The Grothendieck-Teichmüller group $\widehat{GT}$: a survey

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In this short survey, we recall the definition of the *profinite version of the Grothendieck-Teichmüller group* $\widehat{GT}$. The Grothendieck-Teichmüller group was originally defined by Drinfel'd in [D] (as it happens, not in the profinite case, but in the discrete, pro-ℓ and k-pro-unipotent cases; these definition are all analogous). We state most of the results on the profinite version $\widehat{GT}$ known to date, and give references to all the proofs. (It should be noted that there are several deep results and conjectures concerning only the pro-ℓ version of the Grothendieck-Teichmüller group; we do not discuss them here.) One of the most interesting aspects of $\widehat{GT}$ is the fact that it contains the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as a subgroup in a way compatible with natural actions of both groups on arithmetico-geometric objects such as certain covers defined over $\overline{\mathbb{Q}}$ and fundamental groups of varieties defined over $\mathbb{Q}$; the possibility that these two groups are actually isomorphic is an exciting one, suggesting a new way of considering $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It reflects Grothendieck's suggestion in the *Esquisse d'un Programme* of studying the combinatorial properties of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by studying its natural action on what he calls the *Teichmüller tower*, described as follows:

> Il s'agit du système de toutes les multiplicités $\mathcal{M}_{g,n}$ pour g, n variables, liées entre elles par un certain nombre d'opérations fondamentales (telles les opérations de 'bouchage de trous' i.e. de 'gommage' de points marqués, celle de 'recollement', et les opérations inverses), qui sont le reflet en géométrie algébrique absolue de caractéristique zéro (pour le moment) d'opérations géométriques familières du point de vue de la 'chirurgie' topologique ou conforme des surfaces. La principale raison sans doute de cette fascination, c'est que cette structure géométrique très riche sur le système des multiplicités modulaires 'ouvertes' $\mathcal{M}_{g,n}$ se reflète par une structure analogue sur les groupoïdes fondamentaux correspondants, les 'groupoïdes de Teichmüller' $\hat{T}_{g,n}$, et que ces opérations au niveau des $\hat{T}_{g,n}$ ont un caractère suffisamment intrinsèque pour que le groupe de Galois $\mathbb{\Gamma}$ de $\overline{\mathbb{Q}}/\mathbb{Q}$ opère sur toute cette 'tour' de groupoïdes de Teichmüller, en respectant toutes ces structures. Chose plus extraordinaire encore, cette opération est *fidèle* – à vrai dire, elle est fidèle dejà sur le premier 'etage' non trivial de cette tour, à savoir $\hat{T}_{0,4}$ – ce qui signifie aussi, essentiellement, que l'action extérieure de $\mathbb{\Gamma}$ sur le groupe fondamental $\hat{\pi}_{0,3}$ de la droite projective standard $\mathbb{P}^1$ sur $\mathbb{Q}$, privée des trois points 0, 1, ∞, est dejà fidèle. Ainsi **le groupe de Galois $\mathbb{\Gamma}$ se réalise comme un groupe d'automorphismes d'un groupe profini des plus concrets**, respectant d'ailleurs certaines structures essentielles de ce groupe. Il s'ensuit qu'un élément de $\mathbb{\Gamma}$ peut être 'parametré' (de diverses façons équivalentes d'ailleurs) par un élément convenable de ce groupe profini

(un groupe profini libre à deux générateurs), ou par un système de tels éléments, ce ou ces éléments étant d'ailleurs soumis à certaines conditions simples, nécessaires (et sans doute non suffisantes) pour que ce ou ces éléments corresponde(nt) bien à un élément de $\mathbb{\Gamma}$. Une des tâches les plus fascinantes ici, est justement d'appréhender des conditions nécessaires **et** suffisantes sur un automorphisme extérieur de $\hat{\pi}_{0,3}$, i.e. sur le ou les paramètres correspondants, pour qu'il provienne d'un élément de $\mathbb{\Gamma}$ – ce qui fournirait une description 'purement algebrique', en termes de groupes profinis et sans référence à la théorie de Galois des corps de nombres, du groupe de Galois $\mathbb{\Gamma} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$!

Drinfel'd pointed out in [D] the connection between his construction of $\widehat{GT}$ and the kind of group Grothendieck is referring to; he suggests that if one were to construct a tower of Teichmüller fundamental groupoids with *tangential base points*, then the full automorphism group of this tower should be exactly $\widehat{GT}$. We quote the relevant passage here, with the following remarks about the notation: formulae (4.3), (4.4) and (4.10) correspond to the defining relations (I), (II) and (III) of $\widehat{GT}$ (see §1.1), and the group $\hat{\Gamma}$ denotes the profinite completion of the quotient of the Artin braid group B_3 by its center (again, see §1.1).

It is proposed in [G] to consider, for any g and n, the "Teichmüller groupoid" $T_{g,n}$, i.e. the fundamental groupoid of the module stack $\mathcal{M}_{g,n}$ of compact Riemann surfaces X of genus g with n distinguished points $x_1, \ldots, x_n$. The fundamental groupoid differs from the fundamental group in that we choose not one, but several distinguished points. In the present case it is convenient to choose the distinguished points "at infinity" (see §15 of [De]) in accordance with the methods of "maximal degeneration" of the set $(X, x_1, \ldots, x_n)$. Since degeneration of the set $(X, x_1, \ldots, x_n)$ results in decreasing g and n, the groupoids $T_{g,n}$ for different g and n are connected by certain homomorphisms. The collection of all $T_{g,n}$ and all such homomorphisms is called in [G] the Teichmüller tower. It is observed in [G] that there exists a natural homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G$, where G is the group of automorphisms of the profinite analogue of the Teichmüller tower (in which $T_{g,n}$ is replaced by its pro-finite completion $\hat{T}_{g,n}$). It is also stated in [G], as a plausible conjecture, that $\hat{T}_{0,4}$ and $\hat{T}_{1,1}$ in a definite sense generate the whole tower $\{\hat{T}_{g,n}\}$ and that all relations between generators of the tower come from $\hat{T}_{0,4}$, $\hat{T}_{1,1}$, $\hat{T}_{0,5}$ and $\hat{T}_{1,2}$. This conjecture has been proved, apparently, in Appendix B of the physics paper [MS]. In any case, it is easily seen that $\hat{T}_{0,4}$ generates the subtower $\{\hat{T}_{0,n}\}$, and that all relations in $\{\hat{T}_{0,n}\}$ come from $\hat{T}_{0,4}$ and $\hat{T}_{0,5}$. It can be shown that $\widehat{GT}$ is the automorphism group of the tower $\{\hat{T}_{0,n}\}$. Indeed, an automorphism of this tower is uniquely determined by its action on $\hat{T}_{0,4}$, i.e. on $\hat{\Gamma}$. This action is described by an element (λ, f) satisfying (4.3) and (4.4), and (4.10) is necessary and sufficient for the automorphism of $\hat{T}_{0,4}$ to extend to one of $\hat{T}_{0,5}$. Grothendieck's conjecture implies that the group of automorphisms of the tower $\{\hat{T}_{g,n}\}$ that are compatible with the natural homomorphism $\hat{T}_{0,4} \to \hat{T}_{1,1}$ (to a quadruple of points on $\mathbb{P}^1$ is assigned the double covering of $\mathbb{P}^1$ ramified at these points) is also equal to $\widehat{GT}$.

The possibility that $\widehat{GT}$ may provide the full answer to the double question asked by Grothendieck – namely, what is the automorphism group of the Teichmüller tower and is it equal to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ – constitutes the main motivation behind the study of its properties enumerated in this survey.

§1. Group-theoretic properties of $\widehat{GT}$.

§1.1. Basic ingredients. Let us recall the definitions of the Artin braid groups and the mapping class groups.

