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ON THE DE RHAM COHOMOLOGY OF ALGEBRAIC VARIETIES [1]

by A. GROTHENDIECK

... In connection with Hartshorne's seminar on duality, I had a look recently at your joint paper with Hodge on " Integrals of the second kind " ([2]). As Hironaka has proved the resolution of singularities ([3]), the " Conjecture C " of that paper (p. 81) holds true, and hence the results of that paper which depend on it. Now it occurred to me that in this paper, the whole strength of the " Conjecture C " has not been fully exploited, namely that the theory of " integrals of second kind " is essentially contained in the following very simple

Theorem 1. — Let X *be an affine algebraic scheme over the field* $\mathbf{C}$ *of complex numbers; assume* X *regular* (i.e. " *non singular* "). *Then the complex cohomology* $H^{\bullet}(X, \mathbf{C})$ *can be calculated as the cohomology of the algebraic De Rham complex* (i.e. *the complex of differential forms on* X *which are " rational and everywhere defined "*).

This theorem had been checked previously by Hochschild and Kostant when X is an affine homogeneous space under an algebraic linear group, and I think they also raised the question as for the general validity of the result stated in theorem 1.

It will be convenient, for further applications, to give a slightly more general formulation, as follows. If X is any prescheme locally of finite type over a field k, and " smooth " over k, we can consider the complex of sheaves $\Omega^{\bullet}_{X/k}$ of regular differentials on X, the differential operator being of course the exterior differential. Let us consider the *hypercohomology*

$$H^{\bullet}(X) = \mathbf{H}^{\bullet}(X, \Omega^{\bullet}_{X/k}) \tag{1}$$

which we may call the " De Rham cohomology " of X, in contrast to the " Hodge cohomology "

$$H^{\bullet}(X, \Omega^{\bullet}_{X/k}) = \coprod_{p,q} H^q(X, \Omega^p_{X/k}), \tag{2}$$

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(1) This is part of a letter of the author to M. F. ATIYAH, dated Oct. 14, 1963. Some remarks have been added to provide references and further comments. (Except for remark (13), these remarks were written in November 1963.)

(2) M. F. ATIYAH and W. V. D. HODGE, Integrals of the second kind on an algebraic variety, *Annals of Mathematics*, vol. 62 (1955), p. 56-91. This paper is referred to by A-H in the sequel.

(3) H. HIRONAKA, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Annals of Maths.*, vol. 79 (1964), p. 109-326.

which is bigraded, whereas De Rham cohomology has only a simple grading. The two cohomologies are related by the ususal spectral sequence

(3) $$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \Rightarrow H^\bullet(X),$$

which shows for instance that if X is *affine*, then

(4) $$H^\bullet(X) = H^\bullet(\Gamma(X, \Omega^\bullet_{X/k}))$$

i.e. the De Rham cohomology in this case is what one might naively think.

Now assume k to be the field of complex numbers **C**, and let X^h be the complex analytic variety associated to X. Then we may also consider the " analytic De Rham cohomology "

$$\mathbf{H}^\bullet(X^h, \Omega^\bullet_{X^h/\mathbf{C}}),$$

but by Poincaré's lemma the complex of sheaves $\Omega^\bullet_{X^h/\mathbf{C}}$ on X^h is just a resolution of the constant sheaf **C**, therefore the hypercohomology of this complex is just the usual complex cohomology $H^\bullet(X^h, \mathbf{C})$. Now we have a canonical homomorphism of the algebraic into the analytic De Rham cohomology

(5) $$H^\bullet(X) \to H^\bullet(X^h) \simeq H^\bullet(X^h, \mathbf{C}),$$

and we can state:

Theorem 1'. — The homomorphism (5) *is an isomorphism.*

If X is affine, this reduces by (4) to theorem 1. In fact, we can reduce easily theorem 1' to theorem 1. To see this, take a covering $\mathfrak{U}$ of X by affine open sets U_i; then we get a convergent spectral sequence

