ if } & j \neq i. \end{cases} \tag{1.13}$$

The tensor product $N \otimes_A M$ is the k-module defined as the quotient of $k^{(N\times M)}$ by the submodule generated by the following elements (where $n, n' \in N, m, m' \in M, a \in A, \lambda \in k$ and $N \times M$ is identified to a subset of $k^{(N\times M)}$):

$$\begin{cases} (n+n', m) - (n, m) - (n', m) \\ (n, m+m') - (n, m) - (n, m') \\ (na, m) - (n, am) \\ \lambda(n, m) - (n\lambda, m). \end{cases}$$

The image of (n, m) in $N \otimes_A M$ is denoted $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N, m_j \in M$ and:

$$\begin{cases} (n+n') \otimes m = n \otimes m + n' \otimes m \\ n \otimes (m+m') = n \otimes m + n \otimes m' \\ na \otimes m = n \otimes am \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Consider an A-linear map $f : M \to L$. It defines a linear map $\mathrm{id}_N \times f : N \times M \to N \times L$, hence a (A, k)-bilinear map $N \times M \to N \otimes_A L$, and finally a k-linear map

$$\mathrm{id}_N \otimes f : N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g : N \to L$.

Tensor product commutes to direct sum, that is, there are natural isomorphisms:

$$\begin{aligned} (N \oplus N') \otimes_A M &\simeq (N \otimes_A M) \oplus (N' \otimes_A M), \\ N \otimes_A (M \oplus M') &\simeq (N \otimes_A M) \oplus (N \otimes_A M'). \end{aligned}$$

Clearly, we have constructed additive functors

$$\begin{aligned} N \otimes_A \bullet &: \mathrm{Mod}(A) \to \mathrm{Mod}(k), \\ \bullet \otimes_A N &: \mathrm{Mod}(A^{\mathrm{op}}) \to \mathrm{Mod}(k). \end{aligned}$$

Note that if A is commutative, there is an isomorphism: $N \otimes_A M \simeq M \otimes_A N$, given by $n \otimes m \mapsto m \otimes n$ and moreover the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphisms $L \otimes_A (M \otimes_A N) \simeq (L \otimes_A M) \otimes_A N$. One simply writes $L \otimes_A M \otimes_A N$.

There is a natural isomorphism $A \otimes_A M \simeq M$. We shall often write for short

$$M \otimes_k N = M \otimes N.$$

Sometimes, one has to consider various rings. Consider two k-algebras, A_1 and A_2. Then $A_1 \otimes A_2$ has a natural structure of a k-algebra, by setting

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2.$$

An $(A_1 \otimes A_2^{\mathrm{op}})$-module M is also called a (A_1, A_2)-bimodule (a left A_1-module and right A_2-module). Note that the actions of A_1 and A_2 on M commute, that is,

$$a_1 a_2 m = a_2 a_1 m,\ a_1 \in A_1,\ a_2 \in A_2,\ m \in M.$$

Let A_1, A_2, A_3, A_4 denote four k-algebras.

Proposition 1.4.1. *Let ${}_iM_j$ be an $(A_i \otimes A_j^{\mathrm{op}})$-module. Then*

$$\begin{aligned} {}_1M_2 \otimes_{A_2} {}_2M_3 &\ \textit{is an } (A_1 \otimes A_3^{\mathrm{op}})\textit{-module}, \\ \operatorname{Hom}_{A_1}({}_1M_2, {}_1M_3) &\ \textit{is an } (A_2 \otimes A_3^{\mathrm{op}})\textit{-module}, \end{aligned}$$

and there is a natural isomorphism of $A_4 \otimes A_3^{\mathrm{op}}$-modules

$$\operatorname{Hom}_{A_1}({}_1M_4, \operatorname{Hom}_{A_2}({}_2M_1, {}_2M_3)) \simeq \operatorname{Hom}_{A_2}({}_2M_1 \otimes_{A_1} {}_1M_4, {}_2M_3). \tag{1.14}$$

In particular, if A is a k-algebra, M, N are left A-modules and L is a k-module, we have the isomorphisms

$$\begin{aligned} \operatorname{Hom}_A(L \otimes_k N, M) &\simeq \operatorname{Hom}_A(N, \operatorname{Hom}_k(L, M)) \\ &\simeq \operatorname{Hom}_k(L, \operatorname{Hom}_A(N, M)). \end{aligned} \tag{1.15}$$

Proof. We shall only prove (1.15) in the particular case where $A = k$. In this case, $\operatorname{Hom}_A(L \otimes_k N, M)$ is nothing but the k-module of k-bilinear maps from $L \times N$ to M, and a k-bilinear map from $L \times N$ to M defines uniquely a linear map from L to $\operatorname{Hom}_A(N, M)$ and conversely. q.e.d.

Consider the functors

$$\begin{aligned} &\Phi(\,\bullet\,) := L \otimes_k \bullet \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A), \\ &\Psi(\,\bullet\,) := \operatorname{Hom}_k(L, \bullet\,) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A), \\ &\Phi'(\,\bullet\,) := \bullet \otimes_k N \colon \mathrm{Mod}(k) \to \mathrm{Mod}(A), \\ &\Psi'(\,\bullet\,) := \operatorname{Hom}_A(N, \bullet\,) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(k). \end{aligned}$$

The isomorphisms in (1.15) become:

$$\begin{aligned}\operatorname{Hom}_A(\Phi(N), M) &\simeq \operatorname{Hom}_A(N, \Psi(M))\\ \operatorname{Hom}_A(\Phi'(N), M) &\simeq \operatorname{Hom}_A(N, \Psi'(M))\end{aligned}$$

One translates these isomorphisma (see Chapter 2 below) by saying that (Φ, Ψ) and (Φ', Ψ') are pairs of adjoint functors.

Proposition 1.4.2. *If $M' \to M \to M'' \to 0$ is an exact sequence of left A-modules, then the sequence of k-modules $N \otimes_A M' \to N \otimes_A M \to N \otimes_A M'' \to 0$ is exact.*

Proof. By Proposition 1.3.1 (b), it is enough to check that for any k-module L, the sequence

$$0 \to \operatorname{Hom}_k(N \otimes_A M'', L) \to \operatorname{Hom}_k(N \otimes_A M, L) \to \operatorname{Hom}_k(N \otimes_A M', L)$$

is exact. This sequence is isomorphic to the sequence

$$\begin{aligned}0 \to \operatorname{Hom}_k(M'', \operatorname{Hom}_A(N, L)) \to \operatorname{Hom}_k(M, \operatorname{Hom}_A(N, L))&\\ \to \operatorname{Hom}_k(M', \operatorname{Hom}_A(N, L))&\end{aligned}$$

and it remains to apply Proposition 1.3.1 ((b), (i) $\Rightarrow$ (ii)). q.e.d.

Hence, $\bullet \otimes_A M\colon \mathrm{Mod}(A^{\mathrm{op}}) \to \mathrm{Mod}(k)$ and $N \otimes_A \bullet\colon \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ are right exact functors.

Example 1.4.3. $\bullet \otimes_A M$ is not left exact in general. In fact, consider the commutative ring $A = \mathbb{C}[x]$ and the exact sequence of A-modules:

$$0 \to A \xrightarrow{x\cdot} A \to A/xA \to 0.$$

Apply $\bullet \otimes_A A/Ax$. We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.

Definition 1.4.4. (i) An A^{op}-module N is flat if for any exact sequence $0 \to M' \to M \to M''$ in $\mathrm{Mod}(A)$, the sequence $0 \to N \otimes_A M' \to N \otimes_A M \to N \otimes_A M''$ is exact in $\mathrm{Mod}(k)$. In other words, N is flat if and only if the functor $N \otimes_A \bullet$ is exact.

(ii) If N is flat and moreover $N \otimes_A M = 0$ implies $M = 0$, one says that N is faithfully flat.

(iii) Similarly, an A-module N is flat if the functor $\bullet \otimes_A N$ is exact and N is faithfully flat if moreover $M \otimes_A N = 0$ implies $M = 0$.

One proves easily that a projective module is flat (see Exercise 1.2).

1.5 Limits

Definition 1.5.1. Let I be a set.

(i) An order $\leq$ on I is a relation which satisfies: (a) $i \leq i$, (b) $i \leq j$ & $j \leq k$ implies $i \leq k$, (c) $i \leq j$ and $j \leq i$ implies $i = j$.

(ii) The opposite order $(I, \leq^{\mathrm{op}})$ is defined by $i \leq^{\mathrm{op}} j$ if and only if $j \leq i$.

(iii) An order is discrete if $i \leq j$ implies $i = j$.

The following definition will be of constant use.

Definition 1.5.2. Let $(I, \leq)$ be an ordered set.

(i) One says that $(I, \leq)$ is filtrant (one also says "directed") if for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.

(ii) Assume I is filtrant and let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

Let $(I, \leq)$ be a ordered set and let A be a ring. A projective system $\{N_i, v_{ij}\}$ of A-modules indexed by $(I, \leq)$ is the data for each $i \in I$ of an A-module N_i and for each pair i, j with $i \leq j$ of an A-linear map $v_{ij} : N_j \to N_i$, such that for all i, j, k with $i \leq j$ and $j \leq k$:

$$
\begin{aligned}
v_{ii} &= \mathrm{id}_{N_i} \\
v_{ij} \circ v_{jk} &= v_{ik}.
\end{aligned}
$$

Consider the "universal problem": to find an A-module N and linear maps $v_i : N \to N_i$ satisfying $v_{ij} \circ v_j = v_i$ for all $i \leq j$, such that for any

A-module L and linear maps $g_i : L \to N_i$, satisfying $v_{ij} \circ g_j = g_i$ for all $i \leq j$, there is a unique linear map $g : L \to N$ such that $g_i = v_i \circ g$ for all i. If such a family (N, v_i) exists (and we shall show below that it does), it is unique up to unique isomorphism and one calls it the projective limit of the projective system (N_i, v_{ij}), denoted $\varprojlim_i N_i$. This problem is visualized by the diagram:

$$\begin{tikzcd}
& & N_i \\
L \arrow[urr, "g_i"] \arrow[drr, "g_j"'] \arrow[r, dotted] & \varprojlim_k N_k \arrow[ur, "v_i"'] \arrow[dr, "v_j"] & \\
& & N_j. \arrow[uu, "v_{ij}"']
\end{tikzcd}$$

An inductive system $\{M_i, u_{ji}\}$ of A-modules indexed by $(I, \leq)$ is the data for each $i \in I$ of an A-module M_i and for each pair i, j with $i \leq j$ of an A-linear map $u_{ji} : M_i \to M_j$, such that for all i, j, k with $i \leq j$ and $j \leq k$:

$$\begin{aligned} u_{ii} &= \mathrm{id}_{M_i} \\ u_{kj} \circ u_{ji} &= u_{ki}. \end{aligned}$$

Note that a projective system indexed by $(I, \leq)$ is nothing but an inductive system indexed by $(I, \leq^{\mathrm{op}})$.

Consider the "universal problem": to find an A-module M and linear maps $u_i : M_i \to M$ satisfying $u_j \circ u_{ji} = u_i$ for all $i \leq j$, such that for any A-module L and linear maps $f_i : M_i \to L$ satisfying $f_j \circ u_{ji} = f_i$ for all $i \leq j$, there is a unique linear map $f : M \to L$ such that $f_i = f \circ u_i$ for all i. If such a family $(M, u_i)_i$ exists (and we shall show below that it does), it is unique up to unique isomorphism and one calls it the inductive limit of the inductive system (M_i, u_{ji}), denoted $\varinjlim_i M_i$. This problem is visualized by the diagram:

$$\begin{tikzcd}
M_i \arrow[dr, "u_i"'] \arrow[drr, "f_i"] \arrow[dd, "u_{ji}"'] & & \\
& \varinjlim_k M_k \arrow[r, dotted] & L. \\
M_j \arrow[ur, "u_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

Theorem 1.5.3. (i) *The projective limit of the projective system* $\{N_i, v_{ij}\}$ *is the* A*-module*

$$\varprojlim_i N_i = \{\{x_i\}_i \in \prod_i N_i; v_{ij}(x_j) = x_i \text{ for all } i \le j\}.$$

The maps $v_i : \varprojlim_j N_j \to N_i$ *are the natural ones.*

(ii) *The inductive limit of the inductive system* $\{M_i, u_{ij}\}$ *is the* A*-module*

$$\varinjlim_i M_i = (\bigoplus_{i \in I} M_i)/N$$

where N *is the submodule of* $\bigoplus_{i\in I} M_i$ *generated by* $\{x_i - u_{ji}(x_i); x_i \in M_i, i \le j\}$*. The maps* $u_i : M_i \to \varinjlim_j M_j$ *are the natural ones.*

Note that if I is discrete, then $\varinjlim_i M_i = \bigoplus_i M_i$ and $\varprojlim_i N_i = \prod_i N_i$.

The proof is straightforward.

The universal properties on the projective and inductive limit are better formulated by the isomorphisms which characterize $\varprojlim_i N_i$ and $\varinjlim_i M_i$:

$$\mathrm{Hom}_A(L, \varprojlim_i N_i) \xrightarrow{\sim} \varprojlim_i \mathrm{Hom}_A(L, N_i), \tag{1.16}$$

$$\mathrm{Hom}_A(\varinjlim_i M_i, L) \xrightarrow{\sim} \varprojlim_i \mathrm{Hom}_A(M_i, L). \tag{1.17}$$

There are also natural morphisms

$$\varinjlim_i \mathrm{Hom}_A(L, M_i) \to \mathrm{Hom}_A(L, \varinjlim_i M_i) \tag{1.18}$$

$$\varinjlim_i \mathrm{Hom}_A(N_i, L) \to \mathrm{Hom}_A(\varprojlim_i N_i, L). \tag{1.19}$$

One should be aware that morphisms (1.18) and (1.19) are not isomorphisms in general (see Example 1.5.7 below).

Proposition 1.5.4. *Let* $M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i''$ *be a family of exact sequences of* A*-modules, indexed by the set* I*. Then the sequences*

$$\prod_i M_i' \to \prod_i M_i \to \prod_i M_i'' \text{ and } \bigoplus_i M_i' \to \bigoplus_i M_i \to \bigoplus_i M_i''$$

are exact.

The proof is left as an (easy) exercise.

Proposition 1.5.5. (i) *Consider a projective system of exact sequences of A-modules: $0 \to N'_i \xrightarrow{f_i} N_i \xrightarrow{g_i} N''_i$. Then the sequence $0 \to \varprojlim_i N'_i \xrightarrow{f} \varprojlim_i N_i \xrightarrow{g} \varprojlim_i N''_i$ is exact.*

(ii) *Consider an inductive system of exact sequences of A-modules: $M'_i \xrightarrow{f_i} M_i \xrightarrow{g_i} M''_i \to 0$. Then the sequence $\varinjlim_i M'_i \xrightarrow{f} \varinjlim_i M_i \xrightarrow{g} \varinjlim_i M''_i \to 0$ is exact.*

Proof. (i) Since $\varprojlim_i N'_i$ is a submodule of $\prod_i N'_i$, the fact that f is injective follows from Proposition 1.5.4. Let $\{x_i\}_i \in \varprojlim_i N_i$ with $g(\{x_i\}_i) = 0$. Then $g_i(x_i) = 0$ for all i, and there exists a unique $x'_i \in N'_i$ such that $x_i = f_i(x'_i)$. One checks immedialtely that the element $\{x'_i\}_i$ belongs to $\varprojlim_i N'_i$.

(ii) Let L be an A-module. The sequence

$$0 \to \operatorname{Hom}_A(\varinjlim_i M''_i, L) \to \operatorname{Hom}_A(\varinjlim_i M_i, L) \to \operatorname{Hom}_A(\varinjlim_i M'_i, L)$$

is isomorphic to the sequence

$$0 \to \varprojlim_i \operatorname{Hom}_A(M''_i, L) \to \varprojlim_i \operatorname{Hom}_A(M_i, L) \to \varprojlim_i \operatorname{Hom}_A(M'_i, L)$$

and this sequence is exact by (i) and Proposition 1.3.1. Then the result follows, again by Proposition 1.3.1. q.e.d.

One says that "the functor $\varinjlim$ is right exact", and "the functor $\varprojlim$ is left exact". We shall give a precise meaning to these sentences in Chapter 2.

Remark 1.5.6. (i) If all M_i's are submodules of a module M, and if the maps $u_{ji} : M_i \to M_j, (i \leq j)$ are the natural injective morphisms, then $\varinjlim_i M_i \simeq \bigcup_i M_i$.
(ii) If all M_i's are submodules of a module M, and if the maps $v_{ij} : M_j \to M_i, (i \leq j)$ are the natural injective morphisms, then $\varprojlim_i M_i \simeq \bigcap_i M_i$.

Example 1.5.7. Let k be a commutative ring and consider the k-algebra $A := k[x]$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that

$A/I^{n+1} \simeq k[x]^{\leq n}$, where $k[x]^{\leq n}$ denotes the k-module consisting of polynomials of degree less than or equal to n.
(i) For $p \leq n$ there are monomorphisms $u_{pn} : k[x]^{\leq p} \rightarrowtail k[x]^{\leq n}$ which define an inductive system of k-modules. One has the isomorphism

$$k[x] = \varinjlim_{n} k[x]^{\leq n}.$$

Notice that $\mathrm{id}_{k[x]} \notin \varinjlim_{n} \mathrm{Hom}_k(k[x], k[x]^{\leq n})$. This shows that the morphism (1.18) is not an isomorphism in general.
(ii) For $p \leq n$ there are epimorphisms $v_{pn} : A/I^n \twoheadrightarrow A/I^p$ which define a projective system of A-modules whose projective limit is $k[[x]]$, the ring of formal series with coefficients in k.
(iii) For $p \leq n$ there are monomorphisms $I^n \rightarrowtail I^p$ which define a projective system of A-modules whose projective limit is 0.
(iv) We thus have a projective system of complexes of A-modules

$$L_n^\bullet : 0 \to I^n \to A \to A/I^n \to 0.$$

Taking the projective limit, we get the complex $0 \to 0 \to k[x] \to k[[x]] \to 0$ which is no more exact.

Tensor products and inductive limits

Let $\{M_i, u_{ji}\}$ be an inductive system of A-modules, N a right A-module. The family of morphisms $M_i \to \varinjlim_{i} M_i$ defines the family of morphisms $N \otimes_A M_i \to N \otimes_A \varinjlim_{i} M_i$, hence the morphism

$$\varinjlim_{i}(N \otimes_A M_i) \to N \otimes_A \varinjlim_{i} M_i. \tag{1.20}$$

Proposition 1.5.8. *The morphism* (1.20) *is an isomorphism.*

Proof. Let L be a k-module. Consider the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_k(N \otimes_A \varinjlim_{i} M_i, L) &\simeq \mathrm{Hom}_A(\varinjlim_{i} M_i, \mathrm{Hom}_k(N, L)) \\
&\simeq \varprojlim_{i} \mathrm{Hom}_A(M_i, \mathrm{Hom}_k(N, L)) \\
&\simeq \varprojlim_{i} \mathrm{Hom}_k(N \otimes_A M_i, L) \\
&\simeq \mathrm{Hom}_k(\varinjlim_{i}(N \otimes_A M_i), L).
\end{aligned}$$

Then the result follows from Proposition 1.3.6. q.e.d.

Filtrant limit

Lemma 1.5.9. *Assume I is a filtrant ordered set and let $M = \varinjlim_i M_i$.*

(i) *Let $x_i \in M_i$. Then $u_i(x_i) = 0 \Leftrightarrow$ there exists $k \geq i$ with $u_{ki}(x_i) = 0$.*

(ii) *Let $x \in M$. Then there exists $i \in I$ and $x_i \in M_i$ with $u_i(x_i) = x$.*

Proof. We keep the notations of Theorem 1.5.3 (ii). Hence, N denotes the submodule of $\oplus_i M_i$ generated by the elements $\{x_i - u_{ji}(x_i); x_i \in M_i, i \leq j\}$. Let N' denote the subset of $\oplus_i M_i$ consisting of finite sums $\sum_{j \in J} x_j$, $x_j \in M_j$ such that there exists $k \geq j$ for all $j \in J$ with $\sum_{j \in J} u_{kj}(x_j) = 0$. Since I is filtrant, N' is a submodule of $\oplus_i M_i$. Let us show that $N = N'$. The inclusion $N \subset N'$ is obvious since $u_{ji}(x_i) + u_{jj}(-u_{ji}(x_i)) = 0$. Conversely, let $J \subset I$ be a finite set and let $x = \sum_{j \in J} x_j \in N'$. Then

$$\begin{aligned} \sum_{j \in J} x_j &= \sum_{j \in J} x_j - \sum_{j \in J} u_{kj}(x_j) \\ &= \sum_{j \in J} (x_j - u_{kj}(x_j)) \in N'. \end{aligned}$$

(i) follows from the fact that $x \in N' \cap M_i$ if and only if there exists $k \geq i$ with $u_{ki}(x_i) = 0$.
(ii) Let $x \in M$. There exist a finite set $J \subset I$ and $x_j \in M_j$ such that $x = \sum_{j \in J} u_j(x_j)$. Choose i with $i \geq j$ for all $j \in J$. Then

$$x = \sum_{j \in J} u_k u_{ij}(x_j) = u_i(\sum_{j \in J} u_{ij}(x_j)).$$

Setting $x_i = \sum_{j \in J} u_{ij}(x_j)$, the result follows. q.e.d.

Example 1.5.10. Let X be a topological space, $x \in X$ and denote by I_x the set of open neighborhoods of x in X. We endow I_x with the order: $U \leq V$ if $V \subset U$. Given U and V in I_x, and setting $W = U \cap V$, we have $U \leq W$ and $V \leq W$. Therefore, I_x is filtrant.

Denote by $\mathcal{C}^0(U)$ the $\mathbb{C}$-vector space of complex valued continuous functions on U. The restriction maps $\mathcal{C}^0(U) \to \mathcal{C}^0(V), V \subset U$ define an inductive system of $\mathbb{C}$-vector spaces indexed by I_x. One sets

$$\mathcal{C}^0_{X,x} = \varinjlim_{U \in I_x} \mathcal{C}^0(U). \tag{1.21}$$

An element φ of $\mathcal{C}^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U, \varphi_U)/\sim$ with U a neighborhood of x, φ_U a continuous function on U, and $(U,\varphi_U)\sim 0$ if there exists a neighborhood V of x with $V\subset U$ such that the restriction of φ_U to V is the zero function. Hence, a germ of function is zero at x if this function is identically zero in a neighborhood of x.

Proposition 1.5.11. *Consider an inductive system of exact sequences of A-modules indexed by a filtrant ordered set I: $M'_i \xrightarrow{f_i} M_i \xrightarrow{g_i} M''_i$. Then the sequence*

$$\varinjlim_i M'_i \xrightarrow{f} \varinjlim_i M_i \xrightarrow{g} \varinjlim_i M''_i$$

is exact.

Proof. Let $x\in\varinjlim_i M_i$ with $g(x)=0$. There exists $x_i\in M_i$ with $u_i(x_i)=x$, and there exists $j\geq i$ such that $u_{ji}(g_i(x_i))=0$. Hence $g_j(u_{ji}(x_i))=u_{ji}(f_i(x_i))=0$, which implies that there exists $x'_j\in M'_j$ such that $u_{ji}(x_i)=f_j(x'_j)$. Then $x'=u'_j(x'_j)$ satisfies $f(x')=f(u'_j(x'_j))=u_jf_j(x'_j)=u_ju_{ji}(x_i)=x$. q.e.d.

Proposition 1.5.12. *Assume $J\subset I$ and assume that I is filtrant and J is cofinal to I.*

(i) *Let $\{M_i,u_{ij}\}$ be an inductive system of A-modules indexed by I. Then the natural morphism $\varinjlim_{j\in J} M_j\to\varinjlim_{i\in I} M_i$ is an isomorphism.*

(ii) *Let $\{M_i,v_{ji}\}$ be a projective system of A-modules indexed by I. Then the natural morphism $\varprojlim_{i\in I} M_i\to\varprojlim_{j\in J} M_j$ is an isomorphism.*

The proof is left as an exercise.

In particular, assume $I=\{0,1\}$ with $0<1$. Then the inductive limit of the inductive system $u_{10}:M_0\to M_1$ is M_1, and the projective limit of the projective system $v_{01}:M_1\to M_0$ is M_1.

The Mittag-Leffler condition

Recall (Proposition 1.5.4) that a product of exact sequences of A-modules is an exact sequence. Let us give another criterion in order that the projective limit of an exact sequence remains exact. This is a particular case of the so-called "Mittag-Leffler" condition (see [15]).

Proposition 1.5.13. *Let* $0 \to \{M'_n\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M''_n\} \to 0$ *be an exact sequence of projective systems of* A*-modules indexed by* $\mathbb{N}$*. Assume that for each* n*, the map* $M'_{n+1} \to M'_n$ *is surjective. Then the sequence*

$$0 \to \varprojlim_n M'_n \xrightarrow{f} \varprojlim_n M_n \xrightarrow{g} \varprojlim_n M''_n \to 0$$

is exact.

Proof. Let us denote for short by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p.

Let $\{x''_p\}_p \in \varprojlim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n : g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$.
q.e.d.

1.6 Koszul complexes

If L is a finite free k-module of rank n, one denotes by $\bigwedge^j L$ the k-module consisting of j-multilinear alternate forms on the dual space L^* and calls it the j-th exterior power of L. (Recall that $L^* = \mathrm{Hom}_k(L, k)$.)

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq k$. One sets $\bigwedge^0 L = k$.

If $(e_1, \ldots, e_n)$ is a basis of L and $I = \{i_1 < \cdots < i_j\} \subset \{1, \ldots, n\}$, one sets

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \ldots, n\}$, one denotes by $|I|$ its cardinal. The family of e_I's with $|I| = j$ is a basis of the free module $\bigwedge^j L$.

Let M be an A-module and let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be n endomorphisms of M over A which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \le i, j \le n.$$

(Recall the notation $[a, b] := ab - ba$.) Set $M^{(j)} = M \otimes \bigwedge^j k^n$. Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \ldots, e_n)$ the canonical basis of k^n. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \operatorname{Hom}_A(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^{n} \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex, called a Koszul complex and denoted $K^\bullet(M, \varphi)$:

$$0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0.$$

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$\begin{aligned} H^0(K^\bullet(M, \varphi)) &\simeq \operatorname{Ker} \varphi_1 \cap \ldots \cap \operatorname{Ker} \varphi_n, \\ H^n(K^\bullet(M, \varphi)) &\simeq M/(\varphi_1(M) + \cdots + \varphi_n(M)). \end{aligned}$$

Definition 1.6.1. (i) If for each j, $1 \leq j \leq n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M) + \cdots + \varphi_{j-1}(M))$, one says $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence.

(ii) If for each j, $1 \leq j \leq n$, φ_j is surjective as an endomorphism of $\operatorname{Ker} \varphi_1 \cap \ldots \cap \operatorname{Ker} \varphi_{j-1}$, one says $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence.

Theorem 1.6.2. (i) *Assume that $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence. Then $H^j(K^\bullet(M, \varphi)) \simeq 0$ for $j \neq n$.*

(ii) *Assume that $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence. Then $H^j(K^\bullet(M, \varphi)) \simeq 0$ for $j \neq 0$.*

Proof. The proof will be given in Section 4.2. Here, we restrict ourselves to the simple case $n = 2$ for coregular sequences. Hence we consider the complex:

$$0 \to M \xrightarrow{d} M \times M \xrightarrow{d} M \to 0$$

where $d(x) = (\varphi_1(x), \varphi_2(x))$, $d(y, z) = \varphi_2(y) - \varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\operatorname{Ker} \varphi_1$.

Let $(y, z) \in M \times M$ with $\varphi_2(y) = \varphi_1(z)$. We look for $x \in M$ solution of $\varphi_1(x) = y$, $\varphi_2(x) = z$. First choose $x' \in M$ with $\varphi_1(x') = y$. Then $\varphi_2 \circ \varphi_1(x') = \varphi_2(y) = \varphi_1(z) = \varphi_1 \circ \varphi_2(x')$. Thus $\varphi_1(z - \varphi_2(x')) = 0$ and there exists $t \in M$ with $\varphi_1(t) = 0$, $\varphi_2(t) = z - \varphi_2(x')$. Hence $y = \varphi_1(t + x')$, $z = \varphi_2(t + x')$ and $x = t + x'$ is a solution to our problem. q.e.d.

Example 1.6.3. Let k be a field of characteristic 0 and let $A = k[x_1, \dots, x_n]$.
(i) Denote by $x_i\cdot$ the multiplication by x_i in A. We get the complex:

$$0 \to A^{(0)} \xrightarrow{d} \cdots \to A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I x_j \cdot a_I \otimes e_j \wedge e_I.$$

The sequence $(x_1\cdot, \dots, x_n\cdot)$ is a regular sequence in A, considered as an A-module. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to k.
(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a k-linear map on the k-vector space A. Hence we get a Koszul complex

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1\cdot, \dots, \partial_n\cdot)$ is a coregular sequence, and the above complex is exact except in degree 0 where its cohomology is isomorphic to k. Writing dx_j instead of e_j, we recognize the "de Rham complex".

Example 1.6.4. Let $W = W_n(k)$ be the Weyl algebra introduced in Example 1.1.2, and denote by $\cdot\partial_i$ the multiplication on the right by ∂_i. Then $(\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence on W (considered as an W-module) and we get the Koszul complex:

$$0 \to W^{(0)} \xrightarrow{\delta} \cdots \to W^{(n)} \to 0$$

where:

$$\delta(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I a_I \cdot \partial_j \otimes e_j \wedge e_I.$$

This complex is exact except in degree n where its cohomology is isomorphic to $k[x]$ (see Exercise 1.3).

Remark 1.6.5. One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_l}} \wedge \ldots \wedge e_{i_k} & \text{if } e_{i_l} = e_j \end{cases}$$

where $e_{i_1} \wedge \ldots \wedge \widehat{e_{i_l}} \wedge \ldots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \ldots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^{n} \varphi_j(m) e_j \lfloor e_I.$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi) : 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0,$$

This complex is in fact isomorphic to a Koszul complex. Consider the isomorphism

$$* : \bigwedge^{j} k^n \xrightarrow{\sim} \bigwedge^{n-j} k^n$$

which associates $\varepsilon_I m \otimes e_{\hat{I}}$ to $m \otimes e_I$, where $\hat{I} = (1, \ldots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \ldots, n)$ to $I \sqcup \hat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ.

Exercises to Chapter 1

Exercise 1.1. Consider two complexes of A-modules $M_1' \to M_1 \to M_1''$ and $M_2' \to M_2 \to M_2''$. Prove that the two sequences are exact if and only if the sequence $M_1' \oplus M_2' \to M_1 \oplus M_2 \to M_1'' \oplus M_2''$ is exact.

Exercise 1.2. (i) Prove that a free module is projective and flat.
(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (*i.e.,* there exists a module K such that $P \oplus K$ is free).
(iii) Deduce that projective modules are flat.

Exercise 1.3. Let k be a field of characteristic 0, $W := W_n(k)$ the Weyl algebra in n variables.
(i) Denote by $x_i \cdot : W \to W$ the multiplication on the left by x_i on W (hence, the $x_i\cdot$'s are morphisms of right W-modules). Prove that $\varphi = (x_1\cdot, \ldots, x_n\cdot)$ is a regular sequence and calculate $H^j(K^\bullet(W, \varphi))$.
(ii) Denote $\cdot\partial_i$ the multiplication on the right by ∂_i on W. Prove that $\psi = (\cdot\partial_1, \ldots, \cdot\partial_n)$ is a regular sequence and calculate $H^j(K^\bullet(W, \psi))$.
(iii) Now consider the left $W_n(k)$-module $\mathcal{O} := k[x_1, \ldots, x_n]$ and the k-linear map $\partial_i : \mathcal{O} \to \mathcal{O}$ (derivation with respect to x_i). Prove that $\lambda = (\partial_1, \ldots, \partial_n)$ is a coregular sequence and calculate $H^j(K^\bullet(\mathcal{O}, \lambda))$.

Exercise 1.4. Let $A = W_2(k)$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot\partial_2$ and calculate its cohomology.

Exercise 1.5. If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.
(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.
(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.
(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.
(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.
(Hint: (iii) Use formula (1.15). (iv) Prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 1.6. Let k be a field, $A = k[x, y]$ and consider the A-module $M = \bigoplus_{i \geq 1} k[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Exercise 1.7. Let I be a filtrant ordered set and let $M_i, i \in I$ be an inductive sytem of k-modules indexed by I. Let $M = \bigsqcup M_i/\sim$ where $\bigsqcup$ denotes the set-theoretical disjoint union and $\sim$ is the relation $M_i \ni x_i \sim y_j \in M_j$ if there exists $k \geq i, k \geq j$ such that $u_{ki}(x_i) = u_{kj}(y_j)$.

Prove that M is naturally a k-module and is isomorphic to $\varinjlim_i M_i$.

Exercise 1.8. Let I be a filtrant ordered set and let $A_i, i \in I$ be an inductive sytem of rings indexed by I.
(i) Prove that $A := \varinjlim_i A_i$ is naturally endowed with a ring structure.
(ii) Define the notion of an inductive system M_i of A_i-modules, and define the A-module $\varinjlim_i M_i$.
(iii) Let N_i (resp. M_i) be an inductive system of right (resp. left) A_i modules. Prove the isomorphism

$$\varinjlim_i (N_i \otimes_{A_i} M_i) \xrightarrow{\sim} \varinjlim_i N_i \otimes_A \varinjlim_i M_i.$$

Chapter 2

The language of categories

In this chapter we introduce some basic notions of category theory which are of constant use in various fields of Mathematics, without spending too much time on this language. After giving the main definitions on categories and functors, we prove the Yoneda Lemma. We also introduce the notions of representable functors and adjoint functors.

Then we construct inductive and projective limits in categories by using *projective* limits in the category **Set** of sets and give some examples. We also analyze some related notions, in particular those of cofinal categories, filtrant categories and exact functors. Special attention will be paid to filtrant inductive limits in the category **Set**.
Some references: [21], [4], [20], [10], [18], [19].

2.1 Categories and functors

Definition 2.1.1. A category $\mathcal{C}$ consists of:

(i) a family $\mathrm{Ob}(\mathcal{C})$, the objects of $\mathcal{C}$,

(ii) for each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, the morphisms from X to Y,

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map, called the composition, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, and denoted $(f, g) \mapsto g \circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, $f \circ \mathrm{id}_X = f$, $\mathrm{id}_X \circ g = g$.

Note that $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ is characterized by the condition in (b).

Remark 2.1.2. There are some set-theoretical dangers, and one should mention in which "universe" we are working. For sake of simplicity, we shall not enter in these considerations here.

Notation 2.1.3. One often writes $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$ and $f : X \to Y$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. One calls X the source and Y the target of f.

A morphism $f : X \to Y$ is an *isomorphism* if there exists $g : X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$. In such a case, one writes $f : X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.

A morphism $f : X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f : X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f : X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).

Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g : X \rightrightarrows Y$.

One introduces the *opposite category* $\mathcal{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}), \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

A category $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$, denoted $\mathcal{C}' \subset \mathcal{C}$, if: $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any $X, Y \in \mathcal{C}'$ and the composition $\circ$ in $\mathcal{C}'$ is induced by the composition in $\mathcal{C}$. One says that $\mathcal{C}'$ is a *full* subcategory if for all $X, Y \in \mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category.

A category $\mathcal{C}$ is *finite* if the family of all morphisms in $\mathcal{C}$ (hence, in particular, the family of objects) is a finite set.

A category $\mathcal{C}$ is a *groupoid* if all morphisms are isomorphisms.

Examples 2.1.4. (i) **Set** is the category of sets and maps, $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.
(ii) **Rel** is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X, Y) = \mathcal{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. If $f : X \to Y$ and $g : Y \to Z$, $g \circ f$ is the set

$$\{(x, z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x, y) \in f, (y, z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta \subset X \times X$, the diagonal of $X \times X$.
Notice that **Set** is a subcategory of **Rel**, not a full subcategory.

(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall often use the notations **Ab** instead of $\mathrm{Mod}(\mathbb{Z})$ and $\mathrm{Hom}_A(\bullet, \bullet)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\bullet, \bullet)$.

One denotes by $\mathrm{Mod}^f(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.
(iv) One denotes by $C(\mathrm{Mod}(A))$ the category whose objects are the complexes of A-modules and morphisms, morphisms of such complexes.
(v) One associates to a pre-ordered set $(I, \leq)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite order.
(vi) We denote by **Top** the category of topological spaces and continuous maps.

Definition 2.1.5. (i) An object $P \in \mathcal{C}$ is called initial if for all $X \in \mathcal{C}, \mathrm{Hom}_{\mathcal{C}}(P, X) \simeq \{\mathrm{pt}\}$. One often denotes by $\emptyset_{\mathcal{C}}$ an initial object in $\mathcal{C}$.

(ii) One says that P is terminal if P is initial in $\mathcal{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathcal{C}, \mathrm{Hom}_{\mathcal{C}}(X, P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathcal{C}}$ a terminal object in $\mathcal{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathcal{C}$ has a zero object, for any object $X \in \mathcal{C}$, the morphism obtained as the composition $X \to 0 \to X$ is still denoted by $0 \colon X \to X$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 2.1.6. (i) In the category **Set**, $\emptyset$ is initial and $\{\mathrm{pt}\}$ is terminal.
(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.
(iii) The category associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Definition 2.1.7. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ consists of a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and for all $X, Y \in \mathcal{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ the contravariant functor, associated with $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$.

Definition 2.1.8. (i) One says that F is faithful (resp. full, resp. fully faithful) if for X, Y in $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$$

is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathcal{C}'$ there exists $X \in \mathcal{C}$ and an isomorphism $F(X) \simeq Y$.

One defines the product of two categories $\mathcal{C}$ and $\mathcal{C}'$ by :

$$\begin{aligned} \mathrm{Ob}(\mathcal{C} \times \mathcal{C}') &= \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \\ \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}'}((X,X'),(Y,Y')) &= \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}'}(X',Y'). \end{aligned}$$

A bifunctor $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ is a functor on the product category. This means that for $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$, $F(X, \bullet) : \mathcal{C}' \to \mathcal{C}''$ and $F(\bullet, X') : \mathcal{C} \to \mathcal{C}''$ are functors, and moreover for any morphisms $f : X \to Y$ in $\mathcal{C}$, $g : X' \to Y'$ in $\mathcal{C}'$, the diagram below commutes:

$$\begin{tikzcd} F(X,X') \arrow[r, "F(X,g)"] \arrow[d, "F(f,X')"'] & F(X,Y') \arrow[d, "F(f,Y')"] \\ F(Y,X') \arrow[r, "F(Y,g)"] & F(Y,Y') \end{tikzcd}$$

In fact, $(f,g) = (\mathrm{id}_Y, g) \circ (f, \mathrm{id}_{X'}) = (f, \mathrm{id}_{Y'}) \circ (\mathrm{id}_X, g)$.

Examples 2.1.9. (i) $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is a bifunctor.
(ii) If A is a k-algebra, $\bullet \otimes_A \bullet$: $\mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ and $\mathrm{Hom}_A(\bullet, \bullet)$: $\mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ are bifunctors.
(iii) Let A be a ring. Then $H^j(\bullet) : C(\mathrm{Mod}(A)) \to \mathrm{Mod}(A)$ is a functor.
(iv) The forgetful functor $for : \mathrm{Mod}(A) \to \mathbf{Set}$ associates to an A-module M the set M, and to a linear map f the map f.

Definition 2.1.10. Let F_1, F_2 are two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism of functors $\theta : F_1 \to F_2$ is the data for all $X \in \mathcal{C}$ of a morphism $\theta(X) : F_1(X) \to F_2(X)$ such that for all $f : X \to Y$, the diagram below commutes:

$$\begin{tikzcd} F_1(X) \arrow[r, "\theta(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "\theta(Y)"] & F_2(Y) \end{tikzcd}$$

A morphism of functors is visualized by a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"{name=U}] \arrow[r, bend right, "F_2"'{name=D}] & \mathcal{C}' \arrow[from=U, to=D, phantom, "\Downarrow\theta"] \end{tikzcd}$$

Hence, by considering the family of functors from $\mathcal{C}$ to $\mathcal{C}'$ and the morphisms of such functors, we get a new category.

Notation 2.1.11. (i) We denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from $\mathcal{C}$ to $\mathcal{C}'$. One may also use the shorter notation $(\mathcal{C}')^{\mathcal{C}}$.

Examples 2.1.12. Let k be a field and consider the functor

$$\begin{aligned} {}^*\colon \mathrm{Mod}(k)^{\mathrm{op}} &\to \mathrm{Mod}(k), \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k). \end{aligned}$$

Then there is a morphism of functors $\mathrm{id} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(k), \mathrm{Mod}(k))$.
(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (2.5)).