Definition. Let B_n denote the Artin braid group on n strands, generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Let $M(0,n)$ be the quotient of B_n by $\sigma_{n-1}\cdots\sigma_1^2\cdots\sigma_{n-1} = 1$ and $(\sigma_1\cdots\sigma_{n-1})^n = 1$. Let K_n and $K(0,n)$ be the *pure* subgroups of B_n and $M(0,n)$, i.e. kernels of the natural surjections of B_n and $M(0,n)$ onto S_n. Both K_n and $K(0,n)$ are generated by the elements $x_{ij} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$ for $1 \le i < j \le n$. Note that $K(0,5)$ is generated by $x_{i,i+1}$ for $i \in \mathbb{Z}/5\mathbb{Z}$, with relations $x_{i,i+1}$ commutes with $x_{j,j+1}$ if $|i-j| \ge 2$ and $x_{51}x_{23}^{-1}x_{12}x_{34}^{-1}x_{23}x_{45}^{-1}x_{34}x_{51}^{-1}x_{45}x_{12}^{-1} = 1$.

For any discrete group G, let $\hat{G}$ denote its profinite completion and $\hat{G}'$ the derived group of $\hat{G}$. Let F_2 denote the free group on two generators x and y. Let θ be the automorphism of $\hat{F}_2$ defined by $\theta(x) = y$ and $\theta(y) = x$. Let ω be the automorphism of F_2 and $\hat{F}_2$ defined by $\omega(x) = y$ and $\omega(y) = (xy)^{-1}$. Let ρ be the automorphism of $K(0,5)$ and $\hat{K}(0,5)$ given by $\rho(x_{i,i+1}) = x_{i+3,i+4}$. If $\hat{K}(0,5)$ is considered as a subgroup of $\hat{M}(0,5)$, then ρ is just the restriction to $\hat{K}(0,5)$ of the inner automorphism $\mathrm{Inn}(c)$ where $c = (\sigma_4\sigma_3\sigma_2\sigma_1)^3$ of $\hat{M}(0,5)$ (here inner automorphism means $\mathrm{Inn}(\alpha)(x) = \alpha^{-1}x\alpha$). Note that $\hat{M}(0,5)$ is generated by σ_1 and c since $c^{-1}\sigma_1 c = \sigma_4$, $c^{-2}\sigma_1 c^2 = \sigma_2$ and $c^{-4}\sigma_1 c^4 = \sigma_3$ (we also have $c^{-3}\sigma_1 c^3 = \sigma_{51}$). For all $f \in \hat{F}_2$, let $\tilde{f}$ denote the image of f in $\langle x_{12}, x_{23}\rangle$ under the isomorphism $\hat{F}_2 \xrightarrow{\sim} \langle x_{12}, x_{23}\rangle \subset \hat{K}(0,5)$ given by $x \mapsto x_{12}$ and $y \mapsto x_{23}$.

As a final point of notation, whenever we have a homomorphism $\phi : \hat{F}_2 \to G$ for a profinite group G, with $\phi(x) = a$ and $\phi(y) = b$, we write $\phi(f) = f(a,b)$ for all $f \in \hat{F}_2$. In particular, when ϕ is the identity we have $f = f(x,y)$

Definition. Let $\underline{\widehat{GT}}_0$ be the set

$$\{(\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2' \,|(I) \quad \theta(f)f = 1;$$
$$(II) \quad \omega^2(fx^m)\omega(fx^m)fx^m = 1 \text{ where } m = (\lambda-1)/2\}.$$

The couples (λ, f) in $\underline{\widehat{GT}}_0$ induce endomorphisms of $\hat{F}_2$ via $x \mapsto x^\lambda$ and

$y \mapsto f^{-1}y^\lambda f$. Let us define a composition law on the elements of $\underline{\widehat{GT}}_0$ by

$$(\lambda, f)(\mu, g) = \big(\lambda\mu, fF(g)\big),$$

where F is the endomorphism of $\hat{F}_2$ corresponding to the couple (λ, f). Let $\widehat{GT}_0$ denote the set of elements $(\lambda, f) \in \underline{\widehat{GT}}_0$ which have a two-sided inverse in $\underline{\widehat{GT}}_0$ under this law; in particular, they are automorphisms of $\hat{F}_2$. Let $\widehat{GT}$ denote the subset of elements $(\lambda, f) \in \widehat{GT}_0$ satisfying the further relation, which takes place in $\hat{K}(0,5)$:

$$(III) \quad \rho^4(\tilde{f})\rho^3(\tilde{f})\rho^2(\tilde{f})\rho(\tilde{f})\tilde{f} = 1.$$

§1.2. $\widehat{GT}$ is a group.

The proof that $\widehat{GT}_0$ (resp. $\widehat{GT}$) is a group, i.e. that if (λ, f) and $(\mu, g) \in \widehat{GT}_0$ (resp. $\widehat{GT}$) then $(\lambda\mu, fF(g)) \in \widehat{GT}_0$ (resp. $\widehat{GT}$) is not at all immediate. Drinfel'd obtained this as a consequence of the definition of $\widehat{GT}$ as a group of transformations of a certain category; this argument is given more simply in the profinite case in [IM, A.4]. Here we will describe a proof based only on group theory, citing [LS] for parts of the proof. The key to the proof is the fact that the two (resp. three) relations defining $\widehat{GT}_0$ (resp. $\widehat{GT}$) can be interpreted as the relations necessary to make elements of $\widehat{GT}_0$ (resp. $\widehat{GT}$) into automorphisms of $\hat{F}_2$ which extend to automorphisms of $\hat{B}_3$ (resp. $\hat{M}(0,5)$) in a certain way. Let us give the precise statement for $\widehat{GT}_0$ first.

Lemma 1. *Suppose $(\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2'$ gives an automorphism of $\hat{F}_2$ via $x \mapsto x^\lambda$ and $y \mapsto f^{-1}y^\lambda f$. Identify $\hat{F}_2$ with the pure subgroup of $\hat{B}_3$ by setting $x = \sigma_1^2$ and $y = \sigma_2^2$. Then (λ, f) extends to an automorphism of $\hat{B}_3$ via*

$$\sigma_1 \mapsto \sigma_1^\lambda, \quad \sigma_2 \mapsto f^{-1}\sigma_2^\lambda f$$

if and only if $(\lambda, f) \in \widehat{GT}_0$, i.e. if and only if (λ, f) satisfies relations (I) and (II).

The proof of this lemma consists in an easy direct computation of the constraints imposed on the couple (λ, f) by the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and the fact that $(\sigma_1\sigma_2)^3$ is central in $\hat{B}_3$. It is given on p. 335 of [LS].

Now, two pairs (λ, f) and (μ, g) in $\widehat{GT}_0$ give automorphisms of $\hat{B}_3$, so their composition $(\lambda\mu, fF(g))$ gives an automorphism of $\hat{B}_3$, so by lemma 1 it also lies in $\widehat{GT}_0$. An automorphism (λ, f) also has an inverse of the form (μ, g), so $\widehat{GT}_0$ is a group.

Let us give the analogous result for $\widehat{GT}$.

Lemma 2. *A pair $F = (\lambda, f) \in \hat{\mathbf{Z}}^* \times \hat{F}_2'$ gives an automorphism of $\hat{M}(0,5)$ via $\sigma_1 \mapsto \sigma_1^\lambda$ and $c \mapsto c\tilde{f}$ if and only if the couple lies in $\widehat{GT}$, i.e. satisfies (I), (II) and (III).*

Let us sketch the proof of this lemma. The first point is to notice that the subgroup $\langle\sigma_1, \sigma_2\rangle \subset \hat{M}(0,5)$ is isomorphic to $\hat{B}_3$, and that this subgroup is preserved by the proposed action since $\sigma_2 = c^{-2}\sigma_1 c^2 \mapsto \tilde{f}^{-1}c^{-2}\sigma_1^\lambda c^2\tilde{f} = \tilde{f}^{-1}\sigma_2^\lambda\tilde{f}$ and $\tilde{f} \in \langle x_{12} = \sigma_1^2, x_{23} = \sigma_2^2\rangle$. Thus, by lemma 1, $(\lambda, f) \in \widehat{GT}_0$, i.e. it satisfies relations (I) and (II).