(6) $$H^\bullet(X) \Leftarrow E_2^{p,q} = H^p(\mathfrak{U}, \mathscr{H}^q),$$

where $\mathscr{H}^q$ stands for the presheaf $V \rightsquigarrow H^q(V)$ on X, and an analogous spectral sequence for X^h and the covering $\mathfrak{U}^h$ of X^h by the open subsets U_i^h,

$$H^\bullet(X^h) \Leftarrow E_2^{p,q} = H^p(\mathfrak{U}^h, \mathscr{H}^q).$$

Now (5) is associated to a homomorphism of spectral sequences, so we are reduced to prove we have isomorphisms for the terms E_2, which will follow if we know that (5) is an isomorphism, when X is replaced by any prescheme $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_q}$. This reduces us, first to the case when X is contained in an affine scheme, and hence is separated, and in this case the previous open sets are affine, so we are finally reduced to theorem 1.

Besides, if X is complete, then we can prove theorem 1' directly, using the spectral sequence (3) and the analogous one for X^h, and using Serre's GAGA ([4]); thus in this case, the result is elementary i.e. does not use resolution, as does th. 1.

([4]) J.-P. Serre, Géométrie algébrique et géométrie analytique, *Annales de l'Institut Fourier*, vol. VI (1956), p. 1-42.

Now to the proof of theorem 1. As Hironaka's resolution theorems are equally valid on complex analytic spaces, at least locally, one can deduce theorem 1 from a stronger, purely local theorem on complex analytic spaces, as follows:

Theorem 2. — *Let* X *be a reduced complex analytic space,* Y *an analytic closed subset,* $U = X - Y$, *assume* U *non singular and dense in* X, *and that* Y *can be defined locally by one equation. Let* $\Omega^{\bullet}_X(*Y)$ *be the complex of Modules on* X *of " differential forms with polar singularities on* Y *", which on* U *reduces to the complex of holomorphic differential forms on* U, *hence we get canonical homomorphisms*

$$\mathcal{H}^q(\Omega^{\bullet}_X(*Y)) \to R^q f_*(\mathbf{C}_U), \tag{7}$$

where $\mathbf{C}_U$ *is the constant sheaf* $\mathbf{C}$ *on* U *and* $f : U \to X$ *the natural injection. These homomorphisms are isomorphisms.*

(N.B. For a formal definition of the sheaf $\Omega^{\bullet}_X(*Y)$, take the subsheaf of $f_*(\Omega^{\bullet}_U)$ whose sections, on an open set V, are the holomorphic differential forms on $V \cap U$ which are restrictions of meromorphic Kähler differential forms on V) (5).

This theorem is essentially equivalent with the case *b)* of the

Corollary. — *Let* X *be as above, assume moreover that*

$$H^p(X, \Omega^q_X(*Y)) = 0 \qquad \textit{for } p > 0, \textit{ any } q; \tag{8}$$

then the global homomorphisms analogous to (7) :

$$H^q\Gamma(X, \Omega^{\bullet}_X(*Y)) \to H^q(U, \mathbf{C}) \tag{9}$$

are isomorphisms. This conclusion holds in particular in each of the following cases:

a) X *is projective, and* Y *is the support of an ample positive divisor on* X (6).

b) X *is Stein.*

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(5) In fact, what we actually need is that for any p, the coherent sheaf $\mathcal{F} = \Omega^p_U$ on U can be extended, on an open neighbourhood W_y in X of any point $y \in Y$, into a coherent sheaf $\mathcal{E}$ on W_y. In this situation, the " sheaf of meromorphic sections of $\mathcal{F}$, holomorphic on U " is defined as the subsheaf $\mathcal{F}(*Y)$ of $f_*(\mathcal{F})$ whose sections, on an open set V, are the sections ω of $\mathcal{F}$ on $V \cap U$ such that, for every $y \in V \cap Y$, there exists an open neighbourhood $W \subset W_y \cap V$ of y, a section ω' of $\mathcal{E}$ on W, and an integer n, such that $\omega|(W \cap U) = (\omega'/\varphi'^n)|(W \cap U)$, where φ' is the defining equation of Y in X. It is easily checked that these ω are indeed the sections of a subsheaf $\mathcal{F}(*Y)$ of $f_*(\mathcal{F})$, and that this subsheaf does not depend on the choices performed. In the case when $\mathcal{F}$ can be extended globally to a coherent sheaf $\mathcal{E}$ on X, there is a natural isomorphism