In particular we have the notion of an isomorphism of categories. If F is an isomorphism of categories, then there exists $G : \mathcal{C}' \to \mathcal{C}$ such that for all $X \in \mathcal{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is an weaker notion that we introduce below.

Definition 2.1.13. A functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if there exists $G : \mathcal{C}' \to \mathcal{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathcal{C}'}$.

We shall not give the proof of the following important result below.

Theorem 2.1.14. *The functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 2.1.15. (i) Let k be a field and let $\mathcal{C}$ denote the category defined by $\mathrm{Ob}(\mathcal{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathcal{C}}(n, m) = M_{m,n}(k)$, the space of matrices of type (m, n) with entries in a field k (the composition being the usual composition of matrices). Define the functor $F : \mathcal{C} \to \mathrm{Mod}^f(k)$ as follows. To $n \in \mathbb{N}$, $F(n)$ associates $k^n \in \mathrm{Mod}^f(k)$ and to a matrix of type (m, n), F associates the induced linear map from k^n to k^m. Clearly F is fully faithful, and since

any finite dimensional vector space admits a basis, it is isomorphic to k^n for some n, hence F is essentially surjective. In conclusion, F is an equivalence of categories.
(ii) let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathcal{C},\mathcal{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},(\mathcal{C}')^{\mathrm{op}}). \tag{2.1}$$

(iii) Let I, J and $\mathcal{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I\times J,\mathcal{C}) \simeq \mathrm{Fct}(J,\mathrm{Fct}(I,\mathcal{C})) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathcal{C})). \tag{2.2}$$

2.2 The Yoneda Lemma

Definition 2.2.1. Let $\mathcal{C}$ be a category. One defines the categories

$$\mathcal{C}^\wedge = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathbf{Set}), \qquad \mathcal{C}^\vee = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathbf{Set}^{\mathrm{op}}),$$

and the functors

$$\begin{array}{rcll} \mathrm{h}_{\mathcal{C}} & : & \mathcal{C}\to\mathcal{C}^\wedge, & X\mapsto \mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,X) \\ \mathrm{k}_{\mathcal{C}} & : & \mathcal{C}\to\mathcal{C}^\vee, & X\mapsto \mathrm{Hom}_{\mathcal{C}}(X,\,\bullet\,). \end{array}$$

Since there is a natural equivalence of categories

$$\mathcal{C}^\vee \simeq \mathcal{C}^{\mathrm{op},\wedge,\mathrm{op}}, \tag{2.3}$$

we shall concentrate our study on $\mathcal{C}^\wedge$.

Proposition 2.2.2. (The Yoneda lemma.) *For $A\in\mathcal{C}^\wedge$ and $X\in\mathcal{C}$, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A)\simeq A(X)$, functorial with respect to X and A.*

Proof. One constructs the morphism $\varphi\colon \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A)\to A(X)$ by the chain of morphisms: $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A)\to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{C}}(X,X),A(X))\to A(X)$, where the last map is associated with id_X.

To construct $\psi : A(X)\to \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}(X)_{\mathcal{C}},A)$, it is enough to associate with $s\in A(X)$ and $Y\in\mathcal{C}$ a map from $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ to $A(Y)$. It is defined by the chain of maps $\mathrm{Hom}_{\mathcal{C}}(Y,X)\to\mathrm{Hom}_{\mathbf{Set}}(A(X),A(Y))\to A(Y)$ where the last map is associated with $s\in A(X)$.

One checks that φ and ψ are inverse to each other. q.e.d.

Corollary 2.2.3. *The functor $\mathrm{h}_{\mathcal{C}}$ is fully faithful.*

Proof. For X and Y in $\mathcal{C}$, one has $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_\mathcal{C}(X), \mathrm{h}(Y)) \simeq \mathrm{h}_\mathcal{C}(Y)(X) = \mathrm{Hom}_\mathcal{C}(X, Y)$. q.e.d.

One calls $\mathrm{h}_\mathcal{C}$ the Yoneda embedding. Hence, one may consider $\mathcal{C}$ as a full subcategory of $\mathcal{C}^\wedge$.

Corollary 2.2.4. *Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism in $\mathcal{C}$.*

(i) *Assume that for any $Z \in \mathcal{C}$, the map $\mathrm{Hom}_\mathcal{C}(Z, X) \xrightarrow{f\circ} \mathrm{Hom}_\mathcal{C}(Z, Y)$ is bijective. Then f is an isomorphism.*

(ii) *Assume that for any $Z \in \mathcal{C}$, the map $\mathrm{Hom}_\mathcal{C}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_\mathcal{C}(X, Z)$ is bijective. Then f is an isomorphism.*

Proof. (i) By the hypothesis, $\mathrm{h}_\mathcal{C}(f) : \mathrm{h}_\mathcal{C}(X) \to \mathrm{h}_\mathcal{C}(Y)$ is an isomorphism in $\mathcal{C}^\wedge$. Since $\mathrm{h}_\mathcal{C}$ is fully faithful, this implies that f is an isomorphism. (See Exercise 2.2 (ii).)
(ii) follows by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$. q.e.d.

Representable functors

Definition 2.2.5. (i) One says that a functor F from $\mathcal{C}^{\mathrm{op}}$ to **Set** is representable if there exists $X \in \mathcal{C}$ and an isomorphism if $F \simeq \mathrm{h}_\mathcal{C}(X))$ in $\mathcal{C}^\wedge$, or, in other words, if there exists an isomorphism $F(Y) \simeq \mathrm{Hom}_\mathcal{C}(Y, X)$, functorially in $Y \in \mathcal{C}$. Such an object X is called a representative of F.

(ii) Similarly, a functor $G \colon \mathcal{C} \to \mathbf{Set}$ is representable if there exists $X \in \mathcal{C}$ such that $G(Y) \simeq \mathrm{Hom}_\mathcal{C}(X, Y)$, functorially in $Y \in \mathcal{C}$.

It is important to notice that the isomorphisms above determine X up to unique isomorphism.

Representable functors provides a categorical language to deal with universal problems. Let us illustrate this by an example.

Example 2.2.6. Let k be a commutative ring and let M, N, L be three k-modules. Denote by $B(N \times M, L)$ the set of k-bilinear maps from $N \times M$ to L. Then the functor $F \colon L \mapsto B(N \times M, L)$ is representable by $N \otimes_k M$, since $F(L) = B(N \times M, L) \simeq \mathrm{Hom}_k(N \otimes M, L)$.

2.3 Adjoint functors

Definition 2.3.1. Let $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}$ be two functors. One says that (F, G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\operatorname{Hom}_{\mathcal{C}'}(F(\,\bullet\,), \,\bullet\,) \simeq \operatorname{Hom}_{\mathcal{C}}(\,\bullet\,, G(\,\bullet\,)) \tag{2.4}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \operatorname{Hom}_{\mathcal{C}}(F(X), Y)$.

The isomorphism (2.4) gives the isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}'}(F\circ G(\,\bullet\,), \,\bullet\,) &\simeq \operatorname{Hom}_{\mathcal{C}}(G(\,\bullet\,), G(\,\bullet\,)),\\ \operatorname{Hom}_{\mathcal{C}'}(F(\,\bullet\,), F(\,\bullet\,)) &\simeq \operatorname{Hom}_{\mathcal{C}}(\,\bullet\,, G\circ F(\,\bullet\,)).\end{aligned}$$

In particular, we have morphisms $X \to G\circ F(X)$, functorial in $X \in \mathcal{C}$, and morphisms $F \circ G(Y) \to Y$, functorial in $Y \in \mathcal{C}'$. In other words, we have morphisms of functors

$$F \circ G \to \mathrm{id}_{\mathcal{C}'}, \qquad \mathrm{id}_{\mathcal{C}} \to G \circ F. \tag{2.5}$$

Example 2.3.2. Let A be a k-algebra. Let $K \in \operatorname{Mod}(k)$ and let $M, N \in \operatorname{Mod}(A)$. The formula:

$$\operatorname{Hom}_A(N \otimes K, M) \simeq \operatorname{Hom}_A(N, \operatorname{Hom}(K, M)).$$

tells us that the functors $\bullet \otimes K$ and $\operatorname{Hom}(K, \,\bullet\,)$ from $\operatorname{Mod}(A)$ to $\operatorname{Mod}(A)$ are adjoint.
In the preceding situation, denote by *for* : $\operatorname{Mod}(A) \to \operatorname{Mod}(k)$ the "forgetful functor" which, to an A-module M associates the underlying k-module. Applying the above formula with $N = A$, we get

$$\operatorname{Hom}_A(A \otimes K, M) \simeq \operatorname{Hom}(K, \mathit{for}(M)).$$

Hence, the functors $A \otimes \,\bullet\,$ (extension of scalars) and *for* are adjoint.

Example 2.3.3. Let $X, Y, Z \in \mathbf{Set}$. The bijection

$$\operatorname{Hom}_{\mathbf{Set}}(X \times Y, Z) \simeq \operatorname{Hom}_{\mathbf{Set}}(X, \operatorname{Hom}_{\mathbf{Set}}(Y, Z))$$

tells us that $(\,\bullet\, \times Y, \operatorname{Hom}_{\mathbf{Set}}(Y, \,\bullet\,))$ is a pair of adjoint functors.

2.4 Limits

In the sequel, I will denote a category. Let $\mathcal{C}$ be a category. A functor $\alpha\colon I \to \mathcal{C}$ (resp. $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$) is sometimes called an inductive (resp. projective) system in $\mathcal{C}$ indexed by I.

For example, if $(I, \leq)$ is an ordered set, I the associated category, an inductive system indexed by I is the data of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ and for all $i \leq j$, a morphism $X_i \to X_j$ with the natural compatibility conditions.

Assume first that $\mathcal{C}$ is the category **Set** and let us consider projective systems. One sets

$$\varprojlim \beta = \{\{x_i\}_i \in \prod_i \beta(i); \beta(s)(x_j) = x_i \text{ for all } s \in \mathrm{Hom}_I(i,j)\}. \tag{2.6}$$

The next result is obvious.

Lemma 2.4.1. *Let $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$ be a functor and let $X \in \mathbf{Set}$. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{Set}}(X, \varprojlim \beta) \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathbf{Set}}(X, \beta),$$

where $\mathrm{Hom}_{\mathbf{Set}}(X,\beta)$ denotes the functor $I^{\mathrm{op}} \to \mathbf{Set}$, $i \mapsto \mathrm{Hom}_{\mathbf{Set}}(X, \beta(i))$.

Consider now two functors $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ and $\alpha\colon I \to \mathcal{C}$. For $X \in \mathcal{C}$, we get functors from I^{op} to **Set**:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}}(X,\beta)\colon I^{\mathrm{op}} \ni i \mapsto \mathrm{Hom}_{\mathcal{C}}(X, \beta(i)) \in \mathbf{Set}, \\ &\mathrm{Hom}_{\mathcal{C}}(\alpha, X)\colon I^{\mathrm{op}} \ni i \mapsto \mathrm{Hom}_{\mathcal{C}}(\alpha, X) \in \mathbf{Set}. \end{aligned}$$

Definition 2.4.2. (i) Assume that the functor $X \mapsto \varprojlim \mathrm{Hom}_{\mathcal{C}}(X,\beta)$ is representable. We denote by $\varprojlim \beta$ its representative and says that the functor β admits a projective limit in $\mathcal{C}$. In particular,

$$\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim \beta) \simeq \varprojlim \mathrm{Hom}_{\mathcal{C}}(X,\beta). \tag{2.7}$$

(ii) Assume that the functor $X \mapsto \varprojlim \mathrm{Hom}_{\mathcal{C}}(\alpha, X)$ is representable. We denote by $\varinjlim \alpha$ its representative and says that the functor α admits an inductive limit in $\mathcal{C}$. In particular,

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim \alpha, X) \simeq \varprojlim \mathrm{Hom}_{\mathcal{C}}(\alpha, X), \tag{2.8}$$

When $\mathcal{C} = \mathbf{Set}$ this definition of $\varprojlim \beta$ coincides with the former one, in view of Lemma 2.4.1.

Notice that both projective and inductive limits are defined using projective limits in **Set**.

Assume that $\varinjlim \alpha$ exists in $\mathcal{C}$. One gets:

$$\varprojlim \mathrm{Hom}_{\mathcal{C}}(\alpha, \varinjlim \alpha) \simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim \alpha, \varinjlim \alpha)$$

and the identity of $\varinjlim \alpha$ defines a family of morphisms

$$\rho_i \colon \alpha(i) \to \varinjlim \alpha.$$

Consider a family of morphisms $\{f_i \colon \alpha(i) \to X\}_{i \in I}$ in $\mathcal{C}$ satisfying the compatibility conditions

$$f_i = f_j \circ f(s) \text{ for all } s \in \mathrm{Hom}_I(i, j). \tag{2.9}$$

This family of morphisms is nothing but an element of $\varprojlim_i \mathrm{Hom}\,(\alpha(i), X)$, hence by (2.8), an element of $\mathrm{Hom}\,(\varinjlim \alpha, X)$. Therefore, $\varinjlim \alpha$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathcal{C} \text{ and all family of morphisms } \{f_i \colon \alpha(i) \to X\}_{i \in I} \text{ in} \\ \mathcal{C} \text{ satisfying (2.9), all morphisms } f_i\text{'s factorize uniquely through} \\ \varinjlim \alpha. \end{cases} \tag{2.10}$$

Similarly, assume that $\varprojlim \beta$ exists in $\mathcal{C}$. One gets:

$$\varprojlim \mathrm{Hom}_{\mathcal{C}}(\varprojlim \beta, \beta) \simeq \mathrm{Hom}_{\mathcal{C}}(\varprojlim \beta, \varprojlim \beta)$$

and the identity of $\varprojlim \beta$ defines a family of morphisms

$$\rho_i \colon \varprojlim \beta \to \beta(i).$$

Consider a family of morphisms $\{f_i \colon X \to \beta(i)\}_{i \in I}$ in $\mathcal{C}$ satisfying the compatibility conditions

$$f_j = f_i \circ f(s) \text{ for all } s \in \mathrm{Hom}_I(i, j). \tag{2.11}$$

This family of morphisms is nothing but an element of $\varprojlim_i \mathrm{Hom}\,(X, \beta(i))$, hence by (2.7), an element of $\mathrm{Hom}\,(X, \varprojlim \beta, X)$. Therefore, $\varprojlim \beta$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathcal{C} \text{ and all family of morphisms } \{f_i \colon X \to \beta(i)\}_{i \in I} \\ \text{in } \mathcal{C} \text{ satisfying (2.11), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \varprojlim \beta. \end{cases} \tag{2.12}$$

Inductive and projective limits are visualized by the diagrams:

$$
\begin{tikzcd}
\alpha(i) \arrow[dd, "\alpha(s)"'] \arrow[dr, "\rho_i"] \arrow[drr, "f_i"] & & \\
& \varinjlim \alpha \arrow[r, dotted] & X \\
\alpha(j) \arrow[ur, "\rho_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \beta(i) \\
X \arrow[r, dotted] \arrow[urr, "f_i"] \arrow[drr, "f_j"'] & \varprojlim \beta \arrow[ur, "\rho_i"'] \arrow[dr, "\rho_j"] & \\
& & \beta(j) \arrow[uu, "\beta(s)"']
\end{tikzcd}
$$

If $\varphi\colon J \to I$, $\alpha\colon I \to \mathcal{C}$ and $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ are functors, we have natural morphisms:

$$\varinjlim (\alpha \circ \varphi) \to \varinjlim \alpha, \tag{2.13}$$

$$\varprojlim (\beta \circ \varphi) \leftarrow \varprojlim \beta. \tag{2.14}$$

This follows immediately of (2.10) and (2.12).

Proposition 2.4.3. *Let I be a category and assume that $\mathcal{C}$ admits inductive limits indexed by I. Then for any category J, the category $\mathcal{C}^J$ admits inductive limits indexed by I. Moreover, if $\alpha\colon I \to \mathcal{C}^J$ is a functor, then its inductive limit is defined as follows. For $Y \in J$, denote by $\alpha(Y)\colon I \to \mathcal{C}$ the functor $i \mapsto \alpha(i)(Y)$. Then $\varinjlim \alpha \in \mathcal{C}^J$ is given by*

$$(\varinjlim \alpha)(Y) = \varinjlim \alpha(Y),\ Y \in J.$$

Similarly, if $\beta\colon I^{\mathrm{op}} \to \mathcal{C}^J$ is a functor, then $\varprojlim \beta \in \mathcal{C}^J$ is given by

$$(\varprojlim \beta)(Y) = \varprojlim \alpha(Y),\ Y \in J.$$

The proof is obvious.

Recall the equivalence of categories (2.2) and consider a bifunctor $\alpha\colon I \times J \to \mathcal{C}$. It defines a functor $\alpha_J\colon I \to \mathcal{C}^J$ as well as a functor $\alpha_I\colon J \to \mathcal{C}^I$. One easily checks that

$$\varinjlim \alpha \simeq \varinjlim (\varinjlim \alpha_J) \simeq \varinjlim (\varinjlim \alpha_I). \tag{2.15}$$

Similarly, if $\beta\colon I^{\mathrm{op}} \times J^{\mathrm{op}} \to \mathcal{C}$ is a bifunctor, then β defines a functor $\beta_J\colon I^{\mathrm{op}} \to \mathcal{C}^{J^{\mathrm{op}}}$ and a functor $\beta_I\colon J^{\mathrm{op}} \to \mathcal{C}^{I^{\mathrm{op}}}$ and one has the isomorphisms

$$\varprojlim \beta \simeq \varprojlim \varprojlim \beta_J \simeq \varprojlim \varprojlim \beta_I. \tag{2.16}$$

In other words:

$$\varinjlim_{i,j} \alpha(i,j) \simeq \varinjlim_j (\varinjlim_i (\alpha(i,j)) \simeq \varinjlim_i \varinjlim_j (\alpha(i,j)), \tag{2.17}$$

$$\varprojlim_{i,j} \beta(i,j) \simeq \varprojlim_j \varprojlim_i (\beta(i,j)) \simeq \varprojlim_i \varprojlim_j (\beta(i,j)). \tag{2.18}$$

If every functor from I to $\mathcal{C}$ admits an inductive limit, one says that $\mathcal{C}$ admits inductive limits indexed by I. If this property holds for all categories I (resp.
finite categories I), one says that $\mathcal{C}$ admits inductive (resp.
finite inductive) limits, and similarly when replacing I with I^{op} and inductive limits with projective limits.

2.5 Examples

Empty limits.

If I is the empty category and $\alpha\colon I \to \mathcal{C}$ is a functor, then $\varinjlim \alpha$ exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$, and in this case $\varinjlim \alpha \simeq \emptyset_{\mathcal{C}}$. Similarly, $\varprojlim \alpha$ exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has a terminal object $\mathrm{pt}_{\mathcal{C}}$, and in this case $\varprojlim \alpha \simeq \mathrm{pt}_{\mathcal{C}}$.

Terminal object

If I admits a terminal object, say i_o and if $\alpha\colon I \to \mathcal{C}$ and $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ are functor, then

$$\varinjlim \alpha \simeq \alpha(i_o) \qquad \varprojlim \beta \simeq \beta(i_o).$$

This follows immediately of (2.10) and (2.12).

Sums and products

Consider a discrete category I.

Definition 2.5.1. (i) When the category I is discrete, inductive and projective limits are called coproduct and products, denoted $\coprod$ and $\prod$, respectively. Hence, writing $\alpha(i) = X_i$ or $\beta(i) = X_i$, we get for $Y \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(Y, \prod_i X_i) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(Y, X_i),$$
$$\mathrm{Hom}_{\mathcal{C}}(\coprod_i X_i, Y) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

(ii) If I is discrete with two objects, a functor $I \to \mathcal{C}$ is the data of two objects X_0 and X_1 in $\mathcal{C}$ and their coproduct and product (if they exist) are usually denoted by $X_0 \sqcup X_1$ and $X_0 \times X_1$, respectively.

Hence, if $\alpha\colon I \to \mathcal{C}$ is a functor and I is discrete, one denotes by $\coprod \alpha$ or $\coprod_{i\in I} \alpha(i)$ its coproduct and one denotes by $\prod \alpha$ or $\prod_{i\in I} \alpha(i)$ its product.

If $\alpha(i) = X$ for all $i \in I$, one simply denotes this limit by $X^{\coprod I}$ (resp. $X^{\prod I}$). One also writes $X^{(I)}$ and X^I instead of $X^{\coprod I}$ and $X^{\prod I}$, respectively.

Example 2.5.2. In the category **Set**, we have for $I, X, Z \in \mathbf{Set}$:

$$
\begin{aligned}
X^{(I)} &\simeq I \times X, \\
X^{I} &\simeq \mathrm{Hom}_{\mathbf{Set}}(I, X), \\
\mathrm{Hom}_{\mathbf{Set}}(I \times X, Z) &\simeq \mathrm{Hom}_{\mathbf{Set}}(I, \mathrm{Hom}_{\mathbf{Set}}(X, Z)), \\
&\simeq \mathrm{Hom}_{\mathbf{Set}}(X, Z)^{I}.
\end{aligned}
$$

The coproduct and product of two objects are visualized by the diagrams:

$$
\begin{tikzcd}
X_0 \arrow[dr] \arrow[drr] & & & & & X_0 \\
& X_0 \sqcup X_1 \arrow[r, dotted] & X & X \arrow[r, dotted] \arrow[urr] \arrow[drr] & X_0 \times X_1 \arrow[ur] \arrow[dr] & \\
X_1 \arrow[ur] \arrow[urr] & & & & & X_1
\end{tikzcd}
$$

In other words, any pair of morphisms from (resp. to) X_0 and X_1 to (resp. from) X factors uniquely through $X_0 \sqcup X_1$ (resp. $X_0 \times X_1$). If $\mathcal{C}$ is the category **Set**, $X_0 \sqcup X_1$ is the disjoint union and $X_0 \times X_1$ is the product of the two sets X_0 and X_1.

Cokernels and kernels

Consider the category I with two objects and two parallel morphisms other than identities, visualized by

$$\bullet \rightrightarrows \bullet$$

A functor $\alpha\colon I \to \mathcal{C}$ is characterized by two parallel arrows in $\mathcal{C}$:

$$f, g : X_0 \rightrightarrows X_1 \tag{2.19}$$

In the sequel we shall identify such a functor with the diagram (2.19).

Definition 2.5.3. Consider two parallel arrows $f, g : X_0 \rightrightarrows X_1$ in $\mathcal{C}$.

(i) A co-equalizer (one also says a cokernel), if it exists, is an inductive limit of this functor. It is denoted by $\mathrm{Coker}(f, g)$.

(ii) An equalizer (one also says a kernel), if it exists, is a projective limit of this functor. It is denoted by $\mathrm{Ker}(f, g)$.

(iii) A sequence $X_0 \rightrightarrows X_1 \to Z$ (resp. $Z \to X_0 \rightrightarrows X_1$) is exact if Z is isomorphic to the co-equalizer (resp. equalizer) of $X_0 \rightrightarrows X_1$.

(iv) Assume that the category $\mathcal{C}$ admits a zero-object 0. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. A cokernel (resp. a kernel) of f, if it exists, is a cokernel (resp. a kernel) of $f, 0\colon X \rightrightarrows Y$. It is denoted $\mathrm{Coker}(f)$ (resp. $\mathrm{Ker}(f)$).

The co-equalizer L is visualized by the diagram:

$$\begin{tikzcd}
X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr] & X_1 \arrow[r, "k"] \arrow[d, "h"] & L \arrow[dl, dotted] \\
& X &
\end{tikzcd}$$

which means that any morphism $h\colon X_1 \to X$ such that $h \circ f = h \circ g$ factors uniquely through k.

Note that

(2.20) $$k \text{ is an epimorphism.}$$

Indeed, consider a pair of parallel arrows $a, b\colon L \rightrightarrows X$ such that $a \circ k = b \circ k = h$. Then $h \circ f = a \circ k \circ f = a \circ k \circ g = b \circ k \circ g = h \circ g$. Hence h factors uniquely through k, and this implies $a = b$.

Dually, the equalizer K is visualized by the diagram:

$$\begin{tikzcd}
K \arrow[r, "h"] & X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_1 \\
& X \arrow[u, "h"] \arrow[ur] \arrow[ul, dotted] &
\end{tikzcd}$$

and

(2.21) $$h \text{ is a monomorphism.}$$

We have seen that coproducts and co-equalizers (resp. products and equalizers) are particular cases of inductive (resp. projective) limits. One can show that conversely, inductive limits (resp. finite inductive limits) can be obtained as co-equalizer of coproducts (resp. finite coproducts) and projective limits (resp. finite projective limits) can be obtained as equalizer of products (resp. finite products).

In particular, a category $\mathcal{C}$ admits finite projective limits if and only if it satisfies:

(i) $\mathcal{C}$ admits a terminal object,

(ii) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, the product $X \times Y$ exists in $\mathcal{C}$,

(iii) for any parallel arrows in $\mathcal{C}$, $f, g\colon X \rightrightarrows Y$, the equalizer exists in $\mathcal{C}$.

Moreover, if $\mathcal{C}$ admits finite projective limits, a functor $F\colon \mathcal{C} \to \mathcal{C}'$ commutes with such limits if and only if it commutes with the terminal object, (finite) products and kernels.

There is a similar result for finite inductive limits, replacing a terminal object by an initial object, products by coproducts and equalizers by co-equalizers.

Proposition 2.5.4. *The category* **Set** *admits inductive limits. More precisely, if I is a category and $\alpha\colon I \to$* **Set** *is a functor, then*

$$\begin{aligned} \varinjlim \alpha \quad &\simeq \quad (\bigsqcup_{i \in I} \alpha(i))/\sim \ \textit{ where } \sim \textit{ is the equivalence relation } \text{generated } \textit{by} \\ &\qquad \alpha(i) \ni x \sim y \in \alpha(j) \textit{ if there exists } s\colon i \to j \textit{ with } \alpha(s)(x) = y. \end{aligned}$$

In particular, the coproduct in **Set** is the disjoint union, $\coprod = \bigsqcup$.

Proof. Let $S \in$ **Set**. By the definition of the projective limit in **Set** we get:

$$\begin{aligned} \varprojlim \operatorname{Hom}(\alpha, S) \quad \simeq \quad &\{\{p(i,x)\}_{i \in I, x \in \alpha(i)}; p(i,x) \in S, p(i,x) = p(j,y) \\ &\text{if there exists } s\colon i \to j \text{ with } \alpha(s)(x) = y\}. \end{aligned}$$

The result follows. q.e.d.

Notation 2.5.5. In the category **Set** one uses the notation $\bigsqcup$ rather than $\coprod$.

2.6 Exact functors

Let $I, \mathcal{C}$ and $\mathcal{C}'$ be categories and let $\alpha\colon I \to \mathcal{C}$, $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{C}'$ be functors. Recall that if $\mathcal{C}$ and $\mathcal{C}'$ admit inductive (resp. projective) limits indexed by I, there is a natural morphism $\varinjlim (F \circ \alpha) \to F(\varinjlim \alpha)$ (resp. $F(\varprojlim \beta) \to \varprojlim (F \circ \beta)$).

Definition 2.6.1. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) Let I be a category and assume that $\mathcal{C}$ admits inductive limits indexed by I. One says that F commutes with such limits if for any $\alpha\colon I \to \mathcal{C}$, $\varinjlim (F \circ \alpha)$ exits in $\mathcal{C}'$ and is represented by $F(\varinjlim \alpha)$.

(ii) Similarly if I is a category and $\mathcal{C}$ admits projective limits indexed by I, one says that F commutes with such limits if for any $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$, $\varprojlim (F \circ \beta)$ exists and is represented by $F(\varprojlim \beta)$.

Example 2.6.2. Let k be a field, $\mathcal{C} = \mathcal{C}' = \mathrm{Mod}(k)$, and let $X \in \mathcal{C}$. Then the functor $\mathrm{Hom}_k(X, \bullet)$ does not commute with inductive limit if X is infinite dimensional.

Definition 2.6.3. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) Assume that $\mathcal{C}$ admits finite projective limits. One says that F is left exact if it commutes with such limits.

(ii) Assume that $\mathcal{C}$ admits finite inductive limits. One says that F is right exact if it commutes with such limits.

(iii) One says that F is exact if it is both left and right exact.

Proposition 2.6.4. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. Assume that*

(i) *F admits a left adjoint $G\colon \mathcal{C}' \to \mathcal{C}$,*

(ii) *$\mathcal{C}$ admits projective limits indexed by a category I.*

Then F commutes with projective limits indexed by I, that is, $F(\varprojlim_i \beta(i)) \simeq \varprojlim_i F(\beta(i))$.

Proof. Let $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ be a projective system indexed by I and let $Y \in \mathcal{C}'$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}'}(Y, F(\varprojlim_i \beta(i))) &\simeq \mathrm{Hom}_{\mathcal{C}}(G(Y), \varprojlim_i \beta(i)) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(G(Y), \beta(i)) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}'}(Y, F(\beta(i))) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'^\wedge}(Y, \varprojlim_i F(\beta(i))).
\end{aligned}$$

Then the result follows by the Yoneda lemma. q.e.d.

Of course there is a similar result for inductive limits. If $\mathcal{C}$ admits inductive limits indexed by I and F admits a right adjoint, then F commutes with such limits.

Proposition 2.6.5. *Let $\mathcal{C}$ be a category which admits finite inductive and finite projective limits.*

(i) *The functor* $\mathrm{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ *is left exact.*

(ii) *Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a functor. If F admits a right (resp. left) adjoint, then F is right (resp. left) exact.*

(iii) *Let I be a category and assume that $\mathcal{C}$ admits inductive (resp. projective) limits indexed by I. Then the functor* $\varinjlim \colon \mathrm{Fct}(I, \mathcal{C}) \to \mathcal{C}$ (*resp.* $\varprojlim \colon \mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$) *is right (resp. left) exact.*

(iv) *Let I be a discrete category and let A be a ring. Then the functor* $\prod \colon \mathrm{Mod}(A)^I \to \mathrm{Mod}(A)$ *is exact.*

Proof. (i) follows immediately from (1.17) and (1.16).
(ii) is a particular case of Proposition 2.6.4.
(iii) Use the isomorphism (2.15) or (2.16).
(iv) is well-known and obvious. q.e.d.

2.7 Filtrant inductive limits

We shall generalize some notions of Definition 1.5.2 as well as Lemma 1.5.9 and Proposition 1.5.11.

Definition 2.7.1. A category I is called filtrant if it satisfies the conditions (i)–(iii) below.

(i) I is non empty,

(ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,

(iii) for any parallel morphisms $f, g \colon i \rightrightarrows j$, there exists a morphism $h \colon j \to k$ such that $h \circ f = h \circ g$.

One says that I is cofiltrant if I^{op} is filtrant.

The conditions (ii)–(iii) of being filtrant are visualized by the diagrams:

$$\begin{tikzcd} i \arrow[dr, dotted] & \\ & k \\ j \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & j \arrow[d, dotted] \\ & k \end{tikzcd}$$

Of course, if $(I, \leq)$ is a non-empty directed ordered set, then the associated category I is filtrant.

We shall first study filtrant inductive limits in the category **Set**.

Proposition 2.7.2. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtrant. Define the relation $\sim$ on $\coprod_i \alpha(i)$ by $\alpha(i) \ni x_i \sim x_j \in \alpha(j)$ if there exists $s\colon i \to k$ and $t\colon j \to k$ such that $\alpha(s)(x_i) = \alpha(t)(x_j)$. Then*

(i) *the relation $\sim$ is an equivalence relation,*

(ii) $\varinjlim \alpha \simeq \coprod_i \alpha(i)/\sim$.

Proof. (i) The relation $\sim$ is clearly symmetric and reflexive. Let us show it is transitive. Let $x_j \in \alpha(i_j)$, $j = 1, 2, 3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. There exist morphisms visualized by the diagram:

$$\begin{tikzcd}
i_1 \arrow[r, "s_1"] & j_1 \arrow[dr, "u_1"] & & \\
i_2 \arrow[ur, "s_2"] \arrow[dr, "t_2"'] & & k_1 \arrow[r, "v"] & l \\
i_3 \arrow[r, "t_3"] & j_2 \arrow[ur, "u_2"'] & &
\end{tikzcd}$$

such that $\alpha(s_1)x_1 = \alpha(s_2)x_2$, $\alpha(t_2)x_2 = \alpha(t_3)x_3$, and $v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$. Set $w_1 = v \circ u_1 \circ s_1$, $w_2 = v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$ and $w_3 = v \circ u_2 \circ t_3$. Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$.
(ii) follows from Proposition 2.5.4. q.e.d.

Corollary 2.7.3. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtrant.*

(i) *Let S be a finite subset in $\varinjlim \alpha$. Then there exists $i \in I$ such that S is contained in the image of $\alpha(i)$ by the natural map $\alpha(i) \to \varinjlim \alpha$.*

(ii) *Let $i \in I$ and let x and y be elements of $\alpha(i)$ with the same image in $\varinjlim \alpha$. Then there exists $s\colon i \to j$ such that $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$.*

The proof is left as an exercise.

Corollary 2.7.4. *Let A be a ring and denote by for the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$. Then the functor for commutes with filtrant inductive limits. In other words, if I is filtrant and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then*

$$for \circ (\varinjlim_i \alpha(i)) = \varinjlim_i (for \circ \alpha(i)).$$

Inductive limits with values in **Set** indexed by filtrant categories commute with finite projective limits. More precisely:

Proposition 2.7.5. *For a filtrant category I, a finite category J and a functor $\alpha\colon I \times J^{\mathrm{op}} \to \mathbf{Set}$, one has $\varinjlim_i \varprojlim_j \alpha(i,j) \xrightarrow{\sim} \varprojlim_j \varinjlim_i \alpha(i,j)$. In other words, the functor*

$$\varinjlim : \mathrm{Fct}(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with finite projective limits.

Proof. It is enough to prove that $\varinjlim$ commutes with equalizers and with finite products. This verification is left to the reader. q.e.d.

Corollary 2.7.6. *Let A be a ring and let I be a filtrant category. Then the functor $\varinjlim : \mathrm{Mod}(A)^I \to \mathrm{Mod}(A)$ is exact.*

Proof. Let $\alpha\colon I \times J^{\mathrm{op}} \to \mathrm{Mod}(A)$ be a functor. In order to prove that $\varinjlim_i \varprojlim_j \alpha(i,j) \to \varprojlim_j \varinjlim_i \alpha(i,j)$ is an isomorphism, it is enough to check it after applying the functor $for\colon \mathrm{Mod}(A) \to \mathbf{Set}$. Then the result follows from Corollaries 2.7.4 and 2.7.4. q.e.d.

Cofinal functors

Definition 2.7.7. Let I be a filtrant category and let $\varphi\colon J \to I$ be a fully faithful functor. One says that J is cofinal to I (or that $\varphi : J \to I$ is cofinal) if for any $i \in I$ there exists $j \in J$ and a morphism $i \to \varphi(j)$.

Note that the hypothesis implies that J is filtrant.

Proposition 2.7.8. *Assume I is filtrant, $\varphi\colon J \to I$ is fully faithful and $J \to I$ is cofinal. Let $\alpha\colon I \to \mathcal{C}$ (resp. $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$) be a functor. Assume that $\varinjlim \alpha$ (resp. $\varprojlim \beta$) exists in $\mathcal{C}$. Then $\varinjlim (\alpha \circ \varphi)$ (resp. $\varprojlim (\beta \circ \varphi^{\mathrm{op}})$) exists in $\mathcal{C}$ and the natural morphism $\varinjlim (\alpha \circ \varphi) \to \varinjlim \alpha$ (resp. $\varprojlim \beta \to \varprojlim (\beta \circ \varphi^{\mathrm{op}})$) is an isomorphism.*

The proof is left as an exercise.

Remark 2.7.9. In these notes, we have skipped problems related to questions of cardinality and universes, but we should have not. Indeed, the reader will assume that all categories ($\mathcal{C}$, $\mathcal{C}'$ etc.) belong to a given universe $\mathcal{U}$ and that all limits are indexed by $\mathcal{U}$-small categories (I, J, etc.). (We do not give the meaning of "universe" and "small" here.)

Let us give an example which show that, otherwise, we may have troubles. Let $\mathcal{C}$ be a category which admits products and assume there exist $X, Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has more than one element. Set $M = \mathrm{Mor}(\mathcal{C})$, where $\mathrm{Mor}(\mathcal{C})$ denotes the "set" of all morphisms in $\mathcal{C}$, and let $\pi = \mathrm{card}(M)$, the cardinal of the set M. We have $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y)^M$ and therefore $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \geq 2^\pi$. On the other hand, $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \subset \mathrm{Mor}(\mathcal{C})$ which implies $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \leq \pi$.

The "contradiction" comes from the fact that $\mathcal{C}$ does not admit products indexed by such a big set as $\mathrm{Mor}(\mathcal{C})$, or else, $\mathrm{Mor}(\mathcal{C})$ is not small (in general) in the universe to which $\mathcal{C}$ belongs. (The remark was found in [9].)

Exercises to Chapter 2

Exercise 2.1. Prove that the categories **Set** and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent. (Hint: if $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\emptyset) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \emptyset$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 2.2. (i) Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a faithful functor and let f be a morphism in $\mathcal{C}$. Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).
(ii) Assume now that F is fully faithful. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism.

Exercise 2.3. Prove that the category $\mathcal{C}$ is equivalent to the opposite category $\mathcal{C}^{\mathrm{op}}$ in the following cases:
(i) $\mathcal{C}$ denotes the category of finite abelian groups,
(ii) $\mathcal{C}$ is the category **Rel** of relations.

Exercise 2.4. (i) Prove that in the category **Set**, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).
(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

Exercise 2.5. Let $\mathcal{C}$ be a category. We denote by $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ the identity functor of $\mathcal{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that is,

$$\mathrm{End}\,(\mathrm{id}_{\mathcal{C}}) = \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}).$$

Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ is commutative.

Exercise 2.6. In the category **Top**, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism. (Hint: consider a continuous injective map $f\colon X \hookrightarrow Y$ with dense image.)

Exercise 2.7. Let $X, Y \in \mathcal{C}$ and consider the category $\mathcal{D}$ whose arrows are triplets $Z \in \mathcal{C}, f : Z \to X, g : Z \to Y$, the morphisms being the natural one. Prove that this category admits a terminal object if and only if the product $X \times Y$ exists in $\mathcal{C}$, and that in such a case this terminal object is isomorphic to $X \times Y, X \times Y \to X, X \times Y \to Y$. Deduce that if $X \times Y$ exists, it is unique up to unique isomorphism.

Exercise 2.8. (i) Let I be a (non necessarily finite) set and $(X_i)_{i \in I}$ a family of sets indexed by I. Show that $\coprod_i X_i$ is the disjoint union of the sets X_i.
(ii) Construct the natural map $\coprod_i \mathrm{Hom}_{\mathbf{Set}}(Y, X_i) \to \mathrm{Hom}_{\mathbf{Set}}(Y, \coprod_i X_i)$ and prove it is injective.
(iii) Prove that the map $\coprod_i \mathrm{Hom}_{\mathbf{Set}}(X_i, Y) \to \mathrm{Hom}_{\mathbf{Set}}(\prod_i X_i, Y)$ is not injective in general.

Exercise 2.9. Let I and $\mathcal{C}$ be two categories and denote by Δ the functor from $\mathcal{C}$ to $\mathcal{C}^I$ which, to $X \in \mathcal{C}$, associates the constant functor $\Delta(X)\colon I \ni i \mapsto X \in \mathcal{C}$, $(i \to j) \in \mathrm{Mor}(I) \mapsto \mathrm{id}_X$. Assume that any functor from I to $\mathcal{C}$ admits an inductive limit.
(i) Prove that $\varinjlim : \mathcal{C}^I \to \mathcal{C}$ is a functor.
(ii) Prove the formula (for $\alpha : I \to \mathcal{C}$ and $Y \in \mathcal{C}$):

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim_i \alpha(i), Y) \simeq \mathrm{Hom}_{\mathrm{Fct}(I,\mathcal{C})}(\alpha, \Delta(Y)).$$

(iii) Replacing I with the opposite category, deduce the formula (assuming projective limits exist):

$$\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim_i G(i)) \simeq \mathrm{Hom}_{\mathrm{Fct}(I^{\mathrm{op}},\mathcal{C})}(\Delta(X), G).$$

Exercise 2.10. Let $\mathcal{C}$ be a category which admits filtrant inductive limits. One says that an object X of $\mathcal{C}$ is of finite type (resp. of finite presentation) if for any functor $\alpha\colon I \to \mathcal{C}$ with I filtrant, the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ is injective (resp. bijective).
(i) Show that this definition coincides with the classical one when $\mathcal{C} = \mathrm{Mod}(A)$, for a ring A.
(ii) Does this definition coincide with the classical one when $\mathcal{C}$ denotes the category of commutative algebras?