Now, one direction of the proof is easy; if the proposed action of (λ, f) gives an automorphism of $\hat{M}(0,5)$, then in particular it respects the equation $c^5 = 1$, so $(c\tilde{f})^5 = 1$. But since $\rho = \mathrm{Inn}(c)$, we have

$$(c\tilde{f})^5 = c^5\rho^4(\tilde{f})\rho^3(\tilde{f})\rho^2(\tilde{f})\rho(\tilde{f})\tilde{f} = 1,$$

which is exactly relation (III) defining $\widehat{GT}$.

The other direction is more complicated. To show that the action in the statement extends to an automorphism of $\hat{M}(0,5)$, it is necessary to show that it respects all the relations in a presentation of $\hat{M}(0,5)$. It is easiest to use the presentation with generators σ_1, σ_2, σ_3 and σ_4. To compute the action of (λ, f) on the σ_i, we use the fact that they are simply the conjugates of σ_1 by the powers of c; of course, we do not know that (λ, f) is an automorphism yet. But if we show that assuming relation (III), the map defined as follows on the σ_i is an automorphism, then we can compute the image of c directly and find that $c \mapsto c\tilde{f}$, so that also gives an automorphism.

$$\begin{aligned}
\sigma_1 &\mapsto \sigma_1^\lambda \\
\sigma_2 &\mapsto f(x_{23}, x_{12})\sigma_2^\lambda f(x_{12}, x_{23}) \\
\sigma_3 &\mapsto f(x_{34}, x_{45})\sigma_3^\lambda f(x_{45}, x_{34}) \\
\sigma_4 &\mapsto \sigma_4^\lambda \\
\sigma_{51} &\mapsto f(x_{23}, x_{12})f(x_{51}, x_{45})\sigma_{51}^\lambda f(x_{45}, x_{51})f(x_{12}, x_{23}).
\end{aligned}$$

These expressions were first computed by Nakamura (see the appendix of [N]) in a somewhat different (Galois) situation; he proves there that if this action gives an automorphism of $\hat{M}(0,5)$, then f satisfies relation (III). Conversely, assuming that f satisfies relation (III), we need to check that each of the relations between the σ_i are respected by this action. It turns out that all but one are respected simply by virtue of the fact that (λ, f) lies in $\widehat{GT}_0$; the last one is respected by the assumption that f satisfies (III). This computation is done explicitly in [LS], p. 340-342.

As in the case of $\widehat{GT}_0$, we immediately obtain the result we are seeking for:

Corollary. *$\widehat{GT}$ is a group.*

Symmetry. $\widehat{GT}$ is a natural group to look at also because of the following theorem, stating that it consists of *all* automorphisms of $\hat{K}(0,5)$ extending automorphisms of the subgroup $\hat{F}_2 = \langle x_{12}, x_{23}\rangle$ and satisfying certain simple symmetry conditions with respect to the actions of θ and ω on $\hat{F}_2$ and ρ on $\hat{K}(0,5)$, which is essentially equivalent to considering the *outer* actions of S_2 and S_3 on $\hat{F}_2$ and S_5 on $\hat{K}(0,5)$. This theorem is proved in the article [HS] in this volume. For a pair $(\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2'$, we write F for the associated endomorphism of $\hat{F}_2$.

Theorem. *(i) An invertible pair (λ, f) (for the usual composition law) satisfies relation (I) if and only if the images of F and θ commute in $\mathrm{Out}(\hat{F}_2)$.*

(ii) Given an invertible pair (λ, f) satisfying (I), it satisfies relation (II) if and only if the images of F and ω commute in $\mathrm{Out}(\hat{F}_2)$.

(iii) Let (λ, f) be an invertible pair satisfying (I) and (II), so in $\widehat{GT}_0$. Then (λ, f) lies in $\widehat{GT}$ if and only if there exists an automorphism F of $\hat{K}(0,5)$ extending that of $\hat{F}_2$ in the sense that $F(x_{12}) = x_{12}^{\lambda}$ and $F(x_{23}) = f(x_{23}, x_{12})x_{23}^{\lambda} f(x_{12}, x_{23})$, such that F and ρ commute in $\mathrm{Out}(\hat{K}(0,5))$.

§1.3. Cohomological interpretation

Given the definition of $\widehat{GT}$, it is natural to ask oneself if it is possible to give a complete description of the elements $f \in \hat{F}_2'$ satisfying each of the three relations. This makes sense for relations (I) and (III), in which λ does not intervene; we let $\mathcal{E}_I$ and $\mathcal{E}_{III}$ denote the subsets of $\hat{F}_2$ of elements f satisfying relations (I) and (III) respectively. For relation (II), we let $\mathcal{E}_{II} = \{f \in \hat{F}_2' \mid f$ satisfies relation (II) for some value of $\lambda\}$. In fact, it is not difficult to show that if $f \in \hat{F}_2'$ satisfies relation (II) for some λ, then it satisfies relation (II) also for $-\lambda$ but not for any other value (cf. [LS2], Lemma 9).

Let us give a complete description of the sets $\mathcal{E}_I$, $\mathcal{E}_{II}$ and $\mathcal{E}_{III}$. Consider for instance $\mathcal{E}_I$. θ is an automorphism of $\hat{F}_2$ of order 2 and the relation $\theta(f)f = 1$ is a cocycle relation for this automorphism. The set of classes of these cocycles up to coboundaries (i.e. elements $f = \theta(g)^{-1}g \in \hat{F}_2$, all of which satisfy $\theta(f)f = 1$) by the elements of the non-abelian cohomology set $H^1(\langle\theta\rangle, \hat{F}_2)$. Computation of this cohomology set shows that it is trivial (cf. [LS2]), so that we have

$$\mathcal{E}_I = \{\theta(g)^{-1}g \mid g \in \hat{F}_2\},$$

i.e. a complete list of the elements of $\mathcal{E}_I$. The full set of elements of $\mathcal{E}_{II}$ and $\mathcal{E}_{III}$ can be computed similarly. It is necessary to compute the non-abelian cohomology sets $H^1(\langle\omega\rangle, \hat{F}_2)$ and $H^1(\langle\rho\rangle, \hat{K}(0,5))$; then all elements of $\mathcal{E}_{II}$ (resp. $\mathcal{E}_{III}$) are given up to coboundaries $\omega(h)^{-1}h$ for $h \in \hat{F}_2$ (resp. $\rho(k)^{-1}k$ for $k \in \hat{K}(0,5)$) by representative cocycles of these cohomology sets. The result of these computations is given in the following theorem, the main result of [LS2].

Theorem. *Let $(\lambda, f) \in \widehat{GT}$, and let $m = (\lambda-1)/2$. Then there exist elements g and $h \in \hat{F}_2$ and $k \in \hat{K}(0,5)$ such that we have the following equalities, of which the first two take place in $\hat{F}_2$ and the third in $\hat{K}(0,5)$:*

$$(I') \qquad f = \theta(g)^{-1}g$$

$$(II') \qquad fx^m = \begin{cases} \omega(h)^{-1}h & \text{if } \lambda \equiv 1 \ mod\ 3 \\ \omega(h)^{-1}\,xy\,h & \text{if } \lambda \equiv -1 \ mod\ 3 \end{cases}$$

$$(III') \qquad f(x_{12}, x_{23}) = \begin{cases} \rho(k)^{-1}k & \text{if } \lambda \equiv \pm 1 mod\ 5 \\ \rho(k)^{-1}\,x_{34}x_{51}^{-1}x_{45}x_{12}^{-1}\,k & \text{if } \lambda \equiv \pm 2\ mod\ 5. \end{cases}$$

§1.4. Further group-theoretic remarks and questions.

