UNIVERSITY OF KANSAS

Department of Mathematics

Lawrence, Kansas

A General Theory of Fibre Spaces With Structure Sheaf

by

Alexandre Grothendieck

National Science Foundation Research Project

on

Geometry of Function Space

Research Grant NSF-G 1126

Report No. 4

First Edition August, 1955

Second Edition May, 1958

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WITH

STRUCTURE SHEAF

INTRODUCTION. When one tries to state in a general algebraic formalism the various notions of fibre space: general fibre space (without structure group, and maybe not even locally trivial); or fibre bundle with topological structure group G as expounded in the book of Steenrod (The Topology of Fibre Bundles, Princeton University Press); or the "differentiable" and "analytic" (real or complex) variants of these notions; or the notions of algebraic fibre spaces (over an abstract field k) - one is led in a natural way to the notion of fibre space with a structure sheaf $\underline{G}$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space X, with abelian structure group G, as the elements of the first cohomology group of X with coefficients in the sheaf $\underline{G}$ of germs of continuous maps of X into G; the word "continuous" being replaced by "analytic" respectively "regular" if G is supposed an analytic respectively an algebraic group (the space X being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when G is not abelian, to denote by $H^1(X, \underline{G})$ the set of classes of fibre spaces on X with structure sheaf $\underline{G}$, $\underline{G}$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of X into G. Here we develop systematically the notion of fibre space with structure sheaf $\underline{G}$, where $\underline{G}$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $H^1(X, \underline{G})$ of X with coefficients in $\underline{G}$. The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including the sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the last chapter, we define the cohomology set $H^1(X, \underline{G})$ of X with coefficients in the sheaf of groups $\underline{G}$, so that the expected classification theorem for fibre spaces with structure sheaf $\underline{G}$ is valid. We then proceed to a careful study of the exact cohomology sequence associated with an exact sequence of sheaves $e \longrightarrow \underline{F} \longrightarrow \underline{G} \longrightarrow \underline{H} \longrightarrow e$. This is the main part, and in fact the origin, of this paper. Here $\underline{G}$ is any sheaf of groups, $\underline{F}$ a subsheaf of groups, $\underline{H} = \underline{G}/\underline{F}$, and according to various supplementary hypotheses on $\underline{F}$ (such as $\underline{F}$ normal, or $\underline{F}$ normal abelian, or $\underline{F}$ in the center) we get an exact cohomology sequence going from $H^0(X, \underline{F})$ (the group of sections of $\underline{F}$) to $H^1(X, \underline{G})$ respectively $H^1(X, \underline{H})$ respectively $H^2(X, \underline{H})$, with more or less additional algebraic structures involved. The formalism thus developed is quite suggestive, and as it seems useful, in particular in dealing with the problem of classification of fibre bundles with a structure group G in which we consider a subgroup F, or the problem of comparing say the topological and analytic classification for a given analytic structure group G. However, in order to keep this exposition in reasonable bounds, no examples have been given. Some complementary facts, examples, and applications for the notions developed will be given in the future. This report has been written mainly in order to serve the author for future reference; it is hoped that it may serve the same purpose, or as an introduction to the subject, to somebody else.

Of course, as this report consists in a fortunately straightforward adaptation of quite well known notions, no real difficulties had to be overcome and there is no claim for originality whatsoever. Besides, at the moment to give this report for mimeography, I hear that results analogous to those of chapter 5 were known for some years to Mr. Frenkel, who did not publish them till now. The author only hopes that this report is more pleasant to read than it was to write, and is convinced that anyhow an exposition of this sort had to be written.

Remark (added for the second edition). It has appeared that the formalism developed in this report, and specifically the results of Chapter V, are valid (and useful) also in other situations than just for sheaves on a given space X. A generalization for instance is obtained by supposing that a fixed group π is given acting on X as a group of homeomorphisms, and that we restrict our attention to the category of fibre spaces over X (and especially sheaves) on which π operates in a manner compatible with its operations on the base X. (See for instance A. Grothendieck, Sur le mémoire de Weil; Généralisation des fonctions abéliennes, Séminaire Bourbaki Décembre 1956). When X is reduced to a point, one gets (instead of sheaves) sets, groups, homogeneous spaces etc. admitting a fixed group π of operators, which leads to the (commutative and non-commutative) cohomology theory of the group π. One can also replace π by a fixed Lie group (operating on differentiable varieties, on Lie groups, and homogeneous Lie spaces). Or X, π are replaced by a fixed ground field k, and one considers algebraic spaces, algebraic groups, homogeneous spaces defined over k, which leads to a kind of cohomology theory of k. All this suggests that there should exist a comprehensive theory of non-commutative cohomology in suitable categories, an exposition of which is still lacking. (For the "commutative" theory of cohomology, see A. Grothendieck, sur quelques points d'Algèbre Homologique, Tohoku Math. Journal, 1958).

CHAPTER I

GENERAL FIBRE SPACES

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

1.1 Notion of fibre space.

Definition 1.1.1. A fibre space over a space X is a triple (X, E, p) of the space X, a space E and a continuous map p of E into X.

We do not require p to be onto, still less to be open, and if p is onto, we do not require the topology of X to be the quotient topology of E by the map p. For abbreviation, the fibre space (X, E, p) will often be denoted by E only, it being understood that E is provided with the supplementary structure consisting of a continuous map p of E into the space X. X is called the base space of the fibre space, p the projection, and for any $x \in X$, the subspace $p^{-1}(x)$ of E (which is closed if $\{x\}$ is closed) is the fibre of x (in E).

Given two fibre spaces (X, E, p) and (X', E', p'), a homomorphism of the first into the second is a pair of continuous maps $f: X \longrightarrow X'$ and $g: E \longrightarrow E'$, such that $p'g = fp$, i.e. commutativity holds in the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{array}$$

Then g maps fibres into fibres (but not necessarily onto!); furthermore, if p is surjective, then f is uniquely determined by g. The continuous map f of X into X' being given, g will be called also a f-homomorphism of E into E'. If, moreover, E'' is a fibre space over X'', f' a continuous map

$X' \longrightarrow X''$ and $g': E' \longrightarrow E''$ a f'-homomorphism, then $g'g$ is a $f'f$-homomorphism. If f is the identity map of X onto X, we say also X-homomorphism instead of f-homomorphism. If we speak of homomorphisms of fibre spaces over X, without further comment, we will always mean X-homomorphisms.

The notion of isomorphism of a fibre space (X, E, p) onto a fibre space (X', E', p') is clear: it is a homomorphism (f, g) of the first into the second, such that f and g are onto-homeomorphisms.

1.2 Inverse image of a fibre space, inverse homomorphisms. Let (X, E, p) be a fibre space over the space X, and let f be a continuous map of a space X' into X. Then the inverse image of the fibre space E by f is a fibre space E' over X'. E' is defined as the subspace of $X' \times E$ of points (x', y) such that $fx' = py$, the projection p' of E' into the base X' being given by $p'(x', y) = x'$. The map $g(x', y) = y$ of E' into E is then an f-homomorphism, inducing for each $x' \in X'$ a homeomorphism of the fibre of E' over x' onto the fibre of E over $f'.x'$.

Suppose now, moreover, given a continuous map $f': X'' \longrightarrow X'$ of a space X'' into X'. Then there is a canonical isomorphism of the fibre space E'' over X'', inverse image of the fibre space E by ff', and the inverse image of the fibre space E' (considered above) by f' (transitivity of inverse images). If $(x'', y) \in E''$ ($x'' \in X''$, $y \in E$, $ff'x'' = py$), it is mapped by this isomorphism into $(x'', (f'x'', y))$.

Let Y be a subspace of the base X of a fibre space E; consider the injection f of Y into X; the inverse image E' of E by f is called fibre-space induced by E on Y, or the restriction of E to Y, and is denoted by $E|Y$. This is canonically homeomorphic to a subspace of E, namely the set of elements mapped by p into Y; the projection of $E|Y$ into Y is induced by p. By what has been said above, if Z is a subspace of Y, the restriction of $E|Y$ to Z is the restriction $E|Z$ of E to Z.

Again let (X, E, p) and (X', E', p') be two fibre spaces, f a continuous map $X \longrightarrow X'$. An inverse homomorphism associated with f is an X-homomorphism g of the fibre space E_o into E, where E_o denotes the inverse image of the fibre space E' by f. That means that g is a continuous map, of the subspace E_o of $X \times E'$ of pairs (x, y') such that $fx = p'y'$, into E, mapping for any $x \in X$ the fibre of x into E_o (homeomorphic to the fibre of fx in E' !) into the fibre $p^{-1}(x)$ of x in E. For instance, if E is itself the inverse image of E' by f, then there is a canonical inverse homomorphism of E' into E associated with f: the identity ! (Though somewhat trivial, this is the most important case of inverse homomorphisms.)

1.3 Subspace, quotient, product. Let (X, E, p) be a fibre space, E' any subspace of E, then the restriction p' of p to E', defines E' as a fibre space with the same basis X, called a sub-fibre-space of E. So the sub-fibre-spaces of E are in one to one correspondence with the subsets of E; in particular, for them the notions of union, intersection etc. are defined. (Of course, in most cases we are only interested in fibre spaces the projection of which is onto; this imposes then a condition on the subspaces of E considered, which may be fulfilled for two subspaces and not for the intersection.)

Let now R be an equivalence relation in E compatible with the map p, i.e. such that two elements of E congruent mod R have the same image under p. Then p defines a continuous map p' of the quotient space $E' = E/R$ into X, which turns E' into a fibre space with base X, called a quotient fibre space of E. So the latter are in one-to-one correspondence with the equivalence relations in E compatible with p. A quotient fibre space of a quotient fibre space of E is a quotient fibre space.

Let (X, E, p) and (X', E', p') be two fibre spaces, then (p, p') defines a continuous map of $E \times E'$ into $X \times X'$, so that $E \times E'$ appears as a fibre space over $X \times X'$, called the product of the fibre spaces E, E'. The fibre of (x, x') in $E \times E'$ is the product of the fibres of x in E, respectively x' in E'. Suppose now $X = X'$, and consider the inverse image of $E \times E'$ under the diagonal map $X \to X \times X$, we get a fibre space over X, called the fibre product of the fibre spaces E, E' over X, denoted by $E \underset{(X)}{\times} E'$. The fibre of x in this fibre-product is the product of the fibres of x in E respectively E'. Of course, product of an arbitrary family of fibre spaces can be considered, and the usual formal properties hold.