$$\mathcal{F}(*Y) \simeq \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}^n, \mathcal{E}),$$

where $\mathcal{J}$ is a coherent sheaf of ideals on X defining Y. In our case ($\mathcal{F} = \Omega^p_X$) we can take $\mathcal{E} = \Omega^p_X = \bigwedge^p \Omega^1_X$, the sheaf of *Kähler differentials* on X. For the definition of Ω^1_X, see *Séminaire Cartan*, 1960-1961, Exposé 14 : Eléments de calcul infinitésimal.

(6) A slightly different proof shows that the assumption in *a)* can be generalized into the following one: *a')* X is compact algebraic, and $X - Y$ is affine.

Let us show how this implies (8), and more generally for any coherent sheaf $\mathcal{E}$ on X:

$$H^p(X, \mathcal{E}(*Y)) = 0 \qquad \text{for } p > 0. \tag{8'}$$

Let X_0, Y_0 be the schemes over $\mathbf{C}$ that define X, Y (they are uniquely determined up to isomorphisms by GAGA (4), or rather by the extension of the results of GAGA to complete varieties that I gave in Séminaire H. Cartan, 1956-57, Exposé 2 : Sur les faisceaux algébriques et les faisceaux analytiques cohérents). Let $U_0 = X_0 - Y_0$, $f_0 : U_0 \to X_0$

The fact that (8) and isomorphy for (7) imply isomorphy for (9) is standard. That assumption *a)* implies (8) is an immediate consequence of Serre's vanishing theorem and commutation of cohomology of the compact space X with direct limits of sheaves (cf. A-H, lemma 6). An analogous argument holds in case *b)*, as there exists a coherent sheaf $\mathscr{E}$ on X which on U reduces to Ω^q_U, so that we have an isomorphism

$$\Omega^q_X(*Y) \simeq \varinjlim_n \mathscr{H}om_{\mathcal{O}_X}(\mathscr{J}^n, \mathscr{E}),$$

where $\mathscr{J}$ denotes a coherent sheaf of ideals such that $Y = \operatorname{Supp} \mathcal{O}_X/\mathscr{J}$. Using the vanishing of the cohomology groups of the holomorphically convex compact subsets K of X with coefficients in the coherent sheaves $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{J}^n, \mathscr{E})$, it follows as before that the cohomology of these K with coefficients in $\Omega^q_X(*Y)$ vanishes, hence also the cohomology of X with values in the same sheaf.

Theorem 1 is contained in part *a)* of the corollary, as one sees by denoting by U the X of theorem 1, and by X a projective closure of U.

Now to prove theorem 2, use resolution to get a projective birational morphism

$$g : X' \rightarrow X,$$

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the canonical immersion. Let $\mathscr{E}_0$ be the coherent sheaf on X_0 that defines $\mathscr{E}$, equally determined up to unique isomorphism (same reference, which we will abbreviate into " GAGA "). As

$$\mathscr{E}(*Y) \simeq \varinjlim_n \mathscr{H}om_{\mathcal{O}_X}(\mathfrak{J}^n, \mathscr{E}),$$

$\mathfrak{J}$ being a coherent sheaf of ideals defining Y, we get by compactness of X

$$H^p(X, \mathscr{E}(*Y)) \simeq \varinjlim_n H^p(X, \mathscr{H}om_{\mathcal{O}_X}(\mathfrak{J}^n, \mathscr{E})),$$

but by applying GAGA we get

$$H^p(X, \mathscr{H}om_{\mathcal{O}_X}(\mathfrak{J}^n, \mathscr{E})) \simeq H^p(X_0, \mathscr{H}om_{\mathcal{O}_{X_0}}(\mathfrak{J}^n_0, \mathscr{E}_0)),$$

and hence

$$H^p(X, \mathscr{E}(*Y)) \simeq \varinjlim_n H^p(X_0, \mathscr{H}om_{\mathcal{O}_{X_0}}(\mathfrak{J}^n_0, \mathscr{E}_0)) \simeq H^p(X_0, (f_0)_*(\mathscr{E}_0|U_0))$$