Over the last two or three years, many people have asked me if $\widehat{GT}$ possesses certain properties which are known or suspected for the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (which injects into $\widehat{GT}$; see §3.1). Here are several of the more group-theoretic of these questions, some solved, some unsolved, and some only partially solved.

(1) Is the "complex conjugation" element $(-1,1)$ self-centralizing in $\widehat{GT}$? This question was asked by Y. Ihara; its answer turns out to be yes. A computation reduces this result to showing that the only element of $\hat{F}_2$ which is fixed under the automorphism ι given by $\iota(x) = x^{-1}$ and $\iota(y) = y^{-1}$ is the trivial element, in other words that the centralizer of ι in the semi-direct product $\hat{F}_2 \rtimes \langle\iota\rangle$ is exactly $\langle\iota\rangle$; this can be shown in a variety of ways (cf. [LS2]), suggested to me by J-P. Serre.

(2) The derived subgroup $\widehat{GT}'$ of $\widehat{GT}$ is contained in the subgroup $\widehat{GT}^1$ of pairs $(\lambda, f) \in \widehat{GT}$ with $\lambda = 1$. Are these two subgroups equal? This question was also asked by Y. Ihara, and remains unsolved.

(3) Is $\widehat{GT}$ itself a profinite group? This question was asked and (almost immediately) answered by Florian Pop, who noted that indeed $\widehat{GT}$ is the inverse limit of its own images in the (finite) automorphism groups of the quotients $\hat{F}_2/N$, where N runs over the characteristic subgroups of finite

index of $\hat{F}_2$. There are other ways of obtaining $\widehat{GT}$ as an inverse limit, cf. [HS] in this volume.

(4) Is there any torsion in $\widehat{GT}$ apart from the elements of order 2 given by $(-1,1)$ and its conjugates? After a short discussion with Florian Pop, we were able to prove the weak result that any such torsion elements become trivial in the *pro-nilpotent quotient* of $\widehat{GT}$; no stronger result seems to be known. Let us indicate the proof of this result by showing there are any elements of order 2 with $\lambda = 1$ become trivial in the pro-nilpotent quotient of $\widehat{GT}$ (note that any torsion element must have $\lambda = \pm 1$). Suppose there exists $(1, f) \in \widehat{GT}$ such that $(1, f)(1, f) = 1$. Then $f(x, y)f\big(x, f(y, x)yf(x, y)\big) = 1$. We know that $f \in \hat{F}_2'$. Considering this equation modulo the second commutator group $\hat{F}_2'' = [\hat{F}_2, \hat{F}_2']$, we see that modulo $\hat{F}_2''$, we have $f(x, y)^2 = 1$, so since there is no torsion in this group, we must have that $f(x, y) \in \hat{F}_2''$. Working modulo $\hat{F}_2'''$ and so on, we quickly find that f lies in the intersection of all the successive commutator subgroups of $\hat{F}_2$, i.e. that the image of $f(x, y)$ in the nilpotent completion $\hat{F}_2^{\mathrm{nil}}$ is trivial.

(5) Is the outer automorphism group of $\widehat{GT}$ trivial? (F. Pop, absolutely unsolved).

(6) Is it possible to determine the finite quotients of $\widehat{GT}$? (Everyone connected with inverse Galois theory; unsolved except for the obvious remark that $(\lambda, f) \mapsto \lambda$ gives a surjection $\widehat{GT} \to \hat{\mathbb{Z}}^*$, and therefore all abelian groups occur, just as for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

(7) Is the subgroup $\widehat{GT}^1$ of pairs (λ, f) with $\lambda = 1$ profinite free? (Everyone interested in the Shafarevich conjecture; unsolved.)

(8) Less ambitiously than in (7), is the cohomological dimension of $\widehat{GT}^1$ finite? (J-P. Serre, unsolved).

(9) Is it possible to determine the image of, say, the elements $f \in \hat{F}_2$ belonging to pairs $(\lambda, f) \in \widehat{GT}$, in a given finite quotient $\hat{F}_2/N$ of $\hat{F}_2$? (J. Oesterle was the first of several people to ask me this question. Unfortunately, we do not know how to compute this image algorithmically, though there is a non-algorithmic (i.e. ending after a finite number of steps but one doesn't know how many) and very unwieldy procedure for doing it, and also an easier approximative procedure, giving a set which may be too large (cf. [Harbater-Schneps] in this volume). The point is to determine the $\widehat{GT}$-orbits of covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ and compare them with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits.

(10) If $\beta : X \to \mathbb{P}^1\mathbb{C}$ is a Belyi cover, then an element of $\widehat{GT}$ sends it to another Belyi cover $\beta' : X' \to \mathbb{P}^1\mathbb{C}$. Given X, is the curve X' independent

of the choice of β? Certainly this is so whenever the element of $\widehat{GT}$ happens to lie in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

(11) Is it possible to define an action of $\widehat{GT}$ on $\bar{\mathbb{Q}}$ using its action on covers – even an action weaker than an automorphism? For instance, if the answer to question (10) were yes, then letting X range over the elliptic curves, we would obtain a well-defined action of $\widehat{GT}$ on $\bar{\mathbb{Q}}$ by considering their j-invariants. But perhaps it is possible to define such an action (corresponding to the usual Galois action when restricted to the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\widehat{GT}$) without answering question (10).

The answers to questions (1), (3), (4), (5), (8) and of course (10) and (11) are known for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

§2. $\widehat{GT}$-actions

In this section we describe several related objects which are equipped with a $\widehat{GT}$-action (compatible with a natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action when $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is considered as a subgroup of $\widehat{GT}$, cf. §3.2).

To begin with, we should mention the first appearance of $\widehat{GT}$, or at least of its discrete, pro-ℓ and k-pro-unipotent versions. They were first defined by Drinfel'd in [D] as groups of transformations of the associativity and commutativity morphisms of a quasi-triangular quasi-Hopf algebra. In the same article, he showed that the k-pro-unipotent version of $\widehat{GT}$ is an automorphism group of the k-pro-unipotent completions $B_n(k)$ of the Artin braid groups B_n respecting the natural inclusion automorphisms $B_{n-1}(k) \hookrightarrow B_n(k)$ given by $\sigma_i \mapsto \sigma_i$ for $i = 1, \ldots, n-2$. Following ideas of Grothendieck in the *Esquisse d'un Programme*, he further indicated that using the relations between braided tensor category structures and the structure of the fundamental groupoid of the moduli spaces of Riemann surfaces based at "tangential base points", the profinite version of $\widehat{GT}$ should be the automorphism group of the tower of these fundamental groupoids linked by certain natural connecting homomorphisms. All the $\widehat{GT}$-actions described below are essentially concretizations of these ideas of Drinfel'd and of Grothendieck.

§2.1. $\widehat{GT}$-action on groups

(1) Artin braid groups. $\widehat{GT}$ acts on the profinite completions of the Artin braid groups B_n. Up to an easy adaptation to the profinite case (cf.

[IM, A.4] or [LS]), Drinfel'd gave the action on the $\hat{B}_n$ as

$$\sigma_i \mapsto f(\sigma_i^2, y_i)\sigma_i^\lambda f(y_i, \sigma_i^2),$$

where $y_i = \sigma_{i-1} \cdots \sigma_1 \cdot \sigma_1 \cdots \sigma_{i-1}$. The proof that this action really induces an automorphism follows, in his paper, from the role of $\widehat{GT}$ as transformations of a quasi-triangular quasi-Hopf algebra. A direct proof is given in [LS]; indeed the result proved there is actually stronger (see §2.3 below).