1.4 Trivial and locally trivial fibre spaces. Let X and F be two spaces, E the product space, the projection of the product on X defines E as a fibre space over X, called the trivial fibre space over X with fibre F. All fibres are canonically homeomorphic with F. Let us determine the homomorphisms of a trivial fibre space $E = X \times F$ into another $E' = X \times F'$. More generally, we will only assume that the projection of $X \times F$ onto X is the natural one and continuous for the given topology of $X \times F$, which induces on the fibres the given topology (but the topology of $X \times F$ may not be the product topology, for instance: X and F are algebraic varieties with the Zariski topology); same hypothesis on $X \times F'$. Then a homomorphism u of E into E', inducing for each $x \in X$ a continuous map of the fibre of E over x into the fibre of E' over x, defines a function $x \longrightarrow f(x)$ of X into the set of all continuous maps of F into F', and of course the homomorphism is well determined by this map by the formula

$$u(x, y) = (x, f(x).y) \qquad (x \in X, y \in F). \tag{1.4.1}$$

So the homomorphisms of E into E' can be identified with those maps f of X into the set of continuous maps of F into F' such that the map (1.4.1) is continuous. If the topologies of E and E' are the product topologies, this means that $(x, y) \longrightarrow f(x).y$ is continuous; as is well known, if moreover F is locally compact or metrizable, this means also that f is continuous when we take on the set of all continuous maps from F into F' the topology of compact convergence. If we consider a homomorphism v from E' into $E'' = X \times F''$ given by a map g of X into the set of all continuous maps of F' into F'' the homomorphism vu is given by the map $x \longrightarrow g(x)f(x)$. In order that the map (1.4.1) be injective (respectively surjective, bijective) it is necessary and sufficient that for each $x \in X$, $f(x)$ has the same property. In the bijective case, the inverse map is then defined by the function $x \longrightarrow f(x)^{-1}$. It follows that u is an isomorphism onto if and only if for each $x \in X$, f is a homeomorphism of F onto F', and the map $(x, y') \longrightarrow (x, f(x)^{-1}.y')$ continuous. So we get in particular (coming back to the case of trivial fibre spaces):

Proposition 1.4.1. Let $E = X \times F$ and $E' = X \times F'$ be two trivial fibre spaces over X, then the isomorphisms of E onto E' can be identified with the maps f of X into the set of homeomorphisms of F onto F' such that $f(x).y$ and $f(x)^{-1}.y'$ be continuous functions from $X \times F$ into F' respectively $X \times F'$ into F. If $E = E'$, this identification is compatible with the group structures on the set of automorphisms of E respectively the set of maps of X into the group of automorphisms of F.

Two fibre spaces E, E' over X are said to be locally isomorphic if each point x of X has a neighborhood U (which can be assumed open) such that the restrictions of E and E' to U are isomorphic. This is clearly an equivalence relation. A fibre space E over X is said to be locally trivial with fibre F (F being a given space) if it is locally isomorphic to the trivial space $X \times F$.

1.5 Definition of fibre spaces by coordinate transformations.

Let X be a space, (U_i) a covering of X, for each index i let E_i be a fibre space over U_i, and for any couple of indices i, j such that $U_{ij} = U_i \cap U_j \neq \emptyset$, let f_{ij} be a U_{ij}-isomorphism of $E_j | U_{ij}$ onto $E_i | U_{ij}$. On the topological sum $\mathcal{E}$ of the spaces E_i, let us consider the relation

(1.5.1.) $$y_i \in E_i | U_{ij} \text{ and } y_j \in E_j | U_{ij} \text{ are equivalent means } y_i = f_{ij} y_j.$$

This is an equivalence relation, as easily checked, if and only if we have, for each triple (i, j, k) of indices such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$, the relation

(1.5.2.) $$f_{ik} = f_{ij} f_{jk}$$

(where, in order to abbreviate notations, we wrote simply f_{ik} instead of: the isomorphism of $E_k | U_{ijk}$ onto $E_i | U_{ijk}$ induced by f_{ik}; and likewise for f_{ij} and f_{jk}). Supposing this condition satisfied, let E be the quotient space of $\mathcal{E}$ by the preceding equivalence relation. The projections p_i of E_i into U_i define a continuous map of the topological sum $\mathcal{E}$ into X, and this map is compatible with the equivalence relation in $\mathcal{E}$, so that there is a continuous map p of E into X (which is onto if the p_i's are all onto).

Definition 1.5.1. The fibre space over X just constructed is called the fibre space defined by the "coordinate transformations" (f_{ij}) between the fibre spaces E_i.

The identity map of E_i into $\mathcal{E}$ defines a map φ_i of E_i into E, which by virtue of (1.5.1.) is a one to one U_i-homomorphism of E_i onto $E|U_i$. The topology of E (by a well known transitivity property for topologies defined as the finest which ...) is the finest topology on E for which the maps φ_i are continuous. Moreover, it is easy to show that in case the interiors of the U_i's already cover X, the maps φ_i are homeomorphisms into. Henceforth, for simplicity we will only work with open coverings of X, so that the preceding properties are automatically satisfied. Then φ_i can be considered as a U_i-isomorphism of E_i onto $E|U_i$. Clearly

$$f_{ij} = \varphi_i^{-1} \varphi_j \tag{1.5.3.}$$

(where again, in order to abbreviate, we wrote φ_i instead of the restriction of φ_i to $E_i|U_{ij}$, φ_j instead of the restriction of φ_j to $E_j|U_{ij}$). Conversely, let E be a fibre space over X, and suppose that for each i there exists a U_i-isomorphism φ_i of E_i onto $E|U_i$, then (1.5.3.) defines, for each pair (i, j) such that $U_i \cap U_j = U_{ij} \neq \emptyset$, a U_{ij}-isomorphism of $E_j|U_{ij}$ onto $E_i|U_{ij}$, and the system (f_{ij}) satisfies obviously (1.5.2.). Therefore we can consider the fibre space E' defined by the coordinate transformations f_{ij}. Then it is obvious that the map of $\mathcal{E}$ into E defined by the maps φ_i is compatible with the equivalence relation in $\mathcal{E}$, therefore defines a continuous map f of E' into E which is of course an X-homomorphism. Let φ'_i be the natural isomorphism of E_i onto $E'|U_i$ defined above; it is checked at once that the map of $E'|U_i$ into $E|U_i$ induced by f is $\varphi_i \varphi_i'^{-1}$, hence an isomorphism onto. It follows that f itself is an isomorphism of E' onto E, by virtue of the following easy lemma (proof left to the reader):

Lemma 1. Let E, E' be two fibre spaces over X, and f an

X-homomorphism of E into E', such that for any $x \in X$, exists a neighborhood U of x such that f induces an isomorphism of $E|U$ onto (respectively, into) $E'|U$. Then f is an X-isomorphism of E onto (respectively, into) E'.

What precedes shows the truth of:

<u>Proposition</u> <u>1. 5. 1.</u> The open covering (U_i) and the fibre spaces E_i over U_i being given, the fibre spaces over X which can be obtained by means of suitable coordinate transformations (f_{ij}) are exactly those, up to isomorphism, for which $E|U_i$ is isomorphic to E_i for any i.

Consider now two systems of coordinate transformations (f_{ij}), (f'_{ij}) corresponding to the same covering (U_i), and to two systems (E_i), (E'_i) of fibre spaces over the U_i's. Let E be the fibre space defined by (f_{ij}) and E' the fibre space defined by (f'_{ij}); we will determine all homomorphisms of E into E'. If f is such a homomorphism, then for each i, $f_i = \varphi_i'^{-1} f \varphi_i$ (where f stands for the restriction of f to $E|U_i$) is a homomorphism of E_i into E'_i, and the system (f_i) satisfies clearly, for each pair (i, j) such that $U_{ij} \neq \emptyset$:

$$f_i f_{ij} = f'_{ij} f_j \tag{1.5.4}$$

(where we write simply f_i instead of the restriction of f_i to $E_i|U_{ij}$, and likewise for f_j). The homomorphism f is moreover fully determined by the system (f_i) since f_i determines the restriction of f to $E|U_i$; and moreover the system (f_i) subject to (1. 5. 4) can be chosen otherwise arbitrarily, for this relation expresses exactly that the map of the topological sum $\mathcal{E}$ of the E_i's into the topological sum $\mathcal{E}'$ of the E'_i's transforms equivalent points into equivalent points, and therefore defines an X-homomorphism f of E into E'; and it is clear that the system (f_i) is nothing else but the one which is defined as above in terms of the homomorphism f. Of course, in view of lemma 1, in order that f be an isomorphism onto, (respectively, into) it is necessary and sufficient that each f_i be an isomorphism of E_i onto (respectively, into) E'_i. Thus we get:

<u>Proposition</u> 1. 5. 2. Given two fibre spaces over X, E and E', defined by coordinate transformations (f_{ij}) respectively (f'_{ij}) relative to the same open covering (U_i), the X-homomorphisms f of E into E' are in one to one correspondence with systems (f_i) of U_i-homomorphisms $E_i \longrightarrow E'_i$ satisfying (1. 5. 4.). f is an onto-isomorphism if and only if the f_i's are, i. e. E' is isomorphic to E if and only if we can find onto-isomorphisms $f_i: E_i \longrightarrow E'_i$ such that, for any pair (i, j) of indices satisfying $U_{ij} \neq \emptyset$, we have

(1. 5. 5.) $$f'_{ij} = f_i f_{ij} f_j^{-1}$$

(where as usual f_i and f_j stand for restricted maps).

We proceed to the comparison of fibre spaces E, E' defined by coordinate transformations corresponding to different coverings, (U_i) and (U'_i), in particular to the determination of the homomorphisms of E into E' and hence of the X-isomorphisms of E and E', and therefore to the determination of whether E and E' are isomorphic. Let (V_j) be an open covering of X which is a refinement of both preceding coverings; we will show that E and E' are isomorphic to fibre spaces defined by coordinate transformations relative to this same covering (V_j), so that the problem is reduced to one already dealt with.

So let $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ be two open coverings of X, the second finer than the first, that is, any V_j is contained in some U_i, i. e. there exists at least one map $\tau: J \longrightarrow I$ such that $V_j \subset U_{\tau(j)}$ for any $j\in J$. For each $i\in I$, let E_i be a fibre space over U_i, and let $(f_{ii'})$ be a system of coordinate transforms relative to the system (E_i). For each $j\in J$, let $F_j = E_{\tau(j)}|V_j$, and let $g_{jj'}$ be the restriction of $f_{\tau(j),\tau(j')}$ to $F_j|V_{jj'}$; so $g_{jj'}$ is an isomorphism of $F_j|V_{jj'}$ onto $F_j|V_{jj'}$, and the system $(g_{jj'})$ is a system of coordinate transformations, as follows at once from the definition and (1. 5. 2.) applied to the system $(f_{ii'})$. Let F be the fibre space defined by the system of coordinate transformations $(g_{jj'})$; we shall define a canonical X-isomorphism of F onto E. For $j \in J$, let g_j be the injection map of F_j into $E_{\tau(j)}$; it is hence a map of F_j into the topological sum $\mathcal{E}$ of the E_i's; the system (g_j) defines a continuous map g' of the topological sum $\mathcal{F}$ of the f_j's into $\mathcal{E}$, and as easily seen g' maps equivalent points into equivalent points. Hence g' induces a continuous map g of F into E, which clearly is an X-homomorphism. Moreover, for any j, g induces an isomorphism of $F|V_j$ onto $E|V_j$, for if we compose it with the natural isomorphism of $E|U_i$ onto E_i, we get the injection map of $E|V_j$ into E_i (we put $i = \tau(j)$). Now applying lemma 1, we see that g is an isomorphism of F onto E.