Exercise 2.11. Let $\mathcal{C}$ be a category and recall that the category $\mathcal{C}^\wedge$ admits inductive limits. One denotes by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$. Let k be a field and let $\mathcal{C} = \mathrm{Mod}(k)$. Prove that the Yoneda functor $\mathrm{h}_\mathcal{C} \colon \mathcal{C} \to \mathcal{C}^\wedge$ does not commute with inductive limits.

Exercise 2.12. Consider the category I with three objects $\{a, b, c\}$ and two morphisms other than the identities, visualized by the diagram

$$a \leftarrow c \to b.$$

Let $\mathcal{C}$ be a category. A functor $\beta \colon I^{\mathrm{op}} \to \mathcal{C}$ is nothing but the data of three objects X, Y, Z and two morphisms visualized by the diagram

$$X \xrightarrow{f} Z \xleftarrow{g} Y.$$

The fiber product $X \times_Z Y$ of X and Y over Z, if it exists, is the projective limit of β.
(i) Assume that $\mathcal{C}$ admits products (of two objects) and kernels. Prove that the sequence

$$X \times_Z Y \to X \rightrightarrows Y$$

is exact. Here, the two morphisms $X \rightrightarrows Y$ are given by f, g.
(ii) Prove that $\mathcal{C}$ admits finite projective limits if and only if it admits fiber products and a terminal object.

Exercise 2.13. Let I and $\mathcal{C}$ be two categories and let $F, G : I \rightrightarrows \mathcal{C}$ be two functors. Prove the isomorphism:

$$\begin{aligned}&\mathrm{Hom}_{\,\mathrm{Fct}(I,\mathcal{C})}(F, G) \simeq \\ &\qquad \mathrm{Ker}\Big(\prod_{i \in I} \mathrm{Hom}_\mathcal{C}(F(i), G(i)) \rightrightarrows \prod_{(j \to k) \in \mathrm{Mor}(I)} \mathrm{Hom}_\mathcal{C}(F(j), G(k))\Big).\end{aligned}$$

Here, the double arrow is associated with the two maps:

$$\prod_{i \in I} \mathrm{Hom}_\mathcal{C}(F(i), G(i)) \to \mathrm{Hom}_\mathcal{C}(F(j), G(j)) \to \mathrm{Hom}_\mathcal{C}(F(j), G(k)),$$
$$\prod_{i \in I} \mathrm{Hom}_\mathcal{C}(F(i), G(i)) \to \mathrm{Hom}_\mathcal{C}(F(k), G(k)) \to \mathrm{Hom}_\mathcal{C}(F(j), G(k)).$$

Exercise 2.14. Let $\mathcal{C}$ be a category which admits products of two objects and a terminal object, denoted by $\mathrm{pt}_\mathcal{C}$. Let $A \in \mathcal{C}$. Construct a functor

$$\pi_A \colon (\mathbf{Set}^f)^{\mathrm{op}} \to \mathcal{C}, \tag{2.22}$$

such that $\pi_A(\emptyset) \simeq \mathrm{pt}_\mathcal{C}$, $\pi_A(\{\mathrm{pt}\}) \simeq A$ and for S a set with 2 elements, $\pi_A(S) \simeq A \times A$.

Chapter 3

Additive categories

Many results or constructions in the category $\mathrm{Mod}(A)$ of modules over a ring A have their counterparts in other contexts, such as finitely generated A-modules, or graded modules over a graded ring, or sheaves of A-modules, etc. Hence, it is natural to look for a common language which avoids to repeat the same arguments. This is the language of additive and abelian categories.

In this chapter, we give the main properties of additive categories. We expose some basic constructions and notions on complexes such as the shift functor, the homotopy, the mapping cone and the simple complex associated with a double complex. We also construct complexes associated with simplicial objects in an additive category and give a criterion for such a complex to be homotopic to zero.

3.1 Additive categories

Definition 3.1.1. A category $\mathcal{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathcal{C}, \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathbf{Ab}$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathcal{C}$,

(iv) the category $\mathcal{C}$ admits finite coproducts,

(v) the category $\mathcal{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ since it is a group. Note that $\mathrm{Hom}_{\mathcal{C}}(X, 0) = \mathrm{Hom}_{\mathcal{C}}(0, X) = 0$ for all $X \in \mathcal{C}$.

Notation 3.1.2. If X and Y are two objects of $\mathcal{C}$, one denotes by $X \oplus Y$ (instead of $X \sqcup Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

By the definition of a coproduct and a product in a category, for each $Z \in \mathcal{C}$, there is an isomorphism in $\mathrm{Mod}(\mathbb{Z})$:

$$\mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X \oplus Y, Z), \tag{3.1}$$
$$\mathrm{Hom}_{\mathcal{C}}(Z, X) \times \mathrm{Hom}_{\mathcal{C}}(Z, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y). \tag{3.2}$$

Lemma 3.1.3. *Let $\mathcal{C}$ be a category satisfying conditions* (i)–(iii) *in* Definition 3.1.1. *Consider the condition*

(vi) *for any two objects X and Y in $\mathcal{C}$, there exists $Z \in \mathcal{C}$ and morphisms $i_1 \colon X \to Z$, $i_2 \colon Y \to Z$, $p_1 \colon Z \to X$ and $p_2 \colon Z \to Y$ satisfying*

$$p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \tag{3.3}$$
$$p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \tag{3.4}$$
$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z\,. \tag{3.5}$$

Then the conditions (iv), (v) *and* (vi) *are equivalent and the objects $X \oplus Y$, $X \times Y$ and Z are naturally isomorphic.*

Proof. (a) Let us assume condition (iv). The identity of X and the zero morphism $Y \to X$ define the morphism $p_1 \colon X \oplus Y \to X$ satisfying (3.3). We construct similarly the morphism $p_2 \colon X \oplus Y \to Y$ satisfying (3.4). To check (3.5), we use the fact that if $f \colon X \oplus Y \to X \oplus Y$ satisfies $f \circ i_1 = i_1$ and $f \circ i_2 = i_2$, then $f = \mathrm{id}_{X \oplus Y}$.
(b) Let us assume condition (vi). Let $W \in \mathcal{C}$ and consider morphisms $f \colon X \to W$ and $g \colon Y \to W$. Set $h := f \circ p_1 \oplus g \circ p_2$. Then $h \colon Z \to W$ satisfies $h \circ i_1 = f$ and $h \circ i_2 = g$ and such an h is unique. Hence $Z \simeq X \oplus Y$.
(c) We have proved that conditions (iv) and (vi) are equivalent and moreover that if they are satisfied, then $Z \simeq X \oplus Y$. Replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, we get that these conditions are equivalent to (v) and $Z \simeq X \times Y$. q.e.d.

Example 3.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^f(A)$ are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathcal{C}$ is additive, then $\mathcal{C}^{\mathrm{op}}$ is additive.

(iv) Let I be category. If $\mathcal{C}$ is additive, the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$, is additive.
(v) If $\mathcal{C}$ and $\mathcal{C}'$ are additive, then $\mathcal{C} \times \mathcal{C}'$ are additive.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is a morphism of groups. One can prove the following

Proposition 3.1.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. Then F is additive if and only if it commutes with direct sum, that is, for X and Y in $\mathcal{C}$:*

$$\begin{aligned} F(0) &\simeq 0 \\ F(X \oplus Y) &\simeq F(X) \oplus F(Y). \end{aligned}$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.

Example 3.1.6. Let $\mathcal{C}$ be an additive category. One shall be aware that the bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$ is separately additive with respect to each of its argument, but is not additive as a functor on the product category.

Generalization: Let k be a *commutative* ring. One defines the notion of a k-additive category by assuming that for X and Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a k-module and the composition is k-bilinear.

3.2 Complexes in additive categories

Let $\mathcal{C}$ denote an additive category.

Definition 3.2.1. (i) A differential object $(X^\bullet, d_X^\bullet)$ in $\mathcal{C}$ is a sequence of objects X^k and morphisms d^k ($k \in \mathbb{Z}$):

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots . \tag{3.6}$$

(ii) A complex is a differential object $(X^\bullet, d_X^\bullet)$ such that such that $d^k \circ d^{k-1} = 0$ for all $k \in \mathbb{Z}$.

A morphism of differential objects $f^\bullet \colon X^\bullet \to Y^\bullet$ is visualized by a commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ \cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & X^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

One defines naturally the direct sum of two differential objects. Hence, we get a new additive category, the category $Diff(\mathcal{C})$ of differential objects in $\mathcal{C}$. One denotes by $\mathrm{C}(\mathcal{C})$ the full additive subcategory of $Diff(\mathcal{C})$ consisting of complexes.

From mow on, we shall concentrate our study on the category $\mathrm{C}(\mathcal{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathcal{C})(* = b, +, -)$ the full additive subcategory of $\mathrm{C}(\mathcal{C})$ consisting of bounded complexes (resp. bounded below, bounded above).

One considers $\mathcal{C}$ as a full subcategory of $\mathrm{C}^b(\mathcal{C})$ by identifying an object $X \in \mathcal{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0.

Shift functor

Let $X \in \mathrm{C}(\mathcal{C})$ and $k \in \mathbb{Z}$. One defines the shifted complex $X[k]$ by:

$$\begin{cases} (X[k])^n = X^{n+k} \\ d^n_{X[k]} = (-1)^k d^{n+k}_X \end{cases}$$

If $f : X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$ one defines $f[k] : X[k] \to Y[k]$ by $(f[k])^n = f^{n+k}$.

The shift functor $X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $\mathrm{C}(\mathcal{C})$.

Homotopy

Let $\mathcal{C}$ denote an additive category.

Definition 3.2.2. (i) A morphism $f : X \to Y$ in $\mathrm{C}(\mathcal{C})$ is homotopic to zero if for all k there exists a morphism $s^k : X^k \to Y^{k-1}$ such that:

$$f^k = s^{k+1} \circ d^k_X + d^{k-1}_Y \circ s^k.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $\mathrm{C}(\mathcal{C})$ is homotopic to 0 if id_X is homotopic to zero.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd}
X^{k-1} \arrow[r] & X^k \arrow[r, "d_X^k"] \arrow[dl, "s^k"'] \arrow[d, "f^k"] & X^{k+1} \arrow[dl, "s^{k+1}"'] \\
Y^{k-1} \arrow[r, "d_Y^{k-1}"'] & Y^k \arrow[r] & Y^{k+1}
\end{tikzcd}$$

Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Example 3.2.3. The complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero.

Mapping cone

Definition 3.2.4. Let $f : X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $\mathrm{C}(\mathcal{C})$ defined by:

$$\begin{aligned} \mathrm{Mc}(f)^k &= (X[1])^k \oplus Y^k \\ d^k_{\mathrm{Mc}(f)} &= \begin{pmatrix} d^k_{X[1]} & 0 \\ f^{k+1} & d^k_Y \end{pmatrix} \end{aligned}$$

Of course, before to state this definition, one should check that $d^{k+1}_{\mathrm{Mc}(f)} \circ d^k_{\mathrm{Mc}(f)} = 0$. Indeed:

$$\begin{pmatrix} -d_X^{k+2} & 0 \\ f^{k+2} & d_Y^{k+1} \end{pmatrix} \circ \begin{pmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^{k} \end{pmatrix} = 0$$

Notice that although $\mathrm{Mc}(f)^k = (X[1])^k \oplus Y^k$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1] \oplus Y$ in $\mathrm{C}(\mathcal{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

$$\alpha(f) : Y \to \mathrm{Mc}(f), \quad \beta(f) : \mathrm{Mc}(f) \to X[1].$$

and $\beta(f) \circ \alpha(f) = 0$.

If $F : \mathcal{C} \to \mathcal{C}'$ is an additive functor, then $F(\mathrm{Mc}(f)) \simeq \mathrm{Mc}(F(f))$.

The homotopy category $\mathrm{K}(\mathcal{C})$

Let $\mathcal{C}$ be an additive category.

Starting with $\mathrm{C}(\mathcal{C})$, we shall construct a new category by deciding that a morphism of complexes homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X,Y) = \{f : X \to Y; f \text{ is homotopic to } 0\}.$$

If $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathrm{C}(\mathcal{C})$ and if f or g is homotopic to zero, then $g \circ f$ is homotopic to zero. This allows us to state:

Definition 3.2.5. The homotopy category $\mathrm{K}(\mathcal{C})$ is defined by:

$$\begin{aligned} \mathrm{Ob}(\mathrm{K}(\mathcal{C})) &= \mathrm{Ob}(\mathrm{C}(\mathcal{C})) \\ \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X,Y) &= \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X,Y)/Ht(X,Y) \end{aligned}$$

In other words, a morphism homotopic to zero in $\mathrm{C}(\mathcal{C})$ becomes the zero morphism in $\mathrm{K}(\mathcal{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $\mathrm{K}^*(\mathcal{C})$, $(* = b,+,-)$. They are clearly additive categories, endowed with an automorphism, the shift functor $[1] : X \mapsto X[1]$.

3.3 Simplicial constructions

We[1] shall define the simplicial category and use it to construct complexes and homotopies in additive categories.

Definition 3.3.1. (a) The simplicial category, denoted by $\mathbf{\Delta}$, is the category whose objects are the finite totally ordered sets and the morphisms are the order-preserving maps.

(b) We denote by $\mathbf{\Delta}_{inj}$ the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Ob}(\mathbf{\Delta}_{inj}) = \mathrm{Ob}(\mathbf{\Delta})$, the morphisms being the injective order-preserving maps.

For integers n, m denote by $[n,m]$ the totally ordered set $\{k \in \mathbb{Z}; n \leq k \leq m\}$.

Proposition 3.3.2. (i) *the natural functor $\mathbf{\Delta} \to \mathbf{Set}^f$ is faithful,*

(ii) *the full subcategory of $\mathbf{\Delta}$ consisting of objects $\{[0,n]\}_{n \geq -1}$ is equivalent to $\mathbf{\Delta}$,*

(iii) *$\mathbf{\Delta}$ admits an initial object, namely $\emptyset$, and a terminal object, namely $\{0\}$.*

[1] This section has not been treated in 2005/2006

The proof is obvious.

Let us denote by

$$d_i^n\colon [0,n] \to [0,n+1] \qquad (0 \leq i \leq n+1)$$

the injective order-preserving map which does not take the value i. In other words

$$d_i^n(k) = \begin{cases} k & \text{for } k < i, \\ k+1 & \text{for } k \geq i. \end{cases}$$

One checks immediately that

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \text{ for } 0 \leq i < j \leq n+2. \tag{3.7}$$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category $\mathbf{\Delta}_{inj}$ is visualized by

$$\emptyset \xrightarrow{d_0^{-1}} \{0\} \overset{\xrightarrow{d_0^0}}{\underset{\xrightarrow{d_1^0}}{}} \{0,1\} \overset{\xrightarrow{d_0^1}}{\underset{\xrightarrow{d_2^1}}{\xrightarrow{d_1^1}}} \{0,1,2\} \begin{smallmatrix} \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \end{smallmatrix} \tag{3.8}$$

Let $\mathcal{C}$ be an additive category and $F\colon \mathbf{\Delta}_{inj} \to \mathcal{C}$ a functor. We set for $n \in \mathbb{Z}$:

$$F^n = \begin{cases} F([0,n]) & \text{for } n \geq -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_F^n\colon F^n \to F^{n+1}, \quad d_F^n = \sum_{i=0}^{n+1} (-)^i F(d_i^n).$$

Consider the differential object

$$F^\bullet := \cdots \to 0 \to F^{-1} \xrightarrow{d_F^{-1}} F^0 \xrightarrow{d_F^0} F^1 \to \cdots \to F^n \xrightarrow{d_F^n} \cdots . \tag{3.9}$$

Theorem 3.3.3. (i) *The differential object $F^\bullet$ is a complex.*

(ii) *Assume that there exist morphisms $s_F^n\colon F^n \to F^{n-1}$ $(n \geq 0)$satisfying:*

$$\begin{cases} s_F^{n+1} \circ F(d_0^n) = \mathrm{id}_{F^n} & \textit{for } n \geq -1, \\ s_F^{n+1} \circ F(d_{i+1}^n) = F(d_i^{n-1}) \circ s_F^n & \textit{for } i > 0, n \geq 0. \end{cases}$$

Then $F^\bullet$ is homotopic to zero.

Proof. (i) By (3.7), we have

$$\begin{aligned} d_F^{n+1} \circ d_F^n &= \sum_{j=0}^{n+2} \sum_{i=0}^{n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\ &= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\ &= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) \\ &= 0\,. \end{aligned}$$

Here, we have used

$$\begin{aligned} \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) &= \sum_{0 \le i < j \le n+1} (-)^{i+j+1} F(d_i^{n+1} \circ d_j^n) \\ &= \sum_{0 \le j \le i \le n+1} (-)^{i+j+1} F(d_j^{n+1} \circ d_i^n). \end{aligned}$$

(ii) We have

$$\begin{aligned} & s_F^{n+1} \circ d_F^n + d_F^{n-1} \circ s^n \\ & = \sum_{i=0}^{n+1} (-1)^i s_F^{n+1} \circ F(d_i^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\ & = s_F^{n+1} \circ F(d_0^n) + \sum_{i=0}^{n} (-1)^{i+1} s_F^{n+1} \circ F(d_{i+1}^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\ & = \mathrm{id}_{F^n} + \sum_{i=0}^{n} (-1)^{i+1} F(d_i^{n-1} \circ s_F^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\ & = \mathrm{id}_{F^n}. \end{aligned}$$

q.e.d.

3.4 Double complexes

Let $\mathcal{C}$ be as above an additive category. A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathcal{C}$ is the data of

$$\{X^{n,m}, {d'}_X^{n,m}, {d''}_X^{n,m}; (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$$

where $X^{n,m} \in \mathcal{C}$ and the "differentials" ${d'}_X^{n,m} : X^{n,m} \to X^{n+1,m}$, ${d''}_X^{n,m} : X^{n,m} \to X^{n,m+1}$ satisfy:

(3.10) $$ {d'}_X^2 = {d''}_X^2 = 0,\ d' \circ d'' = d'' \circ d'.$$

One can represent a double complex by a commutative diagram:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\arrow[r] & X^{n,m} \arrow[r, "d''^{n,m}"] \arrow[d, "d'^{n,m}"'] & X^{n,m+1} \arrow[r] \arrow[d, "d'^{n,m+1}"'] & {} \\
\arrow[r] & X^{n+1,m} \arrow[r, "d''^{n+1,m}"'] \arrow[d] & X^{n+1,m+1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}$$

One defines naturally the notion of a morphism of double complexes, and one obtains the additive category $\mathrm{C}^2(\mathcal{C})$ of double complexes.

There are two functors $F_I, F_{II} : \mathrm{C}^2(\mathcal{C}) \to \mathrm{C}(\mathrm{C}(\mathcal{C}))$ which associate to a double complex X the complex whose objects are the rows (resp. the columns) of X. These two functors are clearly isomorphisms of categories.

Now consider the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z} \times \mathbb{Z}; X^{n,m} \neq 0, m+n=p\} \text{ is finite} \tag{3.11}$$

and denote by $C^2_f(\mathcal{C})$ the full subcategory of $\mathrm{C}^2(\mathcal{C})$ consisting of objects X satisfying (3.11). To such an X one associates its "total complex" $\mathrm{tot}(X)$ by setting:

$$\begin{aligned}
\mathrm{tot}(X)^p &= \oplus_{m+n=p} X^{n,m}, \\
d^p_{\mathrm{tot}(X)}|_{X^{n,m}} &= d'^{n,m} + (-1)^n d''^{n,m}.
\end{aligned}$$

This is visualized by the diagram:

$$\begin{tikzcd}
X^{n,m} \arrow[r, "(-)^n d''"] \arrow[d, "d'"'] & X^{n,m+1} \\
X^{n+1,m} &
\end{tikzcd}$$

Proposition 3.4.1. *The differential object* $\{\mathrm{tot}(X)^p, d^p_{\mathrm{tot}(X)}\}_{p\in\mathbb{Z}}$ *is a complex* (i.e. $d^{p+1}_{\mathrm{tot}(X)} \circ d^p_{\mathrm{tot}(X)} = 0$) *and* $\mathrm{tot} : C^2_f(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$ *is a functor of additive categories.*

Proof. For $(n,m) \in \mathbb{Z} \times \mathbb{Z}$, one has

$$\begin{aligned}
d \circ d(X^{n,m}) &= d'' \circ d''(X^{n,m}) + d' \circ d'(X^{n,m}) \\
&\quad + (-)^n d'' \circ d'(X^{n,m}) + (-)^{n+1} d' \circ d''(X^{n,m}) \\
&= 0.
\end{aligned}$$

It is left to the reader to check that tot is an additive functor. q.e.d.

Example 3.4.2. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{C}(\mathcal{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j : Z^{-1,j} \to Z^{0,j}$. Then

$$\mathrm{tot}(Z^{\bullet,\bullet}) \simeq \mathrm{Mc}(f^\bullet). \tag{3.12}$$

Bifunctor

Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be additive categories and let $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ be an additive bifunctor (i.e., $F(\,\bullet\,,\,\bullet\,)$ is additive with respect to each argument). It defines an additive bifunctor $\mathrm{C}^2(F) : \mathrm{C}(\mathcal{C}) \times \mathrm{C}(\mathcal{C}') \to \mathrm{C}^2(\mathcal{C}'')$. In other words, if $X \in \mathrm{C}(\mathcal{C})$ and $X' \in \mathrm{C}(\mathcal{C}')$ are complexes, then $\mathrm{C}^2(F)(X, X')$ is a double complex.

Examples 3.4.3. (i) Consider the bifunctor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$. We shall write $\mathrm{Hom}_{\mathcal{C}}^{\bullet,\bullet}$ instead of $\mathrm{C}^2(\mathrm{Hom}_{\mathcal{C}})$. If X and Y are two objects of $\mathrm{C}(\mathcal{C})$, one has

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}}^{\bullet,\bullet}(X,Y)^{n,m} && = \mathrm{Hom}_{\mathcal{C}}(X^{-m}, Y^n), \\ d'^{n,m} &= \mathrm{Hom}_{\mathcal{C}}(X^{-m}, d_Y^n), && d''^{n,m} = \mathrm{Hom}_{\mathcal{C}}((-)^n d_X^{-n-1}, Y^m). \end{aligned}$$

Note that $\mathrm{Hom}_{\mathcal{C}}^{\bullet,\bullet}(X,Y)$ is a double complex in the category $\mathbf{Ab}$, which should not be confused with the group $\mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X,Y)$.
(ii) Consider the bifunctor $\bullet \otimes \bullet : \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$. We shall simply write $\otimes$ instead of $\mathrm{C}^2(\otimes)$. Hence, for $X \in \mathrm{C}^-(\mathrm{Mod}(A^{\mathrm{op}}))$ and $Y \in \mathrm{C}^-(\mathrm{Mod}(A))$, one has

$$(X \otimes Y)^{n,m} = X^n \otimes Y^m,$$
$$d'^{n,m} = d_X^n \otimes Y^m, d'\prime^{n,m} = X^n \otimes d_Y^m.$$

Definition 3.4.4. Let $X \in \mathrm{C}^-(\mathcal{C})$ and $Y \in \mathrm{C}^+(\mathcal{C})$. One sets

$$\mathrm{Hom}_{\mathcal{C}}^{\bullet}(X,Y) = \mathrm{tot}(\mathrm{Hom}_{\mathcal{C}}^{\bullet,\bullet}(X,Y)). \tag{3.13}$$

Exercises to Chapter 3

Exercise 3.1. Let $\mathcal{C}$ be an additive category and let $X \in \mathrm{C}(\mathcal{C})$. By using $\pm d_X^p \colon X^p \to X^{p+1}$ $(p \in \mathbb{Z})$, construct a morphism $d_X \colon X \to X[1]$ in $\mathrm{C}(\mathcal{C})$ and prove that $d_X \colon X \to X[1]$ is homotopic to zero.

Exercise 3.2. Let $\mathcal{C}$ be an additive category, $f, g\colon X \rightrightarrows Y$ two morphisms in $\mathrm{C}(\mathcal{C})$. Prove that f and g are homotopic if and only if there exists a commutative diagram in $\mathrm{C}(\mathcal{C})$

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"'] \arrow[d, equal] & \mathrm{Mc}(f) \arrow[r, "\beta(f)"'] \arrow[d, "u"] & X[1] \arrow[d, equal] \\ Y \arrow[r, "\alpha(g)"'] & \mathrm{Mc}(f) \arrow[r, "\beta(g)"'] & X[1]. \end{tikzcd}$$

In such a case, prove that u is an isomorphism in $\mathrm{C}(\mathcal{C})$.

Exercise 3.3. Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$.
Prove that the following conditions are equivalent:

(a) f is homotopic to zero,

(b) f factors through $\alpha(\mathrm{id}_X)\colon X \to \mathrm{Mc}(\mathrm{id}_X)$,

(c) f factors through $\beta(\mathrm{id}_Y)[-1]\colon \mathrm{Mc}(\mathrm{id}_Y)[-1] \to Y$,

(d) f decomposes as $X \to Z \to Y$ with Z a complex homotopic to zero.

Exercise 3.4. A category with translation $(\mathcal{A}, T)$ is a category $\mathcal{A}$ together with an equivalence $T\colon \mathcal{A} \to \mathcal{A}$. A differential object (X, d_X) in a category with translation $(\mathcal{A}, T)$ is an object $X \in \mathcal{A}$ together with a morphism $d_X\colon X \to T(X)$. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential objects is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[d, "f"] & TX \arrow[d, "T(f)"] \\ Y \arrow[r, "d_Y"] & TY. \end{tikzcd}$$

One denotes by $\mathcal{A}_d$ the subcategory of $(\mathcal{A}, T)$ consisting of differential objects and morphisms of such objects. If $\mathcal{A}$ is additive, one says that a differential object (X, d_X) in $(\mathcal{A}, T)$ is a complex if the composition $X \xrightarrow{d_X} T(X) \xrightarrow{T(d_X)} T^2(X)$ is zero. One denotes by $\mathcal{A}_c$ the full subcategory of $\mathcal{A}_d$ consisting of complexes.
(i) Let $\mathcal{C}$ be a category. Denote by $\mathbb{Z}_d$ the set $\mathbb{Z}$ considered as a discrete category and still denote by $\mathbb{Z}$ the ordered set $(\mathbb{Z}, \leq)$ considered as a category. Prove that $\mathcal{C}^{\mathbb{Z}} := \mathrm{Fct}(\mathbb{Z}_d, \mathcal{C})$ is a category with translation.
(ii) Show that the category $\mathrm{Fct}(\mathbb{Z}, \mathcal{C})$ may be identified to the category of differential objects in $\mathcal{C}^{\mathbb{Z}}$.

(iii) Let $\mathcal{C}$ be an additive category. Show that the notions of differential objects and complexes given above coincide with those in Definition 3.2.1 when choosing $\mathcal{A} = \mathrm{C}(\mathcal{C})$ and $T = [1]$.

Exercise 3.5. Consider the category $\boldsymbol{\Delta}$ and for $n > 0$, denote by

$$s_i^n \colon [0,n] \to [0,n-1] \qquad (0 \leq i \leq n-1)$$

the surjective order-preserving map which takes the same value at i and $i+1$. In other words

$$s_i^n(k) \quad = \quad \begin{cases} k & \text{for } k \leq i, \\ k-1 & \text{for } k > i. \end{cases}$$

Checks the relations:

$$\begin{cases} s_j^n \circ s_i^{n+1} = s_{i-1}^n \circ s_j^{n+1} & \text{for } 0 \leq j < i \leq n, \\ s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^n & \text{for } 0 \leq i < j \leq n, \\ s_j^{n+1} \circ d_i^n = \mathrm{id}_{[0,n]} & \text{for } 0 \leq i \leq n+1, i = j, j+1, \\ s_j^{n+1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^n & \text{for } 1 \leq j+1 < i \leq n+1. \end{cases}$$

Exercise 3.6. Let M be an A-module and let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be n endomorphisms of M over A which satisfy:

$$\varphi_i = 0 \text{ or } \varphi_i = \mathrm{id}_M \,.$$

Prove that the Koszul complex $K^\bullet(M, \varphi)$ is homotopic to zero.

Chapter 4

Abelian categories

In this chapter, we give the main properties of abelian categories and expose some basic constructions on complexes in such categories, such as the snake Lemma. We explain the notion of injective resolutions and apply it to the construction of derived functors, with applications to the functors Ext and Tor.

For sake of simplicity, we shall always argue as if we were working in a full abelian subcategory of $\mathrm{Mod}(A)$ for a ring A. (See Convention 4.1.1 below.) Some important historical references are the book [5] and the paper [14].

4.1 Abelian categories

Convention 4.1.1. In these Notes, when dealing with an abelian category $\mathcal{C}$ (see Definition 4.1.4 below), we shall assume that $\mathcal{C}$ is a full abelian subcategory of a category $\mathrm{Mod}(A)$ for some ring A. This makes the proofs much easier and moreover there exists a famous theorem (due to Freyd & Mitchell) that asserts that this is in fact always the case (up to equivalence of categories).

From now on, $\mathcal{C}$, $\mathcal{C}'$ will denote additive categories.

Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Recall that if $\operatorname{Ker} f$ exists, it is unique up to unique isomorphism, and for any $W \in \mathcal{C}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(W, \operatorname{Ker} f) \to \mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(W, Y) \tag{4.1}$$

is exact in $\mathrm{Mod}(\mathbb{Z})$.

Similarly, if $\operatorname{Coker} f$ exists, then for any $W \in \mathcal{C}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\operatorname{Coker} f, W) \to \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(X, W) \tag{4.2}$$

is exact in $\mathrm{Mod}(\mathbb{Z})$.

Example 4.1.2. Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\mathrm{Mod}^f(A)$ has no kernel.

Let $\mathcal{C}$ be an additive category which admits kernels and cokernels. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. One defines:

$$\begin{aligned} \mathrm{Coim}\, f &= \mathrm{Coker}\, h, \text{ where } h : \mathrm{Ker}\, f \to X \\ \mathrm{Im}\, f &= \mathrm{Ker}\, k, \text{ where } k : Y \to \mathrm{Coker}\, f. \end{aligned}$$

Consider the diagram:

$$\begin{tikzcd} \mathrm{Ker}\, f \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "k"] & \mathrm{Coker}\, f \\ & \mathrm{Coim}\, f \arrow[ur, dotted, "\tilde{f}"] \arrow[r, dotted, "u"] & \mathrm{Im}\, f \arrow[u] & \end{tikzcd}$$

Since $f \circ h = 0$, f factors uniquely through $\tilde{f}$, and $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\mathrm{Ker}\, k = \mathrm{Im}\, f$. We have thus constructed a canonical morphism:

$$\mathrm{Coim}\, f \xrightarrow{u} \mathrm{Im}\, f. \tag{4.3}$$

Examples 4.1.3. (i) If A is a ring and f is a morphism in $\mathrm{Mod}(A)$, then (4.3) is an isomorphism.
(ii) The category **Ban** admits kernels and cokernels. If $f : X \to Y$ is a morphism of Banach spaces, define $\mathrm{Ker}\, f = f^{-1}(0)$ and $\mathrm{Coker}\, f = Y/\overline{\mathrm{Im}\, f}$ where $\overline{\mathrm{Im}\, f}$ denotes the closure of the space $\mathrm{Im}\, f$. It is well-known that there exist continuous linear maps $f : X \to Y$ which are injective, with dense and non closed image. For such an f, $\mathrm{Ker}\, f = \mathrm{Coker}\, f = 0$ although f is not an isomorphism. Thus $\mathrm{Coim}\, f \simeq X$ and $\mathrm{Im}\, f \simeq Y$. Hence, the morphism (4.3) is not an isomorphism.

Definition 4.1.4. Let $\mathcal{C}$ be an additive category. One says that $\mathcal{C}$ is abelian if:

(i) any $f : X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathcal{C}$, the natural morphism $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\mathrm{Ker}\, f \simeq 0$ (resp. $\mathrm{Coker}\, f \simeq 0$). If f is both a monomorphism and an epimorphism, it is an isomorphism.

Examples 4.1.5. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category.
(ii) If A is noetherian, then $\mathrm{Mod}^f(A)$ is abelian.
(iii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 4.1.3 (ii).)
(iv) Let I be category. Then if $\mathcal{C}$ is abelian, the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$, is abelian. For example, if $F, G : I \to \mathcal{C}$ are two functors and $\varphi : F \to G$ is a morphism of functors, define the functor $\mathrm{Ker}\,\varphi$ by $\mathrm{Ker}\,\varphi(X) = \mathrm{Ker}(F(X) \to G(X))$. Then clearly, $\mathrm{Ker}\,\varphi$ is a kernel of φ. One defines similarly the cokernel.
(v) If $\mathcal{C}$ is abelian, then the opposite category $\mathcal{C}^{\mathrm{op}}$ is abelian.

Unless otherwise specified, we assume until the end of this chapter that $\mathcal{C}$ is abelian.

One naturally extends Definition 1.2.1 to abelian categories. Consider a sequence of morphisms $X' \xrightarrow{f} X \xrightarrow{g} X''$ with $g \circ f = 0$ (sometimes, one calls such a sequence a complex). It defines a morphism $\mathrm{Coim}\, f \to \mathrm{Ker}\, g$, hence, $\mathcal{C}$ being abelian, a morphism $\mathrm{Im}\, f \to \mathrm{Ker}\, g$.

Definition 4.1.6. (i) One says that a sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ with $g \circ f = 0$ is exact if $\mathrm{Im}\, f \xrightarrow{\sim} \mathrm{Ker}\, g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\mathrm{Im}\, d^i \xrightarrow{\sim} \mathrm{Ker}\, d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f : X \to Y$ may be decomposed into short exact sequences:

$$0 \to \mathrm{Ker}\, f \to X \to \mathrm{Im}\, f \to 0$$

$$0 \to \mathrm{Im}\, f \to Y \to \mathrm{Coker}\, f \to 0.$$

Proposition 4.1.7. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be a short exact sequence in $\mathcal{C}$. Then the conditions* (i) *to* (iii) *are equivalent.*

(i) *there exists $h : X'' \to X$ such that $g \circ h = \mathrm{id}_{X''}$,*

(ii) *there exists $k : X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$,*

(iii) *there exists $\varphi = (k, g)$ and $\psi = (f + h)$ such that $X \xrightarrow{\varphi} X' \oplus X''$ and $X' \oplus X'' \xrightarrow{\psi} X$ are isomorphisms inverse to each other,* The proof is similar to the case of A-modules and is left as an exercise.

If the conditions of the above proposition are satisfied, one says that the sequence splits.

Note that an additive functor of abelian categories sends split exact sequences into split exact sequences.

Lemma 4.1.8. *(The "five lemma".) Consider a commutative diagram:*

$$\begin{tikzcd}
X^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"'] & X^2 \arrow[r, "\alpha_2"] \arrow[d, "f^2"'] & X^3 \arrow[d, "f^3"'] \\
Y^0 \arrow[r, "\beta_0"'] & Y^1 \arrow[r, "\beta_1"'] & Y^2 \arrow[r, "\beta_2"'] & Y^3
\end{tikzcd}$$

and assume that the rows are exact sequences.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism, and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

According to Convention 4.1.1, we shall assume that $\mathcal{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A. Hence we may choose elements in the objects of $\mathcal{C}$.

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since $\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathcal{C}^{\mathrm{op}}$. q.e.d.

Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor of abelian categories. Since F is additive, $F(0) \simeq 0$ and $F(X \oplus Y) \simeq F(X) \oplus F(Y)$. In other words, F commutes with finite direct sums (and with finite products).

Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. Recall that F is left exact if and only if it commutes with kernels, that is, if and only if for any exact sequence in $\mathcal{C}$, $0 \to X' \to X \to X''$ the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$.

Similarly, F is right exact if and only if it commutes with cokernels, that is, if and only if for any exact sequence in $\mathcal{C}$, $X' \to X \to X'' \to 0$ the sequence $F(X') \to F(X) \to F(X'') \to 0$ is exact.

Lemma 4.1.9. *Let $F\mathcal{C} \to \mathcal{C}'$ be an additive functor.*

(i) *F is left exact if and only if for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact.*

(ii) *F is exact if and only if for any exact sequence $X' \to X \to X''$ in $\mathcal{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact.*

The proof is left as an exercise.

Examples 4.1.10. (i) Let $\mathcal{C}$ be an abelian category. The functor $\mathrm{Hom}_{\mathcal{C}}$ from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathrm{Mod}(\mathbb{Z})$ is left exact.
(ii) Let A be a k-algebra. Let M and N in $\mathrm{Mod}(A)$. It follows from (i) that the functors Hom_A from $\mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$ is left exact.
The functors $\otimes_A$ from $\mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$ is right exact.
If A is a field, all the above functors are exact.
(iii) Let I and $\mathcal{C}$ be two categories with $\mathcal{C}$ abelian. Assume that $\mathcal{C}$ admits inductive limits. Recall that the functor $\varinjlim : \mathrm{Fct}(I, \mathcal{C}) \to \mathcal{C}$ is right exact. If $\mathcal{C} = \mathrm{Mod}(A)$ and I is filtrant, then the functor $\varinjlim$ is exact.

Similarly, if $\mathcal{C}$ admits projective limits, the functor $\varprojlim : \mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ is left exact. If $\mathcal{C} = \mathrm{Mod}(A)$ and I is discrete, the functor $\varprojlim$ (that is, the functor $\prod$) is exact.

4.2 Complexes in abelian categories

We assume that $\mathcal{C}$ is abelian. Notice first that the categories $\mathrm{C}^*(\mathcal{C})$ are clearly abelian for $* = \emptyset, +, -, b$. For example, if $f : X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, the complex Z defined by $Z^n = \mathrm{Ker}(f^n : X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\mathrm{Coker}\, f$.

Let $X \in \mathrm{C}(\mathcal{C})$. One defines the following objects of $\mathcal{C}$:

$$\begin{aligned} Z^k(X) &:= \mathrm{Ker}\, d_X^k \\ B^k(X) &:= \mathrm{Im}\, d_X^{k-1} \\ H^k(X) &:= Z^k(X)/B^k(X) \quad (:= \mathrm{Coker}(B^k(X) \to Z^k(X))) \end{aligned}$$

One calls $H^k(X)$ the k-th cohomology object of X. If $f : X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, then it induces morphisms $Z^k(X) \to Z^k(Y)$ and $B^k(X) \to B^k(Y)$, thus a morphism $H^k(f) : H^k(X) \to H^k(Y)$. Clearly,$H^k(X \oplus Y) \simeq H^k(X) \oplus H^k(Y)$. Hence we have obtained an additive functor:

$$H^k(\,\bullet\,) : \mathrm{C}(\mathcal{C}) \to \mathcal{C}.$$

Notice that:

$$H^k(X) = H^0(X[k]).$$

Lemma 4.2.1. *Let $\mathcal{C}$ be an abelian category and let $f : X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$ homotopic to zero. Then $H^k(f) : H^k(X) \to H^k(Y)$ is the 0 morphism.*

Proof. Let $f^k = s^{k+1} \circ d_X^k + d_Y^{k-1} \circ s^k$. Then $d_X^k = 0$ on $\operatorname{Ker} d_X^k$ and $d_Y^{k-1} \circ s^k = 0$ on $\operatorname{Ker} d_Y^k / \operatorname{Im} d_Y^{k-1}$. Hence $H^k(f) : \operatorname{Ker} d_X^k / \operatorname{Im} d_X^{k-1} \to \operatorname{Ker} d_Y^k / \operatorname{Im} d_Y^{k-1}$ is the zero morphism. q.e.d.

In view of Lemma 4.2.1, the functor $H^0 \colon \mathrm{C}(\mathcal{C}) \to \mathcal{C}$ extends as a functor

$$H^0 \colon \mathrm{K}(\mathcal{C}) \to \mathcal{C}.$$

Definition 4.2.2. One says that a morphism $f : X \to Y$ in $\mathrm{C}(\mathcal{C})$ or in $\mathrm{K}(\mathcal{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic.

In particular, $X \in \mathrm{C}(\mathcal{C})$ is qis to 0 if and only if the complex X is exact.