(2) Genus zero mapping class groups. The group $M(0,n)$ is the quotient of the group $\hat{B}_n$ by the two relations $(\sigma_{n-1} \cdots \sigma_1)^n = 1$ and $\sigma_{n-1} \cdots \sigma_1^2 \cdots \sigma_{n-1} = 1$. In order to show that the action of $\widehat{GT}$ on $\hat{B}_n$ given above passes to the quotient, it suffices to show that in fact the two elements $\omega_n = (\sigma_{n-1} \cdots \sigma_1)^n$ and $y_n = \sigma_{n-1} \cdots \sigma_1^2 \cdots \sigma_{n-1}$ are sent to ω_n^λ and y_n^λ by an element $(\lambda, f) \in \widehat{GT}$ acting on $\hat{B}_n$ as in (1); this is an easy computation (cf. [IM] or [LS]). Note that since the pure mapping class subgroups $K(0,n)$ are characteristic, $\widehat{GT}$ acts on them as well.

The reason it is interesting for us to have $\widehat{GT}$ act on the groups $\hat{K}(0,n)$ and $\hat{M}(0,n)$ is that these groups have geometric significance: they are the (algebraic) fundamental groups of the moduli spaces of Riemann spheres with n *ordered* resp. *unordered* distinct marked points. We denote these moduli spaces by $\mathcal{M}_{0,n}$ and $\mathcal{M}_{0,[n]}$ respectively. (There are many references for the definitions of these moduli spaces; see for instance [Lochak] in this volume.) This remark is developed further in the examination of $\widehat{GT}$-actions on groupoids below.

(2') Dessins d'enfants. Dessins d'enfants are the drawings on topological surfaces obtained by taking $\beta^{-1}([0,1])$ where $\beta : X \to \mathbb{P}^1\mathbb{C}$ is a cover of Riemann surfaces ramified over 0, 1 and ∞ only. They are in bijection with the pairs (β, X) up to isomorphism, and therefore with the set of conjugacy classes of subgroups of finite index of $\hat{F}_2 = \pi_1(\mathbb{P}^1 - \{0,1,\infty\})$. The link with the situation of (2) is given by the fact that $\mathbb{P}^1 - \{0,1,\infty\}$ is isomorphic to the first non-trivial moduli space $\mathcal{M}_{0,4}$, i.e. $K(0,4) \simeq F_2$. The two main problems of the theory of dessins d'enfants are the following: (i) given a dessin, i.e. a purely combinatorial object, find the equations for β and X explicitly; (ii) find a list of (combinatorial? topological? algebraic) invariants of dessins which completely identify their $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits. The second problem can be interestingly weakened from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\widehat{GT}$, but it remains absolutely non-trivial.

(3) Fundamental groups of spheres with n removed points. For every $n \geq 5$, there is a homomorphism $\iota : K(0,n) \to K(0,n-1)$ obtained by removing the n-th strand of the braids; ι passes to the profinite completions.

It is easily checked by direct computation that the $\widehat{GT}$-action on the $\hat{K}(0,n)$ for $n \geq 4$ are compatible with this homomorphism. This remark has the following geometric interpretation. Let $S_{0,n-1}$ denote a topological sphere with $n-1$ punctures. For a suitable choice of base points, we have an exact sequence

$$1 \to \hat{\pi}_1(S_{0,n-1}) \to \hat{\pi}_1(\mathcal{M}_{0,n}) \to \hat{\pi}_1(\mathcal{M}_{0,n-1}) \to 1,$$

where the map between fundamental groups on moduli spaces comes from the map $\mathcal{M}_{0,n}$ is given by erasing the n-th point on every sphere. This exact sequence corresponds to considering the moduli space $\mathcal{M}_{0,n}$ as the *universal curve* over $\mathcal{M}_{0,n-1}$. The map from $\hat{\pi}_1(\mathcal{M}_{0,n}) \simeq \hat{K}(0,5)$ to $\pi_1(\mathcal{M}_{0,n-1}) \simeq \hat{K}(0,n-1)$ is exactly ι. Thus by the exact sequence we obtain a $\widehat{GT}$-action on the kernel of ι, namely on $\hat{\pi}_1(S_{0,n-1})$ (which is, of course, a free group of rank $n-2$). This $\widehat{GT}$-action can be explicitly computed. It is compatible with a (lifting of) the outer $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $\hat{\pi}_1(S_{0,n-1})$ associated to a certain choice of "tangential" sphere with n marked points. Indeed, using tangential base point techniques, Ihara and Nakamura were able to explicitly compute (cf. [IN] and further unpublished work) a $\widehat{GT}_0$-action on $\pi_1(S_{g,n-1})$ where $S_{g,n-1}$ denotes a topological surface of genus $g \geq 0$ with $n-1$ punctures; see the end of this article for the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-compatibility properties of this action.

(4) Higher genus mapping class groups. In the case $g > 0$, we know that there is a $\widehat{GT}$-action on the mapping class groups $\hat{M}(g,n)$ for the following pairs (g,n): $(1,1)$, $(1,2)$, $(2,0)$, and $(2,1)$. These actions are obtained quite simply by the well-known similarities between these groups and Artin braid groups or genus zero mapping class groups. Indeed, the group $M(1,1)$ is isomorphic to B_3 modulo its center (and to $\mathrm{PSL}_2(\mathbb{Z})$). Since the natural action of $\widehat{GT}$ (and even $\widehat{GT}_0$) on $\hat{B}_3$ passes to the quotient mod center, $\hat{M}(1,1)$ inherits it. The group $\hat{M}(1,2)$ is a quotient of a certain subgroup of $\hat{B}_5$ and again, it is easily confirmed that the $\widehat{GT}$-action passes to it. By the Birman-Hilden presentations of $M(2,1)$ and $M(2,0)$, one can show directly that $M(2,1)$ is isomorphic to the quotient of $\hat{B}_6$ by the single relation

$$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4\sigma_5 = (\sigma_1\sigma_2\sigma_3\sigma_4)^5,$$

and that the $\widehat{GT}$-action on $\hat{B}_5$ passes to this quotient and to the group $\hat{M}(2,0)$ which is the quotient of $\hat{M}(2,1)$ by the relation $(\sigma_1\sigma_2\sigma_3\sigma_4)^5 = 1$.

According to Grothendieck's conception, these facts should suffice to build up a $\widehat{GT}$-action on all the $\hat{M}(g,n)$, but this remains an open problem. We note here the important discovery by Nakamura of the explicit form of a *Galois* action on the groups $\hat{M}(g,1)$ and $\hat{M}(g,0)$. This naturally gives rise to a conjecture concerning the $\widehat{GT}$-action; see further remarks on this at the end of this article.

§2.2. $\widehat{GT}$-action on groupoids

(1) Braided tensor categories. Let us first of all mention a certain *braided tensor category* $\mathcal{C}$ on which $\widehat{GT}$ acts. This groupoid has less structure than the quasi-triangular quasi-Hopf algebras used in [D], and there is also another difference with from the abstract braided tensor category whose objects are *bracketing patterns* on objects $V_1, \ldots, V_n$ used in [D]. Namely, the approach here substitutes numbered trivalent trees for bracketing patterns; the combinatorial difference is slight, and this version has the advantage of reflecting more closely the structure at infinity of the moduli spaces (similarly to the braided-tensor category constructions of Moore and Seiberg [MS]). The category $\mathcal{C}$ (suitably completed) possesses the basic elements of structure necessary to ensure a $\widehat{GT}$-action.