As U_0 is affine and X_0 complete and hence separated, $f_0 : U_0 \rightarrow X_0$ is an affine morphism, hence $R^q(f_0)_*(\mathscr{E}_0|U_0) = 0$ for $q > 0$, hence

$$H^p(X_0 (f_0)_*(\mathscr{E}_0|U_0)) \simeq H^p(U_0, \mathscr{E}_0|U_0),$$

which is zero for $p \neq 0$, as U_0 is affine. This proves (8′) and also the formula

(8″) $$H^0(X, \mathscr{E}(*Y)) \simeq H^0(U_0, \mathscr{E}_0|U_0),$$

i.e. the meromorphic sections of $\mathscr{E}$ on X with only polar singularities on Y and holomorphic on $X - Y$, are just the rational sections of $\mathscr{E}_0$ on X_0 regular on U_0, i.e. the sections of $\mathscr{E}|U_0$ on U_0.

If now X is regular and Y has only normal crossings, then (as observed and used below in the proof of theorem 2) the homomorphisms (7) are isomorphisms, as seen by easy explicit computation (A-H, lemma 17), hence the homomorphisms (9) are isomorphisms, hence (using (8″)) theorem 1 holds true for $U_0 = X_0 - Y_0$. But using global resolution for algebraic schemes over **C**, one sees that any regular affine scheme U_0 over **C** is isomorphic to $X_0 - Y_0$, with X_0 projective and regular, and Y_0 a divisor with only normal crossings. This proves theorem 1 independently of theorem 2, using global resolution of algebraic schemes rather than local resolution of complex analytic spaces, and independently also of Grauert-Remmert's result (7), whose application in the proof of theorem 2 is a little bit subtle.

It would be interesting to know if the conclusion (8′) holds true if we assume only X compact, and $U = X - Y$ Stein (which would imply that (9) are still isomorphisms in this case).

with X′ non singular and $Y'=g^{-1}(Y)$ a divisor with only normal crossings (" simple divisor " in the terminology of A-H) for the reduced induced structure. This is possible at least locally on X, which is enough for our purpose. For simplicity let

$$\mathscr{K}^\bullet = \Omega^\bullet_X(*Y), \qquad \mathscr{K}'^\bullet = \Omega^\bullet_{X'}(*Y'),$$

Let $U'=g^{-1}(U)$, we can assume that g induces an *isomorphism*

$$g' : U' \to U.$$

Let $\mathscr{L}^\bullet$ be an injective resolution of $\mathbf{C}_U$, hence an injective resolution $\mathscr{L}'^\bullet$ of $\mathbf{C}_{U'}$. The homomorphism (7) is deduced from a homomorphism of complexes

(10) $$\mathscr{K}^\bullet \to f_*(\mathscr{L}^\bullet),$$

and the homomorphism analogous to (7) for (X′, Y′) is deduced from a homomorphism of complexes

(10′) $$\mathscr{K}'^\bullet \to f'_*(\mathscr{L}'^\bullet),$$

where $f' : U' \to X'$ is the canonical embedding. Besides, one has natural isomorphisms

$$g_*(\mathscr{K}'^\bullet) \xrightarrow{\sim} \mathscr{K}^\bullet, \qquad g_*(f'_*(\mathscr{L}'^\bullet)) \xrightarrow{\sim} f_*(\mathscr{L}^\bullet),$$

and we can assume (10) deduced from (10′) by applying g_*. I contend we have moreover:

(11) $$R^q g_*(\mathscr{K}'^p) = R^q g_*(f'_*(\mathscr{L}'^p)) = 0 \quad \text{for } q>0, \text{ any } p.$$

This is trivial for the second relation, as $f'_*(\mathscr{L}'^p)$ is flasque. As for the first, we have more generally, for any coherent sheaf $\mathscr{E}'$ on X′, and denoting by $\mathscr{E}'(*Y')$ the " sheaf of meromorphic sections of $\mathscr{E}'$ holomorphic on $X'-Y'=U'$ ", the relations

$$R^q g_*(\mathscr{E}'(*Y')) = 0 \qquad \text{for } q>0.$$

To see this, write

$$\mathscr{E}'(*Y') \simeq \varinjlim_n \mathscr{H}om_{\mathscr{O}_X}(\mathscr{J}'^n, \mathscr{E}'),$$

where $\mathscr{J}'$ is a coherent sheaf of ideals defining Y′, so that $\mathscr{E}'(*Y')$ appears as a direct limit of coherent sheaves $\mathscr{E}'_n$. As g is proper, $R^q g_*$ commutes with direct limits of sheaves; on the other hand the comparison theorem of Grauert-Remmert [7] on projective morphisms of analytic spaces tells us that the fiber at a point x of the sheaf $R^q f_*(\mathscr{E}'_n)$

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[7] H. Grauert et R. Remmert, Faisceaux analytiques cohérents sur le produit d'un espace analytique et d'un espace projectif, *C.R. Acad. Sc. Paris*, t. 245, p. 819-822. The proof that follows is essentially the same as the one given in the previous remark [6], except that reference to GAGA is replaced by a reference to the theorem of Grauert-Remmert, which should be viewed as the generalization of GAGA, from the case of a base space reduced to one point, to the case of a ground space an arbitrary complex analytic space. For the general philosophy of the result of Grauert and Remmert, one may read my talk in Cartan's Seminar 1960-61, Exposé 15, Rapport sur les théorèmes de finitude de Grauert et Remmert (especially the remarks on the last page of the exposé).

We sould remark also that the theorem of Grauert and Remmert is implicit in the proof of the isomorphism

$$g_*(\mathscr{K}'^\bullet) \simeq \mathscr{K}^\bullet,$$

(which in our remark [6] above corresponds to the isomorphism (8″)).

can be computed as the q-th cohomology group, on the projective scheme $\widetilde{X}'$ over $\widetilde{X} = \mathrm{Spec}(\mathcal{O}_{X,x})$ which defines X'/X in a neighbourhood of x, of the algebraic coherent sheaf $\widetilde{\mathcal{E}}'_n$ on $\widetilde{X}'$ which corresponds to $\mathcal{E}'_n$. Thus to prove that

$$R^q g_*(\mathcal{E}'(*Y'))_x = \varinjlim_n R^q g_*(\mathcal{E}'_n)_x$$

is zero for $q>0$, we are reduced to the corresponding fact on $\widetilde{X}$, i.e. to prove

$$H^q(\widetilde{X}', \widetilde{f}'_*(\widetilde{\mathcal{E}}'|\widetilde{U}')) = 0 \text{ for } \quad q>0.$$

Now this is easy, for as $\widetilde{Y}$ is defined in $\widetilde{X}$ by one equation, the same holds for $\widetilde{Y}'$ in $\widetilde{X}'$, therefore $\widetilde{f}' : \widetilde{U}' \to \widetilde{X}'$ is affine, and hence $H^q(\widetilde{X}', f'_*(\widetilde{\mathcal{E}}'|\widetilde{U}')) \simeq H^q(\widetilde{U}', \widetilde{\mathcal{E}}'|\widetilde{U}')$, which is zero for $q>0$ as $\widetilde{U}' \simeq \widetilde{U}$ is affine. This establishes (11).