Remark 4.2.3. By Lemma 4.2.1, a complex homotopic to 0 is qis to 0, but the converse is false. For example, a short exact sequence does not necessarily split. One shall be aware that the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 4.2.4. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \oplus_i H^i(X^\bullet)[-i]. \tag{4.4}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

There are exact sequences

$$X^{k-1} \to \operatorname{Ker} d_X^k \to H^k(X) \to 0,$$
$$0 \to H^k(X) \to \operatorname{Coker} d_X^{k-1} \to X^{k+1},$$

which give rise to the exact sequence:

$$0 \to H^k(X) \to \operatorname{Coker}(d_X^{k-1}) \xrightarrow{d_X^k} \operatorname{Ker} d_X^{k+1} \to H^{k+1}(X) \to 0. \tag{4.5}$$

Lemma 4.2.5. (The snake lemma.) *Consider the commutative diagram in $\mathcal{C}$ below with exact rows:*

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' & \end{tikzcd}$$

Then it gives rise to an exact sequence:

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\varphi} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma.$$

The proof is similar to that of Lemma 4.1.8 and is left as an exercise.

Theorem 4.2.6. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathrm{C}(\mathcal{C})$.*

(i) *For each $k \in \mathbb{Z}$, the sequence $H^k(X') \to H^k(X) \to H^k(X'')$ is exact.*

(ii) *For each $k \in \mathbb{Z}$, there exists $\delta^k : H^k(X'') \to H^{k+1}(X')$ making the sequence:*

$$H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \tag{4.6}$$

exact. Moreover, one can construct δ^k functorial with respect to short exact sequences of $\mathrm{C}(\mathcal{C})$.

Proof. The exact sequence in $\mathrm{C}(\mathcal{C})$ gives rise to commutative diagrams with exact rows:

$$\begin{tikzcd} & \operatorname{Coker} d_{X'}^{k-1} \arrow[r, "f"'] \arrow[d, "d_{X'}^k"'] & \operatorname{Coker} d_X^{k-1} \arrow[r, "g"'] \arrow[d, "d_X^k"'] & \operatorname{Coker} d_{X''}^{k-1} \arrow[r] \arrow[d, "d_{X''}^k"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker} d_{X'}^{k+1} \arrow[r, "f"'] & \operatorname{Ker} d_X^{k+1} \arrow[r, "g"'] & \operatorname{Ker} d_{X''}^{k+1} & \end{tikzcd}$$

Then using the exact sequence (4.5), the result follows from Lemma 4.2.5. q.e.d.

Remark 4.2.7. Let us denote for a while by $\delta^k(f,g)$ the map δ^k constructed in Theorem 4.2.6. Then one can prove that $\delta^k(-f,g) = \delta^k(f,-g) = -\delta^k(f,g)$.

Corollary 4.2.8. *In the situation of Theorem 4.2.6, if two of the complexes X', X, X'' are exact, so is the third one.*

Corollary 4.2.9. *Let $f : X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$. Then there is a long exact sequence*

$$\cdots \to H^k(X) \xrightarrow{H^k(f)} H^k(Y) \to H^{k+1}(\mathrm{Mc}(f)) \to \cdots$$

Proof. There are natural morphisms $Y \to \mathrm{Mc}(f)$ and $\mathrm{Mc}(f) \to X[1]$ which give rise to an exact sequence in $\mathrm{C}(\mathcal{C})$:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0. \tag{4.7}$$

Applying Theorem 4.2.6, one finds a long exact sequence

$$\cdots \to H^k(X[1]) \xrightarrow{\delta^k} H^{k+1}(Y) \to H^{k+1}(\mathrm{Mc}(f)) \to \cdots .$$

One can prove that the morphism $\delta^k : H^{k+1}(X) \to H^{k+1}(Y)$ is $H^{k+1}(f)$ up to a sign. q.e.d.

Double complexes

Let $\mathcal{C}$ denote as above an abelian category.

Theorem 4.2.10. *Let $X^{\bullet,\bullet}$ be a double complex such that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$ and are exact for $j > 0$.*

Then $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0}) \simeq H^p(\mathrm{tot}(X^{\bullet,\bullet}))$ for all p.

Proof. We shall only describe the first isomorphism $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ in the case where $\mathcal{C} = \mathrm{Mod}(A)$, by the so-called "Weil procedure".

Let $x^{p,0} \in X^{p,0}$, with $d'x^{p,0} = 0$ which represents $y \in H^p(X^{\bullet,0})$. Define $x^{p,1} = d''x^{p,0}$. Then $d'x^{p,1} = 0$, and the first column being exact, there exists $x^{p-1,1} \in X^{p-1,1}$ with $d'x^{p-1,1} = x^{p,1}$. One can iterate this procedure until getting $x^{0,p} \in X^{0,p}$. Since $d'd''x^{0,p} = 0$, and d' is injective on $X^{0,p}$ for $p > 0$ by the hypothesis, we get $d''x^{0,p} = 0$. The class of $x^{0,p}$ in $H^p(X^{0,\bullet})$ will be the image of y by the Weil procedure. Of course, one has to check that this image does not depend of the various choices we have made, and that it induces an isomorphism.

This can be visualized by the diagram:

$$\begin{tikzcd}
 & & & & x^{0,p} \arrow[r, "d''"] \arrow[d, "d'"'] & 0 \\
 & & & x^{1,p-2} \arrow[r, "d''"] \arrow[d, dotted] & x^{1,p-1} & \\
 & & & {} & & \\
 & x^{p-1,1} \arrow[r, dotted] \arrow[d, "d'"'] & {} & & & \\
x^{p,0} \arrow[r, "d''"] \arrow[d, "d'"'] & x^{p,1} & & & & \\
0 & & & & &
\end{tikzcd}$$

q.e.d.

Proposition 4.2.11. *Let $X^{\bullet,\bullet}$ be a double complex such that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$. Assume that all rows (resp. all columns) of $X^{\bullet,\bullet}$ are exact. Then the complex $\mathrm{tot}(X^{\bullet,\bullet})$ is exact.*

The proof is left as an exercise. Note that if there are only two rows let's say in degrees -1 and 0, then the result follows from Theorem 4.6.4

4.3 Application to Koszul complexes

Consider a Koszul complex, as in §1.6. Keeping the notations of this section, set $\varphi' = \{\varphi_1, \ldots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

$$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi') \tag{4.8}$$

Proposition 4.3.1. *The complex $K^\bullet(M, \varphi)[1]$ is isomorphic to the mapping cone of $-\widetilde{\varphi}_n$.*

Proof. [1] Consider the diagram

$$\begin{tikzcd}
\mathrm{Mc}(\widetilde{\varphi}_n)^p \arrow[r, "d_M^p"'] \arrow[d, "\lambda^p"'] & \mathrm{Mc}(\widetilde{\varphi}_n)^{p+1} \arrow[d, "\lambda^{p+1}"'] \\
K^{p+1}(M, \varphi) \arrow[r, "d_K^{p+1}"'] & K^{p+2}(M, \varphi)
\end{tikzcd}$$

[1]The proof has been skipped in 2005/2006

given explicitly by:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(M\otimes\bigwedge^{p+1}\mathbb{Z}^{n-1})\oplus(M\otimes\bigwedge^{p}\mathbb{Z}^{n-1}) \arrow[r, "{\begin{pmatrix} -d' & 0 \\ -\varphi_n & d' \end{pmatrix}}"'] \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] & M(\otimes\bigwedge^{p+2}\mathbb{Z}^{n-1})\oplus(M\otimes\bigwedge^{p+1}\mathbb{Z}^{n-1}) \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] \\
M\otimes\bigwedge^{p+1}\mathbb{Z}^{n} \arrow[r, "-d"'] & M\otimes\bigwedge^{p+2}\mathbb{Z}^{n}
\end{tikzcd}$$

Then

$$\begin{aligned}
&d_M^p(a\otimes e_J + b\otimes e_K) = -d'(a\otimes e_J) + (d'(b\otimes e_K) - \varphi_n(a)\otimes e_J),\\
&\lambda^p(a\otimes e_J + b\otimes e_K) = a\otimes e_J + b\otimes e_n\wedge e_K.
\end{aligned}$$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by:

$$\sum_J a_J\otimes e_J + \sum_K b_K\otimes e_K \mapsto \sum_J a_J\otimes e_J + \sum_K b_K\otimes e_n\wedge e_K \tag{4.9}$$

with $|J| = p+1$ and $|K| = p$. Any element of $M\otimes\bigwedge^{p+1}\mathbb{Z}^n$ may uniquely be written as in the right hand side of (4.9).
(ii) The diagram commutes. Indeed,

$$\begin{aligned}
\lambda^{p+1}\circ d_M^p(a\otimes e_J + b\otimes e_K) &= -d'(a\otimes e_J) + e_n\wedge d'(b\otimes e_K) - \varphi_n(a)\otimes e_n\wedge e_J\\
&= -d'(a\otimes e_J) - d'(b\otimes e_n\wedge e_K) - \varphi_n(a)\otimes e_n\wedge e_J,\\
d_K^{p+1}\circ\lambda^p(a\otimes e_J + b\otimes e_K) &= -d(a\otimes e_J + b\otimes e_n\wedge e_K)\\
&= -d'(a\otimes e_J) - \varphi_n(a)\otimes e_n\wedge e_J - d'(b\otimes e_n\wedge e_K).
\end{aligned}$$

q.e.d.

Proposition 4.3.2. *There exists a long exact sequence*

$$\cdots\to H^j(K^\bullet(M,\varphi')) \xrightarrow{\varphi_n} H^j(K^\bullet(M,\varphi')) \to H^{j+1}(K^\bullet(M,\varphi)) \to\cdots \tag{4.10}$$

Proof. Apply Proposition 4.3.1 and Corollary 4.2.9. q.e.d.

We can now give a proof to Theorem 1.6.2. Assume for example that $(\varphi_1,\ldots,\varphi_n)$ is a regular sequence, and let us argue by induction on n. The cohomology of $K^\bullet(M,\varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M)+\cdots+\varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Theorem 1.6.2 follows.

4.4 Injective objects

Definition 4.4.1. (i) An object I of $\mathcal{C}$ is injective if $\mathrm{Hom}_{\mathcal{C}}(\,\bullet\,, I)$ is an exact functor.

(ii) One says that $\mathcal{C}$ has enough injectives if for any $X \in \mathcal{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathcal{C}$ iff it is injective in $\mathcal{C}^{\mathrm{op}}$, i.e. if the functor $\mathrm{Hom}_{\mathcal{C}}(P, \,\bullet\,)$ is exact.

(iv) One says that $\mathcal{C}$ has enough projectives if for any $X \in \mathcal{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Example 4.4.2. Let A be a ring. An A-module M is called injective (resp. projective) if it is so in the category $\mathrm{Mod}(A)$. If M is free then it is projective. More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 1.2). One immediately deduces that the category $\mathrm{Mod}(A)$ has enough projectives. One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 1.5).
If k is a field, then any object of $\mathrm{Mod}(k)$ is both injective and projective.

Proposition 4.4.3. *The object $I \in \mathcal{C}$ is injective if and only if, for any $X, Y \in \mathcal{C}$ and any diagram in which the row is exact:*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "k"'] & Y \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd}$$

the dotted arrow may be completed, making the solid diagram commutative.

The proof is similar to that of Proposition 1.3.8.

Lemma 4.4.4. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h \colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 4.1.7. q.e.d.

It follows that if $F \colon \mathcal{C} \to \mathcal{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 4.4.5. *Let X', X'' belong to $\mathcal{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, $X \mapsto F(X) \oplus G(X)$ is exact if and only if F and G are exact. q.e.d.

Applying Lemmas 4.4.4 and 4.4.5, we get:

Proposition 4.4.6. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{C}$ and assume X' and X are injective. Then X'' is injective.*

4.5 Resolutions

In this section, $\mathcal{C}$ denotes an abelian category and $\mathcal{I}_{\mathcal{C}}$ its full additive subcategory consisting of injective objects. We shall asume

(4.11) the abelian category $\mathcal{C}$ admits enough injectives.

Definition 4.5.1. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. We say that $\mathcal{J}$ is cogenerating if for all X in $\mathcal{C}$, there exist $Y \in \mathcal{J}$ and a monomorphism $X \rightarrowtail Y$.

Note that the category of injective objects is cogenerating iff $\mathcal{C}$ has enough injectives.

Notations 4.5.2. Consider an exact sequence in $\mathcal{C}$, $0 \to X \to J^0 \to \cdots \to J^n \to \cdots$ and denote by $J^\bullet$ the complex $0 \to J^0 \to \cdots \to J^n \to \cdots$. We shall say for short that $0 \to X \to J^\bullet$ is a resolution of X. If the J^k's belong to $\mathcal{J}$, we shall say that this is a $\mathcal{J}$-resolution of X. When $\mathcal{J}$ denotes the category of injective objects one says this is an injective resolution.

Proposition 4.5.3. *Assume $\mathcal{J}$ is cogenerating. Then for any $X \in \mathcal{C}$, there exists a $\mathcal{J}$-resolution of X.*

Proof. We proceed by induction. Assume to have constructed:

$$0 \to X \to J^0 \to \cdots \to J^n$$

For $n = 0$ this is the hypothesis. Set $B^n = \mathrm{Coker}(J^{n-1} \to J^n)$ (with $J^{-1} = X$). Then $J^{n-1} \to J^n \to B^n \to 0$ is exact. Embed B^n in an object of $\mathcal{J}$: $0 \to B^n \to J^{n+1}$. Then $J^{n-1} \to J^n \to J^{n+1}$ is exact, and the induction proceeds. q.e.d.

Proposition 4.5.4. (i) *Let $f^\bullet : X^\bullet \to I^\bullet$ be a morphism in $\mathrm{C}^+(\mathcal{C})$. Assume $I^\bullet$ belongs to $\mathcal{C}^+(\mathcal{I}_\mathcal{C})$ and $X^\bullet$ is exact. Then $f^\bullet$ is homotopic to* 0.

(ii) *Let $I^\bullet \in \mathrm{C}^+(\mathcal{I}_\mathcal{C})$ and assume $I^\bullet$ is exact. Then $I^\bullet$ is homotopic to* 0.

Proof. (i) Consider the diagram:

$$\begin{tikzcd} X^{k-2} \arrow[r] & X^{k-1} \arrow[r] \arrow[d, "f^{k-1}"'] \arrow[dl, dotted, "s^{k-1}"'] & X^k \arrow[r] \arrow[d, "f^k"'] \arrow[dl, dotted, "s^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\ I^{k-2} \arrow[r] & I^{k-1} \arrow[r] & I^k \arrow[r] & I^{k+1} \end{tikzcd}$$

We shall construct by induction morphisms s^k satisfying:

$$f^k = s^{k+1} \circ d_X^k + d_I^{k-1} \circ s^k.$$

For $j << 0$, $s^j = 0$. Assume we have constructed the s^j for $j \le k$. Define $g^k = f^k - d_I^{k-1} \circ s^k$. One has

$$\begin{aligned} g^k \circ d_X^{k-1} &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ s^k \circ d_X^{k-1} \\ &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-2} \circ s^{k-1} \\ &= 0. \end{aligned}$$

Hence, g^k factorizes through $X^k/\operatorname{Im} d_X^{k-1}$. Since the complex $X^\bullet$ is exact, the sequence $0 \to X^k/\operatorname{Im} d_X^{k-1} \to X^{k+1}$ is exact. Consider

$$\begin{tikzcd} 0 \arrow[r] & X^k/\operatorname{Im} d_X^{k-1} \arrow[r] \arrow[d, "g^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\ & I^k & \end{tikzcd}$$

The dotted arrow may be completed by Proposition 4.4.3.
(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = \mathrm{id}_X$. q.e.d.

Proposition 4.5.5. (i) *Let $f : X \to Y$ be a morphism in $\mathcal{C}$, let $0 \to X \to X^\bullet$ be a resolution of X and let $0 \to Y \to J^\bullet$ be a complex with the J^k's injective. Then there exists a morphism $f^\bullet : X^\bullet \to J^\bullet$ making the diagram below commutative:*

(4.12)
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & X^\bullet \arrow[d, "f^\bullet"'] \\ 0 \arrow[r] & Y \arrow[r] & J^\bullet \end{tikzcd}$$

(ii) *The morphism* $f^\bullet$ *in* $\mathrm{C}(\mathcal{C})$ *constructed in* (i) *is unique up to homotopy.*

Proof. (i) Let us denote by d_X (resp. d_Y) the differential of the complex $X^\bullet$ (resp. $J^\bullet$), by d_X^{-1} (resp. d_Y^{-1}) the morphism $X \to X^0$ (resp. $Y \to J^0$) and set $f^{-1} = f$.

We shall construct the f^n's by induction. Morphism f^0 is obtained by Proposition 4.4.3. Assume we have constructed $f^0, \dots, f^n$. Let $g^n = d_Y^n \circ f^n : X^n \to J^{n+1}$. The morphism g^n factorizes through $h^n : X^n / \operatorname{Im} d_X^{n-1} \to J^{n+1}$. Since $X^\bullet$ is exact, the sequence $0 \to X^n / \operatorname{Im} d_X^{n-1} \to X^{n+1}$ is exact. Since J^{n+1} is injective, h^n extends as $f^{n+1} : X^{n+1} \to J^{n+1}$.
(ii) We may assume $f = 0$ and we have to prove that in this case $f^\bullet$ is homotopic to zero. Since the sequence $0 \to X \to X^\bullet$ is exact, this follows from Proposition 4.5.4 (i), replacing the exact sequence $0 \to Y \to J^\bullet$ by the complex $0 \to 0 \to J^\bullet$. q.e.d.

4.6 Derived functors

In this section, $\mathcal{C}$ and $\mathcal{C}'$ will denote abelian categories and $F \colon \mathcal{C} \to \mathcal{C}'$ a left exact functor. We shall make the hypothesis

(4.13) the category $\mathcal{C}$ admits enough injectives.

Recall that $\mathcal{I}_\mathcal{C}$ denotes the additive category of injective objects in $\mathcal{C}$.

Lemma 4.6.1. *Assuming* (4.13)*, there exists a functor* $\lambda \colon \mathcal{C} \to \mathrm{K}(\mathcal{I}_\mathcal{C})$ *and for for each* $X \in \mathcal{C}$*, a qis* $X \to \lambda(X)$*, functorially in* $X \in \mathcal{C}$*.*

Proof. (i) Let $X \in \mathcal{C}$ and let $I_X^\bullet \in \mathrm{C}^+(\mathcal{I}_\mathcal{C})$ be an injective resolution of X. The image of $I_X^\bullet$ in $\mathrm{K}^+(\mathcal{C})$ is unique up to unique isomorphism, by Proposition 4.5.5.

Indeed, consider two injective resolutions $I_X^\bullet$ and $J_X^\bullet$ of X. By Proposition 4.5.5 applied to id_X, there exists a morphism $f^\bullet \colon I_X^\bullet \to J_X^\bullet$ making the diagram 4.12 commutative and this morphism is unique up to homotopy, hence is unique in $\mathrm{K}^+(\mathcal{C})$. Similarly, there exists a unique morphism $g^\bullet \colon J_X^\bullet \to I_X^\bullet$ in $\mathrm{K}^+(\mathcal{C})$. Hence, $f^\bullet$ and $g^\bullet$ are isomorphisms inverse one to each other.
(ii) Let $f \colon X \to Y$ be a morphism in $\mathcal{C}$, let $I_X^\bullet$ and $I_Y^\bullet$ be injective resolutions of X and Y respectively, and let $f^\bullet \colon I_X^\bullet \to I_Y^\bullet$ be a morphism of complexes such as in Proposition 4.5.5. Then the image of $f^\bullet$ in $\operatorname{Hom}_{\mathrm{K}^+(\mathcal{I}_\mathcal{C})}(I_X^\bullet, I_Y^\bullet)$ does not depend on the choice of $f^\bullet$ by Proposition 4.5.5.

In particular, we get that if $g \colon Y \to Z$ is another morphism in $\mathcal{C}$ and $I_Z^\bullet$ is an injective resolutions of Z, then $g^\bullet \circ f^\bullet = (g \circ f)^\bullet$ as morphisms in $\mathrm{K}^+(\mathcal{I}_\mathcal{C})$. q.e.d.

Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories and assume that $\mathcal{C}$ has enough injectives. Consider the functors

$$\mathcal{C} \xrightarrow{\lambda} \mathrm{K}^+(\mathcal{I}_{\mathcal{C}}) \xrightarrow{F} \mathrm{K}^+(\mathcal{C}') \xrightarrow{H^n} \mathcal{C}'.$$

Definition 4.6.2. One sets

$$R^nF = H^n \circ F \circ \lambda \tag{4.14}$$

and calls R^nF the n-th right derived functor of F.

By its definition, the receipt to construct $R^nF(X)$ is as follows:

- choose an injective resolution $I_X^\bullet$ of X, that is, construct an exact sequence $0 \to X \to I_X^\bullet$ with $I_X^\bullet \in \mathrm{C}^+(\mathcal{I}_{\mathcal{C}})$,
- apply F to this resolution,
- take the n-th cohomology.

In other words, $R^nF(X) \simeq H^n(F(I_X^\bullet))$.

Note that R^nF is an additive functor from $\mathcal{C}$ to $\mathcal{C}'$ and

$$\begin{aligned} R^nF(X) &\simeq 0 \text{ for } n < 0,\\ R^0F(X) &\simeq F(X),\\ R^nF(X) &\simeq 0 \text{ for } n \neq 0 \text{ if } F \text{ is exact},\\ R^nF(X) &\simeq 0 \text{ for } n \neq 0 \text{ if } X \text{ is injective}. \end{aligned}$$

The first assertion is obvious since $I_X^k = 0$ for $k < 0$, and the second one follows from the fact that F being left exact, then $\mathrm{Ker}(F(I_X^0) \to F(I_X^1)) \simeq F(\mathrm{Ker}(I_X^0 \to I_X^1)) \simeq F(X)$. The third assertion is clear since F being exact, it commutes with $H^j(\,\bullet\,)$. The last assertion is obvious by the construction of $R^jF(X)$.

Definition 4.6.3. An object X of $\mathcal{C}$ such that $R^kF(X) \simeq 0$ for all $k > 0$ is called F-acyclic.

Hence, injective objects are F-acyclic for all left exact functors F.

Theorem 4.6.4. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{C}$. Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to \cdots$$

Sketch of the proof. One constructs an exact sequence of complexes $0 \to X'^{\bullet} \to X^{\bullet} \to X''^{\bullet} \to 0$ whose objects are injective and this sequence is quasi-isomorphic to the sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ in $\mathrm{C}(\mathcal{C})$. Since the objects X'^j are injectice, we get a short exact sequence in $\mathrm{C}(\mathcal{C}')$:

$$0 \to F(X'^{\bullet}) \to F(X^{\bullet}) \to F(X''^{\bullet}) \to 0$$

Then one applies Theorem 4.2.6. q.e.d.

Definition 4.6.5. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. One says that $\mathcal{J}$ is F-injective if:

(i) $\mathcal{J}$ is cogenerating,

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}, X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathcal{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Proposition 4.6.6. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and denote by $\mathcal{I}_F$ the full subcategory of $\mathcal{C}$ consisting of F-acyclic objects. Then $\mathcal{I}_F$ is F-injective.*

Proof. Since injective objects are F-acyclic, hypothesis (4.13) implies that $\mathcal{I}_F$ is co-generating. The conditions (ii) and (iii) in Definition 4.6.5 are satisfied by Theorem 4.6.4. q.e.d.

Examples 4.6.7. (i) If $\mathcal{C}$ has enough injectives, the category $\mathcal{I}$ of injective objects is F-acyclic for all left exact functors F.

(ii) Let A be a ring and let N be a right A-module. The full additive subcategory of $\mathrm{Mod}(A)$ consisting of flat A-modules is projective with respect to the functor $N \otimes_A \bullet$.

Lemma 4.6.8. *Assume $\mathcal{J}$ is F-injective and let $X^{\bullet} \in \mathrm{C}^+(\mathcal{J})$ be a complex qis to zero* (i.e. *$X^{\bullet}$ is exact*)*. Then $F(X^{\bullet})$ is qis to zero.*

Proof. We decompose $X^{\bullet}$ into short exact sequences (assuming that this complex starts at step 0 for convenience):

$$\begin{aligned}
&0 \to X^0 \to X^1 \to Z^1 \to 0\\
&0 \to Z^1 \to X^2 \to Z^2 \to 0\\
&\cdots\\
&0 \to Z^{n-1} \to X^n \to Z^n \to 0
\end{aligned}$$

By induction we find that all the Z^j's belong to $\mathcal{J}$, hence all the sequences:

$$0 \to F(Z^{n-1}) \to F(X^n) \to F(Z^n) \to 0$$

are exact. Hence the sequence

$$0 \to F(X^0) \to F(X^1) \to \cdots$$

is exact. q.e.d.

Theorem 4.6.9. *Assume $\mathcal{J}$ is F-injective and contains the category $\mathcal{I}_{\mathcal{C}}$ of injective objects. Let $X \in \mathcal{C}$ and let $0 \to X \to J^\bullet$ be a resolution of X with $J^k \in \mathcal{J}$. Then for each k, there is an isomorphism $R^kF(X) \simeq H^k(F(J^\bullet))$.*

Proof. Let $0 \to X \to J^\bullet$ be a $\mathcal{J}$-resolution of X and let $0 \to X \to I^\bullet$ be an injective resolution of X. Applying Proposition 4.5.5, there exists $f : J^\bullet \to I^\bullet$ making the diagram below commutative

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\mathrm{id}"] & J^0 \arrow[r, "d_J^0"] \arrow[d, "f^0"] & J^1 \arrow[r, "d_J^1"] \arrow[d, "f^1"] & \cdots \\
0 \arrow[r] & X \arrow[r] & I^0 \arrow[r, "d_I^0"] & I^1 \arrow[r, "d_I^1"] & \cdots
\end{tikzcd}$$

Define the complex $K^\bullet = \mathrm{Mc}(f)$, the mapping cone of f. By the hypothesis, $K^\bullet$ belongs to $\mathrm{C}^+(\mathcal{J})$ and this complex is qis to zero by Corollary 4.2.8. By Lemma 4.6.8, $F(K^\bullet)$ is qis to zero.

On the other-hand, $F(\mathrm{Mc}(f))$ is isomorphic to $\mathrm{Mc}(F(f))$, the mapping cone of $F(f) : F(J^\bullet) \to F(I^\bullet)$. Applying Theorem 4.2.6 to this sequence, we find a long exact sequence

$$\cdots \to H^n(F(J^\bullet)) \to H^n(F(I^\bullet)) \to H^n(F(K^\bullet)) \to \cdots$$

Since $F(K^\bullet)$ is qis to zero, the result follows. q.e.d.

By this result, one sees that in order to calculate the k-th derived functor of F at X, the recipe is as follows. Consider a resolution $0 \to X \to J^\bullet$ of X by objects of $\mathcal{J}$, then apply F to the complex $J^\bullet$, and take the k-th cohomology object.

Proposition 4.6.10. *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ be left exact functors of abelian categories. We assume that $\mathcal{C}$ and $\mathcal{C}'$ have enough injectives.*

(i) *If G is exact, then $R^j(G \circ F) \simeq G \circ R^jF$.*

(ii) *Assume that F is exact. There is a natural morphism $R^j(G \circ F) \to (R^j G) \circ F$.*

(iii) *Let $\mathcal{J}'$ be a G-injective subcategory of $\mathcal{C}'$ and assume that F sends the injective objects of $\mathcal{C}$ in $\mathcal{J}'$. If $X \in \mathcal{C}$ satisfies $R^k F(X) = 0$ for $k \neq 0$, then $R^j(G \circ F)(X) \simeq R^j G(F(X))$.*

(iv) *In particular, let $\mathcal{J}'$ be a G-injective subcategory of $\mathcal{C}'$ and assume that F is exact and sends the injective objects of $\mathcal{C}$ in $\mathcal{J}'$. Then $R^j(G \circ F) \simeq R^j G \circ F$.*

Proof. Let $X \in \mathcal{C}$ and let $0 \to X \to I_X^\bullet$ be an injective resolution of X. Then $R^j(G \circ F)(X) \simeq H^j(G \circ F(I_X^\bullet))$.
(i) If G is exact, the right-hand side is isomorphic to $G(H^j(F(I_X^\bullet))$.
(ii) Consider an injective resolution $0 \to F(X) \to J^\bullet_{F(X)}$ of $F(X)$. By Proposition 4.5.5, there exists a morphism $F(I_X^\bullet) \to J^\bullet_{F(X)}$. Applying G we get a morphism of complexes: $(G \circ F)(I_X^\bullet) \to G(J^\bullet_{F(X)})$. Since $H^j((G \circ F)(I_X^\bullet)) \simeq R^j(G \circ F)(X)$ and $H^j(G(J^\bullet_{F(X)})) \simeq R^j G(F(X))$, we get the result.
(iii) By the hypothesis, $F(I_X^\bullet)$ is qis to $F(X)$ and belongs to $\mathrm{C}^+(\mathcal{J}')$. Hence $R^j G(F(X)) \simeq H^j(G(F(I_X^\bullet)))$.
(iv) is a particular case of (iii). q.e.d.

4.7 Bifunctors

Now consider an additive bifunctor $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ of abelian categories, and assume: F is left exact with respect of each of its arguments (i.e., $F(X, \bullet)$ and $F(\bullet, Y)$ are left exact).

Let $\mathcal{I}_\mathcal{C}$ (resp. $\mathcal{I}_{\mathcal{C}'}$) denote the full additive subcategory of $\mathcal{C}$ (resp. $\mathcal{C}'$) consisting of injective objects.

Definition 4.7.1. (a) The pair $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ is F-injective if $\mathcal{C}$ admits enough injective and for all $I \in \mathcal{I}_\mathcal{C}$, $F(I, \bullet)$ is exact.

(b) If $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ is F-injective, we denote by $R^k F(X, Y)$ the k-th derived functor of $F(\bullet, Y)$ at X, *i.e.,* $R^k F(X, Y) = R^k F(\bullet, Y)(X)$.

Proposition 4.7.2. *Assume that $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ is F-injective.*

(i) *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{C}$ and let $Y \in \mathcal{C}'$. Then there is a long exact sequence in $\mathcal{C}''$:*

$$\cdots \to R^{k-1}F(X'', Y) \to R^k F(X', Y) \to R^k F(X, Y) \to R^k F(X'', Y) \to$$

(ii) *Let* $0 \to Y' \to Y \to Y'' \to 0$ *be an exact sequence in* $\mathcal{C}'$ *and let* $X \in \mathcal{C}$*. Then there is a long exact sequence in* $\mathcal{C}''$*:*

$$\cdots \to R^{k-1}F(X, Y'') \to R^kF(X, Y') \to R^kF(X, Y) \to R^kF(X, Y'') \to$$

Proof. (i) is a particular case of Theorem 4.6.4.
(ii) Let $0 \to X \to I^\bullet$ be an injective resolution of X. By the hypothesis, the sequence in $\mathrm{C}(\mathcal{C}'')$:

$$0 \to F(I^\bullet, Y') \to F(I^\bullet, Y) \to F(I^\bullet, Y'') \to 0$$

is exact. By Theorem 4.2.6, it gives rise to the desired long exact sequence. q.e.d.

Proposition 4.7.3. *Assume that both* $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ *and* $(\mathcal{C}, \mathcal{I}_{\mathcal{C}'})$ *are* F*-injective. Then for* $X \in \mathcal{C}$ *and* $Y \in \mathcal{C}'$*, we have the isomorphism:* $R^kF(X, Y) := R^kF(\,\bullet\,, Y)(X) \simeq R^kF(X, \bullet\,)(Y)$*.*

Moreover if $I_X^\bullet$ *is an injective resolution of* X *and* $I_Y^\bullet$ *an injective resolution of* Y*, then* $R^kF(X, Y) \simeq \mathrm{tot}H^k(F(I_X^\bullet, I_Y^\bullet)$*.*

Proof. Let $0 \to X \to I_X^\bullet$ and $0 \to Y \to I_Y^\bullet$ be injective resolutions of X and Y, respectively. Consider the double complex:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & F(I_X^0, Y) & \longrightarrow & F(I_X^1, Y) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(X, I_Y^0) & \longrightarrow & F(I_X^0, I_Y^0) & \longrightarrow & F(I_X^1, I_Y^0) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(X, I_Y^1) & \longrightarrow & F(I_X^0, I_Y^1) & \longrightarrow & F(I_X^1, I_Y^1) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

The cohomology of the first row (resp. column) calculates $R^kF(\,\bullet\,, Y)(X)$ (resp. $R^kF(X, \bullet\,)(Y)$). Since the other rows and columns are exact by the hypotheses, the result follows from Theorem 4.2.10. q.e.d.

Example 4.7.4. Assume $\mathcal{C}$ has enough injectives. Then

$$R^k\mathrm{Hom}_\mathcal{C} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$$

exists and is calculated as follows. Let $X \in \mathcal{C}$, $Y \in \mathcal{C}$. There exists a qis in $\mathrm{C}^+(\mathcal{C})$, $Y \to I^\bullet$, the I^j's being injective. Then:

$$R^k\mathrm{Hom}_\mathcal{C}(X, Y) \simeq H^k(\mathrm{Hom}_\mathcal{C}(X, I^\bullet)).$$

If $\mathcal{C}$ has enough projectives, and $P^\bullet \to X$ is a qis in $\mathrm{C}^-(\mathcal{C})$, the P^j's being projective, one also has:

$$\begin{aligned} R^k\mathrm{Hom}_{\mathcal{C}}(X,Y) &\simeq H^k\mathrm{Hom}_{\mathcal{C}}(P^\bullet,Y) \\ &\simeq H^k\mathrm{tot}(\mathrm{Hom}_{\mathcal{C}}(P^\bullet,I^\bullet)). \end{aligned}$$

If $\mathcal{C}$ has enough injectives or enough projectives, one sets:

$$\mathrm{Ext}^k_{\mathcal{C}}(\,\bullet\,,\,\bullet\,) = R^k\mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,\,\bullet\,). \tag{4.15}$$

For example, let $A = k[x,y]$, $M = k \simeq A/xA+yA$ and let us calculate the groups $\mathrm{Ext}^j_A(M,A)$. Since injective resolutions are not easy to calculate, it is much simpler to calculate a free (hence, projective) resolution of M. Since (x,y) is a regular sequence of endomorphisms of A (viewed as an A-module), M is quasi-isomorphic to the complex:

$$M^\bullet : 0 \to A \xrightarrow{u} A^2 \xrightarrow{v} A \to 0,$$

where $u(a) = (ya,-xa)$, $v(b,c) = xb+yc$ and the module A on the right stands in degree 0. Therefore, $\mathrm{Ext}^j_A(M,N)$ is the j-th cohomology object of the complex $\mathrm{Hom}_A(M^\bullet,N)$, that is:

$$0 \to N \xrightarrow{v'} N^2 \xrightarrow{u'} N \to 0,$$

where $v' = \mathrm{Hom}\,(v,N)$, $u' = \mathrm{Hom}\,(u,N)$ and the module N on the left stands in degree 0. Since $v'(n) = (xn,yn)$ and $u'(m,l) = ym - xl$, we find again a Koszul complex. Choosing $N = A$, its cohomology is concentrated in degree 2. Hence, $\mathrm{Ext}^j_A(M,A) \simeq 0$ for $j \neq 2$ and $\simeq k$ for $j = 2$.

Example 4.7.5. Let A be a k-algebra. Since the category $\mathrm{Mod}(A)$ admits enough projective objects, the bifunctor

$$\bullet \otimes \bullet\, : \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$$

admits derived functors, denoted $\mathrm{Tor}^A_{-k}(\,\bullet\,,\,\bullet\,)$ or else, $\mathrm{Tor}^k_A(\,\bullet\,,\,\bullet\,)$.

If $Q^\bullet \to N \to 0$ is a projective resolution of the A^{op}-module N, or $P^\bullet \to M \to 0$ is a projective resolution of the A-module M, then :

$$\begin{aligned} \mathrm{Tor}^A_k(N,M) &\simeq H^{-k}(Q^\bullet \otimes_A M) \\ &\simeq H^{-k}(N \otimes_A P^\bullet) \\ &\simeq H^{-k}(\mathrm{tot}(Q^\bullet \otimes_A P^\bullet)). \end{aligned}$$

Exercises to Chapter 4

Exercise 4.1. Let $\mathcal{C}$ be an abelian category which admits inductive limits and such that filtrant inductive limits are exact. Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathcal{C}$ indexed by a set I and let $i_0 \in I$. Prove that the natural morphism $X_{i_0} \to \bigoplus_{i\in I} X_i$ is a mornomorphism.

Exercise 4.2. Let $\mathcal{C}$ be an abelian category.

(i) Prove that a complex $0 \to X \to Y \to Z$ is exact iff and only if for any object $W \in \mathcal{C}$ the complex of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(W,X) \to \mathrm{Hom}_{\mathcal{C}}(W,Y) \to \mathrm{Hom}_{\mathcal{C}}(W,Z)$ is exact.

(ii) By reversing the arrows, state and prove a similar statement for a complex $X \to Y \to Z \to 0$.

Exercise 4.3. Let $\mathcal{C}$ be an abelian category. A square is a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$

A square is Cartesian if moreover the sequence $0 \to V \to X \times Y \to Z$ is exact, that is, if $V \simeq X \times_Z Y$ (recall that $X \times_Z Y = \mathrm{Ker}(f - g)$, where $f - g : X \oplus Y \to Z$). A square is co-Cartesian if the sequence $V \to X \oplus Y \to Z \to 0$ is exact, that is, if $Z \simeq X \oplus_V Y$ (recall that $X \oplus_Z Y = \mathrm{Coker}(f' - g')$, where $f' - g' : V \to X \times Y$).

(i) Assume the square is Cartesian and f is an epimorphism. Prove that f' is an epimorphism.

(ii) Assume the square is co-Cartesian and f' is a monomorphism. Prove that f is a monomorphism.

Exercise 4.4. Let $\mathcal{C}$ be an abelian category and consider two sequences of morphisms $X_i' \xrightarrow{f_i} X_i \xrightarrow{g_i} X_i''$, $i = 1, 2$ with $g_i \circ f_i = 0$. Set $X' = X_1' \oplus X_2'$, and define similarly X, X'' and f, g. Prove that the two sequences above are exact if and only if the sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact.

Exercise 4.5. Let $\mathcal{C}$ be an abelian category and consider a commutative diagram of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X_0' \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X_0'' \arrow[d] \\
0 \arrow[r] & X_1' \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_1'' \arrow[d] \\
0 \arrow[r] & X_2' \arrow[r] & X_2 \arrow[r] & X_2''
\end{tikzcd}$$

Assume that all rows are exact as well as the second and third column. Prove that all columns are exact.

Exercise 4.6. Let $\mathcal{C}$ be an abelian category. To $X \in \mathrm{C}^b(\mathcal{C})$, one associates the new complex $H^\bullet(X) = \bigoplus H^j(X)[-j]$ with 0-differential. In other words

$$H^\bullet(X) := \qquad \cdots \to H^i(X) \xrightarrow{0} H^{i+1}(X) \xrightarrow{0} \cdots$$

(i) Prove that $H^\bullet : \mathrm{C}^b(\mathcal{C}) \to \mathrm{C}^b(\mathcal{C})$ is a well-defined additive functor.
(ii) Give examples which show that in general, $H^\bullet$ is neither right nor left exact.

Exercise 4.7. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n commuting endomorphisms of an A-module M. Let $\varphi' = (\varphi_1, \dots, \varphi_{n-p})$ and $\varphi'' = (\varphi_{n-p+1}, \dots, \varphi_n)$.
Calculate the cohomology of $K^\bullet(M, \varphi)$ assuming that φ' is a regular sequence and φ'' is a coregular sequence.

Exercise 4.8. Let $A = k[x_1, x_2]$. Consider the A-modules: $M' = A/(Ax_1 + Ax_2)$, $M = A/(Ax_1^2 + Ax_1x_2)$, $M'' = A/(Ax_1)$.
(i) Show that the monomorphism $Ax_1 \hookrightarrow A$ induces a monomorphism $M' \hookrightarrow M$and deduce an exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$.
(ii) By considering the action of x_1 on these three modules, show that the sequence above does not split.
(iii) Construct free resolutions of M' and M''.
(iv) Calculate $\mathrm{Ext}^j_A(M, A)$ for all j .

Exercise 4.9. Let $\mathcal{C}$ and $\mathcal{C}'$ be two abelian categories. We assume that $\mathcal{C}'$ admits inductive limits and filtrant inductive limits are exact in $\mathcal{C}'$. Let $\{F_i\}_{i\in I}$ be an inductive system of left exact functors from $\mathcal{C}$ to $\mathcal{C}'$, indexed by a filtrant category$I$.
(i) Prove that $\varinjlim_i F_i$ is a left exact functor.
(ii) Prove that for each $k \in \mathbb{Z}$, $\{R^kF_i\}_{i\in I}$ is an inductive system of functors and $R^k(\varinjlim_i F_i) \simeq \varinjlim_i R^kF_i$.