Unfortunately, the full definition of the groupoid in question is too long to reproduce here, even assuming the definition of a braided tensor category. It can be found in detail in [PS, chapter II]. We content ourselves with a rather brief description. $\mathcal{C}$ is a braided tensor category whose objects are the trivalent trees equipped with a cyclic order on the three edges coming out of each trivalent vertex, and a numbering which consists of a positive integer associated to each tail, except for exactly one distinguished tail numbered 0. The tensor product on such trees is given by attaching the two 0-tails of two trees together and adding a new tail and a cyclic order at the new vertex. The associativity morphisms change the position of an inner edge from horizontal to vertical with respect to the four branches coming out of it (i.e. an H configuration is changed to an I configuration). The commutativity morphisms consist in rigidly switching two branches of the tree which meet at a single vertex. All morphisms of the category are generated (in the sense of composing and taking tensor products) by these morphisms of associativity and commutativity. The local group $\mathrm{Hom}(T, T)$ at a tree T with n tails is exactly the subgroup of the Artin braid group B_n which is the preimage of the subgroup of the permutation group S_n which leaves the tree fixed when acting as permutations on the indices labeling the tails. Thus the local group is larger than K_n only for trees whose indices are not all distinct. In order to define a $\widehat{GT}$-action on such a braided tensor category, it is necessary to take its *profinite completion*, which consists in adding new morphisms, in the sense that each local group is replaced by its profinite completion. Then (very roughly speaking) the action of an element $(\lambda, f) \in \widehat{GT}$ on the profinite category fixes the objects and sends an associativity morphism from T_1 to T_2 to itself precomposed by the image of f in the local group $\widehat{\mathrm{Hom}}(T_1,T_1)$ under a certain natural homomorphism $\hat{F}_2 \to \widehat{\mathrm{Hom}}(T_1,T_1)$, and the image of a commutativity morphism $c(T_1,T_2)$ from T_1 to T_2 is given by $\big(c(T_2,T_1)c(T_1,T_2)\big)^m c(T_1,T_2)$. Note that $c(T_2,T_1)$ is not the inverse of $c(T_1,T_2)$, since the rigid switch of two branches of T_1 performed twice gives a non-trivial morphism from T_1 to itself (visually, one can imagine that in switching, the left-hand branch passes above the right-hand one, so that doing it two times makes a full twist on the third edge connected to the common vertex of the two branches; this is non-trivial). Since all elements of the profinite local groups $\widehat{\mathrm{Hom}}(T,T)$ are generated by the associativity and commutativity moves, this action defines a $\widehat{GT}$-action on the local groups, which turns out to correspond to the $\widehat{GT}$-action on Artin braid groups given above, up to inner automorphisms as one changes from one object to another.

(2) The fundamental groupoids based near infinity of the genus zero moduli spaces. In order to define these groupoids, known as *Teichmüller groupoids*, it is necessary only to define the set of simply connected regions which will serve as base points (a complete and detailed description of them can be found in [PS, I.2].) They are obtained as follows: the neighborhood of each point of *maximal degeneration* in the stable compactification of the genus zero moduli space $\mathcal{M}_{0,n}$ is isomorphic to the $(n-3)$-fold product of a pointed disk with itself; the real locus of this neighborhood naturally falls into 2^{n-3} simply connected pieces. (Simply think of the moduli space $\mathcal{M}_{0,4}$, isomorphic to $\mathbb{P}^1 - \{0,1,\infty\}$: the stable compactification is $\mathbb{P}^1$, the points of maximal degeneration are 0, 1 and ∞, their neighborhoods are pointed disks and the real part of each disk consists of the two segments of the real line on each side of the missing point. This original set of tangential base points was defined by Deligne in [De].) We denote this set of *tangential base points* or *base points near infinity* by $\mathcal{B}_n$, and write $T(0,n) = \pi_1(\mathcal{M}_{0,n};\mathcal{B}_n)$ for the Teichmüller groupoid. Let us give a brief indication of why $\widehat{GT}$ acts on the profinite completion of $T(0,n)$ for $n \geq 4$ (cf. [PS, II] for full proofs). The tangential base points can be visualized as degenerating Riemann spheres with n punctures; they are in bijection with the combinatorial set of trivalent n-tailed trees with tails numbered 1 to n (in any order). Let $\hat{\mathcal{C}}_n$ denote the subgroupoid of the profinite completion $\hat{\mathcal{C}}$ of $\mathcal{C}$ whose objects are the set of trivalent trees with tails numbered from 0 to $n-1$ and whose morphisms are all morphisms of $\hat{\mathcal{C}}$ between these objects. Renumbering the n-th tail to 0 for each tree in $\hat{\mathcal{C}}_n$, we can define a surjective groupoid homomorphism from $\hat{\mathcal{C}}_n$ to $\hat{T}(0,n)$; the associativity and commutativity morphisms have simple geometric interpretations as natural paths on the moduli space $\mathcal{M}_{0,n}$ between tangential base points. The kernel of this homomorphism is not very important; the local groups of $\hat{\mathcal{C}}_n$ are isomorphic to $\hat{K}_n$ and those of $\hat{T}(0,n)$ are the mapping class groups $\hat{K}(0,n)$, and the relations involved in the surjection $\hat{K}_n \to \hat{K}(0,n)$ for all the local groups completely describes the kernel. Once we check that the action of $\widehat{GT}$ on $\hat{\mathcal{C}}$ preserves the subgroupoid $\hat{\mathcal{C}}_n$, which is not difficult, this remark shows that the $\widehat{GT}$-action passes to $\hat{T}(0,n)$.

(3) Higher genus groupoids. As yet, not much is known about a possible $\widehat{GT}$-action on higher genus groupoids. Even the complete structure of the Teichmüller groupoids $T(g,n)$ has not been determined. However, generalizing work by Moore and Seiberg, Lyubashenko recently described an abstract groupoid somewhat in the same vein as $\mathcal{C}$ above, whose objects instead of being trees are trivalent graphs with loops, and which contains besides the associativity and commutativity morphisms, further morphisms related especially to the presence of the loops. The situation is ripe to attempt to determine if there is a $\widehat{GT}$-action on Lyubashenko's groupoid, or conversely if its group of automorphisms can be explicitly determined by analogous simple relations, and the precise relationship of this groupoid with the higher genus Teichmüller groupoids.

(4) The universal Ptolemy-Teichmüller groupoid. The $\widehat{GT}$-action on the profinite completion of a suitable extension of Thompson's group is proved in [Lochak-Schneps] in the companion volume to this one; by the construction of the groupoid, it is closely related to the $\widehat{GT}$-action on the genus zero Teichmüller groupoids. However, the consequences of the various intriguing relations between the Ptolemy-Teichmüller groupoid and higher genus situations are not yet understood.

§**2.3. Towers**

A *tower* of groups or groupoids is simply a collection of groups or groupoids linked by homomorphisms. An automorphism of a tower is a tuple $(\phi_G)_G$ of automorphisms of all the groups (or groupoids) G in the tower, respecting all the linking homomorphisms. The advantage of towers over isolated groups or groupoids in our situation is that instead of obtaining statements of the form "$\widehat{GT}$ is an automorphism group of such-and-such a group", one obtains stronger statements of the form "$\widehat{GT}$ is *the full* group of automorphisms (of a certain type) of such-and-such a tower".

(1) Artin braid group tower. The simplest example is the small tower of Artin braid groups used in the following theorem proved in [LS]. Let $\hat{A}_3$ be the subgroup of $\hat{B}_3$ generated by σ_1^2 and σ_2; this is exactly the preimage of the subgroup $\{1,(23)\}$ under the natural surjection $\hat{B}_3 \to S_3$. Let $\kappa : \hat{A}_3 \hookrightarrow \hat{B}_4$ be defined by $\kappa(\sigma_1^2) = \sigma_2\sigma_1^2\sigma_2$, $\kappa(\sigma_2) = \sigma_3$. Let $\iota : \hat{B}_3 \hookrightarrow \hat{B}_4$ be given by $\iota(\sigma_i) = \sigma_i$ for $i = 1, 2$.