Moreover, (10′) induces an isomorphism for the cohomology sheaves, i.e. Theorem 2 holds true when (X, Y) are replaced by (X′, Y′). This follows from the fact that X′ is non singular and Y′ has only normal crossings, by an elementary explicit calculation (A-H, lemma 17). From this and (11) follows, by a standard argument, that the image of (10′) by g_*, namely (10), induces also an isomorphism for the cohomology sheaves. The proof of theorem 2, and hence of theorem 1, is now complete.

The purely algebraic definition (1) of De Rham cohomology raises a number of further questions. Let for instance

$$f : U \to X$$

be a morphism of regular schemes over **C**; does the corresponding Leray spectral sequence

$$\text{(12)} \qquad H^\bullet(U^h, \mathbf{C}_{U^h}) \Leftarrow H^p(X^h, R^q f_*(\mathbf{C}_{U^h}))$$

admit a purely algebraic definition, valid for any ground field k, at least for k of char. 0 (cf. ([13])). For instance, if f comes from a morphism of schemes $U_0 \to X_0$ over a subfield k_0 of **C**, is it true that the Leray filtration of $H^\bullet(U^h, \mathbf{C}_{U^h})$ comes from a filtration of $H^\bullet(U_0)$? The spectral sequence (12), when f is an open imbedding and X is projective, was used in A-H (p. 84) in order to define the notion of a rational closed form φ " of second kind " on X. However, this definition now makes sense in a purely algebraic way, as it just means that there exists a dense open subset U of X (for the Zariski topology) such that φ is regular on U and the element of $H^\bullet(U)$ it defines is in the image of $H^\bullet(X) \to H^\bullet(U)$ ([8]).

([8]) There is however another spectral sequence of a purely algebraic character, which for the study of the notion of " integrals of the second kind " may be substituted to the one used in A-H, having for abutment the De Rham cohomology $H^\bullet(X)$, X being any algebraic prescheme smooth over a field k. To get it, we observe that every locally free sheaf $\mathcal{E}$ on a regular scheme X has a canonical injective resolution (the " Cousin resolution " or " residue resolution " (cf. Hartshorne's Seminar, Harvard, 1963-64), which as a graded sheaf is just the direct sum of the flasque (even injective) sheaves associated to the local invariants $H^p_x(\mathcal{E}) = \varinjlim_n \mathrm{Ext}^p(\mathcal{O}_{X,x}/\mathfrak{m}^n_x, \mathcal{E}_x)$, Also, for a general regular [9] scheme X over any subfield k of $\mathbf{C}$, a number field say, there is in the usual complex cohomology $H^n(X^h, \mathbf{C}_{X^h})$ a natural sub-vector space $H^{\bullet}(X)$ over k, which gives $H^n(X^h, \mathbf{C}_{X^h})$ by extensions of scalars $K \to \mathbf{C}$. This subspace (if X is projective) contains the subspace $H^0(X, \Omega^n_{X/k})$ of all regular differential forms on X as a subspace of the pure component $H^{n,0}(X^h) = H^0(X^h, \Omega^n_{X^h})$, but even in the case when X is a complete non singular curve or an abelian variety (an elliptic curve, say), and $n = 1$, the position of the whole of $H^n(X)$ in $H^n(X^h, \mathbf{C}_{X^h})$ with respect to $H^n(X^h, \mathbf{C}_{X^h})$ or $H^n(X^h, \mathbf{Z}_{X^h})$ yields an interesting arithmetic invariant, generalizing the " periods " of the regular differential forms. In a way, one has $4g$ " periods " instead of the usual $2g$ one ($2g$ being the first Betti number); one may ask for instance