1.6 The case of locally trivial fibre spaces. The method of the preceding section for constructing fibre spaces over X will be used mainly in the case where we are given a fibre space T over X, and where, given an open covering (U_i) of X, we consider the fibre spaces $E_i = T|U_i$ over U_i and coordinate transformations (f_{ij}) with respect to these. Then f_{ij} is an U_{ij}-automorphism of $T|U_{ij}$. The fibre space defined by the system (f_{ij}) of coordinate transformations will be locally isomorphic (cf. 1.4.) to T, and in virtue of proposition 1.5.1., we obtain in this way exactly (up to isomorphism) all fibre spaces over X which are locally isomorphic to T (by taking the open sets U_i small enough, and then a suitable system (f_{ij})).

In case T is a trivial fibre space, $T = X \times F$, we have $E_i = U_i \times F$, and $E_i|U_{ij} = U_{ij} \times F$. Thus f_{ij} is an automorphism of the trivial fibre space $U_{ij} \times F$, and therefore, in view of proposition 1.4.1. given by a map $x \longrightarrow f_{ij}(x)$ of U_{ij} into the group of homeomorphisms of F onto itself. The equations (1.5.2.) expressing that (f_{ij}) is a system of coordinate transformations then translate into

(1.6.1.) $$f_{ik}(x) = f_{ij}(x)f_{jk}(x) \qquad \text{for } x \in U_{ijk}.$$

Moreover, it must not be forgotten that $x \longrightarrow f_{ij}(x)$ is submitted to the continuity condition of proposition 1.4.1. Such a system then defines in a natural way a fibre space E over X, and by what has been said it follows that this fibre bundle is locally isomorphic to $X \times F$, i.e. <u>locally trivial with fibre F</u>, and that (for suitable choice of the covering and the coordinate transformations), <u>we get thus, up to isomorphism, all locally trivial fibre spaces over X with fibre F</u>.

Let in the same way $T' = X \times F'$, and consider for the same covering (U_i) a system (f_{ij}) and a system (f'_{ij}) of coordinate transformations, the first relative to the fibre F and the second to the fibre F'. Let E and E' be the corresponding fibre spaces over X. The homomorphisms of E into E', by proposition 1.5.2., correspond to homomorphisms f_i of $E_i = U_i \times F$ into $E'_i = U_i \times F'$, satisfying conditions (1.5.4.). Now, (proposition 1.4.1.) such a homomorphism f_i is determined by a map $x \longrightarrow f_i(x)$ of U_i into the set of continuous maps of F into F' by $f_i(x, y) = (x, f_i(x).y)$, subject to the only requirement that $f_i(x).y$ is continuous with respect to the pair $(x, y) \in U_i \times F$. Then the equation (1.5.4.) translates into

(1.6.2.) $$f_i(x)f_{ij}(x) = f'_{ij}(x)f_j(x) \qquad (x \in U_{ij})$$

Thus are determined the homomorphisms of E into E'. In particular, the isomorphisms of E onto E' are obtained by systems (f_i) such that $f_i(x)$ be a homeomorphism of F onto F' for any $x \in U_i$, and that $x \longrightarrow f_i^{-1}(x)$ satisfies the same continuity requirement as $x \longrightarrow f_i(x)$. The compatibility condition (1.6.2.) can then be written

(1.6.3.) $$f'_{ij}(x) = f_i(x)f_{ij}(x)f_j(x)^{-1} \qquad (x \in U_{ij})$$

<u>1.7 Sections of fibre spaces.</u>

<u>Definition 1.7.1.</u> Let (X, E, p) be a fibre space; a <u>section</u> of this fibre space (or, by pleonasm, a section of E over X) is a map x of X into E such that ps is the identity map of X. The set of <u>continuous</u> sections of E is noted $H^o(X, E)$.

It amounts to the same to say that s is a function the value of which at each $x \in X$ is in the fibre of x in E (which depends on x !). The existence of a section implies of course that p is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. - A section of E over a subset Y of X is by definition a section of $E|Y$. If Y is open, we write $H^o(Y, E)$ for the set $H^o(Y, E|Y)$ of all continuous sections of E over Y.

$H^o(X, E)$ as a functor. Let E, E' be two fibre spaces over X, f an X-homomorphism of E into E'. For any section s of E, the composed map fs is a section of E', continuous if s is continuous. We get thus a map, noted f , of $H^o(X, E)$ into $H^o(X, E')$. The usual functor properties are satisfied:

a. If the two fibre spaces are identical and f is the identity, then so is f

b. if f is an X-homomorphism of E into E' and f' an X-homomorphism of E' into E" (E, E', E" fibre spaces over X) then (f'f) = f' f .

Let (X, E, p) be a fibre space, f a continuous map of a space X' into X, and E' the inverse image of E under f. Let s be a section of E' consider the map s' of X' into E' given by s'x' = (x', sfx') (the second member belongs to E', since fx' = psfx' because px = identity), this is a section of E', continuous if s is continuous. Thus we get a canonical map of $H^o(X, E)$ into $H^o(X', E')$ (E' being the inverse image of E by f). In case $X' \subset X$ and f is the inclusion map, therefore $E' = E|X'$, then the preceding map is nothing but the restriction map (of $H^o(X, E)$ into $H^o(X', E)$ if X' open). - We leave to the reader statement and proof of an evident property of transitivity for the canonical maps just considered.

The two sorts of homomorphisms for sets of continuous sections are compatible in the following sense. Let φ be a fixed continuous map of a space X' into X, then to any fibre space E over X corresponds its inverse image E' under φ, which is a fibre space over X'; moreover, given an X-homomorphism f: $E \longrightarrow F$, it defines in a natural way an X'-homomorphism f' of E' into F'. (We could go further and state that, for fixed φ, E' is a "functor" of E by means of the preceding definitions.)

Then the following diagram

$$\begin{array}{ccc} H^o(X, E) & \xrightarrow{f_*} & H^o(X, F) \\ \downarrow & & \downarrow \\ H^o(X', E') & \xrightarrow{f'_*} & H^o(X', F') \end{array}$$

is commutative, where the vertical arrows stand for the canonical homomorphisms defined above. The checking of course is trivial. Particular case: replacing X by an open subset U of X, and taking for X' an open subset V of U and φ the inclusion map $V \longrightarrow U$, we get that for any two fibre spaces E, F over X and X-homomorphism $f: E \longrightarrow F$, the following diagram is commutative:

(1.7.2.)
$$\begin{array}{ccc} H^o(U, E) & \longrightarrow & H^o(U, F) \\ \downarrow & & \downarrow \\ H^o(V, E) & \longrightarrow & H^o(V, F) \end{array}$$

where the vertical arrows are the restriction maps, and the horizontal arrows are the maps defined by f (or, strictly speaking, by the restrictions of f to $E|U$ respectively $E|V$). In words: the "homomorphisms" between spaces of sections over open sets defined by X-homomorphisms of fibre spaces commute with the restriction operators.

<u>Determination</u> <u>of</u> <u>sections</u>. Let us come back to the conditions of the definition 1.5.1.; we keep the notations of that section. Let s be a section of the fibre space E, and for any i let $s_i = \varphi_i^{-1} s$; then s_i is a section of E_i over U_i, and from $s = \varphi_i s_i = \varphi_j s_j$ over U_{ij} we get $s_i = \varphi_i^{-1} \varphi_j s_j = f_{ij} s_j$:

(1.7.3.) $$s_i = f_{ij} s_j$$

where again we write s_i, s_j instead of: restriction of s_i, s_j to U_{ij}. Of course, s is entirely determined by the system (s_i), for s is given over U_i by $s = \varphi_i s_i$. On the other hand, the system (s_i) subject to

(1. 7. 3.) can be otherwise arbitrary, for these conditions express precisely that for $x \in X$, the element $\varphi_i s_i(x)$ of E obtained by taking a U_i containing x does not depend on i, and may therefore be denoted by s(x): Then the φ_i^{-1}s determined by the above definition are of course nothing else than the s_i's we started with. Let us note also that in order that the section s be continuous, it is necessary and sufficient that each s_i be continuous. We thus obtain:

Proposition 1. 7. 1. Let E be the fibre space defined by coordinate transformations (f_{ij}) relative to an open covering (U_i) of X and fibre spaces E_i over U_i. Then there is a canonical one to one correspondence between sections of E and systems (s_i) of sections of E_i over U_i, satisfying conditions (1. 7. 3.). Continuous sections correspond to systems of continuous sections.

Let again, as in section 1. 5, be given two systems (E_i) and (E'_i) of fibre spaces over the U_i's and two corresponding systems of coordinate transformations (f_{ij}) and $f'_{ij})$, let E and E' be the corresponding fibre spaces, and f an X-homomorphism of E into E', defined by virtue of proposition 1. 5. 2., by a system (f_i) of U_i-homomorphisms of E_i into E'_i satisfying (1. 5. 4.). Let s be a section of E, given by a system (s_i) of sections of E_i over U_i. Then the system $(f_i s_i)$ of sections of E'_i over U_i defines the section fs (trivial).

The reader may check, as an exercise, how the canonical maps of spaces of sections considered above in this section, can be made explicit for fibre spaces given by means of coordinate transformations.

CHAPTER II

SHEAVES OF SETS

Throughout this exposition, we will now use the word "section" for "continuous section".

2.1 Sheaves of sets.

Definition 2.1.1. Let X be a space. A *sheaf of sets* on X (or simply a *sheaf*) is a fibre space (E, X, p) with base X, satisfying the condition: each point a of E has an open neighborhood U such that p induces a homeomorphism of U onto an open subset p(U) of X.

This can be expressed by saying that p is an interior map and a local homeomorphism. It should be kept in mind that, even if X is separated, E is not supposed separated (and will in most important instances not be separated).

With the notations of definition 2.1.1, let $x = p(a)$. If f is a section of E such that $fx = a$, then $V = f^{-1}(U) \cap p(U)$ is an open set containing x, and on this neighborhood V of x, f must coincide with the inverse of the homeomorphism $p|U$ of U onto p(U). In particular

Proposition 2.1.1. Two sections of a sheaf E defined in a neighborhood of x and taking the same value at x coincide in some neighborhood of X.

Corollary: Given two sections of E in an open set V, the set of points where they are equal is *open*. (But in general not closed, as would be the case if E were separated!)

2.2 $H^o(A, E)$ for arbitrary $A \subset X$. First let E be an arbitrary fibre space over X. Let A be an arbitrary subset of X; the open neighborhoods of A, ordered by $\supset$, form an ordered filtering set. To each element U of this set is associated a set $H^o(U, E)$: the set of sections of E over U, and if $U \supset V$ (U and V open neighborhoods of A), we have a natural map φ_{VU}: $H^o(U, E) \rightarrow H^o(V, E)$ (restriction map), with the evident transitivity property $\varphi_{WV} \; \varphi_{VU} = \varphi_{WU}$ when $U \supset V \supset W$. Therefore we can consider the direct limit of the family of sets $H^o(U, E)$ for the maps φ_{VU}.

Definition 2.2.1. We put $H^o(A, E) = \varinjlim H^o(U, E)$, (U ranging over the open neighborhoods as explained above). If $A = \{x\}$ $(x \in X)$ we simply write $H^o(x, E)$. The elements of $H^o(A, E)$ are called germs of sections of E in the neighborhood of A.