Theorem. *$\widehat{GT}_0$ is the subgroup of $\mathrm{Aut}(\hat{B}_3)$ defined by properties (i) and (ii) below, and $\widehat{GT}$ is the subgroup of $\widehat{GT}_0$ defined by the additional properties (iii) and (iv).*

(i) $\phi(\sigma_1) = \sigma_1^\lambda$ for some $\lambda \in \mathbb{Z}^*$;

(ii) ϕ preserves permutations, i.e. if $\eta : \hat{B}_3 \to S_3$ is the natural surjection, then $\eta \circ \phi = \eta$ on $\hat{B}_3$;

(iii) there exists an automorphism ϕ' of $\hat{B}_4$ such that $\phi' \circ \iota = \iota \circ \phi$ as maps from $\hat{B}_3$ to $\hat{B}_4$;

(iv) there exists an automorphism ϕ' of $\hat{B}_4$ such that $\phi' \circ \kappa = \kappa \circ \phi$ as maps from $\hat{A}_3 \to \hat{B}_4$. (Note that by (ii), ϕ preserves $\hat{A}_3$, so that the map $\kappa \circ \phi$ is defined on $\hat{A}_3$.)

In other words, $\widehat{GT}$ is the group of automorphisms $(\phi, \phi|_{\hat{A}_3}, \phi')$ of the groups $\hat{B}_3$, $\hat{A}_3$ and $\hat{B}_4$, which send σ_1 to a power of itself, preserve permutations and respect the homomorphisms ι and κ. We note that an analogous tower can be constructed with all the profinite braid groups $\hat{A}_n = \langle \sigma_1^2, \sigma_2, \ldots, \sigma_{n-1} \rangle$ and $\hat{B}_n$ and the inclusion homomorphisms as well as those doubling the first string of every braid; the automorphism group of this tower is still $\widehat{GT}$, reflecting Grothendieck's deep principle that everything is decided on the first two levels.

(2) The tower of the $\hat{T}(0,n)$ with point-erasing homomorphisms. There are natural homomorphisms relating the profinite Teichmüller groupoids $\hat{T}(0,n)$; they do not go *up* the tower as in the Artin braid group tower above, but down from $\hat{T}(0,n)$ to $\hat{T}(0,n-1)$, and are induced by the maps r_i from the moduli space $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n-1}$ given by erasing the i-th point of every sphere with n marked points. The main theorem of [S] states that $\widehat{GT}$ is the automorphism group of this tower; the methods of proof are quite similar. A Teichmüller tower in all genera has not yet been constructed (it should be remarked that in order to equip it with adequate homomorphisms, it is probably necessary to use Riemann surfaces with holes, i.e. with boundary components, rather than just punctures).

§3. Relations between $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\widehat{GT}$.

The main statement is the major result due to Drinfel'd and Ihara that

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a subgroup of $\widehat{GT}$; this result suggests the conjecture that the two groups are in fact isomorphic. Below, we indicate the definition of the inclusion map given in [I1] and the proof of the three relations given in [LS2]. The next step thus consists in adducing evidence for the conjecture by showing that various properties of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are also properties of $\widehat{GT}$. Several such *group-theoretic* properties were considered in §1.4; in §3.2 we explain the compatibility of the $\widehat{GT}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ actions on the various fundamental groups and groupoids described in §2.

§3.1. The injection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT}$.

Let $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ denote the moduli spaces of Riemann spheres with four and five ordered marked points. We need the two following basic facts about $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

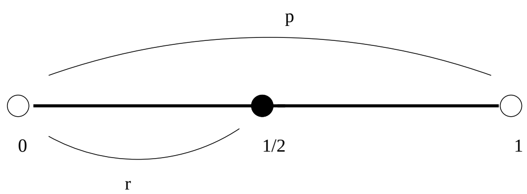
Fact 1. Let $X = \mathcal{M}_{0,4}$ or $\mathcal{M}_{0,5}$. Let

$$\Sigma = \{\text{points defined over } \overline{\mathbb{Q}}\} \cup \{\text{tangential base points}\}.$$

For any α, $\beta \in \Sigma$, the set of *pro-paths* from α to β is given by any path from α to β precomposed with all pro-loops based at α, i.e. elements of the profinite completion of $\pi_1(X;\alpha)$. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of pro-paths of X with base points in Σ; a pro-path from α to β is sent by $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to a pro-path from $\sigma(\alpha)$ to $\sigma(\beta)$. σ fixes the tangential base points.

Fact 2. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on pro-paths of X commutes with the action of $\mathrm{Aut}(X)$. Note that $\mathrm{Aut}(\mathcal{M}_{0,4}) \simeq S_3$, generated by $\Theta(z) = 1 - z$ and $\Omega(z) = 1/1 - z$ and $\mathrm{Aut}(\mathcal{M}_{0,5}) \simeq S_5$; these automorphisms correspond to permuting marked points on spheres.

Let $\hat{F}_2$ be identified with the π_1 of $\mathcal{M}_{0,4}$ based at the tangential base point $\vec{01}$ describing a small open interval along the real axis from 0 towards 1. Consider the paths r and p in the following figure; r goes from the tangential base point $\vec{01}$ to 1/2 and p from $\vec{01}$ to $\vec{10}$.



Now, by fact 1, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on p and since it fixes the endpoints of p, p is sent to itself precomposed with a pro-loop based at $\vec{01}$, i.e. $\sigma(p) = pf_\sigma$ for some $f \in \pi_1(\mathcal{M}_{0,4};\vec{01}) \simeq \hat{F}_2$ (this isomorphism identifies x with a small counterclockwise loop around 0 and y with the $p^{-1}\Theta(x)p$ around 1). Ihara shows that this f_σ actually lies in the derived subgroup $\hat{F}_2'$, and he associates to σ the couple $(\chi(\sigma), f_\sigma)$ where χ is the cyclotomic character. This defines Ihara's homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}^* \times \hat{F}_2'$. Let us show that this homomorphism is injective. There is a natural homomorphism from $\mathbb{Z}^* \times \hat{F}_2'$ to $\mathrm{End}(\hat{F}_2)$ which sends (λ, f) to the endomorphism of $\hat{F}_2$ taking x to x^λ and $y \mapsto f^{-1}y^\lambda f$, so any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is in the kernel of Ihara's map would give the trivial automorphism of $\hat{F}_2$. Now, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{F}_2$ is given by $x \mapsto x^{\chi(\sigma)}$ and $y \mapsto f_\sigma^{-1}y^\lambda f_\sigma$, and this is a lifting of the natural outer Galois action on $\hat{F}_2$ coming from the fact that $\hat{F}_2$ is the algebraic fundamental group of a variety defined over $\mathbb{Q}$, namely $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ (cf. §3.2). So the question is, can an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act trivially on $\hat{F}_2$?

The negative answer to this question is a consequence of Belyi's famous theorem stating that a Riemann surface X has a model defined over $\overline{\mathbb{Q}}$ if and only if there exists a covering of Riemann surfaces $\beta : X \to \mathbb{P}^1\mathbb{C}$ ramified over 0, 1 and ∞ only. As a corollary, we see that no element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can act trivially on all the subgroups of $\hat{F}_2$. For if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then there exists $\alpha \in \overline{\mathbb{Q}}$ on which σ acts non-trivially, and letting N be the finite-index subgroup of $\hat{F}_2$ corresponding to a Belyi cover $\beta : X \to \mathbb{P}^1\mathbb{C}$ where X is the elliptic curve of j-invariant equal to α, we see that σ cannot act trivially on N.