Exercise 4.10. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories. Let $\mathcal{J}$ be an F-injective subcategory of $\mathcal{C}$, and let $Y^\bullet$ be an object of $\mathrm{C}^+(\mathcal{J})$. Assume that $H^k(Y^\bullet) = 0$ for all $k \neq p$ for some $p \in \mathbb{Z}$, and let $X = H^p(Y^\bullet)$. Prove that $R^kF(X) \simeq H^{k+p}(F(Y^\bullet))$.

Exercise 4.11. We consider the following situation: $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ are left exact functors of abelian categories having enough injectives, $\mathcal{J}'$ is an G-injective subcategory of $\mathcal{C}'$ and F sends injective objects of $\mathcal{C}$ in $\mathcal{J}'$.
(i) Let $X \in \mathcal{C}$ and assume that there is $q \in \mathbb{N}$ with $R^kF(X) = 0$ for $k \neq q$. Prove that $R^j(G \circ F)(X) \simeq R^{j-q}G(R^qF(X))$. (Hint: use Exercise 4.10.)
(ii) Assume now that $R^jF(X) = 0$ for $j \neq 0, 1$. Prove that there is a long exact sequence:

$$\cdots \to R^{k-1}G(R^1F(X)) \to R^k(G \circ F)(X) \to R^kG(F(X)) \to \cdots$$

(Hint: construct an exact sequence $0 \to X \to X^0 \to X^1 \to 0$ with X^0 injective and X^1 F-acyclic.)

Exercise 4.12. In the situation of Proposition 4.6.10, let $X \in \mathcal{C}$ and assume that $R^jF(X) \simeq 0$ for $j < n$. Prove that $R^n(F' \circ F)(X) \simeq F'(R^nF(X))$.

Exercise 4.13. Here, we shall use the notation $H^\bullet$ introduced in Exercise 4.6. Assume that k is a field and consider the complexes in $\mathrm{Mod}(k)$:

$$\begin{aligned} X^\bullet &:= \quad X^0 \xrightarrow{f} X^1, \\ Y^\bullet &:= \quad Y^0 \xrightarrow{g} Y^1 \end{aligned}$$

and the double complex

$$X^\bullet \otimes Y^\bullet := \quad \begin{tikzcd} X^0 \otimes Y^0 \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes g"'] & X^1 \otimes Y^0 \arrow[d, "\mathrm{id}\otimes g"'] \\ X^0 \otimes Y^1 \arrow[r, "f\otimes\mathrm{id}"] & X^1 \otimes Y^1. \end{tikzcd}$$

(i) Prove that $\mathrm{tot}(X^\bullet \otimes Y^\bullet)$ and $\mathrm{tot}(H^\bullet(X^\bullet) \otimes Y^\bullet)$ have the same cohomology objects.
(ii) Deduce that $\mathrm{tot}(X^\bullet \otimes Y^\bullet)$ and $\mathrm{tot}(H^\bullet(X^\bullet) \otimes H^\bullet(Y^\bullet))$ have the same cohomology objects.

Exercise 4.14. Assume that k is a field. Let $X^\bullet$ and $Y^\bullet$ be two objects of $\mathrm{C}^b(\mathrm{Mod}(k))$. Prove the isomorphism

$$\begin{aligned} H^p(\mathrm{tot}(X^\bullet \otimes Y^\bullet)) &\simeq \bigoplus_{i+j=p} H^i(X^\bullet) \otimes H^j(Y^\bullet) \\ &\simeq H^p(\bigoplus_i H^i(X^\bullet)[-i] \otimes \bigoplus_j H^j(Y^\bullet)[-j]). \end{aligned}$$

Here, we use the convention that:

$$\begin{aligned} &(A \oplus B) \otimes (C \oplus D) \simeq (A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D) \\ &A[i] \otimes B[j] \sim A \otimes B[i+j]. \end{aligned}$$

(Hint: use the result of Exercise 4.13.)

Chapter 5

Abelian sheaves

In this chapter we expose basic sheaf theory in the framework of topological spaces. Although we restrict our study to topological spaces, it will be convenient to consider morphisms of sites $f\colon X \to Y$ which are not continuous maps from X to Y.
Recall that all along these Notes, k denotes a commutative unitary ring.
Some references: [12], [3], [10], [17], [18].

5.1 Presheaves

Let X be a topological space. The family of open subsets of X is ordered by inclusion and we denote by Op_X the associated category. Hence:

$$\mathrm{Hom}_{\mathrm{Op}_X}(U,V) = \begin{cases} \{\mathrm{pt}\} & \text{if } U \subset V, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the category Op_X admits a terminal object, namely X, and finite products, namely $U \times V = U \cap V$. Indeed,

$$\mathrm{Hom}_{\mathrm{Op}_X}(W,U) \times \mathrm{Hom}_{\mathrm{Op}_X}(W,V) \simeq \mathrm{Hom}_{\mathrm{Op}_X}(W, U \cap V).$$

Definition 5.1.1. (i) One sets $\mathrm{PSh}(X) := \mathrm{Fct}((\mathrm{Op}_X)^{\mathrm{op}}, \mathbf{Set})$ calls an object of this category a presheaf of sets. In other words, a presheaf of sets in a functor from $(\mathrm{Op}_X)^{\mathrm{op}}$ to $\mathbf{Set}$.

(ii) One denotes by $\mathrm{PSh}(k_X)$ the subcategory of $\mathrm{PSh}(X)$ consisting of functors with values in $\mathrm{Mod}(k)$ and calls an object of this category a presheaf of k-modules.

Hence, a presheaf F on X associates to each open subset $U \subset X$ a set $F(U)$, and to an open inclusion $V \subset U$, a map $\rho_{VU} : F(U) \to F(V)$, such that for each open inclusions $W \subset V \subset U$, one has:

$$\rho_{UU} = \mathrm{id}_U, \qquad \rho_{WU} = \rho_{WV} \circ \rho_{VU}.$$

A morphism of presheaves $\varphi : F \to G$ is thus the data for any open set U of a map $\varphi(U) : F(U) \to G(U)$ such that for any open inclusion $V \subset U$, the diagram below commutes:

$$\begin{tikzcd} F(U) \arrow[r, "\varphi(U)"] \arrow[d] & G(U) \arrow[d] \\ F(V) \arrow[r, "\varphi(V)"] & G(V) \end{tikzcd}$$

If F is a presheaf of k-modules, then the $F(U)$'s are k-modules and the maps ρ_{VU} are k-linear.

The category $\mathrm{PSh}(k_X)$ inherits of most of the properties of the category $\mathrm{Mod}(k)$. In particular it is abelian and admits inductive and projective limits. For example, one checks easily that if F and G are two presheaves, the presheaf $U \mapsto F(U) \oplus G(U)$ is the coproduct of F and G in $\mathrm{PSh}(k_X)$. If $\varphi : F \to G$ is a morphism of presheaves, then $(\mathrm{Ker}\,\varphi)(U) \simeq \mathrm{Ker}\,\varphi(U)$ and $(\mathrm{Coker}\,\varphi)(U) \simeq \mathrm{Coker}\,\varphi(U)$ where $\varphi(U) : F(U) \to G(U)$.

More generally, if $i \mapsto F_i, i \in I$ is an inductive system of presheaves, one checks that the presheaf $U \mapsto \varinjlim\limits_i F_i(U)$ is the inductive limit of this system in the category $\mathrm{PSh}(k_X)$ and similarly with projective limits.

Note that for $U \in \mathrm{Op}_X$, the functor $\mathrm{PSh}(k_X) \to \mathrm{Mod}(k)$, $F \mapsto F(U)$ is exact.

Notation 5.1.2. (i) One calls the morphisms ρ_{VU}, the restriction morphisms. If $s \in F(U)$, one better writes $s|_V$ instead of $\rho_{VU}s$ and calls $s|_V$ the restriction of s to V.
(ii) One denotes by $F|_U$ the presheaf on U defined by $V \mapsto F(V)$, V open in U and calls $F|_U$ the restriction of F to U.
(iii) If $s \in F(U)$, one says that s is a section of F on U, and if V is an open subset of U, one writes $s|_V$ instead of $\rho_{VU}(s)$.

Examples 5.1.3. (i) Let $M \in \mathbf{Set}$. The correspondence $U \mapsto M$ is a presheaf, called the constant presheaf on X with fiber M. For example, if $M = \mathbb{C}$, one gets the presheaf of $\mathbb{C}$-valued constant functions on X.
(ii) Let $\mathcal{C}^0(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on U. Then $U \mapsto \mathcal{C}^0(U)$ (with the usual restriction morphisms) is a presheaf of $\mathbb{C}$-vector spaces, denoted $\mathcal{C}^0_X$.

Definition 5.1.4. Let $x \in X$, and let I_x denote the full subcategory of Op_X consisting of open neighborhoods of x. For a presheaf F on X, one sets:

$$F_x = \varinjlim_{U \in I_x^{\mathrm{op}}} F(U). \tag{5.1}$$

One calls F_x the stalk of F at x.

Let $x \in U$ and let $s \in F(U)$. The image $s_x \in F_x$ of s is called the germ of s at x. Note that any $s_x \in F_x$ is represented by a section $s \in F(U)$ for some open neighborhood U of x, and for $s \in F(U), t \in F(V)$, $s_x = t_x$ means that there exists an open neighborhood W of x with $W \subset U \cap V$ such that $\rho_{WU}(s) = \rho_{WV}(t)$. (See Example 1.5.10.)

Proposition 5.1.5. *The functor $F \mapsto F_x$ from $\mathrm{PSh}(k_X)$ to $\mathrm{Mod}(k)$ is exact.*

Proof. The functor $F \mapsto F_x$ is the composition

$$\mathrm{PSh}(k_X) = \mathrm{Fct}(\mathrm{Op}_X^{\mathrm{op}}, \mathrm{Mod}(k)) \to \mathrm{Fct}(I_x^{\mathrm{op}}, \mathrm{Mod}(k)) \xrightarrow{\varinjlim} \mathrm{Mod}(k).$$

The first functor associates to a presheaf F its restriction to the category I_x^{op}. It is clearly exact. Since $U, V \in I_x$ implies $U \cap V \in I_x$, the category I_x^{op} is filtrant and it follows that the functor $\varinjlim$ is exact (see Example 4.1.10 (iii)). q.e.d.

5.2 Sheaves

Let X be a topological space and let Op_X denote the category of its open subsets. Recall that if $U, V \in \mathrm{Op}_X$, then $U \cap V$ is the product of U and V in Op_X.

We shall have to consider families $\mathcal{U} := \{U_i\}_{i \in I}$ of open subsets of U indexed by a set I. One says that $\mathcal{U}$ is an open covering of U if $\bigcup_i U_i = U$. Note that the empty family $\{U_i; i \in I\}$ with $I = \emptyset$ is an open covering of $\emptyset \in \mathrm{Op}_X$.

Let F be a presheaf on X and consider the two conditions below.

- S1 For any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any $s, t \in F(U)$ satisfying $s_i|_{U_i} = t_i|_{U_i}$ for all i, one has $s = t$.

- S2 For any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any family $\{s_i \in F(U_i), i \in I\}$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, there exists $s \in F(U)$ with $s|_{U_i} = s_i$ for all i.

Definition 5.2.1. (i) One says that F is separated if it satisfies S1. One says that F is a sheaf if it satisfies S1 and S2.

(ii) One denotes by $\mathrm{Sh}(X)$ the full subcategory of $\mathrm{PSh}(X)$ whose objects are sheaves.

(iii) One denotes by $\mathrm{Mod}(k_X)$ the full k-additive subcategory of $\mathrm{PSh}(k_X)$ whose objects are sheaves and by $\iota_X : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X)$ the forgetful functor. If there is no risk of confusion, one writes ι instead of ι_X.

(iv) One writes $\mathrm{Hom}_{k_X}(\bullet, \bullet)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(k_X)}(\bullet, \bullet)$.

Note that if F is a sheaf on X and U is open in X, then $F|_U$ is a sheaf on U.

Let $\mathcal{U}$ be an open covering of $U \in \mathrm{Op}_X$ and let $F \in \mathrm{PSh}(k_X)$. One sets

$$F(\mathcal{U}) = \mathrm{Ker}(\prod_{V \in \mathcal{U}} F(V) \rightrightarrows \prod_{V', V'' \in \mathcal{U}} F(V' \cap V'')). \tag{5.2}$$

Here the two arrows are associated with $\prod_{V \in \mathcal{U}} F(V) \to F(V') \to F(V' \cap V'')$ and $\prod_{V \in \mathcal{U}} F(V) \to F(V'') \to F(V' \cap V'')$.

In other words, a section $s \in F(\mathcal{U})$ is the data of a family of sections $\{s_V \in F(V); V \in \mathcal{U}\}$ such that for any $V', V'' \in \mathcal{U}$,

$$s_{V'}|_{V' \cap V''} = s_{V''}|_{V' \cap V''}.$$

Therefore, if F is a presheaf, there is a natural map

$$F(U) \to F(\mathcal{U}). \tag{5.3}$$

The next result is obvious.

Proposition 5.2.2. *A presheaf F is separated (resp. is a sheaf) if and only if for any $U \in \mathrm{Op}_X$ and any open covering $\mathcal{U}$ of U, the natural map $F(U) \to F(\mathcal{U})$ is injective (resp. bijective).*

One can consider $\mathcal{U}$ as a category. Assuming that $\mathcal{U}$ is stable by finite intersection, we have

$$F(\mathcal{U}) \simeq \varprojlim_{V \in \mathcal{U}} F(V). \tag{5.4}$$

Note that if F is a sheaf of sets, then $F(\emptyset) = \{\mathrm{pt}\}$. If F is a sheaf of k-modules, then $F(\emptyset) = 0$. If $\{U_i\}_{i \in I}$ is a family of disjoint open subsets, then $F(\bigsqcup_i U_i) = \prod_i F(U_i)$.

If F is a sheaf on X, then its restriction $F|_U$ to an open subset U is a sheaf.

In the sequel, we shall concentrate on sheaves of k-modules.

Notation 5.2.3. (i) Let F be a sheaf on X. One can define its support, denoted by $\operatorname{supp} F$, as the complementary of the union of all open subsets U of X such that $F|_U = 0$. Note that $F|_{X\setminus \operatorname{supp} F} = 0$.
(ii) Let $s \in F(U)$. One can define its support, denoted by $\operatorname{supp} s$, as the complementary of the union of all open subsets U of X such that $s|_U = 0$.

The next result is extremely useful. It says that to check that a morphism of sheaves is an isomorphism, it is enough to do it at each stalk.

Proposition 5.2.4. *Let $\varphi : F \to G$ be a morphism of sheaves.*

(i) *φ is a monomorphism if and only if, for all $x \in X$, $\varphi_x : F_x \to G_x$ is injective.*

(ii) *φ is an isomorphism if and only if, for all $x \in X$, $\varphi_x : F_x \to G_x$ is an isomorphism.*

Proof. (i) The condition is necessary by Proposition 5.1.5. Assume now φ_x is injective for all $x \in X$ and let us prove that $\varphi : F(U) \to G(U)$ is injective. Let $s \in F(U)$ with $\varphi(s) = 0$. Then $(\varphi(s))_x = 0 = \varphi_x(s_x)$, and φ_x being injective, we find $s_x = 0$ for all $x \in U$. This implies that there exists an open covering $U = \cup_i U_i$, with $s|_{U_i} = 0$, and by S1, $s = 0$.
(ii) The condition is clearly necessary. Assume now φ_x is an isomorphism for all $x \in X$ and let us prove that $\varphi : F(U) \to G(U)$ is surjective. Let $t \in G(U)$. There exists an open covering $U = \cup_i U_i$ and $s_i \in F(U_i)$ such that $t|_{U_i} = \varphi(s_i)$.

Then, $\varphi(s_i)|_{U_i\cap U_j} = \varphi(s_j)|_{U_i\cap U_j}$, hence by (i), $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ and by S2, there exists $s \in F(U)$ with $s|_{U_i} = s_i$. Since $\varphi(s)|_{U_i} = t|_{U_i}$, we have $\varphi(s) = t$, by S1. q.e.d.

Examples 5.2.5. (i) The presheaf $\mathcal{C}^0_X$ is a sheaf.
(ii) Let $M \in \mathrm{Mod}(k)$. The presheaf of locally constant functions on X with values in M is a sheaf, called the constant sheaf with stalk M and denoted M_X. Note that the constant presheaf with stalk M is not a sheaf except if $M = 0$.
(iii) Let X be a topological space in (a) below, a real manifold of class C^∞ in (b)–(d), a complex analytic manifold in (e)–(h). We have the classical sheaves:
(iii)–(a) k_X: k-valued locally constant functions, (iii)–(b) $\mathcal{C}^\infty_X$: complex valued functions of class $\mathcal{C}^\infty$,
(iii)–(c) $\mathcal{D}b_X$: complex valued distributions,
(iii)–(d) $\mathcal{C}^{\infty,(p)}_X$: p-forms of class $\mathcal{C}^\infty$, also denoted Ω^p_X,

(iii)–(e) $\mathcal{O}_X$: holomorphic functions,
(iii)–(f) Ω_X^p: holomorphic p-forms (hence, $\Omega_X^0 = \mathcal{O}_X$).

(iv) On a topological space X, the presheaf $U \mapsto \mathcal{C}_X^{0,b}(U)$ of continuous bounded functions is not a sheaf in general. To be bounded is not a local property and axiom S2 is not satisfied.

(v) Let $X = \mathbb{C}$, and denote by z the holomorphic coordinate. The holomorphic derivation $\frac{\partial}{\partial z}$ is a morphism from $\emptyset_X$ to $\emptyset_X$. Consider the presheaf:

$$F : U \mapsto \mathcal{O}(U)/\frac{\partial}{\partial z}\mathcal{O}(U),$$

that is, the presheaf $\mathrm{Coker}(\frac{\partial}{\partial z} : \mathcal{O}_X \to \mathcal{O}_X)$. For U an open disk, $F(U) = 0$ since the equation $\frac{\partial}{\partial z} f = g$ is always solvable. However, if $U = \mathbb{C} \setminus \{0\}$, $F(U) \neq 0$. Hence the presheaf F does not satisfy axiom S1.

(vi) If F is a sheaf on X and U is open, then $F|_U$ is sheaf on U.

5.3 Sheaf associated with a presheaf

Consider the forgetful functor

$$\iota_X : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X) \tag{5.5}$$

which, to a sheaf F associates the underlying presheaf. In this section, we shall rapidly construct a left adjoint to this functor.

When there is no risk of confusion, we shall often omit the symbol ι_X. In other words, we shall identify a sheaf and the underlying presheaf.

Theorem 5.3.1. *The forgetful functor ι_X in* (5.5) *admits a left adjoint*

$$^a : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X). \tag{5.6}$$

More precisely, one has the isomorphism, functorial with respect to $F \in \mathrm{PSh}(k_X)$ and $G \in \mathrm{Mod}(k_X)$

$$\mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(F, \iota_X G) \simeq \mathrm{Hom}_{\,k_X}(F^a, G). \tag{5.7}$$

Moreover (5.7) *defines a morphism of presheaves $\theta : F \to F^a$ and $\theta_x : F_x \to F^a_x$ is an isomorphism for all $x \in X$.*

Note that if F is locally 0, then $F^a = 0$. If F is a sheaf, then $\theta : F \to F^a$ is an isomorphism.

If F is a presheaf on X, the sheaf F^a is called the sheaf associated with F.

Proof. Define:

$$F^a(U) = \begin{array}{l}\{s : U \to \bigsqcup_{x \in U} F_x \ ; \ s(x) \in F_x \text{ such that, for all } x \in U, \\ \text{there exists } V \ni x,\ V \text{ open in } U, \text{ and there exists } t \in \\ F(V) \text{ with } t_y = s(y) \text{ for all } y \in V\}.\end{array}$$

Define $\theta : F \to F^a$ as follows. To $s \in F(U)$, one associates the section of F^a:

$$(x \mapsto s_x) \in F^a(U).$$

One checks easily that F^a is a sheaf and any morphism of presheaves $\varphi : F \to G$ with G a sheaf will factorize uniquely through θ. In particular, any morphism of presheaves $\varphi : F \to G$ extends uniquely as a morphism of sheaves $\varphi^a : F^a \to G^a$, and $F \mapsto F^a$ is functorial. q.e.d.

Example 5.3.2. Let $M \in \mathrm{Mod}(k)$. Then the sheaf associated with the constant presheaf $U \mapsto M$ is the sheaf M_X of M-valued locally constant functions.

Theorem 5.3.3. (i) *The category* $\mathrm{Mod}(k_X)$ *admits projective limits and such limits commute with the functor* ι_X *in* (5.5). *More precisely, if* $\{F_i\}_{i \in I}$ *is a projective system of sheaves, its projective limit in* $\mathrm{PSh}(k_X)$ *is a sheaf and is a projective limit in* $\mathrm{Mod}(k_X)$.

(ii) *The functor* ^{a} *in* (5.6) *commutes with kernels. More precisely, let* $\varphi : F \to G$ *be a morphism of presheaves and let* $\varphi^a : F^a \to G^a$ *denote the associated morphism of sheaves. Then*

$$(\operatorname{Ker} \varphi)^a \simeq \operatorname{Ker} \varphi^a. \tag{5.8}$$

(iii) *The category* $\mathrm{Mod}(k_X)$ *admits inductive limits. If* $\{F_i\}_{i \in I}$ *is an inductive system of sheaves, its inductive limit is the sheaf associated with its inductive limit in* $\mathrm{PSh}(k_X)$.

(iv) *The functor* ^{a} *commutes with inductive limits (in particular, with cokernels). More precisely, if* $\{F_i\}_{i \in I}$ *is an inductive system of of presheaves, then*

$$\varinjlim_i (F_i^a) \simeq (\varinjlim_i F_i)^a, \tag{5.9}$$

where $\varinjlim$ *on the left (resp. right) is the inductive limit in the category of sheaves (resp. of presheaves).*

(v) *The category* $\mathrm{Mod}(k_X)$ *is abelian and the functor* ^{a} *is exact.*

(vi) *The functor* $\iota_X : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X)$ *is fully faithful and left exact.*

(vii) *Filtrant inductive limits are exact in* $\mathrm{Mod}(k_X)$.

Let $\varphi : F \to G$ is a morphism of sheaves and let $\iota_X\varphi : \iota_X F \to \iota_X G$ denote the underlying morphism of presheaves. Then $\mathrm{Ker}\,\iota_X\varphi$ is a sheaf and coincides with $\iota_X \mathrm{Ker}\,\varphi$. On the other-hand, one shall be aware that $\mathrm{Coker}\,\iota_X\varphi$ is not necessarily a sheaf. The cokernel in the category of sheaves is the sheaf associated with this presheaf. In other words, the functor ι_X : $\mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X)$ is left exact, but not right exact in general.

Proof. (i) Let $\mathcal{U}$ be an open covering of an open subset U. Since $F \mapsto F(\mathcal{U})$ commutes with projective limits, $(\varprojlim_i F_i)(U) \xrightarrow{\sim} (\varprojlim_i F_i)(\mathcal{U})$. Hence a projective limit of sheaves in the category $\mathrm{PSh}(k_X)$ is a sheaf. One has, for $G \in \mathrm{Mod}(k_X)$:

$$\begin{aligned}
\varprojlim_i \mathrm{Hom}_{k_X}(G, F_i) &\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{PSh}(k_X)}(G, F_i) \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(k_X)}(G, \varprojlim_i F_i) \\
&\simeq \mathrm{Hom}_{k_X}(G, \varprojlim_i F_i).
\end{aligned}$$

It follows that $\varprojlim_i F_i$ is a projective limit in the category $\mathrm{Mod}(k_X)$.

(ii) The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,\varphi \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[d] \\
0 \arrow[r] & \mathrm{Ker}(\varphi^a) \arrow[r] & F^a \arrow[r] & G^a
\end{tikzcd}$$

defines the morphism $\mathrm{Ker}\,\varphi \to \mathrm{Ker}\,\varphi^a$, hence, the morphism $\psi : (\mathrm{Ker}\,\varphi)^a \to \mathrm{Ker}\,\varphi^a$. Since the functor $F \mapsto F_x$ commutes both with Ker and with ^{a}, ψ_x is an isomorphism for all x and it remains to apply Proposition 5.2.4.

(iii)–(iv) Let $G \in \mathrm{Mod}(k_X)$ and let $\{F_i\}_{i\in I}$ be an inductive system of of presheaves. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{k_X}((\varinjlim_i F_i)^a, G) &\simeq \mathrm{Hom}_{\mathrm{PSh}(k_X)}(\varinjlim_i F_i, G) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{PSh}(k_X)}(F_i, G) \\
&\simeq \varprojlim_i \mathrm{Hom}_{k_X}(F_i^a, G).
\end{aligned}$$

(v) By (i) and (iii), the category $\mathrm{Mod}(k_X)$ admits kernels and cokernels. Let $\varphi : F \to G$ be a morphism of sheaves and denote by $\iota_X\varphi$ the underlying morphism of presheaves. Using (5.9) we get that $\mathrm{Coim}\,\varphi := \mathrm{Coker}(\mathrm{Ker}\,\varphi \to F)$ is isomorphic to the sheaf associated with $\mathrm{Coim}(\iota_X\varphi)$. Using (5.8) we get that $\mathrm{Im}\,\varphi := \mathrm{Ker}(F \to (\mathrm{Coker}\,\iota_X\varphi)^a)$ is isomorphic to the sheaf associated with $\mathrm{Im}(\iota_X\varphi)$. The isomorphism of presheaves $\mathrm{Coim}(\iota_X\varphi) \xrightarrow{\sim} \mathrm{Im}(\iota_X\varphi)$ yields the isomorphism of the associated sheaves. Hence $\mathrm{Mod}(k_X)$ is abelian.

The functor $F \mapsto F^a$ is exact since it commutes with kernels by by (5.9) and with cokernels by (5.8).

(vi) The functor ι_X is fully faithful by definition. Since it admits a left adjoint, it is left exact.

(vii) Filtrant inductive limits are exact in the category $\mathrm{Mod}(k)$, whence in the category $\mathrm{PSh}(k_X)$. Then the result follows since ^{a} is exact. q.e.d.

Recall that the functor $F \mapsto F^a$ commutes with the functors of restriction $F \mapsto F|_U$, as well as with the functor $F \mapsto F_x$.

Proposition 5.3.4. (i) *Let $\varphi : F \to G$ be a morphism of sheaves and let $x \in X$. Then $(\mathrm{Ker}\,\varphi)_x \simeq \mathrm{Ker}\,\varphi_x$ and $(\mathrm{Coker}\,\varphi)_x \simeq \mathrm{Coker}\,\varphi_x$. In particular the functor $F \mapsto F_x$, from $\mathrm{Mod}(k_X)$ to $\mathrm{Mod}(k)$ is exact.*

(ii) *Let $F' \xrightarrow{\varphi} F \xrightarrow{\psi} F''$ be a complex of sheaves. Then this complex is exact if and only if for any $x \in X$, the complex $F'_x \xrightarrow{\varphi_x} F_x \xrightarrow{\psi_x} F''_x$ is exact.*

Proof. (i) The result is true in the category of presheaves. Since $\iota_X\,\mathrm{Ker}\,\varphi \simeq \mathrm{Ker}\,\iota_X\varphi$ and $\mathrm{Coker}\,\varphi \simeq (\mathrm{Coker}\,\iota_X\varphi)^a$, the result follows.

(ii) By Proposition 5.2.4, $\mathrm{Im}\,\varphi \simeq \mathrm{Ker}\,\psi$ if and only if $(\mathrm{Im}\,\varphi)_x \simeq (\mathrm{Ker}\,\psi)_x$ for all $x \in X$. Hence the result follows from (i). q.e.d.

By this statement, the complex of sheaves above is exact if and only if for each section $s \in F(U)$ defined in an open neighborhood U of x and satisfying $\psi(s) = 0$, there exists another open neighborhood V of x with $V \subset U$ and a section $t \in F'(V)$ such that $\varphi(t) = s|_V$.

On the other hand, a complex of sheaves $0 \to F' \to F \to F''$ is exact if and only if it is exact as a complex of presheaves, that is, if and only if, for any $U \in |Op_X$, the sequence $0 \to F'(U) \to F(U) \to F''(U)$ is exact.

Examples 5.3.5. Let X be a real analytic manifold of dimension n. The (augmented) de Rham complex is

$$0 \to \mathbb{C}_X \to \mathcal{C}_X^{\infty,(0)} \xrightarrow{d} \cdots \to \mathcal{C}_X^{\infty,(n)} \to 0 \tag{5.10}$$

where d is the differential. This complex of sheaves is exact. The same result holds with the sheaf $\mathcal{C}_X^\infty$ replaced with the sheaf $\mathcal{C}_X^\omega$ or the sheaf $\mathcal{D}b_X$.
(ii) Let X be a complex manifold of dimension n. The (augmented) holomorphic de Rham complex is

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n \to 0 \tag{5.11}$$

where d is the holomorphic differential. This complex of sheaves is exact.

Definition 5.3.6. Let $U \in \mathrm{Op}_X$. We denote by $\Gamma(U;\bullet) : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k)$ the functor $F \mapsto F(U)$.

Proposition 5.3.7. *The functor $\Gamma(U;\bullet)$ is left exact.*

Proof. The functor $\Gamma(U;\bullet)$ is the composition

$$\mathrm{Mod}(k_X) \xrightarrow{\iota_X} \mathrm{PSh}(k_X) \xrightarrow{\lambda_U} \mathrm{Mod}(k),$$

where λ_U is the functor $F \mapsto F(U)$. Since ι_X is left exact and λ_U is exact, the result follows. q.e.d.

The functor $\Gamma(U;\bullet)$ is not exact in general. Indeed, consider Example 5.2.5 (v). Recall that $X = \mathbb{C}$, z is a holomorphic coordinate and $U = X \setminus \{0\}$. Then the sequence of sheaves $0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{\partial_z} \mathcal{O}_X \to 0$ is exact. Applying the functor $\Gamma(U;\bullet)$, the sequence one obtains is no more exact.

5.4 Internal operations

[1]

Internal hom

Definition 5.4.1. Let $F, G \in \mathrm{PSh}(k_X)$. One denotes by $\mathcal{H}om_{\,\mathrm{PSh}(k_X)}(F,G)$ or simply $\mathcal{H}om\,(F,G)$ the presheaf on X, $U \mapsto \mathrm{Hom}_{\,\mathrm{PSh}(k_U)}(F|_U, G|_U)$ and calls it the "internal hom" of F and G.

Proposition 5.4.2. *Let $F, G \in \mathrm{Mod}(k_X)$. Then the presheaf $\mathcal{H}om\,(F,G)$ is a sheaf.*

[1]The proofs in this section may be skipped

Proof. Let $U \in \mathrm{Op}_X$ and let $\mathcal{U}$ be an open covering of U. Let us show that $\mathcal{H}om\,(F,G)(U) \simeq \mathcal{H}om\,(F,G)(\mathcal{U})$. In other words, we shall prove that the sequence below is exact, (in these formulas, we write Hom instead of Hom_{k_W}, W open, for short):

$$0 \to \mathrm{Hom}_{k_X}(F|_U, G|_U) \to \prod_{V \in \mathcal{U}} \mathrm{Hom}_{k_X}(F|_V, G|_V)$$
$$\rightrightarrows \prod_{V', V'' \in \mathcal{U}} \mathrm{Hom}_{k_X}(F|_{V' \cap V''}, G|_{V' \cap V''}).$$

(i) Let $\varphi \in \mathrm{Hom}\,(F|_U, G|_U)$ and assume that $\varphi|_V : F|_V \to G|_V$ is zero for all $V \in \mathcal{U}$. Then for $V \in \mathcal{U}$, any $W \in \mathrm{Op}_U$ and any $s \in F(W)$, $\varphi(s)|_{W \cap V} = 0$. Since $\{W \cap V; V \in \mathcal{U}\}$ is a covering of W, $\varphi(s) \in G(W)$ is zero. This implies $\varphi = 0$.
(ii) Let $\{\varphi_V\}$ belong to $\prod_{V \in \mathcal{U}} \mathrm{Hom}\,(F|_V, G|_V)$. Assume that

$$\varphi_{V'}|_{V' \cap V''} = \varphi_{V''}|_{V' \cap U V''}$$

for any $V', V'' \in \mathcal{U}$. Let $W \in \mathrm{Op}_U$. Then $\{\varphi_V\}$ defines a commutative diagram

$$F(W) \xrightarrow{a_W} \prod_{V \in \mathcal{U}} G(W \cap V) \rightrightarrows \prod_{V', V'' \in \mathcal{U}} G(W \cap V' \cap V'')$$

where a_W is given by $F(W) \ni s \mapsto \varphi_V(s|_{W \cap V})$. Since G is a sheaf, a_W factors uniquely as

$$F(W) \xrightarrow{\psi(W)} G(W) \to \prod_{V \in \mathcal{U}} G(W \cap V).$$

It is easy to see that $\psi : \mathrm{Op}_U \ni W \mapsto \psi(W) \in \mathrm{Hom}\,(F(W), G(W))$ defines an element of $\mathrm{Hom}_{k_U}(F|_U, G|_U)$. q.e.d.

The functor $\mathrm{Hom}_{k_X}(\,\bullet\,,\,\bullet\,)$ being left exact, it follows that

$$\mathcal{H}om\,(\,\bullet\,,\,\bullet\,) : \mathrm{Mod}(k_X)^{\mathrm{op}} \times \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$$

is left exact. Note that

$$\mathrm{Hom}_{k_X}(\,\bullet\,,\,\bullet\,) \simeq \Gamma(X;\,\bullet\,) \circ \mathcal{H}om\,(\,\bullet\,,\,\bullet\,).$$

Since a morphism: $\varphi : F \to G$ defines a k-linear map $F_x \to G_x$, we get a natural morphism $(\mathcal{H}om\,(F,G))_x \to \mathrm{Hom}\,(F_x, G_x)$. In general, this map is neither injective nor surjective.

Tensor product

Definition 5.4.3. Let $F, G \in \mathrm{Mod}(k_X)$.

(i) One denotes by $F \overset{\mathrm{psh}}{\otimes} G$ the presheaf on X, $U \mapsto F(U) \otimes_k G(U)$.

(ii) One denotes by $F \otimes_{k_X} G$ the sheaf associated with the presheaf $F \overset{\mathrm{psh}}{\otimes} G$ (see Definition 5.4.1) and calls it the tensor product of F and G. If there is no risk of confusion, one writes $F \otimes G$ instead of $F \otimes_{k_X} G$.

Proposition 5.4.4. *Let $F, G, K \in \mathrm{Mod}(k_X)$. There are natural isomorphisms:*

$$\begin{aligned}
&\mathrm{Hom}_{k_X}(G \otimes F, H) \simeq \mathrm{Hom}_{k_X}(G, \mathcal{H}om\,(F, H)),\\
&\mathcal{H}om\,(G \otimes F, H) \simeq \mathcal{H}om\,(G, \mathcal{H}om\,(F, H)).
\end{aligned}$$

Proof. (i) Let us define a map $\lambda\colon \mathrm{Hom}\,(G, \mathcal{H}om\,(F, H)) \to \mathrm{Hom}\,(F \overset{\mathrm{psh}}{\otimes} G, H)$. For $U \in \mathrm{Op}_X$, we have the chain of morphisms

$$\begin{aligned}
\mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(G, \mathcal{H}om\,(F, H)) &\to \mathrm{Hom}_k(G(U), \mathcal{H}om\,(F, H)(U))\\
&\to \mathrm{Hom}_k(G(U), \mathrm{Hom}_k(F(U), H(U)))\\
&\simeq \mathrm{Hom}_k(G(U) \otimes F(U), H(U)).
\end{aligned}$$

Since these morphisms are functorial with respect to U, they define λ.

Let us define a map $\mu\colon \mathrm{Hom}_{k_X}(F \overset{\mathrm{psh}}{\otimes} G, H) \to \mathrm{Hom}_{k_X}(G, \mathcal{H}om\,(F, H))$. For $V \to U$ in $\mathcal{C}_X$, we have the chain of morphisms

$$\begin{aligned}
\mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(F \overset{\mathrm{psh}}{\otimes} G, H) &\to \mathrm{Hom}_k(F(V) \overset{\mathrm{psh}}{\otimes} G(V), H(V))\\
&\simeq \mathrm{Hom}_k(G(V), \mathrm{Hom}_k(F(V), H(V)))\\
&\to \mathrm{Hom}_k(G(U), \mathrm{Hom}_k(F(V), H(V))).
\end{aligned}$$

Since these morphism are functorial with respect to $V \subset U$, they define μ.

It is easily checked that λ and μ are inverse to each other.
(ii) Applying Theorem 5.3.1 and Proposition 5.4.2, we get the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{k_X}(G \otimes F, H) &\simeq \mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(G \overset{\mathrm{psh}}{\otimes} F, H)\\
&\simeq \mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(G, \mathcal{H}om\,(F, H))\\
&\simeq \mathrm{Hom}_{k_X}(G, \mathcal{H}om\,(F, H)).
\end{aligned}$$

(iii) The second isomorphism follows from the first one. q.e.d.

Example 5.4.5. Let $\mathcal{C}_X^\infty$ denote as above the sheaf of real valued $\mathcal{C}^\infty$-functions on a real manifold X. If V is a finite $\mathbb{R}$-dimensional vector space (e.g., $V = \mathbb{C}$), then the sheaf of V-valued $\mathcal{C}^\infty$-functions is nothing but $\mathcal{C}_X^\infty \otimes_{\mathbb{R}_X} V_X$.

The functor

$$\bullet \otimes \bullet : \mathrm{Mod}(k_X) \times \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$$

is the composition of the right exact functor $\overset{\mathrm{psh}}{\otimes}$ and the exact functor ^{a}. This functor is thus right exact and if k is a field, it is exact. Note that for $x \in X$ and $U \in \mathrm{Op}_X$:

(i) $(F \otimes G)_x \simeq F_x \otimes G_x$,

(ii) $\mathcal{H}om\,(F, G)|_U \simeq \mathcal{H}om\,(F|_U, G|_U)$,

(iii) $\mathcal{H}om\,(k_X, F) \simeq F$,

(iv) $k_X \otimes F \simeq F$.

5.5 Direct and inverse images

Let $f : X \to Y$ be a continuous map. We denote by f^t the inverse image of a set by f. Hence, we set for $V \subset Y$: $f^t(V) := f^{-1}(V)$ and $f^t : \mathrm{Op}_Y \to \mathrm{Op}_X$ is a functor. Note that f^t commutes with products and coverings, that is, it satisfies

$$\begin{cases} \text{for any } U, V \in \mathrm{Op}_Y,\ f^t(U \cap V) = f^t(U) \cap f^t(V), \\ \text{for any } V \in \mathrm{Op}_Y \text{ and any covering } \mathcal{V} \text{ of } V,\ f^t(\mathcal{V}) \text{ is a covering} \\ \text{of } f^t(V). \end{cases} \tag{5.12}$$

Definition 5.5.1. Let $U \hookrightarrow X$ be an open embedding. One denotes by $j_U^t : \mathrm{Op}_U \to \mathrm{Op}_X$ the functor $\mathrm{Op}_V \ni V \mapsto V \in \mathrm{Op}_X$.

The functor j_U^t satisfies (5.12). It is a motivation to introduce the following definition which extends the notion of a continuous map.

Definition 5.5.2. A morphism of sites $f : X \to Y$ is a functor $f^t : \mathrm{Op}_Y \to \mathrm{Op}_X$ which satisfies (5.12).

One shall be aware that we do not ask that f^t commutes with finite projective limits. In particular, we do not ask $f^t(Y) = X$.

Note that if $f : X \to Y$ and $g : Y \to Z$ are morphisms of sites, then $g \circ f : X \to Z$ is also a morphism of sites.

Definition 5.5.3. Let $f : X \to Y$ be a morphism of sites and let $F \in \mathrm{PSh}(k_X)$. One defines $f_*F \in \mathrm{PSh}(k_Y)$, the direct image of F by f, by setting: $f_*F(V) = F(f^t(V))$.

Proposition 5.5.4. *Let $F \in \mathrm{Mod}(k_X)$. Then $f_*F \in \mathrm{Mod}(k_Y)$. In other words, the direct image of a sheaf is a sheaf.*

Proof. Let $V \in \mathrm{Op}_Y$ and let $\mathcal{V}$ be an open covering of V. Then $f^t(\mathcal{V})$ is an open covering of $f^t(V)$ and we get

$$f_*F(V) \simeq F(f^t(V)) \simeq F(f^t(\mathcal{V})) \simeq f_*F(\mathcal{V}).$$

q.e.d.

Definition 5.5.5. Let $f : X \to Y$ be a morphism of sites.