If A is open, we find of course nothing else but the set of continuous sections of E over A, already denoted by $H^o(A, E)$. -If $A \supset B$, there is a natural map, again noted φ_{BA}, of $H^o(A, E)$ into $H^o(B, E)$, (definition left to the reader). When A and B are both open, this is the usual restriction map (therefore it will in general still be called restriction map); when A is open, then this is the natural homomorphism of $H^o(A, E)$ into the direct limit of all $H^o(A', E)$ corresponding to open neighborhoods A' of B. Of course $A \supset B \supset C$ implies $\varphi_{CB} \varphi_{BA} = \varphi_{CA}$.

Let $\Gamma(A, E)$ be the set of continuous sections of E over the arbitrary set $A \subset X$, then the restriction maps $H^o(U, E) = \Gamma(U, E) \rightarrow \Gamma(A, E)$ (U, open neighborhood of A) define a natural map of $\varinjlim H^o(U, E) = H^o(A, E)$ into $\Gamma(A, E)$. In particular, there is a natural map $H^o(x, E) \rightarrow E_x$, where E_x is the fibre of x in E (value at x of a germ of section in a neighborhood of x). This of course, though frequently an onto-map, will seldom be one-to-one. However:

Proposition 2.2.1. If E is a sheaf on X, then for $x \in X$, the canonical map $H^o(x, E) \rightarrow E_x$ is bijective (i.e. one-to-one and onto). If A is any subset of X, then the canonical map $H^o(A, E) \rightarrow \Gamma(A, E)$ is one-to-one; it is moreover onto if A admits a fundamental system of paracompact neighborhoods.

The one-to-one parts are contained in Proposition 2.1.1 and its corollary. The first onto-assertion results at once from definition 2.1.1. Now let f be a continuous section of E over A; for any $x \in A$ let g_x be a continuous section of E on an open neighborhood V_x of x in X, such that $g_x(x) = f(x)$ (these exist by first part of proposition 2.2.1.). Moreover, by the first part of proposition 2.1.1. applied to $E|A$, we can suppose V_x small enough so that on $V_x \cap A$, g_x and f coincide. We can suppose that $U = \bigcup V_x$ is a paracompact neighborhood of A. Let $(V_i)_{i \in I}$ be an open locally finite covering of U finer than (V_x), that is each V_i is contained in some V_x. Then for each V_i exists $g_i \in H^o(V_i, E)$ such that g_i and f coincide on $V_i \cap A$. U being paracompact, we can find an open covering (V_i') of U such that the relative closure of V_i' in U be contained in V_i. For each $x \in A$, there exists an open neighborhood W_x of x in U meeting only a finite number among the V_i's; taking W_x small enough, we can assume that $x \notin \overline{V_i'}$ implies $V_i' \cap W_x = \emptyset$, and $x \in \overline{V_i'}$ implies $W_x \subset V_i$. Moreover, by virtue of proposition 2.1.1., we can suppose that the corresponding g_i's are identical on W_x since they take the same value f(x) at x. Therefore whenever a V_j' encounters W_x, then g_i is defined on W_x and does not depend on the choice of i, so that we can denote it by h_x. It follows that in $W_x \cap W_y$, h_x and h_y are the same, therefore, the h_x are the restrictions of a unique section h of E over $W = \bigcup W_x$. This is a continuous section of E on an open neighborhood of A, and we see at once that its restriction to A is f. This ends the proof.

Remark. The last part of proposition 2.2.1. becomes false if we drop the paracompactness restriction. Let for instance X be an infinite set, with the topology in which the open sets are the complements of all finite sets (such spaces are significant in algebraic topology, for instance: irreducible algebraic curve with the Zariski topology). Let F be a discrete space; consider the trivial fibre space $X \times F$. This is a sheaf; its sections on a set A are the locally constant maps of A into F (cf. section 2.6. below, example a). Let A be a finite subset of X; it is seen at once that any open neighborhood of A is homeomorphic to X and hence connected; therefore a section of E on such a neighborhood is constant; but sections on A can have arbitrary distinct values at the points of A and therefore will not in general be restrictions of sections defined in a neighborhood of A.

2.3 Definition of a sheaf by systems of sets.

As we noticed in the preceding section, any fibre space E (and in particular any sheaf) determines sets $H^o(U, E)$ (for instance for any open $U \subset X$) and maps $H^o(U, E) \longrightarrow H^o(V, E)$ for $U \supset V$, satisfying an evident transitivity property. Proposition 2.2.1. suggests that conversely such a system should define a sheaf. Indeed, let $\mathcal{V}$ be an open covering of X, and suppose defined a function $U \longrightarrow E_U$ on the set of open sets which are small of order $\mathcal{V}$ (i.e. contained in some set element of $\mathcal{V}$), each E_U being a set. Suppose given moreover, if U and V are $\mathcal{V}$-small and $U \supset V$, a map $\varphi_{VU}: E_U \longrightarrow E_V$, these maps satisfying the transitivity condition

$$\varphi_{WV}\,\varphi_{VU} = \varphi_{WU} \quad \text{(if } U \supset V \supset W\text{)}. \tag{2.3.1.}$$

For any $x \in X$, let $E_x = \varinjlim E_U$, U ranging over the ordered filtering set of open neighborhoods of x (ordered by $\supset$). Let E be the union of the E_x's, and p the map of E into X mapping E_x in x. Define in E a topology as follows: for any $f \in E_U$, and $x \in U$, we consider the canonical image f_x of f in the direct limit E_x of the sets $E_{U'}$ corresponding to all open neighborhoods U' of x. Let O(f) be the set of all elements $f_x \in E$ when x ranges over U. When U and $f \in E_U$ vary, we get a family of subsets O(f) of E, which generate a topology on E. It is easily checked that (E, X, p) form a sheaf, that is that p is continuous, interior and a local homeomorphism.

Definition 2.3.1. The sheaf E thus defined is called the sheaf defined by the system of sets E_U amd maps φ_{VU}.

Consider now an open set $U \subset X$, $\mathcal{V}$-small; for any $f \in E_U$, the map $x \longrightarrow f_x$ is clearly a section of the sheaf E, and moreover continuous, which we denote by $\tilde{f}$. We get thus a natural map $f \longrightarrow \tilde{f}$ of E_U into $H^o(U, E)$.

Proposition 2.3.1. In order that $f \longrightarrow \tilde{f}$ be a one-to-one map, it is necessary and sufficient that for any open covering (U_i) of U, and two elements f, g of E_U, $\varphi_{U_iU} f = \varphi_{U_iU} g$ for each i implies f = g. In order that $f \longrightarrow \tilde{f}$ be onto, it is necessary and sufficient that for any open covering (U_i) of U, and any system $(f_i) \in \prod E_{U_i}$ satisfying

$$\varphi_{U_i \cap U_j, U_i} f_i = \varphi_{U_i \cap U_j, U_j} f_j \quad \text{when } U_i \cap U_j \neq \emptyset \tag{2.3.2.}$$

there exists a $f \in E_U$ such that $f_i = \varphi_{U_iU} f$ for each i.

Corollary. In order that $f \longrightarrow \tilde{f}$ be bijective, it is necessary and sufficient that for any open covering (U_i) of U, the natural map $E_U \longrightarrow \prod E_{U_i}$ (the components of which are the maps φ_{U_iU}) be a one-to-one map of E_U onto the subset of the product of all (f_i) satisfying condition (2.3.2.).

Proof left to the reader, as well as the proof of the following:

Proposition 2.3.2. Let E be a sheaf on X, consider the system of sets $H^o(U, E)$ and of restriction maps φ_{VU}: $H^o(U, E) \longrightarrow H^o(V, E)$ for $U \supset V$ (U, V open sets). Then the sheaf E' defined by these data (definition 2.3.1.) is canonically isomorphic to E, this isomorphism, transforming for each $x \in X$, $E'_x = \varinjlim H^o(U, E) = H^o(x, E)$ into E_x, being the isomorphism considered in proposition 2.2.1.

The two preceding propositions show essential equivalence of the notion of sheaf on the space X, and the notion of a system of sets (E_U) (U open $\subset$ X) and of maps φ_{VU} for $U \supset V$, satisfying conditions (2.3.1.) and the condition of corollary of proposition 2.3.1. Both pictures are of importance, the second more intuitive, but the first often technically more simple.

Exercise. Given a system of sets E_U (U open and $\mathcal{V}$-small) and of homomorphisms φ_{VU} $(U \supset V)$ satisfying (2.3.1.), prove that if we restrict to those U which are $\mathcal{V}'$-small (where $\mathcal{V}'$ is an open covering of X finer than $\mathcal{V}$), the sheaf defined by this new system is canonically isomorphic to the sheaf defined by the first.

2.4 Permanence properties.

Let E be a sheaf on the space X, and let f be a continuous map of a space X' into X, then the inverse image of the fibre space E by f (cf 1.2.) is again a sheaf. In particular, if $X' \subset X$, E induces a sheaf on X'. If E is a sheaf on X, F a sheaf on Y, then $E \times F$ is a sheaf on $X \times Y$; therefore, if E and F are two sheaves on X, then their fibre-product $E \underset{X}{\times} F$ (cf. 1.3) is again a sheaf; this extends to the product of a finite number of sheaves.

Under the conditions of 1.5. suppose that the fibre spaces E_i on the open sets U_i are sheaves, then the fibre space E obtained by means of coordinate transforms f_{ij} is again a sheaf. This results at once from the more general remark: if E is a fibre space such that each $x \in X$ has a neighborhood U such that $E|U$ be a sheaf, then E is a sheaf (trivial).

2.5 Subsheaf, quotient sheaf. Homomorphisms of sheaves.

Proposition 2.5.1. Let E be a sheaf on the space X. In order that a subset F of E, considered as a fibre space over X, be a sheaf, it is necessary and sufficient that it be open. In order that the quotient of E by an equivalence relation R compatible with the fibering, be a sheaf, it is necessary and sufficient that the set of equivalent pairs (z, z') be open in the fibered product $E \underset{X}{\times} E$.

These conditions can be stated also equivalently: if a section f of E in a neighborhood of $x \in X$ is such that $fx \in F$, then $fy \in F$ for y in a neighborhood of x; if two sections f, g of E in a neighborhood of $x \in X$ are such that fx and gx are equivalent mod R, then fy and gy are equivalent mod R for y in a neighborhood of x.

Proposition 2.5.2. Let E be a sheaf on X, E' a sheaf on X', f a continuous map of X into X' and g a map from E into E' such that p'g = fp (p, p' being the projections of E, e'). In order for g to be an f-homomorphism (i.e. to be continuous) it is necessary and sufficient that for any section s of E over an open set U, gs be a section of E' over f(U).

Corollary 1. Let f be a bijective X-homomorphism of a sheaf E in a sheaf F, then f is an isomorphism of E onto F.

Corollary 2. Let E, F be two sheaves on X, f an X-homomorphism of E into F. Then f is an interior map, and f(E) is a subsheaf of F. The quotient of E by the equivalence relation defined by the map f is again a sheaf, and f defines an isomorphism of this quotient onto the sheaf f(E).