Now we sketch part of the proof that the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ under this injective homomorphism actually lies in $\widehat{GT}$, i.e. that the couples obtained this way satisfy relations (I), (II) and (III). We restrict ourselves to showing relation (I) and relation (III) in the case $\lambda \equiv \pm 1 \bmod 5$ here and refer to [LS2] for complete details; these two cases certainly give the essence of the proof. Let us do relation (I). The automorphisms θ of $\hat{F}_2$ and Θ of $\mathcal{M}_{0,4}$ introduced earlier are related as follows: for all $z \in \hat{F}_2$, we have $p^{-1}\Theta(z)p = \theta(z)$. Now, by fact 1, we have $\sigma(r) = rg$ for some $g \in \hat{F}_2$, so applying σ to $p = \Theta(r)^{-1}r$ and using fact 2, we have $pf_\sigma = \Theta(rg)^{-1}rg = \Theta(g)^{-1}\Theta(r)^{-1}rg = \Theta(g)^{-1}pg = p\theta(g)^{-1}g$, and therefore $f_\sigma = \theta(g)^{-1}g$. But such an element obviously satisfies relation (I).

Let us show it for relation (III), $\lambda \equiv \pm 1 \bmod 5$. First, we put any sphere with 5 marked points $(x_1, \ldots, x_5)$ into the standard form $(0, y_1, 1, \infty, y_2)$ via an isomorphism (i.e. a unique element of $\mathrm{PSL}_2(\mathbb{C})$). So we give points on $\mathcal{M}_{0,5}$ by pairs (y_1, y_2). Let $\tilde{p}$ denote the following path from the tangential base point (ϵ, μ) on $\mathcal{M}_{0,5}$ to $(1-\epsilon, \mu)$, where ϵ is a small positive parameter and μ is a negative parameter of very large absolute value: the path $\tilde{p}$ sends ϵ to $1-\epsilon$ along the real axis and does not move μ. Then $\sigma(\tilde{p}) = \tilde{p}\tilde{f}$. Let Z be the point on $\mathcal{M}_{0,5}$ corresponding to the sphere with five marked points $(1,\zeta,\zeta^2,\zeta^3,\zeta^4)$ where $\zeta=\exp(2\pi i/5)$. In standard form, this point is approximately given by the pair $(.38,-.62)$ and the path v slides ϵ to $.38$ and λ to $-.62$ along the real axis without intersections. Let $P=(12345)^2\in S_5\simeq \mathrm{Aut}(\mathcal{M}_{0,5})$. Then P is related to the automorphism ρ of $M(0,5)$ defined in §1.1 by $\tilde p^{-1}P\tilde p=\rho$, and since P fixed Z, the path $P(v)$ goes from $(1-\epsilon,\lambda)$ to Z and $\tilde p=P(v)^{-1}v$. If $\lambda\equiv\pm1$ mod 5, then σ fixes the point Z, so by fact 1, $\sigma(v)=vk$ for some $k\in\hat\pi_1(\mathcal{M}_{0,5};(\epsilon,\mu))\simeq\hat K(0,5)$. Applying σ to this equality, we obtain

$$\tilde p\tilde f_\sigma=P(vk)^{-1}vk=P(k)^{-1}P(v)^{-1}vk=P(k)^{-1}\tilde pk=\tilde p\rho(k)^{-1}k,$$

so $\tilde f_\sigma=\rho(k)^{-1}k$. This $\tilde f_\sigma$ obviously satisfies relation (III). The proofs of relation (II) and of relation (III) for $\lambda\equiv\pm2$ mod 5 are analogous, all showing that the elements f_σ trivially satisfy the relations defining $\widehat{GT}$.

§3.2. Compatibility of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\widehat{GT}$-actions

For every variety X defined over $\mathbb{Q}$, there is an exact sequence

$$1\to\hat\pi_1(X\otimes\overline{\mathbb{Q}})\to\hat\pi_1(X)\to\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to1,$$

for a suitable choice of base points of the *algebraic* or *geometric* fundamental group $\hat\pi_1(X\otimes\overline{\mathbb{Q}})$ and the *arithmetic* fundamental group $\hat\pi_1(X)$ (cf. [Oort] in this volume for details on this exact sequence). Furthermore, this sequence is split if X has a rational point, which is certainly the case when X is one of the moduli spaces $\mathcal{M}_{g,n}$. Thus we have canonical homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\hookrightarrow\mathrm{Out}\big(\hat\pi_1(\mathcal{M}_{g,n})\big)$ which lift to non-canonical homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\hookrightarrow\mathrm{Aut}\big(\hat\pi_1(\mathcal{M}_{g,n})\big)$, corresponding to sections of the exact sequence, i.e. to rational points on $\mathcal{M}_{g,n}$. The canonical homomorphisms into the outer automorphism groups are not known to be injective except when $g=0$, although it is plausible. Their liftings into the true automorphism groups are known to be injective for $g=0$ and also for $g>0$ and $n=0$ or 1 (cf. Nakamura's article in this volume). In the case $g=0$, it is natural to ask whether it is possible to choose one of these injections to make the following diagram commute:

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[rr] \arrow[dr] & & \widehat{GT} \arrow[dl] \\
& \mathrm{Aut}\big(\hat\pi_1(\mathcal{M}_{0,n})\big), &
\end{tikzcd}$$

where the injective maps $\widehat{GT} \hookrightarrow \mathrm{Aut}\big(\hat{\pi}_1(\mathcal{M}_{0,n})\big)$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT}$ were explained earlier. The answer is not only that it is possible to do so, but even that there is a natural and elegant geometric explanation for the right choice of section; it corresponds to a "tangential" point on $\mathcal{M}_{0,n}$. This was understood by Ihara and Matsumoto (cf. [IM]). This compatibility of the $\widehat{GT}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions extends to all the genus zero Teichmüller groups and groupoids of the $\mathcal{M}_{0,n}$ based at the tangential base points, almost by construction.

What about the higher genus situation? In (3) of §2.1, we mentioned that Ihara and Nakamura computed a $\widehat{GT}$-action on the profinite fundamental groups of all topological surfaces with punctures. Their action is compatible with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on the exact sequence

$$1 \to \hat{\pi}_1(S_{g,n-1}) \to \hat{\pi}_1(\mathcal{M}_{g,n}) \to \hat{\pi}_1(\mathcal{M}_{g,n-1}) \to 1,$$

arising from every "tangential" Riemann-surface with n-marked points, cf. §6 of Nakamura's article in this volume.

Does the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the $\hat{\pi}_1(\mathcal{M}_{g,n})$ extend to a $\widehat{GT}$-action? Thanks to Nakamura's discovery of the long-awaited and elusive explicit $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the higher mapping class groups $\hat{M}(g,1)$ and $\hat{M}(g,0)$, we can at least give an explicit conjecture on the form of such a $\widehat{GT}$-action. Indeed, Nakamura's Galois action is given on the generators of $\hat{M}(g,1)$ and $\hat{M}(g,0)$ by expressions *in which the elements of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *act via their components* $(\chi(\sigma), f_\sigma)$, so in order to express the conjectured $\widehat{GT}$-action which should of course be compatible with the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we simply replace $\chi(\sigma)$ by λ and f_σ by f in Nakamura's expressions. It remains to show that this action on the generators extends to an automorphism of the mapping class groups, i.e. respects their defining relations, without recourse to the knowledge that an action exists a priori, as for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This question remains open, as does the possibility that instead of $\widehat{GT}$, we should be considering some subgroup of $\widehat{GT}$ defined by one or more additional relations and still containing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Bibliography

This *very restricted* bibliography lists only articles concerned with the profinite version of $\widehat{GT}$. We encourage the reader to consult the beautiful survey article "Galois rigidity of profinite fundamental groups" by H. Nakamura, covering many related aspects of what we loosely term *anabelian algebraic geometry*, including some discussion of $\widehat{GT}$. This article appeared in Japanese in Sugaku (Math. Soc. of Japan), vol. 47 (1995), p. 1-17, and its English translation should appear this year in "Sugaku Expositions", in the AMS translation. Its bibliography contains a list of articles dealing with a wide selection of themes, for instance the pro-ℓ version of $\widehat{GT}$ (a subject rich with results and conjectures unavailable in the profinite case), the study of Galois actions on various fundamental groups, the anabelian conjecture and so on.

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