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($x \in X$, $p = \dim \mathcal{O}_{X,x}$). If X is of dimension 1, this is just the resolution by " repartitions ", $C^0(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{R}_X$ the sheaf of rational sections of E, $C^1(\mathcal{E}) \simeq C^0(\mathcal{E})/\mathcal{E} \simeq \coprod_{x \in F} (\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{R}_X)/\mathcal{E}_x$ (x runs over the set F of closed points of X) = the sheaf of additive Cousin data on X relative to $\mathcal{E}$. This resolution $C^{\bullet}(\mathcal{E})$ is functorial in $\mathcal{E}$ with respect to arbitrary homomorphisms of abelian sheaves (not necessarily $\mathcal{O}_X$-linear), hence for every complex Ω of abelian sheaves on X, whose components are locally free sheaves of modules (the differential operator not necessarily linear) an injective resolution $C'(\Omega^{\bullet}_X)$ of $\Omega^{\bullet}_X$ which can be used to compute the hypercohomology $\mathbf{H}^{\bullet}(X, \Omega^{\bullet}_X)$. This complex seems to be quite convenient for the study of the De Rham cohomology. It yields a spectral sequence

$$\mathbf{H}^{\bullet}(X, \Omega^{\bullet}_X) \Leftarrow E_1^{p,q} = \coprod_{x \in X^p} \mathbf{H}_x^{p+q}(\Omega^{\bullet}_X),$$

where X^p is the set of all $x \in X$ such that $\dim \mathcal{O}_{X,x} = p$. If $\Omega^{\bullet}_X$ is the complex of regular differentials on X, then one can define (using residues of differentials) a canonical homomorphisms:

$$\text{(R)} \qquad H^{q-p}(\boldsymbol{k}(x)) \to \mathbf{H}_x^{p+q}(\Omega^{\bullet}_X),$$

where for any extension K of k, we denote by $H^{\bullet}(K)$ the De Rham cohomology of K/k (which, if K is of finite type and separable over k, is just the direct limit of the De Rham cohomologies of affine models of K smooth over k). Moreover, if k is of char. 0, one shows by a transcendental argument using theorem 1, that the homomorphisms (R) are isomorphisms. Therefore, in the case X smooth over k of characteristic zero, wo get the spectral sequence

$$(S^{\bullet}) \qquad H^{\bullet}(X) \Leftarrow E_1^{p,q} = \coprod_{x \in X^p} H^{q-p}(\boldsymbol{k}(x)).$$

(If $k = \mathbf{C}$, this spectral sequence can be interpreted geometrically as the direct limit of the spectral sequences, for the usual complex cohomology of X, arising from filtrations of X by Zariski closed subsets of decreasing dimension). In the corresponding filtration of $H^{2p}(X)$, the last factor occuring is the subspace of $H^{2p}(X)$ which is " algebraic ", i.e. generated by algebraic cycles of codimension p. Although E_1 is pretty big, it seems plausible that already E_2 is of finite dimension over the groundfield k; this can be checked at least if $\dim X \leqslant 2$, or for $E_2^{p,q}$ for small values of p, q.

[9] It is likely that " regular " is not needed, provided we replace cohomology by *homology*; namely that for any algebraic scheme X over a field of characteristic zero, X not necessarily regular, one can define (in a purely algebraic way) " singular " *homology groups* $H_i(X)$, which in the case $k = \mathbf{C}$ will coïncide with the ususal singular homology groups with arbitrary supports of X^h (dual to the cohomology with compact supports of X). If for instance X is imbedded in a regular algebraic scheme X' everywhere of dimension n, one should define

$$\text{(H)} \qquad H_i(X) = \mathbf{H}_i^{2n-i}(X', \Omega^{\bullet}_{X'/k})$$

(where the right-hand side denotes hypercohomology with supports in X); of course one should check that this is independant of the chosen imbedding, a problem very much of the same type as the invariance problem dealt with in Hartshorne's Harvard Séminar 1963-64, and which calls for a common generalization. The residue resolution of $\Omega^{\bullet}_{X'/k}$ then gives rise to a spectral sequence, generalizing the spectral sequence (S) defined in the previous remark [8]

$$(S_{\bullet}) \qquad H_{\bullet}(X) \Leftarrow E^1_{p,q} = \coprod_{x \in X_p} H^{p-q}(\boldsymbol{k}(x)),$$

where now X_p denotes the set of points $x \in X$ such that the dimension of the closure of x be p, and $H^{p-q}(\boldsymbol{k}(x))$ denotes birational De Rham cohomology as before.

if Schneider's theorem generalizes in some way to this larger set of periods ([10]).