(i) Let $G \in \mathrm{PSh}(k_Y)$. One defines $f^\dagger G \in \mathrm{PSh}(k_X)$ by setting for $U \in \mathrm{Op}_X$:

$$f^\dagger G(U) = \varinjlim_{U \subset f^t(V)} G(V).$$

(ii) Let $G \in \mathrm{Mod}(k_Y)$. One defines $f^{-1}G \in \mathrm{Mod}(k_X)$, the inverse image of G by f, by setting

$$f^{-1}G = (f^\dagger G)^a.$$

Note that if f is a continuous map, one has for $x \in X$:

$$(f^{-1}G)_x \simeq (f^\dagger G)_x \simeq G_{f(x)}. \tag{5.13}$$

Example 5.5.6. Let $U \hookrightarrow X$ be an open embedding and let $F \in \mathrm{Mod}(k_X)$. Then $i_U^\dagger F \simeq j_{U*}F$ is already a sheaf. Hence:

$$i_U^\dagger = i_U^{-1} \simeq j_{U*}. \tag{5.14}$$

Theorem 5.5.7. *Let $f : X \to Y$ be a morphism of sites.*

(i) *The functor $f^{-1}\colon \mathrm{Mod}(k_Y) \to \mathrm{Mod}(k_X)$ is left adjoint to the functor $f_*\colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_Y)$. In other words, we have for $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_Y)$:*

$$\mathrm{Hom}_{k_X}(f^{-1}G, F) \simeq \mathrm{Hom}_{k_Y}(G, f_*F).$$

(ii) *The functor f_* is left exact and commutes with projective limits.*

(iii) *The functor* f^{-1} *is exact and commutes with inductive limits.*

(iv) *There are natural morphisms of functors* $\mathrm{id} \to f_* f^{-1}$ *and* $f^{-1} f_* \to \mathrm{id}$.

Proof. (i) First we shall prove that the functor $f^\dagger \colon \mathrm{PSh}(k_Y) \to \mathrm{PSh}(k_X)$ is left adjoint to $f_* \colon \mathrm{PSh}(k_X) \to \mathrm{PSh}(k_Y)$. In other words, we have an isomorphism, functorial with respect to $F \in \mathrm{PSh}(k_X)$ and $G \in \mathrm{PSh}(k_Y)$:

$$\mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(f^\dagger G, F) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(k_Y)}(G, f_* F). \tag{5.15}$$

A section $\varphi \in \mathrm{Hom}_{\,\mathrm{PSh}(k_Y)}(G, f_* F)$ is a family of maps

$$\{\varphi_V : G(V) \to F(f^t(V))\}_{V \in \mathrm{Op}_Y}$$

compatible with the restriction morphisms. Equivalently, this is a family of maps

$$\{\varphi_U : G(V) \to F(U); U \subset f^t(V)\}_{V \in \mathrm{Op}_Y, U \in \mathrm{Op}_X}$$

compatible with the restriction morphisms. Hence it gives a family of maps

$$\{\psi_V : \varinjlim_{U \subset f^t(V)} G(V) \to F(U)\}_{U \in \mathrm{Op}_X}$$

which defines $\psi \in \mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(f^\dagger G, F)$. Clearly, the correspondence $\varphi \mapsto \psi$ is an isomorphism.

Using the isomorphism (5.15) we get the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{k_Y}(G, f_* F) &\simeq \mathrm{Hom}_{\,\mathrm{PSh}(k_Y)}(G, f_* F) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(f^\dagger G, F) \\ &\simeq \mathrm{Hom}_{k_X}((f^\dagger G)^a, F) = \mathrm{Hom}_{k_X}(f^{-1} G, F). \end{aligned}$$

(ii) With the exception of the fact that f^{-1} is left exact, the other assertions follow by the adjunction property. If f is a continuous map, f^{-1} is left exact by (5.13). Let us give a proof in the general case. Let

$$0 \to G' \to G \to G'' \tag{5.16}$$

be an exact sequence of sheaves and let $U \in \mathrm{Op}_X$, $V \in \mathrm{Op}_Y$ with $U \subset f^t(V)$. The sequence $0 \to G'(V) \to G(V) \to G''(V)$ is exact.

Consider the category $(\mathrm{Op}_Y)^U := \{V \in \mathrm{Op}_Y, U \subset f^t(V)\}$. The sequence (5.16) defines an exact sequence of functors from $((\mathrm{Op}_Y)^U)^{\mathrm{op}}$ to $\mathrm{Mod}(k)$. The category $((\mathrm{Op}_Y)^U)^{\mathrm{op}}$ is either filtrant or empty. Indeed, if $U \subset f^t(V_1)$ and $U \subset f^t(V_2)$, then $U \subset f^t(V_1 \cap V_2)$. It follows that the sequence

$$0 \to \varinjlim_{U \subset f^t(V)} G'(V) \to \varinjlim_{U \subset f^t(V)} G(V) \to \varinjlim_{U \subset f^t(V)} G''(V)$$

is exact. Hence the functor $G \mapsto f^\dagger \iota_Y G$ from $\mathrm{Mod}(k_Y)$ to $\mathrm{PSh}(k_X)$ is left exact. Since the functor ^{a} is exact, the result follows. q.e.d.

Consider morphisms of sites $f : X \to Y$, $g : Y \to Z$ and $g \circ f : X \to Z$.

Proposition 5.5.8. *One has natural isomorphisms of functors*

$$\begin{aligned} g_* \circ f_* &\simeq (g \circ f)_*, \\ f^{-1} \circ g^{-1} &\simeq (g \circ f)^{-1}. \end{aligned}$$

Proof. The functoriality of direct images is clear by its definition. The functoriality of inverse images follows by adjunction. q.e.d.

Note that inverse image commutes with the functor ^{a} (see Exercise 5.9) and inverse image commutes with tensor product (see Exercise 5.10 (ii)).

Let $V \in \mathrm{Op}_Y$, set $U = f^t(V)$ and denote by f_V the restriction of f to V:

(5.17)
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "i_U"] \arrow[r, "f_V"] & V. \arrow[u, "i_V"'] \end{tikzcd}$$

Proposition 5.5.9. *Let $F \in \mathrm{Mod}(k_X)$. Then $i_V^{-1} f_* F \simeq f_{V*} i_U^{-1} F$.*

Proof. Consider the morphisms of presites:

(5.18)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_U"'] & Y \arrow[d, "j_V"] \\ U \arrow[r, "f_V"] & V. \end{tikzcd}$$

By Proposition 5.5.8, one has $j_{V*} f_* F \simeq f_{V*} j_{U*} F$. q.e.d.

We denote by a_X the canonical map $a_X = X \to \{\mathrm{pt}\}$. (Recall that $\{\mathrm{pt}\}$ is the set with one element.) There is a natural equivalence of categories

$$\begin{aligned} \mathrm{Mod}(k_{\mathrm{pt}}) &\xrightarrow{\sim} \mathrm{Mod}(k), \\ F &\mapsto \Gamma(\mathrm{pt}; F). \end{aligned}$$

In the sequel, we shall identify these two categories.

Examples 5.5.10. (i) Let $F \in \mathrm{Mod}(k_X)$. Then:

$$\Gamma(X; F) \simeq a_{X*} F.$$

(ii) Let $M \in \mathrm{Mod}(k)$. Recall that M_X denotes the sheaf associated with the presheaf $U \mapsto M$. Hence:

$$M_X \simeq a_X^{-1} M_{\{\mathrm{pt}\}}.$$

Let $f\colon X \to Y$ be a continuous map. Since $a_X = a_Y \circ f$, we get

$$M_X \simeq f^{-1}M_Y.$$

(iii) Let $x \in X$ and denote by $i_x : \{x\} \hookrightarrow X$ the embedding. Then

$$i_x^{-1}F \simeq F_x.$$

(iv) Let $i_U : U \hookrightarrow X$ be the inclusion of an open subset of X and let F be a sheaf on X. Then $\Gamma(V; i_U^{-1}F) \simeq \Gamma(V; F)$ for $V \in \mathrm{Op}_U$.
(v) Let $X = Y \bigsqcup Y$, the disjoint union of two copies of Y. Let $f : X \to Y$ be the natural map which induces the identity one each copy of Y. Then $f_*f^{-1}G \simeq G \oplus G$. In fact, if V is open in Y, then we have the isomorphisms

$$\begin{aligned} \Gamma(V; f_*f^{-1}G) &\simeq \Gamma(V \sqcup V; f^{-1}G) \\ &\simeq \Gamma(V; G) \oplus \Gamma(V; G). \end{aligned}$$

Example 5.5.11. Let $f : X \to Y$ be a morphism of complex manifolds. To each open subset $V \subset Y$ is associated a natural "pull-back" map:

$$\Gamma(V; \mathcal{O}_Y) \to \Gamma(V; f_*\mathcal{O}_X)$$

defined by:

$$\varphi \mapsto \varphi \circ f$$

We obtain a morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$, hence a morphism:

$$f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

For example, if X is closed in Y and f is the injection, $f^{-1}\mathcal{O}_Y$ will be the sheaf on X of holomorphic functions on Y defined in a neighborhood of X. If f is smooth (locally on X, f is isomorphic to a projection $Y \times Z \to Y$), then $f^{-1}\mathcal{O}_Y$ will be the sub-sheaf of $\mathcal{O}_X$ consisting of functions locally constant in the fibers of f.

Examples 5.5.12. (i) Let $i_S : S \hookrightarrow X$ be the embedding of a closed subset S of X. Then the functor i_{S*} is exact.
(ii) Let $i_U : U \hookrightarrow X$ be the embedding of an open subset U of X. Let $F \in \mathrm{Mod}(k_X)$ and let $x \in X$. Then

$$(i_{U*}i_U^{-1}F)_x \simeq \varinjlim_{V \ni x} \Gamma(U \cap V; F).$$

Notation 5.5.13. On $X \times Y$, we denote by q_1 and q_2 the first and second projection, respectively. If $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_Y)$ we set:

$$F \boxtimes G = q_1^{-1}F \otimes q_2^{-1}G.$$

One can recover the functor $\otimes$ from $\boxtimes$. Denote by $\delta : X \hookrightarrow X \times X$ the diagonal embedding and let F_1 and F_2 be in $\mathrm{Mod}(k_X)$. We have:

$$\delta^{-1}(F_1 \boxtimes F_2) = \delta^{-1}(q_1^{-1}F_1 \otimes q_2^{-1}F_2) \simeq F_1 \otimes F_2.$$

5.6 Sheaves associated with a locally closed subset

Let Z be a subset of X. One denotes by

$$i_Z : Z \hookrightarrow X \tag{5.19}$$

the inclusion morphism. One endows Z with the induced topology, and for $F \in \mathrm{Mod}(k_X)$, one sets:

$$\begin{aligned} F|_Z &= i_Z^{-1}F, \\ \Gamma(Z;F) &= \Gamma(Z;F|_Z). \end{aligned}$$

If $Z := U$ is open, these definitions agree with the previous ones. The morphism $F \to i_{Z*}i_Z^{-1}F$ defines the morphism $a_{X*}F \to a_{X*}i_{Z*}i_Z^{-1}F \simeq a_{Z*}i_Z^{-1}F$, hence the morphism:

$$\Gamma(X;F) \to \Gamma(Z;F). \tag{5.20}$$

One denotes by $s|_Z$ the image of a section s of F on X by this morphism.

Replacing X with an open set U containing Z in (5.20), we get the morphism $\Gamma(U;F) \to \Gamma(Z;F)$. Denote by I_Z the category of open subsets containing Z (the morphisms are the inclusions). Then $(I_Z)^{\mathrm{op}}$ is filtrant, F defines a functor $(I_Z)^{\mathrm{op}} \to \mathrm{Mod}(k)$ and we get a morphism

$$\varinjlim_{U \supset Z} \Gamma(U;F) \to \Gamma(Z;F). \tag{5.21}$$

The morphism (5.21) is injective. Indeed, if a section $s \in \Gamma(U;F)$ is zero in $\Gamma(Z;F)$, then $s_x = 0$ for all $x \in Z$, hence $s = 0$ on an open neighborhood of Z. One shall be aware that the morphism (5.21) is not an isomorphism in general. There is a classical result which asserts that if X is paracompact (*e.g.,* if X is locally compact and countable at infinity) and Z is closed, then (5.21) is bijective.

A subset Z of a topological space X is relatively Hausdorff if two distinct points in Z admit disjoint neighborhoods in X. If $Z = X$, one says that X is Hausdorff.

Proposition 5.6.1. *Let Z be compact subset of X, relatively Hausdorff in X and let $F \in \mathrm{Mod}(k_X)$. Then the natural morphism* (5.21) *is an isomorphism.*

Proof. [2] Let $s \in \Gamma(Z;F|_Z)$. There exist a finite family of open subsets $\{U_i\}_{i=1}^n$ covering Z and sections $s_i \in \Gamma(U_i;F)$ such that $s_i|_{Z\cap U_i} = s|_{Z\cap U_i}$.

[2] The proof may be skipped

Moreover, we may find another family of open sets $\{V_i\}_{i=1}^n$ covering Z such that $Z \cap \bar{V}_i \subset U_i$. We shall glue together the sections s_i on a neighborhood of Z. For that purpose we may argue by induction on n and assume $n = 2$. Set $Z_i = Z \cap \bar{V}_i$. Then $s_1|_{Z_1 \cap Z_2} = s_2|_{Z_1 \cap Z_2}$. Let W be an open neighborhood of $Z_1 \cap Z_2$ such that $s_1|_W = s_2|_W$ and let $W_i (i = 1, 2)$ be an open subset of U_i such that $W_i \supset Z_i \setminus W$ and $W_1 \cap W_2 = \emptyset$. Such W_i's exist thanks to the hypotheses. Set $U'_i = W_i \cup W$, $(i = 1, 2)$. Then $s_1|_{U'_1 \cap U'_2} = s_2|_{U'_1 \cap U'_2}$. This defines $t \in \Gamma(U'_1 \cup U'_2; F)$ with $t|_Z = s$. q.e.d.

The case of open subsets

Let U be an open subset. Recall that we have the morphisms of sites

$$\begin{aligned} j_U \colon X \to U, &\qquad U \supset V \mapsto V \subset X, \\ i_U \colon U \to X, &\qquad X \supset V \mapsto U \cap V \subset U. \end{aligned}$$

Hence, we have the pairs of adjoint functors (i_U^{-1}, i_{U*}) and (j_U^{-1}, j_{U*}). Clearly, there is an isomorphism of functors $j_{U*} \simeq i_U^{-1} \colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_U)$.

Definition 5.6.2. (i) One defines the functor $i_{U!} \colon \mathrm{Mod}(k_U) \to \mathrm{Mod}(k_X)$ by setting $i_{U!} := j_U^{-1}$.

(ii) For $F \in \mathrm{Mod}(k_X)$, one sets $F_U := i_{U!} i_U^{-1} F = j_U^{-1} j_{U*} F$.

(iii) For $F \in \mathrm{Mod}(k_X)$, one sets $\Gamma_U F := i_{U*} i_U^{-1} F = i_{U*} j_{U*} F$.

(iv) One sets $k_{XU} := (k_X)_U$ for short.

Hence, we have the functors

$$\begin{tikzcd} \mathrm{Mod}(k_X) \arrow[r, "i_U^{-1}" description] & \mathrm{Mod}(k_U), \arrow[l, shift right=2, "i_{U!}"'] \arrow[l, shift left=2, "i_{U*}"] \end{tikzcd}$$

and the pairs of adjoint functors

$$(i_U^{-1}, i_{U*}) \qquad (i_{U!}, i_U^{-1}).$$

Note that $i_{U!} i_U^{-1} F \to F$ defines the morphism $F_U \to F$ and $F \to i_{U*} i_U^{-1} F$ defines the morphism $F \to \Gamma_U F$.

Moreover

$$\begin{cases} (F_U)_x \simeq & F_x \text{ if } x \in U, \\ & 0 \text{ otherwise.} \end{cases} \tag{5.22}$$

It follows that the functors $i_{U!} \colon \mathrm{Mod}(k_U) \to \mathrm{Mod}(k_X)$ and $(\bullet)_U \colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$ are exact, and the functor $\Gamma_U(\bullet) \colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$ is left exact.

Proposition 5.6.3. *Let $U \subset X$ be an open subset and let $F \in \mathrm{Mod}(k_X)$.*

(i) *We have $F_U \simeq F \otimes k_{XU}$.*

(ii) *We have $\Gamma(U; F) \simeq \mathrm{Hom}_{k_X}(k_{XU}, F)$.*

(iii) *Let V be another open subset. Then $(F_U)_V = F_{U\cap V}$.*

(iv) *Let U_1 and U_2 be two open subsets of X. Then the sequence below is exact:*

$$0 \to F_{U_1\cap U_2} \xrightarrow{\alpha} F_{U_1} \oplus F_{U_2} \xrightarrow{\beta} F_{U_1\cup U_2} \to 0. \tag{5.23}$$

Here $\alpha = (\alpha_1, \alpha_2)$ and $\beta = \beta_1 - \beta_2$ are induced by the natural morphisms $\alpha_i : F_{U_1\cap U_2} \to F_{U_i}$ and $\beta_i : F_{U_i} \to F_{U_1\cup U_2}$.

Proof. The proofs of (i), (iii), (iv) are obvious, using (5.22).
(ii) We have the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{k_X}(k_{XU}, F) &= \mathrm{Hom}_{k_X}(j_U^{-1} j_{U*} k_X, F) \\ &\simeq \mathrm{Hom}_{k_U}(j_{U*} k_X, j_{U*} F) \\ &\simeq \mathrm{Hom}_{k_U}(k_U, F|_U) \simeq F(U). \end{aligned}$$

q.e.d.

The case of closed subsets

Definition 5.6.4. Let S be a closed subset of X.

(i) For $F \in \mathrm{Mod}(k_X)$, one sets $F_S = i_{S*} i_S^{-1} F$.

(ii) One sets $k_{XS} := (k_X)_S$ for short.

Note that $F \to i_{S*} i_S^{-1} F$ defines the morphism $F \to F_S$. Moreover

$$(F_S)_x \simeq \begin{cases} F_x \text{ if } x \in S, \\ 0 \text{ otherwise.} \end{cases} \tag{5.24}$$

Proposition 5.6.5. *Let $S \subset X$ be a closed subset and let $F \in \mathrm{Mod}(k_X)$.*

(i) *Set $U := X \setminus S$. Then the sequence $0 \to F_U \to F \to F_S \to 0$ is exact in $\mathrm{Mod}(k_X)$.*

(ii) *The functor $(\,\bullet\,)_S \colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$, $F \mapsto F_S$ is exact.*

(iii) *We have* $F_S \simeq F \otimes k_{XS}$.

(iv) *Let* S' *be another closed subset. Then* $(F_S)_{S'} = F_{S \cap S'}$.

(v) *Let* S_1 *and* S_2 *be two closed subsets of* X. *Then the sequence below is exact:*

$$0 \to F_{S_1 \cup S_2} \xrightarrow{\alpha} F_{S_1} \oplus F_{S_2} \xrightarrow{\beta} F_{S_1 \cap S_2} \to 0. \tag{5.25}$$

Here $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = \beta_1 - \beta_2$ *are induced by the natural morphisms* $\alpha_i : F_{S_1 \cup S_2} \to F_{S_i}$ *and* $\beta_i : F_{S_i} \to F_{S_1 \cap S_2}$.

Proof. The proof is obvious, using (5.24). q.e.d.

The case of locally closed subsets

A subset Z of X is locally closed if there exists an open neighborhood U of Z such that Z is closed in U. Equivalently, $Z = S \cap U$ with U open and S closed in X. In this case, one sets

$$F_Z := (F_U)_S.$$

One checks easily that this definition depends only on Z, not on the choice of U and S. Moreover, (5.24) still holds with Z instead of S.

5.7 Locally constant and locally free sheaves

Locally constant sheaves

Definition 5.7.1. (i) Let M be a k-module. Recall that the sheaf M_X is the sheaf of locally constant M-valued functions on X. It is also the sheaf associated with the constant presheaf $U \mapsto M$.

(ii) A sheaf F on X is constant if it is isomorphic to a sheaf M_X, for some $M \in \mathrm{Mod}(k)$.

(iii) A sheaf F on X is locally constant if there exists an open covering $X = \bigcup_i U_i$ such that $F|_{U_i}$ is a constant sheaf of U_i.

Recall that a morphism of sheaves which is locally an isomorphism is an isomorphism of sheaves. However, given two sheaves F and G, it may exist an open covering $\{U_i\}_{i \in I}$ of X and isomorphisms $F|_{U_i} \xrightarrow{\sim} G|_{U_i}$ for all $i \in I$, although these isomorphisms are not induced by a globally defined isomorphism $F \to G$.

Example 5.7.2. Consider $X = \mathbb{R}$ and consider the $\mathbb{C}$-valued function $t \mapsto \exp(t)$, that we simply denote by $\exp(t)$. Consider the sheaf $\mathbb{C}_X \cdot \exp(t)$ consisting of functions which are locally a constant multiple of $\exp(t)$. Clearly $\mathbb{C}_X \cdot \exp(t)$ is isomorphic to the constant sheaf $\mathbb{C}_X$, hence, is a constant sheaf.

Examples 5.7.3. (i) If X is not connected it is easy to construct locally constant sheaves which are not constant. Indeed, let $X = U_1 \sqcup U_2$ be a covering by two non-empty open subsets, with $U_1 \cap U_2 = \emptyset$. Let $M \in \mathrm{Mod}(k)$ with $M \neq 0$. Then the sheaf which is 0 on U_1 and M_{U_2} on U_2 is locally constant and not constant.
(ii) Let $X = Y = \mathbb{C} \setminus \{0\}$, and let $f : X \to Y$ be the map $z \mapsto z^2$, where z denotes a holomorphic coordinate on $\mathbb{C}$. If D is an open disk in Y, $f^{-1}D$ is isomorphic to the disjoint union of two copies of D. Hence, the sheaf $f_*k_X|_D$ is isomorphic to k_D^2, the constant sheaf of rank two on D. However, $\Gamma(Y; f_*k_X) = \Gamma(X; k_X) = k$, which shows that the sheaf f_*k_X is locally constant but not constant.
(iii) Let $X = \mathbb{C} \setminus \{0\}$ with holomorphic coordinate z and consider the differential operator $P = z\frac{\partial}{\partial z} - \alpha$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Let us denote by K_α the kernel of P acting on $\mathcal{O}_X$.

Let U be an open disk in X centered at z_0, and let $A(z)$ denote a primitive of α/z in U. We have a commutative diagram of sheaves on U:

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "z\frac{\partial}{\partial z} - \alpha"] \arrow[d, "\exp(-A(z))"'] & \mathcal{O}_X \arrow[d, "\frac{1}{z}\exp(-A(z))"] \\
\mathcal{O}_X \arrow[r, "\frac{\partial}{\partial z}"] & \mathcal{O}_X
\end{tikzcd}$$

Therefore, one gets an isomorphism of sheaves $K_\alpha|_U \xrightarrow{\sim} \mathbb{C}_X|_U$, which shows that K_α is locally constant, of rank one.

On the other hand, $f \in \mathcal{O}(X)$ and $Pf = 0$ implies $f = 0$. Hence $\Gamma(X; K_\alpha) = 0$, and K_α is a locally constant sheaf of rank one on $\mathbb{C} \setminus \{0\}$ which is not constant.

Let us show that all locally constant sheaf on the interval $[0,1]$ are constant.

Recall that for $M \in \mathrm{Mod}(k)$, M_X is the constant sheaf with stalk M.

Lemma 5.7.4. *Let $M, n \in \mathrm{Mod}(k)$. Then*

(i) $(M \otimes N)_X \simeq M_X \otimes N_X$,

(ii) $(\mathrm{Hom}\,(M, N))_X \simeq \mathcal{H}om_{k_X}(M_X, N_X)$.

The proof is left as an exercise.

Lemma 5.7.5. *let $X = U_1 \cup U_2$ be a covering of X by two open sets. Let F be a sheaf on X and assume that:*

(i) *$U_{12} = U_1 \cap U_2$ is connected and non empty,*

(ii) *$F|_{U_i}$ $(i = 1, 2)$ is a constant sheaf.*

Then F is a constant sheaf.

Proof. By the hypothesis, there is $M_i \in \mathrm{Mod}(k)$ and isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} (M_i)_X|_{U_i}$ $(i = 1, 2)$. Since $U_1 \cap U_2$ is non empty and connected, $M_1 \simeq M_2$ and we may assume $M_1 = M_2 = M$. define the isomorphism $\theta_{12} = \theta_1 \circ \theta_2^{-1} \colon M_X|_{U_1 \cap U_2} \xrightarrow{\sim} M_X|_{U_1 \cap U_2}$. Since $U_1 \cap U_2$ is connected and non empty, $\Gamma(U_1 \cap U_2; \mathcal{H}om\,(M_X, M_X)) \simeq \mathrm{Hom}\,(M, M)$ by Lemma 5.7.4. Hence, θ_{12} defines an invertible element of $\mathrm{Hom}\,(M, M)$. Using the map $\mathrm{Hom}\,(M, M) \to \Gamma(X; \mathcal{H}om\,(M_X, M_X))$, we find that θ_{12} extends as an isomorphism $\theta : M_X \simeq M_X$ all over X. Now define the isomorphisms: $\alpha_i : F|_{U_i} \xrightarrow{\sim} (M_X)|_{U_i}$ by $\alpha_1 = \theta_1$ and $\alpha_2 = \theta|_{U_2} \circ \theta_2$. Then α_1 and α_2 will glue together to define an isomorphism $F \xrightarrow{\sim} M_X$. q.e.d.

Proposition 5.7.6. *Let I denote the interval* $[0, 1]$.

(i) *Let F be a locally constant sheaf on I. Then F is a constant sheaf.*

(ii) *In particular, if $t \in I$, the morphism $\Gamma(I; F) \to F_t$ is an isomorphism.*

(iii) *Moreover, if $F = M_I$ for a k-module M, then the composition*

$$M \simeq F_0 \xleftarrow{\sim} \Gamma(I; M_I) \xrightarrow{\sim} F_1 \simeq M$$

is the identity of M.

Proof. (i) We may find a finite open covering $U_i, (i = 1, \ldots, n)$ such that F is constant on U_i, $U_i \cap U_{i+1}$ $(1 \leq i < n)$ is non empty and connected and $U_i \cap U_j = \emptyset$ for $|i - j| > 1$. By induction, we may assume that $n = 2$. Then the result follows from Lemma 5.7.5.
(ii)–(iii) are obvious. q.e.d.

Locally free sheaves

A sheaf of k-algebras (or, equivalently, a k_X-algebra) $\mathcal{A}$ on X is a sheaf of k-modules such that for each $U \subset X$, $\mathcal{A}(U)$ is endowed with a structure of a k-algebra, and the operations (addition, multiplication) commute to the restriction morphisms. A sheaf of $\mathbb{Z}$-algebras is simply called a sheaf of rings. If $\mathcal{A}$ is a sheaf of rings, one defines in an obvious way the notion of a sheaf F of (left) $\mathcal{A}$-modules (or simply, an $\mathcal{A}$-module) as follows: for each open set $U \subset X$, $F(U)$ is an $\mathcal{A}(U)$-module and the action of $\mathcal{A}(U)$ on $F(U)$ commutes to the restriction morphisms. One also naturally defines the notion of an $\mathcal{A}$-linear morphism of $\mathcal{A}$-modules. Hence we have defined the category $\mathrm{Mod}(\mathcal{A})$ of $\mathcal{A}$-modules.

Examples 5.7.7. (i) Let A be a k-algebra. The constant sheaf A_X is a sheaf of k-algebras.
(ii) On a topological space, the sheaf $\mathcal{C}^0_X$ is a $\mathbb{C}_X$-algebra. If X is open in $\mathbb{R}^n$, the sheaf $\mathcal{C}^\infty_X$ is a $\mathbb{C}_X$-algebra. The sheaf $\mathcal{D}b_X$ is a $\mathcal{C}^\infty_X$-module.
(iii) If X is open in $\mathbb{C}^n$, the sheaf $\mathcal{O}_X$ is a $\mathbb{C}_X$-algebra.

The category $\mathrm{Mod}(\mathcal{A})$ is clearly an additive subcategory of $\mathrm{Mod}(k_X)$. Moreover, if $\varphi : F \to G$ is a morphism of $\mathcal{A}$-modules, then $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ will be $\mathcal{A}$-modules. One checks easily that the category $\mathrm{Mod}(\mathcal{A})$ is abelian, and the natural functor $\mathrm{Mod}(\mathcal{A}) \to \mathrm{Mod}(k_X)$ is exact and faithful (but not fully faithful). Moreover, the category $\mathrm{Mod}(\mathcal{A})$ admits inductive and projective limits and filtrant inductive limits are exact. Now consider a sheaf of rings $\mathcal{A}$.

Definition 5.7.8. (i) A sheaf $\mathcal{L}$ of $\mathcal{A}$-modules is locally free of rank k (resp. of finite rank) if there exists an open covering $X = \cup_i U_i$ such that $\mathcal{L}|_{U_i}$ is isomorphic to a direct sum of k copies (resp. to a finite direct sum) of $\mathcal{A}|_{U_i}$.

(ii) A locally free sheaf of rank one is called an invertible sheaf.

We shall construct locally constant and locally free sheaves by gluing sheaves in the §5.8.

5.8 Gluing sheaves

Let X be a topological space, and let $X = \bigcup_{i\in I} U_i$ be an open covering of X. One sets $U_{ij} = U_i \cap U_j, U_{ijk} = U_{ij} \cap U_k$. First, consider a sheaf F on X, set $F_i = F|_{U_i}$, $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$, $\theta_{ji} = \theta_j \circ \theta_i^{-1}$. Then clearly:

$$\left.\begin{aligned} \theta_{ii} &= \text{id on } U_i, \\ \theta_{ij} \circ \theta_{jk} &= \theta_{ik} \text{ on } U_{ijk}. \end{aligned}\right\} \tag{5.26}$$

The family of isomorphisms $\{\theta_{ij}\}$ satisfying conditions (5.26) is called a 1-cocycle. Let us show that one can reconstruct F from the data of a 1-cocycle.

Theorem 5.8.1. *Let $X = \bigcup_{i\in I} U_i$ be an open covering of X and let F_i be a sheaf on U_i. Assume to be given for each pair (i, j) an isomorphism of sheaves $\theta_{ji} : F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$, these isomorphisms satisfying the conditions* (5.26).

Then there exists a sheaf F on X and for each i isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that $\theta_j = \theta_{ji} \circ \theta_i$. Moreover, $(F, \{\theta_i\}_{i\in I})$ is unique up to unique isomorphism.

Clearly, if the F_i's are locally constant, then F is locally constant.

Sketch of proof. (i) Existence. For each open subset V of X, define $F(V)$ as the submodule of $\prod_{i\in I} F_i(V \cap U_i)$ consisting of families $\{s_i\}_i$ such that for any $(i, j) \in I \times I$, $\theta_{ji}(s_i|_{V\cap U_{ji}}) = s_j|_{V\cap U_{ji}}$. One checks that the presheaf so obtained is a sheaf, and the isomorphisms θ_i's are induced by the projections $\prod_{k\in I} F_k(V \cap U_k) \to F_i(V \cap U_i)$.
(ii) Unicity. Let $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$ and $\lambda_i : G|_{U_i} \xrightarrow{\sim} F_i$. Then the isomorphisms $\lambda_i^{-1} \circ \theta_i : F|_{U_i} \to G|_{U_i}$ will glue as an isomorphism $G \xrightarrow{\sim} F$ on X, by Proposition 5.4.2. q.e.d.

Example 5.8.2. Assume k is a field, and recall that $k^\times$ denote the multiplicative group $k \setminus \{0\}$. Let $X = \mathbb{S}^1$ be the 1-sphere, and consider a covering of X by two open connected intervals U_1 and U_2. Let $U_{12}^{\pm}$ denote the two connected components of $U_1 \cap U_2$. Let $\alpha \in k^\times$. One defines a locally constant sheaf L_α on X of rank one over k by gluing k_{U_1} and k_{U_2} as follows. Let $\theta_\varepsilon : k_{U_1}|_{U_{12}^{\varepsilon}} \to k_{U_2}|_{U_{12}^{\varepsilon}}$ ($\varepsilon = \pm$) be defined by $\theta_+ = 1$, $\theta_- = \alpha$.

Assume that $k = \mathbb{C}$. One can give a more intuitive description of the sheaf L_α as follows. Let us identify $\mathbb{S}^1$ with $[0, 2\pi]/\sim$, where $\sim$ is the relation which identifies 0 and 2π. Choose $\beta \in \mathbb{C}$ with $\exp(i\beta) = \alpha$. If $\beta \notin \mathbb{Z}$, the function $\theta \mapsto \exp(i\beta\theta)$ is not well defined on $\mathbb{S}^1$ since it does not take the same value at 0 and at 2π. However, the sheaf $\mathbb{C}_X \cdot \exp(i\beta\theta)$ of functions which are a constant multiple of the function $\exp(i\beta\theta)$ is well-defined on each of the intervals U_1 and U_2, hence is well defined on $\mathbb{S}^1$, although it does not have any global section.

Example 5.8.3. Consider an n-dimensional real manifold X of class $\mathcal{C}^\infty$, and let $\{X_i, f_i\}$ be an atlas, that is, the X_i are open subsets of X and $f_i : X_i \xrightarrow{\sim} U_i$ is a $\mathcal{C}^\infty$-isomorphism with an open subset U_i of $\mathbb{R}^n$. Let $U^i_{ij} = f_i(X_{ij})$ and denote by f_{ji} the map

$$f_{ji} = f_j|_{X_{ij}} \circ f_i^{-1}|_{U^i_{ij}} : U^i_{ij} \to U^j_{ij}. \tag{5.27}$$

The maps f_{ji} are called the transition functions. They are isomorphisms of class $\mathcal{C}^\infty$. Denote by J_f the Jacobian matrix of a map $f : \mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$. Using the formula $J_{g\circ f}(x) = J_g(f(x)) \circ J_f(x)$, one gets that the locally constant function on X_{ij} defined as the sign of the Jacobian determinant $\det J_{f_{ji}}$ of the f_{ji}'s is a 1-cocycle. It defines a sheaf locally isomorphic to $\mathbb{Z}_X$ called the orientation sheaf on X and denoted by or_X.

Remark 5.8.4. In the situation of Theorem 5.8.1, if $\mathcal{A}$ is a sheaf of k-algebras on X and if all F_i's are sheaves of $\mathcal{A}|_{U_i}$ modules and the isomorphisms θ_{ji} are $\mathcal{A}|_{U_{ij}}$-linear, the sheaf F constructed in Theorem 5.8.1 will be naturally endowed with a structure of a sheaf of $\mathcal{A}$-modules.

Example 5.8.5. (i) Let $X = \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. Then $\Omega_X := \Omega^1_X$ is locally free of rank one over $\mathcal{O}_X$. Since $\Gamma(X; \Omega_X) = 0$, this sheaf is not globally free.
(ii) Consider the covering of X by the two open sets $U_1 = \mathbb{C}$, $U_2 = X \setminus \{0\}$. One can glue $\mathcal{O}_X|_{U_1}$ and $\mathcal{O}_X|_{U_2}$ on $U_1 \cap U_2$ by using the isomorphism $f \mapsto z^p f$ ($p \in \mathbb{Z}$). One gets a locally free sheaf of rank one. For $p \neq 0$ this sheaf is not free.

Exercises to Chapter 5

Exercise 5.1. Let S (resp. U) be a closed (resp. an open) subset of X and let $F \in \mathrm{Mod}(k_X)$.
(i) Prove the isomorphism $\Gamma(X; F_S) \simeq \Gamma(S; F|_S)$.
(ii) Construct the morphism $\Gamma(X; F_U) \to \Gamma(U; F)$ and prove that it is not an isomorphism in general.

Exercise 5.2. Assume that $X = \mathbb{R}$, let S be a non-empty closed interval and let $U = X \setminus S$.
(i) Prove that the natural map $\Gamma(X; k_X) \to \Gamma(X; k_{XS})$ is surjective and deduce that $\Gamma(X; k_{XU}) \simeq 0$.
(ii) Let $x \in \mathbb{R}$.Prove that the morphism $k_X \to k_{X\{x\}}$ does not split.

Exercise 5.3. Let $F \in \mathrm{Mod}(k_X)$. Define $\widetilde{F} \in \mathrm{Mod}(k_X)$ by $\widetilde{F} = \bigoplus_{x \in X} F_{\{x\}}$. (Here, $F_{\{x\}} \in \mathrm{Mod}(k_X)$ and the direct sum is calculated in $\mathrm{Mod}(k_X)$, not in $\mathrm{PSh}(k_X)$.) Prove that F_x and $\widetilde{F}_x$ are isomorphic for all $x \in X$, although F and $\widetilde{F}$ are not isomorphic in general.

Exercise 5.4. Let $Z = Z_1 \sqcup Z_2$ be the disjoint union of two sets Z_1 and Z_2 in X.
(i) Assume that Z_1 and Z_2 are both open (resp. closed) in X. Prove that $k_{XZ} \simeq k_{XZ_1} \oplus k_{XZ_2}$.
(ii) Give an example which shows that (i) is no more true if one only assume that Z_1 and Z_2 are both locally closed.

Exercise 5.5. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $S = \{(x, y) \in X; xy \geq 1\}$, and let $f : X \to Y$ be the map $(x, y) \mapsto y$. calculate $f_* k_{XS}$.

Exercise 5.6. Let $f : X \to Y$ be a continuous map, and let Z be a closed subset of Y. Construct the natural isomorphism $f^{-1} k_{YZ} \xrightarrow{\sim} k_{X(f^{-1}Z)}$.

Exercise 5.7. Assume that X is a compact space and let $\{F_i\}_{i \in I}$ be a filtrant inductive system of sheaves on X. Prove the isomorphism $\varinjlim_i \Gamma(X; F_i) \xrightarrow{\sim} \Gamma(X; \varinjlim_i F_i)$.

Exercise 5.8. Let S be a set endowed with the discrete topology, let $p : X \times S \to X$ denote the projection and let $F \in \mathrm{Mod}(k_{X \times S})$ and set $F_s = F|_{X \times \{s\}}$. Prove that $p_* F \simeq \prod_{s \in S} F_s$.

Exercise 5.9. Let $f : X \to Y$ be a continuous map and let $G \in \mathrm{PSh}(k_Y)$. Prove the isomorphism $(f^\dagger G)^a \xrightarrow{\sim} f^{-1}(G^a)$.

Exercise 5.10. (i) Let $f : X \to Y$ be a morphism of sites, let $F \in \mathrm{Mod}(k_X)$ and let $G \in \mathrm{Mod}(k_Y)$. Prove that there is a natural isomorphism in $\mathrm{Mod}(k_Y)$

$$\mathcal{H}om_{k_Y}(G, f_* F) \xrightarrow{\sim} f_* \mathcal{H}om_{k_X}(f^{-1}G, F).$$

(ii) Let $G_1, G_2 \in \mathrm{Mod}(k_Y)$. Prove that there is a natural isomorphism in $\mathrm{Mod}(k_X)$

$$f^{-1}(G_1 \otimes_{k_Y} G_2) \xrightarrow{\sim} f^{-1}G_1 \otimes_{k_X} f^{-1}G_2.$$

Exercise 5.11. Let $X = \bigcup_i U_i$ be an open covering of X and let $F \in \mathrm{PSh}(k_X)$. Assume that $F|_{U_i}$ is a sheaf for all $i \in I$. Prove that F is a sheaf. (Hint: compare F with the sheaf constructed in Theorem 5.8.1.)

Exercise 5.12. Let M be a k-module and let X be an open subset of $\mathbb{R}^n$. Let F be a presheaf such that for any non empty convex open subsets $U \subset X$, there exists an isomorphism $F(U) \simeq M$ and this isomorphism is compatible to the restriction morphisms for $V \subset U$. Prove that the associated sheaf is locally constant.

Exercise 5.13. Prove Lemma 5.7.4.

Exercise 5.14. Assume k is a field, and let L be a locally constant sheaf of rank one over k (hence, L is locally isomorphic to the sheaf k_X). Set $L^* = \mathcal{H}om\,(L, k_X)$.
(i) Prove the isomorphisms $L^* \otimes L \xrightarrow{\sim} k_X$ and $k_X \xrightarrow{\sim} \mathcal{H}om\,(L, L)$.
(ii) Assume that k is a field, X is connected and $\Gamma(X; L) \neq 0$. Prove that $L \simeq k_X$. (Hint: $\Gamma(X; L) \simeq \Gamma(X; \mathcal{H}om\,(k_X, L))$.)

Chapter 6

Cohomology of sheaves

We first show that the category of abelian sheaves has enough injective objects. This allows us to derive all left exact functors we have constructed.

Then, using the tools on simplicial complexes of §3.3, we construct resolutions of sheaves using open or closed Čech coverings.

Next we prove an important theorem which asserts that the cohomology of constant sheaves is a homotopy invariant. Finally, we apply this result to calculate the cohomology of some classical manifolds.
Some references: [12], [3], [10], [17], [18].