Consider now a system (E_U, φ_{VU}) as in section 2.3., defining a sheaf E Suppose given for each U a subset E'_U of E_U, such that $U \supset V$ implies $\varphi_{VU}(E'_U) \subset E'_V$. Let φ'_{VU} be the map of E'_U into E'_V defined by φ_{VU}, then the system (E'_U, φ'_{VU}) defines a sheaf E'. For any $x \in X$, the fibres of x in E respectively E' are given by

$$E_x = \varinjlim E_U \qquad E'_x = \varinjlim E'_U$$

the direct limit being taken in the ordered filtering set of open neighborhoods of x. Therefore, we have a natural injection $E'_x \subset E_x$, and hence $E' \subset E$. It is easily checked that the injection of E' into E is a homomorphism, (a particular case of a general characterization of homomorphisms to be given below), so that by corollary 1 above, E' is isomorphic to a subsheaf of E. Suppose that the conditions of proposition 2.3.1. corollary, are satisfied, which insure that $E_U = H^0(U, E)$. Then clearly the canonical maps $E'_U \longrightarrow H^0(U, E') \subset H^0(U, E)$ are one-to-one. Proposition 2.3.1. yields that in order that they be onto, it is necessary and sufficient that any $f \in E_U$, such that each $x \in U$ has an open neighborhood V in U such that $\varphi_{VU} f \in E'_V$, be contained in E'_U; or shortly speaking that the property, for an element f of an E_U to belong to the subset E'_U, be a property of local character. If conversely we start with an arbitrary subsheaf E' of E, and denote by E'_U the subset $H^0(U, E')$ of $E_U = H^0(U, E)$, then these E'_U clearly satisfy to the conditions $\varphi_{VU} E'_U \subset E'_V$, and the subsheaf of E defined by them is nothing else but E'.

Now let E, F be two sheaves on X defined by systems (E_U, φ_{VU}) and (F_U, ψ_{VU}). Suppose given for any U a map $f_U: E_U \longrightarrow F_U$, such that $U \supset V$ implies $\psi_{VU} f_U = f_U \varphi_{VU}$. Then this system of maps defines, for each $x \in X$, a map f_x of $E_x = \varinjlim E_U$ into $F_x = \varinjlim F_U$, hence a map f of E into F. It is checked easily (using for instance proposition 2.5.2.) that f is a homomorphism of E into F. Moreover, f(E) is nothing else but the subsheaf of F defined by the subsets $f_U(E_U)$ of the F_U. For any open U, the following diagram is commutative:

$$\begin{array}{ccc} E_U & \xrightarrow{fu} & F_U \\ \downarrow & & \downarrow \\ H^o(U, E) & \xrightarrow{f_*} & H^o(U, F) \end{array}$$

In particular, if the vertical maps are bijective, we see that the maps f_U can be identified with the maps $f_*: H^o(X, E) \longrightarrow H^o(X, F)$ defined by the homomorphism f. Conversely, if we start with an arbitrary homomorphism f of E into F, then the homomorphism defined by the system of maps f_U of $E_U = H^o(U, E)$ into $F_U = H^o(U, F)$ is precisely f.

2.6. Some examples.

a. Constant and locally constant sheaves Let F be a discrete space, then the trivial fibre space $X \times F$ is clearly a sheaf on X; a sheaf isomorphic to such a sheaf is called constant. The sections of this sheaf on a set $A \subset X$ are the continuous maps of A in the discrete set F, i.e. the maps of A in F which are locally constant. If for instance A is connected, these reduce to the constant maps of A into F. Inverse images and products of simple sheaves are simple.

A sheaf E on X is called locally simple, if each $x \in X$ has a neighborhood U such that $E|U$ be simple. Thus a locally simple sheaf on X is nothing else but a covering space of X in the classical sense (but not restricted of course to be connected). Inverse images and products of locally simple sheaves in finite number are locally simple.

b. Sheaf of germs of maps. Let X be a space, E a set. Consider for any open $U \subset X$ the set $\mathcal{F}(U, E)$ of all maps of U into E; if $U \supset V$, we have a natural map of $\mathcal{F}(U, E)$ into $\mathcal{F}(V, E)$, the restriction map. The transitivity condition of section 2.3 is clearly satisfied, and also the condition of proposition 2.3.1., corollary. Therefore the sets $\mathcal{F}(U, E)$ can be identified with the sets of sections $H^o(U, \mathcal{F})$ of a well determined sheaf $\mathcal{F}$, the elements of which are called germs of maps of X into E. If $A \subset X$, then the elements of $H^o(A, \mathcal{F})$ are called germs of maps of a neighborhood of A into E. If now E is a topological space, we can consider for any U the subset C(U, E) of $\mathcal{F}(U, E)$ of the continuous maps of U into E. As continuity is a condition of local character, it follows by section 2.5 that the sets C(U, E) are the sets of sections of a well determined subsheaf of $\mathcal{F}$, which is called the sheaf of germs of continuous maps of X into E. (If we take on E the coarsest topology, we find again the first sheaf.) Suppose now that E is a fibre space over X, then consider for any U the subset $H^o(U, E)$ of C(U, E) of continuous sections of E. The property of being a section is again of local character, so we see that the sets $H^o(U, E)$ are sets of sections of a well determined subsheaf of the sheaf of germs of continuous maps of X into E: the sheaf of germs of sections of the fibre space E. If this sheaf is denoted by $\widetilde{E}$, then $H^o(A, \widetilde{E})$ is nothing else but the set of germs of sections of E in the neighborhood of A, as defined in definition 2.2.1.

Of course, specializing the spaces X and E, we can define a great number of other subsheaves of the sheaf of germs of maps of X into E (germs of differentiable maps, germs of analytic maps, germs of maps which are L^p etc.).

c. Sheaf of germs of homomorphisms of a fibre space into another. Let E and F be two fibre spaces over X, and for any open $U \subset X$ let H_U be the set of homomorphisms of $E|U$ into $F|U$. If V is an open set contained in U, there is an evident natural map of restriction $H_U \longrightarrow H_V$. The condition of transitivity as well as the condition of proposition 2.3.1. corollary, are satisfied, so that the sets H_U appear as the sets $H^o(U, H)$ of sections of a well determined sheaf on X, the elements of which are called germs of homomorphisms of E into F. A section of this sheaf over X is a homomorphism of E into F.

d. *Sheaf of germs of subsets.* Let X be a space, for any open set $U \subset X$ let P(U) be the set of subsets of U. If $U \supset V$, consider the map $A \longrightarrow A \cap V$ of P(U) into P(V). Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets P(U) appear as the sets $H^0(U, P(X))$ of sections of a well determined sheaf on X, the elements of which are called *germs of sets in X*. Any condition of a local character on subsets of X defines a subsheaf of P(X), for instance the sheaf of *germs of closed sets* (corresponding to the relatively closed sets in U), or if X is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.

CHAPTER III

GROUP BUNDLES AND SHEAVES OF GROUPS

3.1 Fibre spaces with composition law.

Let E be a fibre space over X, provided with the supplement of structure defined by a homomorphism of the fibre product $E \underset{X}{\times} E$ (cf. 1.3.) into E, or what is the same, a law of composition defined in each fibre F_x such that the corresponding global map $E \underset{X}{\times} E \longrightarrow E$ be continuous. This will be called shortly "fibre space with composition law". If E and F are fibre spaces with composition law, then their fibre product $E \underset{X}{\times} F$ is in a natural way a fibre space with composition law, the composition law in each fibre being the product of those in the fibre of E and of F. (This extends for products of arbitrary families of fibre spaces.) In the same way, if E is a fibre space with composition law on X, its inverse image under a continuous map f of a space X' into X is again in a natural way a fibre space with composition law on X', the fibre at x' being isomorphic to the fibre of E at fx'. Associativity of the pruduct, and transitivity of inverse images, clearly hold in this modified context. Let E (respectively, E') be a fibre space with composition law on X (respectively, X'), f a continuous map of X into X'. An f-homomorphism of E into E' is an f-homomorphism in the sense of 1.2., with the supplementary condition that it be, for each $x \in X$, a homomorphism of the fibre of E over x (provided with its composition law) into the fibre of E' over $x' = fx$. The product of two homomorphisms between fibre spaces with composition law is again a homomorphism. In particular, if X' = X and f is the identity, we have the notion of X-homomorphism, or simply homomorphism, of a fibre space with composition law on X into another. Let again X' and f be arbitrary; an inverse homomorphism associated with f of E' into E is by definition a homomorphism of the inverse image of E' under f into E;

for each $x \in X$, it hence defines a homomorphism of the fibre of E' over fx into the fibre of E over x.

If E is a fibre space with composition law on X, then $H^{0}(X, E)$ is provided with a natural composition law $H^{0}(X, E) \times H^{0}(X, E) \longrightarrow H^{0}(X, E)$; of course the first member is nothing else than $H^{0}(X, E \times_{X} E)$, and therefore the homomorphism $E \times_{X} E \longrightarrow E$ gives the desired map. We conclude at once that more generally, for each $A \subset X$, $H^{0}(A, E)$ is provided with a natural composition law. If $B \subset A$, then the restriction map $H^{0}(A, E) \longrightarrow H^{0}(B, E)$ is a homomorphism. Also the natural maps $H^{0}(\{x\}, E) \longrightarrow E_{x}$ (E_{x} the fibre of E over x) are homomorphisms, and hence onto-isomorphisms if E is a sheaf (proposition 2.2.1.). If now f is a homomorphism of E into a second fibre space with composition law F, then the maps $f : H^{0}(A, E) \longrightarrow H^{0}(A, F)$ are homomorphisms.

Under the conditions of 2.3., suppose that the sets E_{U} are provided with a composition law, and that the maps φ_{VU} are homomorphisms. Take on $E_{x} = \varinjlim E_{U}$ (U open neighborhood of x) the composition law which is obtained in terms of those in E_{U}. By what has been said at the end of section 2.5., the map of $E \times_{X} E$ into E obtained by these composition laws is a homomorphism, therefore E appears as a sheaf with composition law. The natural maps $E_{U} \longrightarrow H^{0}(U, E)$ are homomorphisms, and hence isomorphisms onto in case the system (E_{U}, φ_{VU}) satisfies the conditions of proposition 2.3.1., corollary. In particular, if E is any fibre space with composition law, then the sheaf of germs of sections of E, defined by the sets $H^{0}(U, E)$, is again a sheaf with composition law, the set of sections of which on the open set U is $H^{0}(U, E)$; this identification being compatible with the composition laws. Again, suppose given a second system of sets F_{U} with composition laws, and of homomorphisms $F_{U} \longrightarrow F_{V}$ for $V \subset U$, thus defining a sheaf F with composition law, and suppose given for each U a <u>homomorphism</u> $E_{U} \longrightarrow F_{U}$ satisfying the commutativity condition as at the end of section 2.5. Then the corresponding homomorphism of E into F (same reference) is a homomorphism in the sense of sheaves with composition law.

A fibre space with composition law is called <u>trivial</u> if it is isomorphic to a fibre space $X \times F$, where F is a topological space with a continuous composition law $F \times F \longrightarrow F$ (the composition law on the fibres of $X \times F$ being of course the one given on F). Hence also the notion of locally trivial fibre space with composition law.