If X is projective and smooth over a field k of char. 0, then Lefschetz's principle and Hodge's theory imply that the spectral sequence (3) degenerates: all differentials d_r $(r \geq 1)$ vanish. The same holds in any characteristic if X is an abelian variety, and presumably every time when the Hodge cohomology gives the " correct " Betti numbers (namely those given by étale $\mathbf{Z}_l$-adic cohomology ([11]), l prime to the characteristic). Thus, in these favorable cases, the bigraded group associated to the natural filtration of $H^{\bullet}(X)$ is just the Hodge cohomology (2). But in contrast to what happens in the transcendent set-up, there is no " natural " splitting available for this filtration, i.e. no natural isomorphism between De Rham and Hodge cohomology (as one can see already in the case when X is an elliptic curve). The extensions thus obtained seem to be again interesting arithmetic invariants. For instance, the $H^1(X)$ of an abelian variety over any field K is an extension of $\mathfrak{t}_{X^*} = H^1(X, \mathcal{O}_X)$ (the tangent space of the dual abelian variety X^* at the origin) by $\mathfrak{t}_X$ (the dual of the tangent space of X at the origin). This may be viewed as an extension of R-modules, where R is the ring of endomorphisms of X. If the characteristic is $p > 0$, it seems unlikely that this extension should always split. It is probably tied up in some way with the following extensions of algebraic finite groups

$$0 \to G(\mathfrak{t}_X) \to {}_pX \to D(G(\mathfrak{t}_{X^*})) \to 0,$$

where ${}_pX = \mathrm{Ker}\,(p.\mathrm{id}_X)$, $G(\mathfrak{t}_X)$ and $G(\mathfrak{t}_{X^*})$ are the radicial groups of height one associated to the restricted p-Lie algebras $\mathfrak{t}_X$ and $\mathfrak{t}_{X^*}$, and D denotes Cartier dual. In the same way, if we are dealing with an abelian scheme X over an arbitrary ground scheme S instead of a ground field, (say S = some modular scheme), we get an extension of locally free sheaves of $\mathfrak{t}_{X^*}$ by $\check{\mathfrak{t}}_X$, hence a canonical cohomology class

$$\xi_{X/S} \in H^1(S, \check{\mathfrak{t}}_{X^*} \otimes \check{\mathfrak{t}}_X),$$

which probably is not always zero, not even in char. 0 and with S proper.

Quite generally, if $f: X \to S$ is a smooth morphism of schemes, one can generalize definition (1) to introduce coherent sheaves on S:

$$\text{(13)} \qquad H^n(f) = \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}),$$

([10]) In fact, J.-P. Serre pointed out to me that for an algebraic curve over **C**, these " periods of differentials of the second kind " are rather classical invariants. Thus, for an elliptic curve defined by the periods ω_1, ω_2 one defines classically the integrals

$$\eta_i = \int_0^{\omega_i} \eta,$$

(where $x = \wp z$, $y = \wp' z$, and $\eta = \frac{xdx}{y}$ is a differential of the second kind which, together with the invariant differential ω, forms a basis of $H^1(X)$ = differentials of second kind mod. exact differentials). The only known general algebraic relation among the η_i and ω_i is

$$\omega_1\eta_2 - \eta_1\omega_2 = 2i\pi.$$

Schneider's theorem states that if X is algebraic (i.e. its coefficients g_2 and g_3 are algebraic), then ω_1 and ω_2 are transcendental, and it is believed that if X has no complex multiplication, then ω_1 and ω_2 are algebraically independent. This conjecture extends in an obvious way to the set of periods $(\omega_1, \omega_2, \eta_1, \eta_2)$ and can be rephrased also for curves of any genus, or rather for abelian varieties of dimension g, involving $4g$ periods.

([11]) Cf. M. Artin, *Grothendieck Topologies*, Spring, 1962, Harvard University, or M. Artin et A. Grothendieck, *Cohomologie étale des Schémas*, Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-64.