6.1 Cohomology of sheaves

A sheaf F of k-modules is injective if it is an injective object in the category $\mathrm{Mod}(k_X)$.

Lemma 6.1.1. (i) *Let X and Y be two topological spaces and let $f : X \to Y$ be a morphism of sites. Assume that $F \in \mathrm{Mod}(k_X)$ is injective. Then f_*F is injective in $\mathrm{Mod}(k_Y)$.*

(ii) *Let $i_U : U \hookrightarrow X$ be an open embedding and let $F \in \mathrm{Mod}(k_X)$ be injective. Then $i_U^{-1}F$ is injective in $\mathrm{Mod}(k_U)$.*

Proof. (i) follows immediately from the adjunction formula:

$$\mathrm{Hom}_{k_X}(f^{-1}(\cdot), F) \simeq \mathrm{Hom}_{k_Y}(\cdot, f_*F)$$

and the fact that the functor f^{-1} is exact.
(ii) follows from (i) since $i_U^{-1} \simeq j_{U*}$. q.e.d.

Theorem 6.1.2. *The category $\mathrm{Mod}(k_X)$ admits enough injectives.*

Proof. (i) When $X = \{\mathrm{pt}\}$, the result follows from $\mathrm{Mod}(k_X) \simeq \mathrm{Mod}(k)$.
(ii) Assume X is discrete. Then for $F, G \in \mathrm{Mod}(k_X)$, the natural morphism

$$\mathrm{Hom}_{k_X}(G, F) \to \prod_{x \in X} \mathrm{Hom}_k(G_x, F_x)$$

is an isomorphism. Since products are exact in $\mathrm{Mod}(k)$, it follows that a sheaf F is injective as soon as each F_x is injective. For each $x \in X$, choose an injective module I_x together with a monomorphism $F_x \rightarrowtail I_x$ and define the sheaf F^0 on X by setting $\Gamma(U, F^0) = \prod_{x \in U} I_x$. Since the topology on X is discrete, $(F^0)_x = I_x$. Therefore the sequence $0 \to F \to F^0$ is exact and F^0 is injective.
(iii) Let $\hat{X}$ denote the set X endowed with the discrete topology, and let $f : \hat{X} \to X$ be the identity map. Let $F \in \mathrm{Mod}(k_X)$. There exists an injective sheaf G^0 on $\hat{X}$ and a monomorphism $0 \to f^{-1}F \to G^0$. Then f_*G^0 is injective in $\mathrm{Mod}(k_X)$ and the sequence $0 \to f_*f^{-1}F \to f_*G^0$ is exact. To conclude, notice that the morphism $F \to f_*f^{-1}F$ is a monomorphism, since on an open subset U of X it is defined by $F(U) \to \prod_{x \in U} F_x$. q.e.d.

It is now possible to derive all left exact functors defined on the category of sheaves, as well as the bifunctors Hom_{k_X} and $\mathcal{H}om_{k_X}$. The derived functors of these two bifunctors are respectively denoted by $\mathrm{Ext}^j_{k_X}$ and $\mathcal{E}xt^j_{k_X}$. Recall that, for $G, F \in \mathrm{Mod}(k_X)$, the k-module $\mathrm{Ext}^j_{k_X}(G, F)$ is calculated as follows. Choose an injective resolution $F^\bullet$ of F. Then

$$\mathrm{Ext}^j_{k_X}(G, F) \simeq H^j(\mathrm{Hom}_{k_X}(G, F^\bullet)). \tag{6.1}$$

Let $F \in \mathrm{Mod}(k_X)$ and let U (resp. S, resp. Z) be an open (resp. a closed, resp. a locally closed) subset of X.

As usual, one denotes by $i_Z : Z \hookrightarrow X$ the embedding of Z in X and by a_Z the map $Z \to \{\mathrm{pt}\}$. Note that $a_Z = a_X \circ i_Z$. Recall that for a sheaf F on X, we have set: $\Gamma(Z; F) = \Gamma(Z; F|_Z)$. One sets

$$H^j(U; F) \;=\; R^j\Gamma(U; \cdot)(F). \tag{6.2}$$

By (6.1), we have

$$H^j(U; F) \simeq \mathrm{Ext}^j_{k_X}(k_{XU}, F). \tag{6.3}$$

Proposition 6.1.3. (i) *If U is open in X, then $H^j(U; F) \simeq H^j(U; F|_U)$.*

(ii) *If S is closed in X, then $H^j(X; F_S) \simeq H^j(S; F|_S)$.*

(iii) *If K is compact and relatively Hausdorff in X, then the natural morphism* $\varinjlim_{U \supset K} H^j(U; F) \to H^j(K; F)$ *is an isomorphism.*

Proof. (i) We have the chain of isomorphisms:

$$\begin{aligned} H^j(U; F) &\simeq R^j(a_{U*} i_U^{-1})F \simeq (R^j a_{U*}) i_U^{-1} F \\ &\simeq H^j(U; F|_U). \end{aligned}$$

The second isomorphism follows from the fact that i_U^{-1} is exact and sends injective sheaves to injective sheaves (Lemma 4.6.10 (iv)).
(ii) We have the chain of isomorphisms:

$$\begin{aligned} H^j(X; F_S) &\simeq (R^j a_{X*})(i_{S*} i_S^{-1})F \simeq R^j(a_X \circ i_{S*}) i_S^{-1} F \\ &\simeq H^j(S; F|_S). \end{aligned}$$

The second isomorphism follows from the fact that i_{S*} is exact and sends injective sheaves to injective sheaves (Lemma 4.6.10 (iv)).
(iii) The result is true for $j = 0$ by Proposition 5.6.1. Consider an injective resolution $F \to F^\bullet$ of F. Then

$$\begin{aligned} H^j(K; F) &\simeq H^j(\Gamma(K; F^\bullet)) \simeq H^j(\varinjlim_{U \supset K} \Gamma(U; F^\bullet)) \\ &\simeq \varinjlim_{U \supset K} H^j(\Gamma(U; F^\bullet)) \simeq \varinjlim_{U \supset K} H^j(U; F). \end{aligned}$$

q.e.d.

Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. Applying a left exact functor Ψ to it, we obtain a long exact sequence

$$0 \to \Psi(F') \to \Psi(F) \to \cdots \to R^j\Psi(F) \to R^j\Psi(F'') \to R^{j+1}\Psi(F') \to \cdots$$

For example, applying the functor $\Gamma(X; \cdot)$, we get a long exact sequence

$$H^{k-1}(X; F'') \to H^k(X; F') \to H^k(X; F) \to H^k(X; F'') \to \cdots \tag{6.4}$$

There are similar results, replacing $\Gamma(X; \cdot)$ with other functors, such as f_*.

Proposition 6.1.4. *Let S_1 and S_2 be two closed subsets of X and set $S_{12} = S_1 \cap S_2$. Let $F \in \mathrm{Mod}(k_X)$. There is a long exact sequence*

$$\begin{aligned} \cdots \to H^j(X; F_{S_1 \cup S_2}) \to H^j(X; F_{S_1}) \oplus H^j(X; F_{S_2}) &\to H^j(X; F_{S_{12}}) \\ &\to H^{j+1}(X; F_{S_1 \cup S_2}) \to \cdots \end{aligned}$$

Proof. Apply the functor $\Gamma(X;\cdot)$ to the exact sequence of sheaves $0 \to F_{S_1\cup S_2} \to F_{S_1} \oplus F_{S_2} \to F_{S_{12}} \to 0$. q.e.d.

Proposition 6.1.5. *Let U_1 and U_2 be two open subsets of X and set $U_{12} = U_1 \cap U_2$. Let $F \in \mathrm{Mod}(k_X)$. There is a long exact sequence*

$$\cdots \to H^j(U_1 \cup U_2; F) \to H^j(U_1; F) \oplus H^j(U_2; F) \to H^j(U_{12}; F) \\ \to H^{j+1}(U_1 \cup U_2; F) \to \cdots$$

Proof. Apply the functor $\mathrm{Hom}_{k_X}(\,\bullet\,, F)$ to the exact sequence of sheaves $0 \to k_{XU_{12}} \to k_{XU_1} \oplus k_{XU_2} \to k_{U_1\cup U_2} \to 0$ and use (6.3). q.e.d.

6.2 Čech complexes for closed coverings

Let I be a finite totally ordered set. In this section, we shall follow the same notations as for Koszul complexes. For $J \subset I$, we denote by $|J|$ its cardinal and for $J = \{i_0 < \cdots < i_p\}$, we set

$$e_{i_0} \wedge \cdots \wedge e_{i_p} \in \bigwedge^{p+1} \mathbb{Z}^{|I|}.$$

Let $\mathcal{S} = \{S_i\}_{i\in I}$ be a family indexed by I of closed subsets of X and let $F \in \mathrm{Mod}(k_X)$. For $J \subset I$ we set

$$S_J := \cap_{j\in J} S_j, \quad S = \bigcup_{i\in I} S_i$$
$$F^p_{\mathcal{S}} := \bigoplus_{|J|=p+1} F_{S_J} \otimes e_J, \quad F^{-1}_{\mathcal{S}} = F_S.$$

For $J \subset I_{\mathrm{ord}}$ and $a \in I, a \notin J$, we denote by $J_a = J \cup \{a\}$ the ordered subset of the ordered set I and we denote by $r_{J,a}\colon F_{S_J} \to F_{S_{Ja}}$ the natural restriction morphism.

We set

$$\delta_{J,a} := r_{J,a}(\,\bullet\,) \otimes e_a \wedge \bullet : F_{S_J} \otimes e_J \to F_{S_{Ja}} \otimes e_{Ja}. \tag{6.5}$$

The morphisms $\delta_{J,a}$ $(J \subset I,\ a \notin J)$ in (6.5) define the morphisms

$$d^p : F^p_{\mathcal{S}} \to F^{p+1}_{\mathcal{S}}.$$

Clearly, $d^{p+1} \circ d^p = 0$ and we obtain a complex

$$F^\bullet_{\mathcal{S}} := 0 \to F_S \xrightarrow{d^{-1}} F^0_{\mathcal{S}} \xrightarrow{d^0} F^1_{\mathcal{S}} \xrightarrow{d^1} \cdots . \tag{6.6}$$

Proposition 6.2.1. *Consider a family* $\mathcal{S} = \{S_i\}_{i\in I}$ *of closed subsets of* X *indexed by a finite totally ordered set* I. *Then the complex* (6.6) *is exact.*

Proof. It is enough to check that the stalk of the complex (6.6) at each $x \in X$ is exact. Hence, we may assume that x belongs to all S_i's. Set $M = F_x$. Then the complex (6.6) is a Koszul complex $K^\bullet(M, \varphi)$ where $\varphi = \{\varphi_i\}_{i\in I}$ and all φ_i are id_M. The sequence φ being both regular and coregular, this complex is exact. q.e.d.

Example 6.2.2. Assume that $X = S_0 \cup S_1 \cup S_2$, where the S_i's are closed subsets. We get the exact complex of sheaves

$$0 \to F \xrightarrow{d^{-1}} F_{S_0} \oplus F_{S_1} \oplus F_{S_2} \xrightarrow{d^0} F_{S_{12}} \oplus F_{S_{02}} \oplus F_{S_{01}} \xrightarrow{d^1} F_{S_{012}} \to 0.$$

Let us denote by

$$s_i \colon F \to F_{S_i}, \quad s_{ij}^a \colon F_{S_a} \to F_{S_{ij}}, \quad s_{\widehat{k}} \colon F_{S_{ij}} \to F_{S_{012}}\, (a, i, j, k) \in \{0, 1, 2\}),$$

the natural morphisms. Then

$$d^{-1} = \begin{pmatrix} s_0, \\ s_1, \\ s_2 \end{pmatrix}, \quad d^0 = \begin{pmatrix} 0 & -s_{12}^1 & s_{12}^2 \\ s_{02}^0 & 0 & -s_{02}^2 \\ -s_{01}^0 & s_{01}^1 & 0 \end{pmatrix} \quad d^1 = (s_{\widehat{2}}, -s_{\widehat{0}}, s_{\widehat{1}}).$$

6.3 Invariance by homotopy

In this section, we shall prove that the cohomology of locally constant sheaves is an homotopy invariant. First, we define what it means.

In the sequel, we denote by I the closed interval $I = [0, 1]$.

Definition 6.3.1. Let X and Y be two topological spaces.

(i) Let f_0 and f_1 be two continuous maps from X to Y. One says that f_0 and f_1 are homotopic if there exists a continuous map $h : I \times X \to Y$ such that $h(0, \cdot) = f_0$ and $h(1, \cdot) = f_1$.

(ii) Let $f : X \to Y$ be a continuous map. One says that f is a homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. In such a case one says that X and Y are homotopic.

(iii) One says that a topological space X is contractible if X is homotopic to a point $\{x_0\}$.

If $f_0, f_1 : X \rightrightarrows Y$ are homotopic, one gets the diagram

$$\begin{tikzcd} X \simeq \{t\} \times X \arrow[r, hook, "i_t"] & I \times X \arrow[r, "h"] \arrow[d, "p"] & Y \\ & X & \end{tikzcd} \tag{6.7}$$

where $t \in I$, $i_t : X \simeq \{t\} \times X \to I \times X$ is the embedding, p is the projection and $f_t = h \circ i_t$, $t = 0, 1$.

One checks easily that the relation "f_0 is homotopic to f_1" is an equivalence relation.

A topological space is contractible if and only if there exist $g : \{x_0\} \to X$ and $f : X \to \{x_0\}$ such that $f \circ g$ is homotopic to id_X. Replacing x_0 with $g(x_0)$, this means that there exists $h : I \times X \to X$ such that $h(0, x) = \mathrm{id}_X$ and $h(1, x)$ is the map $x \mapsto x_0$. Note that contractible implies non empty.

Examples 6.3.2. (i) Let V be a real vector space. A non empty convex set in V as well as a closed non empty cone are contractible sets.
(ii) Let $X = \mathbb{S}^{n-1}$ be the unit sphere of the Euclidian space $\mathbb{R}^n$ and let $Y = \mathbb{R}^n \setminus \{0\}$. The embedding $f \colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy equivalence. Indeed, denote by $g \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ the map $x \mapsto x/||x||$. Then $g \circ f = \mathrm{id}_X$ and $f \circ g$ is homotopic to id_Y. The homotopy is given by the map $h(x, t) = (t/||x|| + 1 - t)x$.

Statement of the main theorem

Let $f : X \to Y$ be a continuous map and let $G \in \mathrm{Mod}(k_Y)$. Remark that $a_X \simeq a_Y \circ f$. The morphism of functors $\mathrm{id} \to f_* \circ f^{-1}$ defines the morphism $R^j a_{Y*} \to R^j(a_{Y*} \circ f_* \circ f^{-1}) \simeq R^j(a_{X*} \circ f^{-1})$. Using Theorem 4.6.10, we get the morphism $R^j(a_{X*} \circ f^{-1}) \to R^j(a_{X*}) \circ f^{-1}$, from which we deduce the natural morphisms:

$$f^{\sharp j} : H^j(Y; G) \to H^j(X; f^{-1}G). \tag{6.8}$$

If $g : Y \to Z$ is another morphism, we have:

$$g^{\sharp j} \circ f^{\sharp j} = (f \circ g)^{\sharp j} \tag{6.9}$$

The aim of this section is to prove:

Theorem 6.3.3 (Invariance by homotopy Theorem). [1] *Let $f_0, f_1 : X \rightrightarrows Y$ be two homotopic maps, and let G be a locally constant sheaf on Y.*

[1]The proof of this theorem may be skipped

Consider the two morphisms $f_t^{\sharp j} : H^j(Y;G) \to H^j(X; f_t^{-1}G)$, *for* $t = 0, 1$. *Then there exists an isomorphism* $\theta^j \colon H^j(X; f_0^{-1}G) \to H^j(X; f_1^{-1}G)$ *such that* $\theta^j \circ f_0^{\sharp j} = f_1^{\sharp j}$.

If $G = M_Y$ *for some* $M \in \mathrm{Mod}(k)$, *then, identifying* $f_t^{-1}M_Y$ *with* M_X $(t = 0, 1)$, *we have* $f_1^{\sharp j} = f_0^{\sharp j}$.

Proof of the main theorem

In order to prove Theorem 6.3.3, we need several preliminary results.

For $F \in \mathrm{Mod}(k_I)$ and a closed interval $[0, t] \subset I$, we write $H^j([0, t]; F)$ instead of $H^j([0, t]; F|_{[0,t]})$ for short.

Lemma 6.3.4. *Let* $F \in \mathrm{Mod}(k_I)$. *Then:*

(i) *For* $j > 1$, *one has* $H^j(I; F) = 0$.

(ii) *If* $F(I) \to F_t$ *is an epimorphism for all* $t \in I$, *then* $H^1(I; F) = 0$.

Proof. Let $j \geq 1$ and let $s \in H^j(I; F)$. For $0 \leq t_1 \leq t_2 \leq 1$, consider the morphism:

$$f_{t_1,t_2} : H^j(I; F) \to H^j([t_1, t_2]; F)$$

and let

$$J = \{t \in [0, 1];\ f_{0,t}(s) = 0\}.$$

Since $H^j(\{0\}; F) = 0$ for $j \geq 1$, we have $0 \in J$. Since $f_{0,t}(s) = 0$ implies $f_{0,t'}(s) = 0$ for $0 \leq t' \leq t$, J is an interval. Since $H^j([0, t_0]; F) = \varinjlim_{t>t_0} H^j([0, t]; F)$, this interval is open. It remains to prove that J is closed. For $0 \leq t \leq t_0$, consider the Mayer-Vietoris sequence (Proposition 6.1.4):

$$\cdots \to H^j([0, t_0]; F) \to H^j([0, t]; F) \oplus H^j([t, t_0]; F) \to H^j(\{t\}; F) \to \cdots$$

For $j > 1$, or else for $j = 1$ assuming $H^0(I; F) \to H^0(\{t\}; F)$ is surjective, we obtain:

$$H^j([0, t_0]; F) \simeq H^j([0, t]; F) \oplus H^j([t, t_0]; F). \tag{6.10}$$

Let $t_0 = \sup \{t; t \in J\}$. Then $f_{0,t}(s) = 0$, for all $t < t_0$. On the other hand,

$$\varinjlim_{t<t_0} H^j([t, t_0]; F) = 0.$$

Hence, there exists $t < t_0$ with $f_{t,t_0}(s) = 0$. By (6.10), this implies $f_{0,t_0}(s) = 0$. Hence $t_0 \in J$. q.e.d.

Recall that the maps $p\colon I\times X\to X$ and $i_t\colon X\to I\times X$ are defined in (6.7). We also introduce the notation $I_x := I\times\{x\}$.

Lemma 6.3.5. *Let $G\in\mathrm{Mod}(k_{I\times X})$. Then $(R^jp_*G)_x\simeq H^j(I_x;G|_{I_x})$.*

Proof. Let $G^\bullet$ be an injective resolution of G. We have the isomorphisms

$$\begin{aligned}(R^jp_*G)_x &\simeq (H^j(p_*G^\bullet))_x\simeq H^j((p_*G^\bullet)_x)\\ &\simeq H^j(\varinjlim_{x\in U}\Gamma(I\times U;G^\bullet))\\ &\simeq \varinjlim_{x\in U}H^j(I\times U;G)\simeq H^j(I_x;G|_{I_x}).\end{aligned}$$

where the last isomorphism follows from Proposition 6.1.3 (iii). q.e.d.

Lemma 6.3.6. *Let $G\in\mathrm{Mod}(k_{I\times X})$ be a locally constant sheaf. Then the natural morphism $p^{-1}p_*G\to G$ is an isomorphism.*

Proof. One has

$$\begin{aligned}(p^{-1}p_*G)_{(t,x)} &\simeq (p_*G)_x\\ &\simeq \Gamma(I_x;G|_{I_x})\simeq G_{(t,x)}.\end{aligned}$$

Here the last isomorphism follows from Proposition 5.7.6. q.e.d.

Lemma 6.3.7. *Let $F\in\mathrm{Mod}(k_X)$. Then $F\xrightarrow{\sim}p_*p^{-1}F$ and $(R^jp_*)p^{-1}F=0$ for $j\geq 1$.*

Proof. Let $x\in X$ and let $t\in I$. Using Lemma 6.3.5 one gets the isomorphism $((R^jp_*)p^{-1}F)_x\simeq H^j(I_x;p^{-1}F|_{I_x})$. Then this group is 0 for $j>0$ by Lemma 6.3.4 and is isomorphic to $(p^{-1}F)_{t,x}\simeq F_x$ for $j=0$. q.e.d.

Lemma 6.3.8. *Let $F\in\mathrm{Mod}(k_X)$.*

(i) *The morphisms $p^{\sharp j}: H^j(X;F)\to H^j(I\times X;p^{-1}F)$ are isomorphisms.*

(ii) *The morphisms $i_t^{\sharp j}: H^j(I\times X;p^{-1}F)\to H^j(X;F)$ are isomorphisms and do not depend on $t\in I$.*

Proof. (i) We know that $R^jp_*(p^{-1}F)=0$ for $j\geq 1$. Applying Proposition 4.6.10 (iii) to the functors a_{X*},p_* and the object $p^{-1}F$, we get that $R^j(a_{X*}p_*)(p^{-1}F)\simeq R^ja_{X*}(p_*p^{-1}F)\simeq R^ja_{X*}F$. Hence $p^{\sharp j}$ is an isomorphism.
(ii) By (6.9), $i_t^{\sharp j}\circ p^{\sharp j}$ is the identity, and $p^{\sharp j}$ is an isomorphism by (i). Hence, $i_t^{\sharp j}$ which is the inverse of $p^{\sharp j}$ does not depend on t. q.e.d.

End of the proof of Theorem 6.3.3. (i) Since $h^{-1}G$ is locally constant, the morphism $p^{-1}p_*h^{-1}G \to h^{-1}G$ is an isomorphism by Lemma 6.3.6. By Lemma 6.3.8 (ii),

$$i_t^{\sharp j}\colon H^j(I \times X; p^{-1}p_*h^{-1}G) \to H^j(X; i_t^{-1}p^{-1}p_*h^{-1}G)$$

is an isomorphism. Therefore

$$i_t^{\sharp j}\colon H^j(I \times X; h^{-1}G) \to H^j(X; i_t^{-1}h^{-1}G)$$

is an isomorphism. By (6.9), $f_t^{\sharp j} = i_t^{\sharp j} \circ h^{\sharp j}$. Set $\theta = i_0^{\sharp j} i_1^{\sharp j^{-1}}$. Then

$$f_0^{\sharp j} = i_0^{\sharp j} \circ h^{\sharp j} = i_0^{\sharp j} i_1^{\sharp j^{-1}} i_1^{\sharp j} \circ h^{\sharp j} = \theta \circ f_0^{\sharp j}.$$

(ii) If $G = M_Y$, then $h^{-1}G \simeq M_{I\times X} = p^{-1}M_X$ and $i_t^{\sharp j}$ does not depend on t by Lemma 6.3.8. q.e.d.

Applications of Theorem 6.3.3

Corollary 6.3.9. *Assume $f : X \to Y$ is a homotopy equivalence and let G be a locally constant sheaf on Y. Then $H^j(X, f^{-1}G) \simeq H^j(Y; G)$.*

In other words, the cohomology of locally constant sheaves on topological spaces is a homotopy invariant.

Proof. Let $g : Y \to X$ be a map such that $f \circ g$ and $g \circ f$ are homotopic to the identity of Y and X, respectively. Consider $f^{\sharp j} : H^j(Y; G) \to H^j(X; f^{-1}G)$ and $g^{\sharp j} : H^j(X; f^{-1}G) \to H^j(Y; G)$. Then: $(f \circ g)^{\sharp j} = g^{\sharp j} \circ f^{\sharp j} \simeq \mathrm{id}_X^{\sharp j} = \mathrm{id}$ and $(g \circ f)^{\sharp j} = f^{\sharp j} \circ g^{\sharp j} \simeq \mathrm{id}_Y^{\sharp j} = \mathrm{id}$. q.e.d.

Corollary 6.3.10. *If X is contractible and $M \in \mathrm{Mod}(k)$, then $\Gamma(X; M_X) \simeq M$ and $H^j(X; M_X) \simeq 0$ for $j > 0$.*

We shall apply this result together with the technique of Mayer-Vietoris sequences to calculate the cohomology of various spaces. We shall follow the notations of Section 5.6.

Theorem 6.3.11. *Let $X = \bigcup_{i\in I} Z_i$ be a finite covering of X by closed subsets satisfying the condition*

(6.11) *for each non empty subset $J \subset I, Z_J$ is contractible or empty.*

Let F be a locally constant sheaf on X. Then $H^j(X; F)$ is isomorphic to the j-th cohomology object of the complex

$$\Gamma(X; F^\bullet_{\mathcal{Z}}) := 0 \to \Gamma(X; F^0_{\mathcal{Z}}) \xrightarrow{d} \Gamma(X; F^1_{\mathcal{Z}}) \to \cdots$$

Proof. Recall (Proposition 6.1.3) that if Z is closed in X, then $\Gamma(X; F_Z) \simeq \Gamma(Z; F|_Z)$. Therefore the sheaves $F^p_{\mathcal{Z}}$ $(p \geq 0)$ are acyclic with respect to the functor $\Gamma(X; \cdot)$, by Corollary 6.3.10. Applying Proposition 6.2.1, the result follows from Theorem 4.6.9. q.e.d.

Corollary 6.3.12. (A particular case of the universal coefficients formula.) *In the situation of Theorem 6.3.11, let M be a flat k-module. Then for all j there are natural isomorphisms $H^j(X; M_X) \simeq H^j(X; k_X) \otimes M$.*

Proof. The k-module $H^j(X; M_X)$ is the j-th cohomology object of the complex $\Gamma(X; M^\bullet_{\mathcal{Z}})$. Clearly,

$$\Gamma(X; M^\bullet_{\mathcal{Z}}) \simeq \Gamma(X; k^\bullet_{\mathcal{Z}}) \otimes M$$

Since M is flat we have for any bounded complex of modules $N^\bullet$:

$$H^j(N^\bullet \otimes M) \simeq H^j(N^\bullet) \otimes M.$$

Hence,

$$H^j(\Gamma(X; M^\bullet_{\mathcal{Z}})) \simeq H^j(\Gamma(X; k^\bullet_{\mathcal{Z}})) \otimes M$$

To conclude, apply Theorem 6.3.11 to both sides. q.e.d.

Proposition 6.3.13. (A particular case of the Künneth formula.) *Let X and Y be two topological spaces which both admit finite closed coverings satisfying condition (6.11). Let F (resp. G) be a locally constant sheaf on X with fiber M, (resp. on Y with fiber N). Assume that k is a field. Then there are natural isomorphisms:*

$$H^p(X \times Y; F \boxtimes G) \simeq \bigoplus_{i+j=p} H^i(X; F) \otimes H^j(Y; G).$$

Sketch of the proof. First, notice that $F \boxtimes G$ is locally constant on $X \times Y$. Next, denote by $\mathcal{S} = \{S_i\}_{i \in I}$ (resp. $\mathcal{Z} = \{Z_j\}_{j \in J}$) a finite covering of X (resp. Y) satisfying condition (6.11). Since the product of two contractible sets is clearly contractible, the covering $\mathcal{S} \times \mathcal{Z} = \{S_i \times Z_j\}_{(i,j) \in I \times J}$ is a finite covering of $X \times Y$ satisfying condition (6.11). Then $H^p(X \times Y; F \boxtimes G)$ is the p-th cohomology object of the complex

$$\Gamma(X \times Y; (F \boxtimes G)^\bullet_{\mathcal{S} \times \mathcal{Z}}).$$

One checks that this complex is the simple complex associated with the double complex

$$\Gamma(X \times Y; F^\bullet_{\mathcal{S}} \boxtimes G^\bullet_{\mathcal{Z}})$$

and this double complex is isomorphic to

$$\Gamma(X;F^\bullet_{\mathcal{S}})\otimes\Gamma(Y;G^\bullet_{\mathcal{Z}})$$

It remains to apply the result of Exercise 4.14. q.e.d.

It may be convenient to reformulate the Künneth formula by saying that $H^p(X\times Y;F\boxtimes G)$ is isomorphic to the p-th cohomology object of the complex

$$(\bigoplus_i H^i(X;F)[-i])\otimes(\bigoplus_j H^j(Y;G)[-j]).$$

(See Exercise 4.14.)

6.4 Cohomology of some classical manifolds

Here, k denotes as usual a commutative unitary ring and M denotes a k-module.

Example 6.4.1. Let X be the circle $\mathbb{S}^1$ and let Z_j's be a closed covering by intervals such that the Z_{ij}'s are single points and $Z_{012}=\emptyset$. Applying Theorem 6.3.11, we find that if F is a locally constant sheaf on X, the cohomology groups $H^j(X;F)$ are the cohomology objects of the complex:

$$0\to F_{Z_0}\oplus F_{Z_1}\oplus F_{Z_2}\xrightarrow{d}F_{Z_{12}}\oplus F_{Z_{20}}\oplus F_{Z_{01}}\to 0.$$

Recall Example 5.8.2: $\mathbb{S}^1=U_1\cup U_2$, $U_1\cap U_2$ has two connected components U_{12}^+ and U_{12}^-, k is a field, $\alpha\in k^\times$ and L_α denotes the locally constant sheaf of rank one over k obtained by gluing k_{U_1} and k_{U_2} by the identity on U_{12}^+ and by multiplication by $\alpha\in k^\times$ on U_{12}^-.

Then for $j=0$ (resp. for $j=1$), $H^j(\mathbb{S}^1;L_\alpha)$ is the kernel (resp. the cokernel) of the matrix $\begin{pmatrix}0&-1&1\\1&0&-\alpha\\-1&1&0\end{pmatrix}$ acting on k^3. (See Example 6.2.2.) Note that these kernel and cokernel are zero except in case of $\alpha=1$ which corresponds to the constant sheaf k_X.

It follows that if M is a k-module, then $H^j(\mathbb{S}^1;M_{\mathbb{S}^1})\simeq M$ for $j=0,1$ and 0 otherwise.

Example 6.4.2. Consider the topological n-sphere $\mathbb{S}^n$. Recall that it can be defined as follows. Let $\mathbb{E}$ be an $\mathbb{R}$-vector space of dimension $n+1$ and denote by $\dot{\mathbb{E}}$ the set $\mathbb{E}\setminus\{0\}$. Then

$$\mathbb{S}^n\simeq\dot{\mathbb{E}}/\mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers and $\mathbb{S}^n$ is endowed with the quotient topology. (See Definition 7.4.4 below.) In other words, $\mathbb{S}^n$ is the set of all half-lines in $\mathbb{E}$. If one chooses an Euclidian norm on $\mathbb{E}$, then one may identify $\mathbb{S}^n$ with the unit sphere in $\mathbb{E}$.

We have $\mathbb{S}^n = \bar{D}^+ \cup \bar{D}^-$, where $\bar{D}^+$ and $\bar{D}^-$ denote the closed hemispheres, and $\bar{D}^+ \cap \bar{D}^- \simeq \mathbb{S}^{n-1}$. Let us prove that for $n \geq 1$:

$$H^j(\mathbb{S}^n; M_{\mathbb{S}^n}) = \begin{cases} M & j = 0 \text{ or } j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

Consider the Mayer-Vietoris long exact sequence

$$\begin{aligned} \to H^j(\bar{D}^+; M_{\bar{D}^+}) \oplus H^j(\bar{D}^-; M_{\bar{D}^-}) &\to H^j(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) \\ &\to H^{j+1}(\mathbb{S}^n; M_{\mathbb{S}^n}) \to \cdots \end{aligned} \tag{6.13}$$

Then the result follows by induction on n since the closed hemispheres being contractible, their cohomology is concentrated in degree 0.

Let $\mathbb{E}$ be a real vector space of dimension $n + 1$, and let $X = \mathbb{E} \setminus \{0\}$. Assume $\mathbb{E}$ is endowed with a norm $|\cdot|$. The map $x \mapsto x((1-t) + t/|x|)$ defines an homotopy of X with the sphere $\mathbb{S}^n$. Hence the cohomology of a constant sheaf with stalk M on $V \setminus \{0\}$ is the same as the cohomology of the sheaf $M_{\mathbb{S}^n}$

As an application, one obtains that the dimension of a finite dimensional vector space is a topological invariant. In other words, if V and W are two real finite dimensional vector spaces and are topologically isomorphic, they have the same dimension. In fact, if V has dimension n, then $V \setminus \{0\}$ is homotopic to $\mathbb{S}^{n-1}$.

Notice that $\mathbb{S}^n$ is not contractible, although one can prove that any locally constant sheaf on $\mathbb{S}^n$ for $n \geq 2$ is constant.

Example 6.4.3. Denote by a the antipodal map on $\mathbb{S}^n$ (the map deduced from $x \mapsto -x$) and denote by $a^{\sharp n}$ the action of a on $H^n(\mathbb{S}^n; M_{\mathbb{S}^n})$. Using (6.13) and Remark 4.2.7, one deduces the commutative diagram:

$$\begin{tikzcd} H^{n-1}(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) \arrow[r, "u"] \arrow[d, "a^{\sharp n-1}"'] & H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \arrow[d, "a^{\sharp n}"'] \\ H^{n-1}(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) \arrow[r, "-u"] & H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \end{tikzcd} \tag{6.14}$$

For $n = 1$, the map a is homotopic to the identity (in fact, it is the same as a rotation of angle π). By (6.14), we deduce:

$$a^{\sharp n} \text{ acting on } H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \text{ is } (-)^{n+1}. \tag{6.15}$$

Example 6.4.4. Let $\mathbb{T}^n$ denote the n-dimensional torus, $\mathbb{T}^n \simeq (\mathbb{S}^1)^n$. Using the Künneth formula, one gets that (if k is a field) $\bigoplus_j H^j(\mathbb{T}^n; k_{\mathbb{T}^n}) \simeq (k \oplus k[-1])^{\otimes n}$. For example, $H^j(\mathbb{T}^2; k_{\mathbb{T}^2})$ is k for $j = 0, 2$, is k^2 for $j = 1$ and is 0 otherwise.

Let us recover this result (when $n = 2$) by using Mayer-Vietoris sequences. One may represent $\mathbb{T}^2$ as follows. Consider the two cylinders $Z_0 = \mathbb{S}^1 \times I_0$, $Z_1 = \mathbb{S}^1 \times I_1$ where $I_0 = I_1 = [0, 1]$. Then

$$\mathbb{T}^2 \simeq (Z_0 \sqcup Z_1)/\sim$$

where $\sim$ is the relation which identifies $\mathbb{S}^1 \times \{0\} \subset Z_0$ with $\mathbb{S}^1 \times \{0\} \subset Z_1$ and $\mathbb{S}^1 \times \{1\} \subset Z_0$ with $\mathbb{S}^1 \times \{1\} \subset Z_1$. Then

$$Z_{01} := Z_0 \cap Z_1 \simeq \mathbb{S}^1 \sqcup \mathbb{S}^1.$$

We have a short exact sequence of sheaves

$$0 \to k_{\mathbb{T}^2} \xrightarrow{\alpha} k_{Z_0} \oplus k_{Z_1} \xrightarrow{\beta} k_{Z_{01}} \to 0$$

Write $H^j(X)$ instead of $H^j(X; k_X)$ for short. Applying the functor $\Gamma(\mathbb{T}^2; \cdot)$, we get the long exact sequence of Theorem 6.1.4:

$$\begin{aligned} 0 \to H^0(\mathbb{T}^2) \xrightarrow{\alpha_0} H^0(Z_0) \oplus H^0(Z_1) \xrightarrow{\beta_0} H^0(Z_{01}) \to H^1(\mathbb{T}^2) \\ \xrightarrow{\alpha_1} H^1(Z_0) \oplus H^1(Z_1) \xrightarrow{\beta_1} H^1(Z_{01}) \to H^2(\mathbb{T}^2) \to 0. \end{aligned}$$

Although we know the groups $H^j(Z_0), H^j(Z_1)$ and $H^j(Z_{01})$ (since Z_0 and Z_1 are homotopic to $\mathbb{S}^1$), this sequence does not allow us to conclude, unless we know the morphisms α or β. Now we remark that identifying Z_0 and Z_1 to $\mathbb{S}^1$, the morphism β is given by the matrix $\begin{pmatrix} \mathrm{id}_{\mathbb{S}^1} & -\mathrm{id}_{\mathbb{S}^1} \\ -\mathrm{id}_{\mathbb{S}^1} & \mathrm{id}_{\mathbb{S}^1} \end{pmatrix}$. Then one easily recovers that $H^j(\mathbb{T}^2; k_{\mathbb{T}^2})$ is k for $j = 0, 2$ and k^2 for $j = 1$.

Exercises to Chapter 6

Exercise 6.1. Let $\{F_i\}_{i \in I}$ be a family of sheaves on X. Assume that this family is locally finite, that is, each $x \in X$ has an open neighborhood U such that all but a finite number of the $F_i|_U$'s are zero.
(i)Prove that $(\prod_i F_i)_x \simeq \prod_i (F_i)_x$.
(ii) Prove that if each F_i is injective, then $\prod_i F_i$ is injective.
(iii) Let $F_i^\bullet$ be an injective resolution of F_i. Prove that $\prod_i F_i^\bullet$ is an injective resolution of $\prod_i F_i$.
(iv) Prove that $H^j(X; \prod_i F_i) \simeq \prod_i H^j(X; F_i)$.

Exercise 6.2. In this exercise, we shall admit the following theorem: for any open subset U of the complex line $\mathbb{C}$, one has $H^j(U;\mathcal{O}_{\mathbb{C}}) \simeq 0$ for $j > 0$.

Let ω be an open subset of $\mathbb{R}$, and let $U_1 \subset U_2$ be two open subsets of $\mathbb{C}$ containing ω as a closed subset.
(i) Prove that the natural map $\mathcal{O}(U_2 \setminus \omega)/\mathcal{O}(U_2) \to \mathcal{O}(U_1 \setminus \omega)/\mathcal{O}(U_1)$ is an isomorphism. One denote by $\mathcal{B}(\omega)$ this quotient.
(ii) Construct the restriction morphism to get the presheaf $\omega \to \mathcal{B}(\omega)$, and prove that this presheaf is a sheaf (the sheaf $\mathcal{B}_{\mathbb{R}}$ of Sato's hyperfunctions on $\mathbb{R}$).
(iii) Prove that the restriction morphisms $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\omega)$ are surjective.
(iv) Let Ω an open subset of $\mathbb{C}$ and let $P = \sum_{j=1}^m a_j(z)\frac{\partial}{\partial z}^j$ be a holomorphic differential operator (the coefficients are holomorphic in Ω). Recall the Cauchy theorem which asserts that if Ω is simply connected and if $a_m(z)$ does not vanish on Ω, then P acting on $\mathcal{O}(\Omega)$ is surjective. Prove that if ω is an open subset of $\mathbb{R}$ and if P is a holomorphic differential operator defined in a open neighborhood of ω, then P acting on $\mathcal{B}(\omega)$ is surjective

Exercise 6.3. Let X and Y be two topological spaces, S and S' two closed subsets of X and Y respectively, $f : S \simeq S'$ a topological isomorphism.

Define $X \sqcup_S Y$ as the quotient space $X \sqcup Y/\sim$ where $\sim$ is the relation which identifies $x \in X$ and $y \in Y$ if $x \in S$, $y \in S'$ and $f(x) = y$. One still denotes by X, Y, S the images of X, Y, $S \sqcup S'$ in $X \sqcup_S Y$.
(i) Let F be a sheaf on $X \sqcup_S Y$. Write the long exact Mayer-Vietoris sequence associated with X, Y, S.
(ii) Application (a). Let $\mathbb{S}^n$ denote the unit sphere of the Euclidian space $\mathbb{R}^{n+1}$, B the intersection of $\mathbb{S}^n$ with an open ball of radius ε $(0 < \varepsilon << 1)$ centered in some point of $\mathbb{S}^n$, Σ its boundary in $\mathbb{S}^n$. Set $X = \mathbb{S}^n \setminus B$, $S = \Sigma$ and let Y and S' be a copy of X and S, respectively. Calculate $H^j(X \sqcup_S Y; k_{X\sqcup_S Y})$.
(iii) Application (b). Same question by replacing the sphere $\mathbb{S}^n$ by the torus $\mathbb{T}^2$ embedded into $\mathbb{R}^3$.

Exercise 6.4. Let $X = \mathbb{R}^4$ and consider the locally closed subset $Z = \{(x, y, z, t) \in \mathbb{R}^4; t^4 = x^2 + y^2 + z^2; t > 0\}$. Denote by $f : Z \hookrightarrow X$ the natural injection. Calculate $(R^j f_* k_Z)_0$ for $j \geq 0$.