3.2 Group bundles and sheaves of groups.

Definition 3.2.1. A group bundle E over the topological space X is a fibre space with composition law over X (cf. 3.1.) such that for each $x \in X$, the fibre E_x of E be a group, the unit of which depends continuously on x, and that the map of E into itself which on each fibre E_x reduces to $z \longrightarrow z^{-1}$ be continuous. If E is moreover a sheaf, E is called a sheaf of groups.

The definition implies that each E_x contains at least one element, the unit element e_x. $x \longrightarrow e_x$ is a section of E simply noted e (or O if the groups are written additively). Besides, in the case E is a sheaf, it is easily seen that the conditions that $x \longrightarrow e_x$ be a section, or $z \longrightarrow z^{-1}$ continuous, are equivalent.

With the definitions introduced in 3.1., we see at once that the product of group bundles is a group-bundle, therefore the product of a finite number of sheaves of groups is a sheaf of groups; the inverse image of a group bundle by a continuous map is a group bundle, in particular the inverse image of a sheaf of groups is a sheaf of groups. If E is a group bundle, then $H^0(A, E)$ is a group for any non void $A \subset X$. Conversely, under the conditions of 2.3. suppose that the sets E_U are groups and the maps φ_{VU} are group-homomorphisms, then the corresponding sheaf E, with the composition law defined in the last section, is a sheaf of groups. For instance, if Γ is a topological group, the sheaf $\mathcal{C}(x, \Gamma)$ of germs of continuous maps of X into Γ can be considered as a sheaf of groups.

Now consider again the construction of 1.5. With the notations of this section, suppose that the fibre spaces E_i are fibre spaces with composition law, and the isomorphisms f_{ij} are isomorphisms for this structure. Then on each fibre E_x of the fibre space defined by the E_i and f_{ij}, there is a natural composition law, obtained from the fibre E_{ix}, over x, of any E_i such that $x \in U_i$, by the natural map of E_{ix} onto E_x

(and this law does not depend on the choice of i). It is seen at once that E becomes in this way a fibre space with composition law. If the E_i's are group bundles, then E is a group bundle, for $E|U_i$ is isomorphic to E_i, in particular, if the E_i's are sheaves of groups, so is E. In order that a trivial fibre space X >< F (ct. end of section 3.1.) be a group bundle, it is necessary and sufficient that F be a topological group. If F is a discrete group, then X >< F is a sheaf of groups; a sheaf of groups isomorphic to such a product is called a constant sheaf of groups. Hence the notion of a locally constant sheaf of groups.

3.3 Sub-group-bundles and quotient-bundles. Subsheaves and quotient sheaves.

Let G be a group-bundle over X. A sub-group-bundle of G is by definition a subspace F of G such that for each $x \in X$, $F_x = F \cap G_x$ be a subgroup of G_x (fibre of G over x). Then the induced fibre space structure, and the group-law induced on each fibre F_x of F by G_x, turn F into a group-bundle. This structure can also be characterized by the fact that the injection $F \longrightarrow G$ be a homomorphism of group bundles. Hence for each $A \subset X$, there is a natural homomorphism $H^o(A, G) \longrightarrow H^o(A, G)$, and of course $H^o(A, G)$ is even a subgroup of $H^o(A, G)$. Let z, $z' \in G$, we will call z and z' congruent mod F, if and only if they belong to the same fibre G_x, and if $z' \in z.G_x$. This is clearly an equivalence relation compatible with the fibering of G, (cf. 1 3.) the quotient is a fibre space over X denoted by G/F. Thus for any $x \in X$, the fibre of G/F is the homogenous space G_x/F_x. The natural maps $G_x \longrightarrow G_x/F_x$ combine into a natural homomorphism of the fibre space G into the fibre space G/F, hence for any non-void $A \subset X$ a natural map $H^o(A, G) \longrightarrow H^o(A, G/F)$. There is a distinguished element in each $H^o(A, G/F)$, namely the image of the unit element e of $H^o(A, G)$, this image is still denoted by e. The inverse image of the distinguished element of $H^o(A, G/F)$ is nothing else than $H^o(A, F)$. We express this fact by saying that the sequence of maps (between sets each of which has a distinguished element)

$$H^o(A, F) \longrightarrow H^o(A, G) \longrightarrow H^o(A, G/F) \tag{3.3.1.}$$

is <u>exact.</u> More generally, two elements of $H^o(A, G)$ have the same image in $H^o(A, G/F)$ if and only if they are congruent mod the subgroup $H^o(A, F)$.

We say that the sub-group-bundle F is <u>normal</u>, if each F_x is normal in G_x. Then on each fibre G_x / F_x of G/F we can consider the structure of quotient group; as seen at once, this turns G/F into a group-bundle, such a group-bundle is called a <u>quotient group-bundle</u> of G. The natural homomorphism $G \to G/F$ is a homomorphism of group-bundles, therefore the corresponding map $H^o(A, G) \longrightarrow H^o(A, G/F)$ is a homomorphism. Besides, the distinguished element of $H^o(A, G/F)$ is nothing but its unit element, so that exactness of 3.3.1. means that the kernel of the second homomorphism is the image of the first, i.e. here $H^o(A, F)$ itself; in particular the latter is a <u>normal</u> subgroup of $H^o(A, G)$.

Suppose now that G is a <u>sheaf</u> of groups. A subset of G which is at the same time a subsheaf (i.e. an open subset, cf. proposition 2.5.1.) and a sub-group-bundle is called a <u>subsheaf of groups</u> or simply a <u>subsheaf</u> (if no confusion can arise) of G. It follows readily from proposition 2.5.1. that <u>then G/F is also a sheaf</u>, and hence, if F is normal, G/F is again a sheaf of groups.

Let us consider again the conditions of 2.3., supposing now that, as in 3.2., the sets E_U are groups and the maps φ_{VU} homomorphisms, so that the corresponding sheaf E is a sheaf of groups. Let for any U be given a <u>subgroup</u> F_U of E_U such that $\varphi_{VU} F_U \subset F_V$ for $V \subset U$. Then on the one hand these F_U define a subsheaf F of E as seen in 2.5., on the other hand F can be considered itself as a sheaf of groups in the same way as E. It is seen at once that F is a subsheaf of groups of E, and the induced structure is the one just considered. This gives an interpretation of the notion of subsheaf of groups in the second aspect of the notion of sheaf. If the subgroups F_U are <u>normal</u> in E_U, then F is normal in E. Moreover, let then $H_U = E_U / F_U$, and consider for any pair $V \subset U$ the homomorphism $H_U \to H_V$ obtained from φ_{VU} by passing to the quotient. The usual condition of transitivity is satisfied, so that we get a sheaf of groups H. On the other hand, the homomorphisms $E_U \longrightarrow H_U$ define a natural homomorphism $E \to H$ (cf. end of 2.5.). <u>This is an onto-homomorphism the kernel of which is F</u>. By proposition 2.5.2. this proves that H is isomorphic canonically to the quotient sheaf G/F. However, it should be noted that even if the systems (E_U) and (F_U) satisfy the conditions of the corollary of proposition 2.3.1, i.e. even if $E_U = H^0(U, E)$, $F_U = H^0(U, F)$, the same will <u>not</u> be true for the system (H_U), i.e. <u>in general the natural homomorphism</u> $H^0(U, E)/H^0(U, F) \longrightarrow H^0(U, E/F)$ (which is injective by virtue of exactness of 3.3.1) <u>is not surjective</u>. The study of the exact extent to which this is so is the starting point of the theory of sheaves and their cohomology groups.

3.4. <u>Fibre space with group bundle of operators</u>.

<u>Definition 3.4.1.</u> Let G be a group bundle on X, and A any fibre space on X. We say that G <u>operates at left</u> on A, or that <u>A is a fibre space with left group bundle of operators</u>, if we are given a homomorphism of fibre spaces $G \underset{X}{\times} A \longrightarrow A$ such that for each $x \in X$, the corresponding map $G_x \times A_x \longrightarrow A_x$ (where G_x and A_x are respectively the fibres of G and A at x) defines A_x as a set with group G_x of left operators.

If the map in question is denoted by $(g, a) \longrightarrow g.a$, this means that $g.(g'.a) = (gg').a$, and $e.a = a$ ($g, g' \in G_x$, $a \in A_x$, e the unit of G_x). In the analogous way we define the notion of fibre space A with <u>right</u> group bundle of operators G: this means that we are given a homomorphism of fibre spaces $A \times G \longrightarrow A$, which for each $x \in X$ defines A_x as a set with group G_x of right operators, i.e. such that $(a.g).g' = a(gg')$ and $a.e = a$ (where the map $A_x \times G_x \longrightarrow A_x$ is now noted by $(a, g) \longrightarrow a.g$). The qualification of "left" or "right" is omitted when no confusion can arise, and we restrict our statements for either sort of operations, the symmetric statement being left to the reader. It should be noted that definition 3.4.1. implies that <u>the fibres A_x all have at least one point</u>.

Suppose A is a fibre space with bundle G of left operators. Then the homomorphism $G \times A \longrightarrow A$ defines, for any non empty subset U of X, a map $H^0(U, G \underset{X}{\times} A) \longrightarrow H^0(U, A)$, i.e. $H^0(U, G) \times H^0(U, A) \longrightarrow H^0(U, A)$, and it is seen at once that this map defines $H^0(U, A)$ as a set with group $H^0(U, G)$ of left operators.

Definition 3.4.2. Let A be a fibre space with group bundle G of operators. We say that G operates faithfully in A (or A is a fibre space with faithful group bundle G of operators) if for any $x \in X$, G_x operates faithfully in A_x. We say that G is transitive in A if for each $x \in X$, G_x is transitive in A_x. A is called principal in the large sense if G is faithful, and if the map from the subspace M of $A \underset{X}{\times} A$ of pairs (a, a') of elements of A_x congruent mod. G_x into G, mapping (a, a') into the unique $g \in G_x$ such that $a' = g.a$ (respectively $a' = a.g$, according as G operates at left or at right) is continuous. A is called principal if it is principal in the large sense, and if G is transitive on A.

For instance, if G and A are sheaves, and if G is faithful, then A is principal in the large sense. For from 2.5.1. follows that M is a subsheaf of $A \underset{X}{\times} A$, and from 2.5.2. that the map $M \longrightarrow G$ is a homomorphism.

Let again A be any fibre space with group bundle G of (for instance left) operators. Two elements a, a' of A are called congruent mod G, if they are in the same fibre A_x and there are congruent mod G_x. This is clearly an equivalence relation compatible with the fibering (cf. 1.3.), the quotient fibre space is denoted by A/G. Consider the natural homomorphism $A \longrightarrow A/G$, it defines for any non-empty subset U of X a natural map

$$H^o(U, A) \longrightarrow H^o(U, A/G) \qquad (3.4.1.)$$

and it is trivial that two elements of $H^o(U, A)$ contruent mod the group of permutations $H^o(U, G)$ have same image in $H^o(U, A/G)$. The converse is true for open U if A is principal in the large sense (definition 3.4.2.), and hence in particular if A and G are both sheaves, and G is faithful. Moreover, it follows from proposition 2.5.1. that then A/G is itself a sheaf. Thus we get:

Proposition 3.4.1. Suppose A is a sheaf with a faithful sheaf G of operators Then A is principal in the large sense (definition 3.4.2.) and A/G is again a sheaf. Moreover two elements of $H^o(U, A)$ have same image under the map (3.4.1.) if and only if they are congruent under the group $H^{o}(U, G)$. Let A be a fibre space with group-bundle G of operators. A section f of A/G is well determined when we know its "inverse image" A_f in A (identifying of course, in this terminology, the section f with its image $f(x) \subset A/G$). A_f is stable under G, which operates transitively on A_f; this is also a sufficient condition for a subspace $A' \subset A$ of A to be obtained from a section f of A / G, provided we admit also discontinuous sections. Suppose now A and G are sheaves, then any A_f (f section of A /G) is a subsheaf of A on which G operates transitively; and conversely, it is seen at once that a subsheaf A' of A, on which G operates transitively, defines a section of A /G. Supposing now G faithful, and applying proposition 3.4.1., we get

Proposition 3.4.2. Let A be a sheaf with faithful sheaf G of operators. Then there is a one-to-one correspondence between sections of A / G, and subsheaves of A stable and principal under G (to the section f corresponding its inverse image in A).

Particular case: the regular representation. Let G be a bundle of groups, F a sub-bundle of groups, then the homomorphism $F \underset{X}{\times} G \longrightarrow G$ defined by the multiplication in G defines G as a fibre space with F as left group bundle of operators, and the analogous homomorphism $G \times F \longrightarrow G$ defines G as a fibre space with F as right group bundle of operators; the corresponding operations of F on G are called left and right regular representations of F into G (by "abus de langage"). For both, G is a fibre space with group bundle F principal in the large sense (definition 3.4.2.), and hence principal if and only if F = G. The quotient of G by F operating at right is nothing else but the fibre space G/ F of 3.3. Moreover, G /F can be considered as a fibre space with G as bundle of left operators, by taking the homomorphism $G \times G/F \longrightarrow G/F$ which in each fibre reduces to the natural map $G_x \times G_x/F_x \longrightarrow G_x/F_x$ (it is trivial that the map $G \times G/F \longrightarrow G/F$ thus obtained is continuous, hence a homomorphism); G is of course transitive on G/F, but in general not faithful. (It is easy to see that when G is a sheaf of groups, then any sheaf A on which G operates transitively and in which there is given a fixed section e, is isomorphic canonically to a quotient G/F.)

3.5 *The sheaf of germs of automorphisms.*

Let A be any fibre space on X. Consider as in 2.5. the sheaf E of germs of endomorphisms of E. For any open set $U \subset X$, $H^o(U, E)$ is the set of endomorphisms of $A|U$, and therefore provided with a natural associative composition law, moreover the restriction maps $H^o(U, E) \longrightarrow H^o(V, E)$ (for $V \subset U$) are clearly homomorphisms for this composition law. Therefore, as has been seen in 3.1. , E can be considered as a *sheaf with composition law.* $H^o(X, E)$ has a unit e: the identity endomorphism of A, therefore each E_x has a unit. Consider now for any non-empty $U \subset X$ the subset G_U of $E_U = H^o(U, E)$ of all *automorphisms* of $A|U$, clearly this is a subgroup of $H^o(U, E)$ and the restriction map $H^o(U, E) \longrightarrow H^o(V, E)$ maps G_U into G_V $(V \subset U)$, and the conditions of proposition 2.3.1. corollary are satisfied. The latter and section 3.3. show that G_U is the $H^o(U, G)$ of a well determined subsheaf G of E, which is a sheaf of groups for the induced structure, called the *sheaf of germs of automorphisms of A.*

There is a natural homomorphism $E \times A \longrightarrow A$, which in each fibre is defined as follows: let $f_x \in E_x$ and $a_x \in A_x$, let f be an endomorphism of $A|U$ (U open neighborhood of x) such that f_x be the class of f, consider $f.a_x \in A_x$, this does not depend on the particular choice of f and may therefore be noted $f_x . a_x$. It is obvious that the map $E \times A \longrightarrow A$ thus defined is continuous, i.e. a homomorphism. Consider the homomorphism induced in $G \times A$; this obviously defines A as a fibre space with G as sheaf of left operators (definition 3.4.1.). Thus *any fibre space A is canonically a fibre space with a sheaf G(A) of left operators*, where G(A) is the sheaf of germs of automorphisms of A.

Now let G be any group bundle on X, let $\overline{G}$ be the sheaf of germs of sections of G, which again is a sheaf of groups. Let A be any fibre space on which G operates at left, *we will define a corresponding homomorphism of $\overline{G}$ into the sheaf* G(A) *of germs of automorphisms of A.* Let $U \subset X$ be open non-empty; we must define a homomorphism of $H^o(U, \overline{G}) = H^o(U, G)$ into the group $H^o(U, G(A))$ of automorphisms of $A|U$, such that for $V \subset U$, the usual commutativity condition be satisfied. Therefore, with $g \in H^o(U, G)$ we associate the map $a \longrightarrow g.a$ of $A|U$ onto itself which on each fibre A_x reduces to $a \longrightarrow g_x . a$ (where g_x is the value of g at x), this map is clearly continuous, hence an endomorphism, and as the same is true for the map defined by $g^{-1} \in H^o(U, G)$ and as these two endomorphisms compose to the identity, we see that $a \longrightarrow g.a$ is an automorphism of $E|U$. Clearly we thus get a homomorphism of the group $H^o(U, G)$ into the group of automorphisms of $A|U$, and also the commutativity relation for a $V \subset U$ clearly holds. Clearly if the natural maps $\overline{G}_x \longrightarrow G_x$ are onto, (for instance if G is itself a sheaf (i.e. $G = \overline{G}$) the homomorphism $\overline{G} \longrightarrow G(A)$ already determines the operations of G on A). Moreover, if G is a sheaf, any homomorphism of G into G(A) can be defined in this way, for we have seen that G(A) operates at left on A, and therefore a homomorphism $G \longrightarrow G(A)$ defines G as a sheaf of left operators on A. So we get:

Proposition 3.5.1. Let G be a sheaf of groups and A any fibre space on X, then it is equivalent to give a homomorphism of G into the sheaf G(A) of germs of automorphisms of A, or to define on A a structure of fibre space with G as sheaf of left operators. The homomorphism $G \longrightarrow G(A)$ is injective if and only if G operates faithfully on A.

Suppose still that G is a sheaf of groups operating at left (for instance) on a fibre space A. Then G operates also in a natural way at left on the sheaf $\overline{A}$ of germs of sections of A. To define this we must for any open non-empty U, define a homomorphism of $H^o(U, G)$ into the group of automorphisms of $\overline{A}|U$, the obvious details are left to the reader.

Putting together the definitions of the two preceding paragraphs we get: if A is any fiber space with group bundle G of left (for instance) operators, then the corresponding sheaf $\overline{A}$ of germs of sections has $\overline{G}$ as sheaf of left operators. We have a natural homomorphism $\overline{A} \longrightarrow \overline{(A/G)}$, associated to the canonical homomorphism $A \longrightarrow A/G$, and obviously two elements of $\overline{A}$ congruent under $\overline{G}$ have the same image, hence a natural homomorphism

$$\overline{A}/\overline{G} \longrightarrow \overline{(A/G)}.$$

It follows from what was said previous to proposition 3.4.1. that this map is injective if A is principal in the large sense (definition 3.4.2). It may not be surjective; surjectivity here means that any germ of a section of

A/G can be lifted into a germ of a section of A. This will be true in the most important cases.

3.6 Particular cases.

a The sheaf of germs of isomorphisms of a fibre space onto another. Let A, A' be two fibre spaces on X, and E the sheaf of germs of homomorphisms of A into A' (cf. 2.5.), we can consider as in 3.5. the subsheaf G(A, A') of germs of isomorphisms of A onto A'. This is not a sheaf of groups if $A \neq A'$. However, let G(A) respectively G(A') be the sheaves of germs of automorphisms of A respectively A', and suppose A and A' locally isomorphic. Then G(A, A') is both a sheaf with G(A) as sheaf of right operators and G(A') as sheaf of left operators, and principal (definition 3.4.2.) in both structures. The operations are respectively composition of germs of homomorphisms at right, or at left; the formal definitions are obvious and left to the reader, as well as the fact that G(A, A') is principal.

b On certain subsheaves of the sheaf G(A) of germs of automorphisms of A. In the next chapter, we will be interested in a given subsheaf of G(A) when A is some standard reference fibre space. We will take here for A a product space $X \times F$ (where F is a topological space). If as usual we take on $X \times F$ the product topology, then an automorphism of $A|U$ can be identified with a map g of U into the group Γ (f) of homeomorphisms of f onto itself such that the maps $(x, y) \longrightarrow g(x).y$ and $(x, y) \longrightarrow g(x)^{-1}y$ of $U \times F$ into F be continuous. (cf. 1.4.1.) Anyhow, whatever the topology of $A = X \times F$, inducing on the fibres the topology of F, an isomorphism of $A|U$ is defined by a map g of U into Γ, by the formula above Therefore an element of G(A) is a germ of a map into Γ, and any subsheaf of G(A) may be considered as a sheaf of germs of maps of X into Γ (F).

Suppose for instance that X and F are manifolds of class $C^m(0 \leq m \leq +\infty)$, then we can take the subsheaf $G^m(A)$ the sections of which, on each open non-empty U, are the homomorphisms of $A|U = U \times F$ which are isomorphisms of class m of the manifold $U \times F$. This definition extends if "class C^m" is replaced by "real analytic", or "complex analytic", or "algebraic". In the latter case, we take algebraic varieties over an arbitrary field, with their usual Zariski topology, (but it should be noted that the topology of $U \times F$ is then no longer the product of the topologies of U and F).

One can suppose also that we are given a topological group Γ operating on F (i.e. there is given a continuous map $\Gamma \times F \longrightarrow F$ such that this map defines Γ as a set with group Γ of left operators), then <u>any</u> continuous map f of U into Γ defines an endomorphism of $U \times F$, namely $(x, y) \longrightarrow f(x).y$; f is uniquely determined by the latter if and only if Γ operates faithfully on F. Thus is defined a natural homomorphism of the sheaf $\mathcal{C}(X, F)$ of germs of continuous maps of X into Γ (which is a <u>sheaf of groups</u> because G is a topological group) into a sheaf G(A), injective if and only if Γ is faithful on F, the image of which will be noted $G_{\Gamma}(A)$. Any subsheaf of $\mathcal{C}(X, F)$ therefore operates on A and defines a subsheaf of G(A).

<u>Definition 3.5.1.</u> Let F be a topological space with a faithful topological group Γ of operators. Consider the product space $A = X \times F$ as a fibre space over X, suppose given a sheaf of operators G of A. We say that the fibre space A with sheaf of operators G has the <u>structure group</u> Γ, if G is a subsheaf of the sheaf $\mathcal{C}(X, \Gamma)$ of germs of continuous maps of X into Γ, containing the germs of constant maps of X into Γ (i.e. the constant sheaf $X \times G$), (the given homomorphism of G into the sheaf G(A) of germs of automorphisms of A being the one induced by the natural homomorphism of $\mathcal{C}(X, \Gamma)$ into G(A)).