Exercise 6.5. Let p, q, n be integers ≥ 1 with $n = p + q$ and let $X = \mathbb{R}^n$ endowed with the coordinates $x = (x_1, \ldots, x_n)$. Set $S_0 = \{x \in X; \sum_{i=1}^n x_i^2 = 1\}$, $S_1 = \{x \in X; \sum_{i=1}^p x_i^2 + 2\sum_{i=p+1}^n x_i^2 = 1\}$, $S = S_0 \cup S_1$. Calculate $H^j(S; k_S)$ for all j.

Exercise 6.6. Let $\gamma = \{(x, y, z, t) \in X = \mathbb{R}^4; x^2 + y^2 + z^2 = t^2\}$ and let $U = X \setminus \gamma$.

(i) Show that γ is contractible.
(ii) Calculate $H^j(X; k_{XU})$ for all j. (Recall that there exists an exact sequence $0 \to k_{XU} \to k_X \to k_{X\gamma} \to 0$.)

Exercise 6.7. A closed subset Z of a space X is called a retract of X if there exists a continuous map $f\colon X \to Z$ which induces the identity on Z. Show that $\mathbb{S}^1$ is not a retract of the closed disk $\bar{D}$ in $\mathbb{R}^2$.
(Hint: denote by $\iota\colon \mathbb{S}^1 \hookrightarrow \bar{D}$ the embedding and assume that there exists a continuous map $f\colon \bar{D} \to \mathbb{S}^1$ such that the composition $f \circ \iota$ is the identity. We get that the composition

$$H^1(\mathbb{S}^1; \mathbb{Z}_{\mathbb{S}^1}) \xrightarrow{f^{\sharp 1}} H^1(\bar{D}; \mathbb{Z}_{\bar{D}}) \xrightarrow{\iota^{\sharp 1}} H^1(\mathbb{S}^1; \mathbb{Z}_{\mathbb{S}^1})$$

is the identity.)

Exercise 6.8. Consider the unit ball $B_{n+1} = \{x \in \mathbb{E}; |x| \leq 1\}$ and consider a map $f : B_{n+1} \to B_{n+1}$. Prove that f has at least one fixed point. (Hint: otherwise, construct a map $g : B_{n+1} \to \mathbb{S}^n$ which induces the identity on $\mathbb{S}^n$ and use the same argument as in Exercise 6.7.)
(Remark: the result of this exercise is known as the Brouwer's Theorem.)

Chapter 7

Homotopy and fundamental groupoid

In this chapter we study locally constant sheaves of sets and sheaves of k-modules and introduce the fundamental group of locally connected topological spaces. We define the monodromy of a locally constant sheaf and prove the equivalence between the category of representations of the fundamental group and that of locally constant sheaves. [1]

Some references: [11], [7], [13], [3], [22]. In this chapter, we shall admit some results treated with all details in [11].

7.1 Fundamental groupoid

Let us recall some classical notions of topology. We denote as usual by I the closed interval $[0,1]$ and by $\mathbb{S}^1$ the circle. Note that topologically $\mathbb{S}^1 \simeq I/\sim$ where $\sim$ is the equivalence relation on I which identifies the two points 0 and 1. We shall also consider the space

$$D := I \times I/\sim$$

where $\sim$ is the equivalence relation which identifies $I \times \{0\}$ to a single point (denoted a_0) and $I \times \{1\}$ to a single point (denoted a_1). Note that topologically, D is isomorphic to the closed unit disk, or else, to $I \times I$.

Let X denote a topological space.

Definition 7.1.1. (i) A path from x_0 to x_1 in X is a continuous map $\sigma : I \to X$, with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The two points x_0 and x_1 are called the ends of the path.

[1] This chapter will not be treated during the course 2005/2006

(ii) Two paths σ_0 and σ_1 are called homotopic if there exists a continuous function $\varphi : I \times I \to X$ such that $\varphi(i,t) = \sigma_i(t)$ for $i = 0, 1$.

(iii) If the two paths have the same ends, x_0 and x_1, one says they are homotopic with fixed ends if moreover $\varphi(s, 0) = x_0, \varphi(s, 1) = x_1$ for all s. This is equivalent to saying that there exists a continuous function $\psi : D \to X$ such that $\psi(i,t) = \sigma_i(t)$ for $i = 0, 1$.

(iv) A loop in X is continuous map $\gamma : \mathbb{S}^1 \to X$. One can also consider a loop as a path γ such that $\gamma(0) = \gamma(1)$. A trivial loop is a constant map $\gamma : \mathbb{S}^1 \to \{x_0\}$. Two loops are homotopic if they are homotopic as paths.

It is left to the reader to check that "homotopy" is an equivalence relation.

If σ is a path from x_0 to x_1 and τ a path from x_1 to x_2 one can define a new path $\tau\sigma$ (in this order) from x_0 to x_2 by setting $\tau\sigma(t) = \sigma(2t)$ for $0 \leq t \leq 1/2$ and $\tau\sigma(t) = \tau(2t-1)$ for $1/2 \leq t \leq 1$.

If σ is a path from x_0 to x_1, one can define the path σ^{-1} from x_1 to x_0 by setting $\sigma^{-1}(t) = \sigma(1-t)$.

Let us denote by $[\sigma]$ the homotopy class of a path σ. It is easily checked that the homotopy class of $\tau\sigma$ depends only on the homotopy classes of σ and τ. Hence, we can define $[\tau][\sigma]$ as $[\tau\sigma]$. The next result is left as an exercise.

Lemma 7.1.2. *The product $[\sigma][\tau]$ is associative, and $[\sigma\sigma^{-1}]$ is the homotopy class of the trivial loop at x_0.*

By this lemma, the set of homotopy classes of loops at x_0 is a group.

Definition 7.1.3. The set of homotopy classes of loops at x_0 endowed with the above product is called the fundamental group of X at x_0 and denoted $\pi_1(X; x_0)$.

Definition 7.1.4. Let X be a topological space.

(i) X is arcwise connected (or "path connected") if given x_0 and x_1 in X, there exists a path with ends x_0 and x_1,

(ii) X is simply connected if any loop in X is homotopic to a trivial loop,

(iii) X is locally connected (resp. locally arcwise connected, resp. locally simply connected) if each $x \in X$ has a neighborhood system consisting of connected (resp. arcwise connected, resp. simply connected) open subsets.

Clearly, if X is arcwise connected, it is connected. If X is locally arcwise connected and connected, it is arcwise connected.

Example 7.1.5. In $\mathbb{R}^2$ denote by X the union of the graph of the function $y = \sin(1/x)$, $x > 0$, the interval $\{(x, y); x = 0, -1 \leq y \leq 1\}$ and the interval $\{(x, y); y = 0, x \geq 0$. Then X is arcwise connected but not locally arcwise connected.

In the sequel, we shall make the hypothesis

$$X \text{ is locally arcwise connected.} \tag{7.1}$$

Assume (7.1). If σ is a path from x_0 to x_1 in X, then the map $\gamma \mapsto \sigma^{-1}\gamma\sigma$ defines an isomorphism

$$\pi_1(X; x_0) \simeq \pi_1(X; x_1).$$

Hence, if X is connected, all groups $\pi_1(X; x)$ are isomorphic for $x \in X$.

Definition 7.1.6. The fundamental groupoid $\Pi_1(X)$ is the category given by

$$\begin{cases} \mathrm{Ob}(\Pi_1(X)) = X, \\ \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1) = \{\text{the set of homotopy classes of paths from} \\ x_0 \text{ to } x_1\}. \end{cases}$$

Note that for $x \in X$, $\mathrm{Hom}_{\Pi_1(X)}(x, x) = \pi_1(X, x)$.

Consider a continuous map $f : X \to Y$. If γ is a path in X, then $f \circ \gamma$ is a path in Y, and if two paths γ_0 and γ_1 are homotopic in X, then $f \circ \gamma_0$ and $f \circ \gamma_1$ are are homotopic in Y. Hence, we get a functor:

$$f_* : \Pi_1(X) \to \Pi_1(Y). \tag{7.2}$$

In particular, if $i_U : U \hookrightarrow X$ denotes the embedding of an open subset U of X, we get the functor

$$i_{U*} : \Pi_1(U) \to \Pi_1(X). \tag{7.3}$$

Proposition 7.1.7. *Let $f_0, f_1 : X \to Y$ be two continuous maps and assume f_0 and f_1 are homotopic. Then the two functors f_{0*} and f_{1*} are isomorphic.*

In particular, if $f : X \to Y$ is a homotopy equivalence, then the two groupoids $\Pi_1(Y)$ and $\Pi_1(X)$ are equivalent.

Proof. Let $h : I \times X \to Y$ be a continuous map such that $h(i, \cdot) = f_i(\cdot)$, $i = 0, 1$, and let $\gamma : I \to X$ be a path. Then $h \circ \gamma : I \times I \to Y$ defines a homotopy between $f_0 \circ \gamma$ and $f_1 \circ \gamma$. q.e.d.

Examples 7.1.8. (i) A contractible space is simply connected. This follows from Proposition 7.1.7.
(ii) One has $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ and $1 \in \mathbb{Z}$ corresponds to the identity map, considered as a loop in the space $X = \mathbb{S}^1$. We refer to [11] a for proof. As a corollary, one gets $\pi_1(\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{Z}$.

Remark 7.1.9. (i) Assume (7.1) and moreover X is non empty and connected. Then all objects $x \in \Pi_1(X)$ are isomorphic and after choosing $x_0 \in X$, one sets $\pi_1(X) = \pi_1(X; x_0)$. One calls $\pi_1(X)$ the fundamental group of X.
(ii) Remark that X being arcwise connected, it is simply connected if and only if $\pi_1(X) \simeq \{1\}$.
(iii) It is easily seen that if X is simply connected, two paths γ and τ with the same ends are homotopic.

Let X and Y be two topological spaces satisfying (7.1). Denote by p_i the projection from $X \times Y$ to X and Y respectively. These projections define functors $p_{1*} : \Pi_1(X \times Y) \to \Pi_1(X)$ and $p_{2*} : \Pi_1(X \times Y) \to \Pi_1(Y)$, hence a functor

$$(p_{1*} \times p_{2*})\Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y). \tag{7.4}$$

Proposition 7.1.10. *The functor in* (7.4) *is an equivalence.*

Proof. (i) The functor in (7.4) is obviously essentially surjective.
(ii) Let $x_0, x_1 \in X$, $y_0, y_1 \in Y$. Let us show the isomorphism

$$\mathrm{Hom}_{\Pi_1(X\times Y)}((x_0, y_0), (x_1, y_1)) \xrightarrow{\sim} \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1) \times \mathrm{Hom}_{\Pi_1(Y)}(y_0, y_1) \tag{7.5}$$

The map in (7.5) is surjective. Indeed, if σ is a path in X and τ is a path in Y, the path $\sigma \times \tau$ in $X \times Y$ satisfies

$$(p_{1*} \times p_{2*})[\sigma \times \tau] = [\sigma] \times [\tau].$$

(iii) The map in (7.5) is injective. Indeed, let γ_0 and γ_1 be two paths in $X \times Y$. Assume that $p_{1*}\gamma_0$ is homotopic to $p_{1*}\gamma_1$ and $p_{2*}\gamma_0$ is homotopic to $p_{2*}\gamma_1$. The product of these two homotopies defines an homotopy from γ_0 to γ_1. q.e.d.

7.2 Monodromy of locally constant sheaves

Locally constant sheaves

Remark 7.2.1. We shall work here with sheaves of k-modules, but many results remain true without any change for sheaves of sets.

Let $M \in \mathrm{Mod}(k)$. Recall that a constant sheaf F with stalk M on X is a sheaf isomorphic to the sheaf M_X of locally constant functions with values in M. We shall denote by $\mathrm{LCSH}(k_X)$ (resp. $\mathrm{CSH}(k_X)$) the full additive subcategory of $\mathrm{Mod}(k_X)$ consisting of locally constant (resp. of constant) sheaves.

Denote as usual by a_X the map $X \to \mathrm{pt}$.

Proposition 7.2.2. (i) *Assume X is connected and non empty. Then the two functors*

$$\mathrm{CSH}(k_X) \underset{a_X^{-1}}{\overset{a_{X*}}{\rightleftarrows}} \mathrm{Mod}(k)$$

are equivalences of categories, inverse one to each other. In particular, the category $\mathrm{CSH}(k_X)$ is abelian and if M and N are two k-modules, there is an isomorphism $\mathcal{H}om_{k_X}(M_X, N_X) \simeq (\mathrm{Hom}_k(M, N))_X$.

(ii) *Assume X is locally connected. Then the category $\mathrm{LCSH}(k_X)$ is abelian.*

Proof. (i) If F is a constant sheaf on X, and if one sets $M = \Gamma(X; F)$, then $F \simeq M_X$. Therefore, if $M \in \mathrm{Mod}(k)$, then $a_{X*}a_X^{-1}M \simeq M$ and $a_X^{-1}a_{X*}M_X \simeq M_X$.
(ii) Let $\varphi : F \to G$ is a morphism of locally constant sheaf, and let $x \in X$. If U is sufficiently small connected open neighborhood of x, the restriction to U of $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ will be constant sheaves, by (i). Hence, $\mathrm{LCSH}(k_X)$ is a full additive subcategory of an abelian category (namely $\mathrm{Mod}(k_X)$) admitting kernels and cokernels. This implies it is abelian. q.e.d.

Lemma 7.2.3. *Let F be a locally constant sheaf on $X = I \times I$. Then F is a constant sheaf.*

Proof. Since $I \times I$ is compact, there exists finite coverings of I by intervals $\{U_i\}_{1\le i\le N_0}$ and $\{V_j\}_{1\le j\le N_1}$ such that $F|_{U_i\times V_j}$ is a constant sheaf. Since $(U_i \times V_j) \cap (U_{i+1} \times V_j)$ is connected, the argument of the proof of Proposition 5.7.6 shows that $F|_{I\times V_j}$ is a constant sheaf for all j. Since $(I \times V_j) \cap (I \times V_{j+1})$ is connected, the argument of the proof of Proposition 5.7.6 shows that F is constant. q.e.d.

Representations

For a group G and a ring k one defines the category $\mathrm{Rep}(G, \mathrm{Mod}(k))$ of representations of G in $\mathrm{Mod}(k)$ as follows. An object is a pair (M, μ_M) with $M \in \mathrm{Mod}(k)$ and $\mu_M \in \mathrm{Hom}\,(G, \mathbb{G}l(M))$. A morphism $\mu_f : (M, \mu_M) \to (N, \mu_N)$ is a k-linear map $f : M \to N$ which satisfies $\mu_N \circ f = f \circ \mu_M$ (i.e.

$\mu_N(g) \circ f = f \circ \mu_M(g)$ for any $g \in G$). Note that $\mathrm{Rep}(G, \mathrm{Mod}(k))$ contains the full abelian subcategory $\mathrm{Mod}(k)$, identified to the trivial representations of G.

We identify G with a category $\mathcal{G}$ with one object c, the morphisms being given by $\mathrm{Hom}_{\mathcal{G}}(c, c) = G$. One gets that

$$\mathrm{Rep}(G, \mathrm{Mod}(k)) \simeq \mathrm{Fct}(\mathcal{G}, \mathrm{Mod}(k)). \tag{7.6}$$

Now we shall consider the category $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$, a generalization of the category of representations $\mathrm{Rep}(\pi_1(X), \mathrm{Mod}(k))$. In fact, if X is connected, non empty and satisfies (7.7), then all $x \in \Pi_1(X)$ are isomorphic, and the groupoids $\Pi_1(X)$ is equivalent to the group $\pi_1(X, x_0)$ identified to the category with one object x_0 and morphisms $\pi_1(X, x_0)$. In this case, the two categories $\mathrm{Rep}(\pi_1(X), \mathrm{Mod}(k))$ and $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ are equivalent.

Monodromy

In this section, we make the hypothesis (7.1) that we recall

$$X \text{ is locally arcwise connected.} \tag{7.7}$$

Definition 7.2.4. (i) One calls an object of $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ a representation of the groupoid $\Pi_1(X)$ in $\mathrm{Mod}(k)$.

(ii) Let $\theta \in \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$. One says that θ is a trivial representation if θ is isomorphic to a constant functor Δ_M which associates the module M to any $x \in X$, and id_M to any $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$.

Let $\mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k))$ be the full subcategory of $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ consisting of trivial representations. Then the functor $M \mapsto \Delta_M$ from $\mathrm{Mod}(k)$ to $\mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k))$ is an equivalence of categories.

Let F be a locally constant sheaf of k-modules on X. Let γ be a path from x_0 to x_1. We shall construct an isomorphism

$$\mu(F)(\gamma) : F_{x_0} \simeq F_{x_1}.$$

Since $\gamma^{-1}F$ is a locally constant sheaf on $[0, 1]$, it is a constant sheaf. We get the isomorphisms, which define $\mu(F)(\gamma)$:

$$F_{x_0} \simeq (\gamma^{-1}F)_0 \xleftarrow{\sim} \Gamma(I; \gamma^{-1}F) \xrightarrow{\sim} (\gamma^{-1}F)_1 \simeq F_{x_1}. \tag{7.8}$$

Example 7.2.5. (i) Let $X = I$, denote by t a coordinate on I, and consider the constant sheaf $F = \mathbb{C}_X \exp(\alpha t)$. Then $\mu(F)(I) \colon F_0 \xrightarrow{\sim} F_1$ is the multiplication by $\exp(\alpha)$.

(ii) Let $X = \mathbb{S}^1 \simeq [0, 2\pi]/\sim$ (where $\sim$ identifies 0 and 2π) and denote by *theta* a coordinate on $\mathbb{S}^1$. Consider the constant locally constant sheaf $F = \mathbb{C}_X \exp(i\beta\theta)$. (See Example 5.8.2.) Let γ be the identity loop. Then $\mu(F)(\gamma) : F_{x_0} \xrightarrow{\sim} F_{x_0}$ is the multiplication by $\exp(2i\pi\beta)$.

Lemma 7.2.6. *The isomorphism $\mu(F)(\gamma)$ depends only on the homotopy class of γ in X.*

Proof. Let φ be a continuous function $D \to X$ such that $\varphi(i, t) = \gamma_i(t)$, $i = 0, 1$. The sheaf $\varphi^{-1}F$ is constant by Lemma 7.2.3. The isomorphisms $\mu(F)(\gamma_i)$ $(i = 0, 1)$ are described by the commutative diagram:

$$\begin{tikzcd} F_{x_0} \simeq (\varphi^{-1}F)_{a_0} \arrow[d, "\sim"'] & \Gamma(D; \varphi^{-1}F) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\sim"'] & (\varphi^{-1}F)_{a_1} \simeq F_{x_1} \arrow[d, "\sim"'] \\ F_{x_0} \simeq (\gamma_i^{-1}F)_0 & \Gamma(I; \gamma_i^{-1}F) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (\gamma_i^{-1}F)_1 \simeq F_{x_1} \end{tikzcd} \tag{7.9}$$

This shows that $\mu(F)(\gamma_0) = \mu(F)(\gamma_1)$. q.e.d.

If τ is another path from x_1 to x_2, then:

$$\mu(F)(\gamma\tau) = \mu(F)(\gamma) \circ \mu(F)(\tau).$$

Hence we have constructed a functor of $\mu(F) \colon \Pi_1(X) \to \mathrm{Mod}(k)$ given by $\mu(F)(x) = F_x$, $\mu(F)([\gamma]) = \mu(F)(\gamma)$ where γ is a representative of $[\gamma]$. This correspondence being functorial in F, we get a functor

$$\mu : \mathrm{LCSH(k_X)} \to \mathrm{Fct(\Pi_1(X), Mod(k))}. \tag{7.10}$$

Definition 7.2.7. The functor μ in (7.10) is called the monodromy functor.

The functor μ is also "functorial" with respect to the space X. More precisely, let $f : X \to Y$ be a continuous map, and assume that both X and Y are locally arcwise connected. We have the commutative (up to isomorphism) diagram of categories and functors:

$$\begin{tikzcd} \mathrm{LCSH}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\mu"] & \mathrm{LCSH}(k_X) \arrow[d, "\mu"] \\ \mathrm{Fct}(\Pi_1(Y), \mathrm{Mod}(k)) \arrow[r, "f^{-1}"] & \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k)) \end{tikzcd} \tag{7.11}$$

Theorem 7.2.8. *Assume* (7.7). *The functor μ in* (7.10) *is fully faithful.*

Proof. (i) μ is faithful. Let $\varphi, \psi : F \to G$ be morphisms of locally constant sheaves and assume that $\mu(\varphi) \simeq \mu(\psi)$. This implies that $\varphi_{x_0} \simeq \psi_{x_0} : F_{x_0} \to G_{x_0}$ for any $x_0 \in X$, and the sheaf $\mathcal{H}om\,(F, G)$ being locally constant, this implies that $\varphi = \psi$ on the connected component of x_0.
(ii) μ is full. Consider a morphism $u : \mu(F) \to \mu(G)$. It defines a morphism $\varphi_{x_0} : F_{x_0} \to G_{x_0}$. To each $x_1 \in X$ we get a well defined morphism $\varphi_{x_1} : F_{x_1} \to G_{x_1}$, given by $\varphi_{x_1} = \mu(G)(\sigma) \circ \varphi_{x_0} \circ \mu(F)(\sigma^{-1})$, where σ is any path from x_0 to x_1. This isomorphism does not depend on the choice of σ by the hypothesis. Since F and G are locally constant, each $x \in X$ has an open neighborhood U_x such that the morphism $\varphi_x : F_x \to G_x$ extends as a morphism $\varphi_{U_x} : F|_{U_x} \to G|_{U_x}$. These morphisms will glue to each other and define a morphism $\varphi : F \to G$ with $\mu(\varphi) = u$. q.e.d.

Proposition 7.2.9. *Assume* (7.7) *and X is non empty and connected. Then the functor μ in* (7.10) *induces an equivalence*

$$\mu_0 : \mathrm{CSH}(k_X) \xrightarrow{\sim} \mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k)) \simeq \mathrm{Mod}(k). \tag{7.12}$$

Proof. (i) First, let us show that μ_0 takes its values in $\mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k))$. Let F be a constant sheaf. We may assume that $F = M_X$, for a module M, and we shall show that $\mu_0(F)$ is the constant functor $x \mapsto M$.

Let $\sigma : I \to X$ be a path with $x_0 = \sigma(0) = \sigma(1) = x_1$. It follows from Lemma 7.2.3 (iii) that the morphism $\mu(F)(\gamma)\colon M \simeq F_{x_0} \xrightarrow{\sim} F_{x_1} \simeq M$ is the identity.
(ii) By Theorem 7.2.8, μ_0 is fully faithful.
(iii) Since any $M \in \mathrm{Mod}(k)$ defines a constant sheaf, μ_0 is essentially surjective. q.e.d.

Corollary 7.2.10. *Assume that X is connected, locally arcwise connected and simply connected. Then any locally constant sheaf F on X is a constant sheaf.*

Proof. By the hypothesis, $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k)) \simeq \mathrm{Mod}(k)$. Let F be a locally constant sheaf. Then $\mu(F) \in \mathrm{Mod}(k)$ and there exists $G \in \mathrm{CSH}(\mathrm{k_X})$ such that $\mu(F) \simeq \mu(G)$. Since μ is fully faithful, this implies $F \simeq G$. q.e.d.

7.3 The Van Kampen theorem

In this section, we shall assume (7.7) and also

$$\begin{cases} \text{there exists an open covering stable by finite intersec-} \\ \text{tions by connected and simply connected subsets.} \end{cases} \tag{7.13}$$

Theorem 7.3.1. *Assume* (7.7) *and* (7.13). *The functor* μ *in* (7.10) *is an equivalence of categories.*

Proof. (i) By Proposition 7.2.8, it remains to show that μ is essentially surjective.
(ii) By Corollary 7.2.10, the theorem is true if X is connected and simply connected.
(iii) Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X as in (7.13). The functors $i_{U_i *}\colon \Pi_1(U_i) \to \Pi_1(X)$ define functors λ_i from $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ to $\mathrm{Fct}(\Pi_1(U_i), \mathrm{Mod}(k))$. Let $G \in \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ and set $G_i = \lambda_i(G)$. Using the result in (ii) for U_i, we find sheaves F_i such that $\mu(F_i) = G_i$. Setting $\lambda_{ji} = \lambda_j \circ \lambda_i^{-1}$, and using the result in (ii) for U_{ij}, we get isomorphisms $\theta_{ji} : F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$, and the cocycle condition (5.26) will be clearly satisfied. Applying Theorem 5.8.1, we find a sheaf F on X, which will be locally isomorphic to the F_i's, hence, will be locally constant.

It remains to show that $\mu(F) \simeq G$. For any U_i as above, $\lambda_i(\mu(F)) \simeq G_i$ and $\lambda_i(G)$ are isomorphic in $\mathrm{Fct}(\Pi_1(U_i), \mathrm{Mod}(k))$. Hence, the result follows from Lemma 7.3.2 below. q.e.d.

Lemma 7.3.2. *Let* $\mathcal{U} = \{U_i\}_{i\in I}$ *be an open covering of* X. *Consider the functor*

$$\lambda = \prod_{i\in I} \lambda_i\colon\ \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k)) \to \prod_{i\in I} \mathrm{Fct}(\Pi_1(U_i), \mathrm{Mod}(k)).$$

If $G_1, G_2 \in \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ *satisfy* $\lambda(G_1) \simeq \lambda(G_2)$, *then* $G_1 \simeq G_2$.

Proof. (i) For each $x \in X$, $G_1(x) \simeq G_2(x)$, since $x \in U_i$ for some i.
(ii) Let $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$ and let γ be a path which represents $[\gamma]$. Assume γ is contained in some U_i, and denote by $[\gamma']$ the corresponding element in $\mathrm{Hom}_{\Pi_1(U_i)}(x, y)$. Then for any $G \in \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$, one has $G([\gamma]) = \lambda_i(G)([\gamma'])$. Hence, $G_1([\gamma]) = G_2([\gamma])$ in this case.
(iii) We may decompose γ as $\gamma = \gamma_1 \cdot \cdots \cdot \gamma_n$, each γ_j $(1 \le j \le n)$ being contained in some U_{i_j}. By the hypothesis, for $1 \le j \le n$, there exist isomorphisms $G_1([\gamma_j]) \simeq G_2([\gamma_j])$. Since $G_\nu([\gamma]) = G_\nu([\gamma_n]) \cdots G_\nu([\gamma_1])$ for $\nu = 1, 2$, the result follows. q.e.d.

Corollary 7.3.3. *Assume* (7.7)*and* (7.13) *and* X *is connected. Then* X *is simply connected if and only if any locally constant sheaf on* X *is constant.*

Proof. By Theorem 7.3.1 and Proposition 7.2.9, any representation of $\pi_1(X)$ is trivial if and only if any locally constant sheaf is constant. It remains to notice that if G is a group such that any representation of G is trivial, then $G = \{1\}$. (See Exercise 7.3.) q.e.d.

Example 7.3.4. A locally constant sheaf of $\mathbb{C}$-vector spaces of finite rank on X is called a local system. Hence, it is now possible to classify all local systems on the space $X = \mathbb{R}^2 \setminus \{0\}$. In fact, $\pi_1(X) \simeq \mathbb{Z}$, hence, $\mathrm{Hom}\,(\pi_1(X), \mathbb{G}l(\mathbb{C}^n)) = \mathbb{G}l(\mathbb{C}^n)$. A local system F of rank n is determined, up to isomorphism, by its monodromy $\mu(F) \in \mathbb{G}l(\mathbb{C}^n)$. The classification of such sheaves is thus equivalent to that of invertible $n \times n$ matrices over $\mathbb{C}$ up to conjugation, a well known theory (Jordan-Hölder decomposition). In particular, when $n = 1$, $\mathbb{G}l(\mathbb{C}) = \mathbb{C}^\times$.

Hence, a local system of rank one is determined, up to isomorphism, by its monodromy $\alpha \in \mathbb{C}^\times$.

We shall deduce a particular case of the Van Kampen theorem.

Theorem 7.3.5. *Let X be a connected, locally arcwise connected, and locally simply connected space. Let $X = \{U_i\}_{i\in I}$ be an open covering stable by finite intersection, the U_i's being connected. Assume that there exists x which belongs to all U_i's and identify each groupoids $\Pi_1(U_i)$ with the group $\pi_1(U_i) = \pi_1(U_i, x)$. Then $\pi_1(X) \simeq \varinjlim_{i\in I} \pi_1(U_i)$.*

Sketch of proof. There is a natural morphism of groups

$$\varinjlim_{i} \pi_1(U_i) \to \pi_1(X). \tag{7.14}$$

Let $M \in \mathrm{Mod}(k)$. One has

$$\begin{aligned} \mathrm{Hom}\,(\pi_1(X), \mathbb{G}l(M) &\simeq \varprojlim_i \mathrm{Hom}\,(\pi_1(U_i), \mathbb{G}l(M)) \\ &\simeq \mathrm{Hom}\,(\varinjlim_i \pi_1(U_i), \mathbb{G}l(M)), \end{aligned}$$

where the first isomorphism follows from Theorem 7.3.1. To conclude, remark that if $u\colon G_1 \to G_2$ is a morphism of groups which induces an isomorphism $\mathrm{Hom}\,(G_2, \mathbb{G}l(M)) \xrightarrow{\sim} \mathrm{Hom}\,(G_1, \mathbb{G}l(M))$ for all $M \in \mathrm{Mod}(\mathbb{Z})$, then u is an isomorphism (see Exercise 7.3). q.e.d.

7.4 Coverings

Let S be a set. We endow it with the discrete topology. Then $X \times S \simeq \bigsqcup_{s\in S} X_s$ where $X_s = X \times \{s\}$ is a copy of X, and the coproduct is taken in the category of topological spaces. In particular each X_s is open.

Definition 7.4.1. (i) A continuous map $f : Z \to X$ is a trivial covering if there exists a non empty set S, a topological isomorphism $h : Z \xrightarrow{\sim} X \times S$ where S is endowed with the discrete topology, and $f = p \circ h$ where $p : X \times S \to X$ is the projection.

(ii) A continuous map $f : Z \to X$ is a covering [2] if f is surjective and any $x \in X$ has an open neighborhood U such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a trivial covering.

(iii) If $f : Z \to X$ is a covering, a section u of f is a continuous map $u : X \to Z$ such that $f \circ u = \mathrm{id}_X$. A local section is a section defined on an open subset U of X.

(iv) A morphism of coverings $f : Z \to X$ to $f' : Z' \to X$ is a continuous map $h : Z \to Z'$ such that $f = f' \circ h$.

Hence, we have defined the category $\mathrm{Cov}(X)$ of coverings above X, and the full subcategory of trivial coverings. Roughly speaking, a covering is locally isomorphic to a trivial covering.

Notation 7.4.2. Let $f : Z \to X$ be a covering. One denotes by $\mathrm{Aut}\,(f)$ the group of automorphisms of this covering, that is, the group of isomorphisms of the object $(f : Z \to X) \in \mathrm{Cov}(X)$.

The definition of a covering is visualized as follows.

$$\begin{tikzcd}
\bigsqcup_{s\in S} U_s \arrow[dr, "p"'] & f^{-1}U \arrow[l, "\sim"', "h"] \arrow[r] \arrow[d, "f"] & Z \arrow[d, "f"] \\
& U \arrow[r] & X
\end{tikzcd}$$

If X is connected and S is finite for some x, S will be finite for all x, with the same cardinal, say n. In this case one says that f is a finite (or an n-)covering.

Example 7.4.3. Let $Z = \mathbb{C} \setminus \{0, \pm i, \pm i\sqrt{2}\}$, $X = \mathbb{C} \setminus \{0, 1\}$ and let $f : Z \to X$ be the map $z \mapsto (z^2 + 1)^2$. Then f is a 4-covering.

Many coverings appear naturally as the quotient of a topological space by a discrete group.

Definition 7.4.4. Let X be a locally compact topological space and let G be a group, that we endow with the discrete topology. We denote by e the unit in G.

[2] "revêtement" in French, not to be confused with "recouvrement".

(i) An action μ of G on X is a map $\mu : G \times X \to X$ such that:

 (a) for each $g \in G$, $\mu(g) : X \to X$ is continuous,

 (b) $\mu(e) = \mathrm{id}_X$,

 (c) $\mu(g_1 \circ g_2) = \mu(g_1) \circ \mu(g_2)$.

 In the sequel, we shall often write $g \cdot x$ instead of $\mu(g)(x)$, for $g \in G$ and $x \in X$.

(ii) Let $x \in X$. The orbit of x in X is the subset $G \cdot x$ of X.

(iii) One says that G acts transitively on X if for any $x \in X$, $X = G \cdot x$.

(iv) One says that G acts properly on X if for any compact subset K of X, the set $G_K = \{g \in G; g \cdot K \cap K = \emptyset\}$ is finite.

(v) One says that G acts freely if for any $x \in X$, the group $G_x = \{g \in G; g \cdot x = x\}$ is trivial, that is, is reduced to $\{e\}$.

If a group G acts on X, it defines an equivalence relation on X, namely, $x \sim y$ if and only if there exists $g \in G$ with $x = g \cdot y$. One denotes by X/G the quotient space, endowed with the quotient topology.

Theorem 7.4.5. *Assume that a discrete group G acts properly and freely on a locally compact space X. Then $p : X \to X/G$ is a covering. Moreover, if X is connected, then* $\mathrm{Aut}\,(p) = G$.

For the proof, we refer to [11].

Examples 7.4.6. (i) The map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ (where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation) is a covering, and the map $t \mapsto \exp(2i\pi t) : \mathbb{R} \to \mathbb{S}^1$ induces an isomorphism $h : \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \mathbb{S}^1$ such that $\exp(2i\pi t) = h \circ p$. Therefore, $t \mapsto \exp(2i\pi t) : \mathbb{R} \to \mathbb{S}^1$ is a covering.

Similarly, the map $p : \mathbb{C} \to \mathbb{C}/2i\pi\mathbb{Z}$ is a covering and the map $z \mapsto \exp(z) : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ induces an isomorphism $h : \mathbb{C}/2i\pi\mathbb{Z} \xrightarrow{\sim} \mathbb{C} \setminus \{0\}$ such that $\exp(z) = h \circ p$. Therefore, $z \mapsto \exp(z) : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering.
(ii) Consider the group H_n of n-roots of unity in $\mathbb{C}$, that is, the subgroup of $\mathbb{C}^\times$ generated by $\exp(2i\pi/n)$. Then $p : \mathbb{S}^1 \to \mathbb{S}^1/H_n$ is a covering and the map $z \mapsto z^n : \mathbb{S}^1 \to \mathbb{S}^1$ induces an isomorphism $h : \mathbb{S}^1/H_n \xrightarrow{\sim} \mathbb{S}^1$ such that $z^n = h \circ p$. Therefore, $z \mapsto z^n : \mathbb{S}^1 \to \mathbb{S}^1$ is an n-covering.
(iii) The projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a covering, and there is are isomorphisms $\mathbb{R}^n/\mathbb{Z}^n \simeq (\mathbb{R}/\mathbb{Z})^n \simeq (\mathbb{S}^1)^n$.

Example 7.4.7. The projective space of dimension n, denoted $\mathbb{P}^n(\mathbb{R})$, is constructed as follows. Let $\mathbb{E}$ be an $n+1$-dimensional $\mathbb{R}$-vector space. Then $\mathbb{P}^n(\mathbb{R})$ is the set of lines in $\mathbb{E}$, in other words,

$$\mathbb{P}^n(\mathbb{R}) = \dot{\mathbb{E}}/\mathbb{R}^\times, \tag{7.15}$$

where $\dot{\mathbb{E}} = \mathbb{E} \setminus \{0\}$, and $\mathbb{R}^\times$ is the multiplicative group of non-zero elements of $\mathbb{R}$.

Identifying $\mathbb{E}$ with $\mathbb{R}^{n+1}$, a point $x \in \mathbb{R}^{n+1}$ is written $x = (x_0, x_1, \cdots, x_n)$ and $x \in \mathbb{P}^n(\mathbb{R})$ may be written as $x = [x_0, x_1, \cdots, x_n]$, with the relation $[x_0, x_1, \cdots, x_n] = [\lambda x_0, \lambda x_1, \cdots, \lambda x_n]$ for any $\lambda \in \mathbb{R}^\times$. One says that $[x_0, x_1, \cdots, x_n]$ are homogeneous coordinates.

The map $\mathbb{R}^n \to \mathbb{P}^n(\mathbb{R})$ given by $(y_1, \cdots, y_n) \mapsto [1, y_1, \cdots, y_n]$ allows us to identify $\mathbb{R}^n$ to the open subset of $\mathbb{P}^n(\mathbb{R})$ consisting of the set $\{x = [x_0, x_1, \cdots, x_n]; x_0 \neq 0\}$.

In the sequel, we shall often write for short $\mathbb{P}^n$ instead of $\mathbb{P}^n(\mathbb{R})$. Since $\mathbb{S}^n = \dot{\mathbb{E}}/\mathbb{R}^+$, we get $\mathbb{P}^n \simeq \mathbb{S}^n/a$ where a is the "antipodal" relation on $\mathbb{S}^n$ which identifies x and $-x$. The map a defines an action of the group $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{S}^n$, and this action is clearly proper and free. Denote by

$$\gamma : \mathbb{S}^n \to \mathbb{P}^n \tag{7.16}$$

the natural map. This is a 2-covering and $\mathbb{P}^n \simeq \mathbb{S}^n/(\mathbb{Z}/2\mathbb{Z})$.

Coverings and locally constant sheaves

We make hypothesis (7.1), that is, X is locally arcwise connected.

Let $f\colon Z \to X$ be a covering. We associates a sheaf of sets F_f on X as follows. For U open in X, $F_f(U)$ is the set of sections of $f|_U\colon f^{-1}(U) \to U$. Since, locally, f is isomorphic to the projection $U \times S \to U$, the sheaf F is locally isomorphic the the constant sheaf with values in S. We have thus constructed a functor

$$\Phi\colon \ \mathrm{Cov}(X) \to \mathrm{LCSH(X)}. \tag{7.17}$$

Proposition 7.4.8. *Assume* (7.1). *Then the functor* Φ *in* (7.17) *is an equivalence.*

Sketch of proof of the proof. We shall construct a quasi-inverse Ψ to Φ. Let $F \in \mathrm{LCSH(X)}$. Consider the set $Z_F = \bigsqcup_{x\in X} F_x$. We endow Z_F with the following topology. A basis of open subsets for this topology is given by the sets $U \times M$ such that U is open in X, $F|_U \simeq M_U$ is a constant sheaf with stalk M (hence, $F_x \simeq M$) and M is endowed with the discrete topology.
q.e.d.

Note that the functor Φ induces an equivalence between trivial coverings and constant sheaves.

Note that Theorem 7.3.1 is also true when replacing sheaves of k-modules with sheaves of sets. Hence, assuming (7.7) and (7.13) we get the equivalences

$$\mathrm{Cov}(X) \simeq \mathrm{LCSH(X)} \simeq \mathrm{Fct}(\Pi_1(\mathrm{X}), \mathbf{Set}).$$

Exercises to Chapter 7

Exercise 7.1. Classify all locally constant sheaves of $\mathbb{C}$-vector spaces on the space $X = \mathbb{S}^1 \times \mathbb{S}^1$.

Exercise 7.2. Let $\gamma = \{(x, y, z, t) \in X = \mathbb{R}^4; x^2 + y^2 + z^2 = t^2\}$ $\dot{\gamma} = \gamma \setminus \{0\}$. Classify all locally constant sheaves of rank one of $\mathbb{C}$-vector spaces on $\dot{\gamma}$. (Hint: one can use the fact that $\dot{\gamma}$ is homotopic to its intersection with the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$.)

Exercise 7.3. (i) Let G be a group and assume that all representation of G in $\mathrm{Mod}(k)$ are trivial. Prove that $G = \{1\}$.
(ii) Let $u\colon G_1 \to G_2$ is a morphism of groups which induces an isomorphism $\mathrm{Hom}\,(G_2, \mathbb{G}l(M)) \xrightarrow{\sim} \mathrm{Hom}\,(G_1, \mathbb{G}l(M))$ for all $M \in \mathrm{Mod}(\mathbb{Z})$. Prove that u is an isomorphism.
(Hint: (i) use the free k-module $k[G]$ generated over k by the element $g \in G$.)

Exercise 7.4. Assume X satisfies (7.1), $X = U_1 \cup U_2$, U_1 and U_2 are connected and simply connected and $U_1 \cap U_2$ is connected.
(i) Prove that X is simply connected.
(ii) Deduce that for $n > 1$, the sphere $\mathbb{S}^n$ as well as $\mathbb{R}^{n+1} \setminus \{0\}$ are simply connected.

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