Of course, G determines already the image of Γ in the group $\Gamma(F)$ of automorphisms of F, but maybe not the topology on this image which corresponds to the topology of Γ We now give some examples of sheaves of operators corresponding to a structure group Γ; i.e. interesting special subsheaves of $\mathcal{C}(X, \Gamma)$, containing $X \times \Gamma$ (germs of constant maps of X into Γ), where Γ is a given topological group. There are of course these two extreme sheaves. If now Γ is a Lie group and X a manifold of class C^m, we can consider the sheaf $\mathcal{C}^m(X, \Gamma)$ of germs of maps of class C^m of X into Γ. Of course, if Γ operates differentiably on a manifold F of class C^m, the sheaf of germs of automorphisms of $X \times F$ corresponding to $\mathcal{C}^m(X, \Gamma)$ is contained in the sheaf $G^m(X \times F)$ defined above. If X is real analytic we can consider also the sheaf of germs of analytic maps of X into Γ ; and if Γ is a <u>complex</u> Lie group, X complex analytic, we can consider the sheaf of germs of complex analytic maps of X into Γ. Analogous definitions for algebraic varieties hold with the only difference that we cannot suppose here $\Gamma \times F \longrightarrow F$ to be continuous for the product topology on $\Gamma \times F$, it has only to be assumed a <u>regular</u> map (i. e. rational and defined everywhere); Then the sheaf $\underline{G}$ of germs of regular maps from X into the algebraic group Γ operates as sheaf of germs of automorphisms on the algebraic trivial fibre space $X \times F$. The fact noted above that $\mathcal{C}^m(X, \Gamma)$ is mapped into $G^m(A)$ extends in an obvious way to the three further examples.

CHAPTER IV

FIBRE SPACES WITH STRUCTURE SHEAF

4.1 <u>The definition</u>. Let Φ be a fibre space with faithful sheaf $\underline{G}$ of groups of left operators on the space X. Let E be a fibre space on X locally isomorphic to Φ. As seen in 3.6. a., the sheaf $G(\Phi, E)$ of germs of isomorphisms of Φ onto E is a principal sheaf under the sheaf $G(\Phi)$ of germs of automorphisms of Φ, and therefore a principal sheaf in the large sense (definition 3.4.2) under the given sheaf $\underline{G}$ (which can be considered as a subsheaf of $G(\Phi)$). We can therefore consider the quotient sheaf $G(\Phi, E)/\underline{G}$.

<u>Definition 4.1.1.</u> Let Φ be a fibre space wth faithful sheaf of groups of left operators $\underline{G}$ on the space X. A <u>fibre space of structure type</u> Φ is a fibre space E on X, locally isomorphic to the fibre space Φ, together with a section of the sheaf $G(\Phi, E)/\underline{G}$ quotient of the sheaf $G(\Phi, E)$ of germs of isomorphisms of Φ onto E by the sheaf $\underline{G}$ of right operators. $\underline{G}$ is called the <u>structure sheaf</u> of E.

By what has been said in 3.4., given E locally isomorphic to Φ, it is equivalent to give a section of $G(\Phi, E)/\underline{G}$, or a subsheaf P of $G(\Phi, E)$ which is stable under $\underline{G}$ and principal under G, i.e. such that given a section f of P over an open set U, the other sections of P over the same U are exactly those of the type f.g, where g is a section of $\underline{G}$ over U; or equivalently, calling an isomorphism $f: \Phi|U \longrightarrow E|U$ <u>compatible with the structure sheaf</u> G or a <u>coordinate map</u>, when it is a section of P, this means that if f is admissible then all other admissible isomorphisms of $\Phi|U$ onto $E|U$ are those of the type f.g, where g is an automorphism of $\Phi|U$ "belonging" to $\underline{G}$ (i.e. defined by a section of $\underline{G}$ on U).

Definition 4.1.2. The subsheaf P of $G(\Phi, E)$ defining the structure of type Φ on E is called the principal sheaf associated with E (remember this is a principal sheaf under $\underline{G}$).

The notion of isomorphism of a fibre space E with structure of type Φ onto another E' is clear; we will not discuss here the notion of homomorphism. Now let U be a subset of X, then $\Phi|U$ has $\underline{G}|U$ as a sheaf of right operators, and if E is a fibre space with structure type Φ over X, then $E|U$ can be considered as a fibre space of structure type $\Phi|U$, structure sheaf $\underline{G}|U$, called the restriction of E to U. (More generally the notion of inverse image of a fibre space of structure type Φ, under a continuous map $X' \longrightarrow X$, is defined in an evident way.) Let E, E' be two fibre spaces of structure type Φ, f an isomorphism for the underlying structures of fibre spaces only of E onto E'; then in order that f be also an isomorphism with respect to the structures of type Φ, it is necessary and sufficient that each $x \in X$ has an open neighborhood U such that the restriction of f to $E|U$ be an isomorphism in this stronger sense of $E|U$ onto $E'|U$. Thus the notion of isomorphism of E onto E' is of local nature, or equivalently: we can consider the sheaf G(E, E') of germs of isomorphisms of E onto E', which is a subsheaf of the sheaf $G(E_o, E'_o)$ of germs of isomorphisms of E_o onto E'_o, (where E_o, E'_o are the underlying fibre spaces - without structure sheaf - of E, E'), and for any open $U \subset X$, the isomorphisms of $E|U$ onto $E'|U$ are exactly the sections of G(E, E').

Let $\underline{G}'$ be the sheaf of germs of automorphisms of the fibre space Φ which commute to $\underline{G}$; this is a subsheaf of groups of the sheaf $G(\Phi)$ of germs of automorphisms of the fibre space Φ. For notational convenience, however, we will consider $\underline{G}'$ as a sheaf of right operators, by putting $z.g' = g'^{-1}.z$ for $z \in \Phi_x$, $g' \in \underline{G}'_x$ $(x \in X)$. Thus the commutation with $\underline{G}$ means that $(g.z).g' = g.(z.g')$ for $z \in \Phi_x$, $g \in \underline{G}_x$, $g' \in \underline{G}'_x$. Let now E be any fibre space of structure type Φ, let E_o be the underlying fibre space, we will define on E_o a canonical structure of fibre space with G' as sheaf of groups of right operators, i.e. define a right representation of $\underline{G}'$ into the sheaf $G(E_o)$ of germs of automorphisms of E_o. Therefore we only need define, for each open $U \subset X$, a right representation of $H^o(U, \underline{G}')$ into the group of germs of automorphisms of $E_o|U$, such that the compatibility condition with respect to the restriction maps be satisfied (cf. 3.1.). We can also restrict to open sets U small enough in order that there exist a coordinate map f: $\Phi|U \longrightarrow E|U$; now such a coordinate map, being an isomorphism of $\Phi|U$ onto $E_o|U$, transforms the sheaf $\underline{G}'|U$ of right operators on $\Phi|U$ into a sheaf of right operators on $E_o|U$, and the right representation of $\underline{G}'|U$ into $G(E|U)$ thus obtained does not depend on the particular choide of f, be cause another coordinate map is of the form f. g (g section of $\underline{G}$ on U) and g commutes with $\underline{G}'$. It should be noted that the operations of $\underline{G}'$ on E commute with the germs of automorphisms of E.

4.2. Some examples.

a. Let again Φ be a fibre space with faithful sheaf $\underline{G}$ of left operators. Then Φ itself is provided in a natural way with a structure of fibre space of structure type Φ : a section of $G(\Phi, \Phi)/\underline{G}$ is defined by taking the image of the identity section of $G(\Phi, \Phi)$ (i.e. the identity automorphism of Φ). Φ will always be considered as fibre space of structure type Φ in the above way; a fibre space E of structure type Φ is called trivial if it is isomorphic to Φ (this notion should be distinguished from that introduced in 1.4. !). Obviously $\underline{G}$ (considered as a sheaf of germs of automorphisms of the underlying fibre space of Φ) is nothing else but the sheaf of germs of automorphisms of Φ (considered with its structure of type Φ). The principal bundle associated with Φ is $\underline{G}_r$ (i.e. $\underline{G}$ on which $\underline{G}$ operates by the right regular representation). More generally, let E be any fibre space of structure type Φ , then the associated principal bundle is nothing else but the sheaf of germs of isomorphisms of Φ onto E (isomorphism in the sense of the structures of type Φ on Φ and E).

b. Let $\underline{G}$ be any sheaf of groups, consider $\Phi = \underline{G}_\ell$, i.e. $\underline{G}$ on which $\underline{G}$ operates by the left regular representation. We will interpret the notion of fibre space of structure type $\underline{G}_\ell$. Therefore, we notice that the sheaf $\underline{G}'$ of germs of automorphisms of the underlying fibre space G_o of G which commute to the operations of $\underline{G}$ on G_o (by left translations) is isomorphic to $\underline{G}$ operating on G_o by the right regular representation; and G_o is clearly a principal fibre space under the sheaf of groups $\underline{G}$ operating by right translations. Therefore, as seen at end of section 4.1., for any fibre space E of structure type $\underline{G}_\ell$. $\underline{G}$ operates on the right on E, thus E turns into a fibre space with $\underline{G}$ as sheaf of right operators, denoted by $E_{\underline{G}}$. Of course $E_{\underline{G}}$ will be principal too (being locally isomorphic to G_r as a fibre space with sheaf $\underline{G}$ of right operators). Conversely, this structure of principal fibre space under $\underline{G}$ determines the structure of type $\underline{G}$ of E, for it is seen at once that the principal associated bundle P of E is nothing else but the sheaf of germs of isomorphisms of G_r onto $E_{\underline{G}}$; and moreover, given *any* principal fibre space $E_{\underline{G}}$ under $\underline{G}$, the sheaf P of germs of isomorphisms of $\underline{G}_r$ onto $E_{\underline{G}}$ defines on E a structure of type $\underline{G}_\ell$. So we see that *the notion of fibre space of type $\underline{G}_1$ is equivalent to the notion of principal fibre space over $\underline{G}$ (operating at right).*

c. Let Φ be any fibre space on X, and $\underline{G}$ the sheaf of germs of all automorphisms of Φ. Then a fibre space E_o on X is the underlying fibre space of a fibre space E of structure type Φ if and only if E_o is locally isomorphic to Φ, and then E is obviously unique; thus the notion of fibre space of structure type Φ here reduces to the notion of fibre space on X locally isomorphic to Φ. If, at the opposite, $\underline{G}$ reduces to the sheaf of unit groups, then a structure of type Φ on E, i.e. a section of $G(\Phi, E)/\underline{G} = G(\Phi, E)$, is nothing else but an isomorphism of Φ onto E.

d. Let F be a topological space, and let $\Phi = X >< F$. Then each of the special sheaves of automorphisms considered in section 3.6.b. defines a corresponding notion of fibre space with structure sheaf; in particular, we have notion of *fibered manifold of class C^m*, of *analytic* (real or complex) *fibre spaces*, of *algebraic fibre spaces*. Let Γ be a topological group operating on F; suppose that we define on Φ a structure of fibre space with sheaf $\underline{G}$ of left operators, admitting Γ as a *structure group* (definition 3.5.1.); then Γ is called also a structure group of any fibre space E of structure type Φ. Of course, the mere giving of the group of operators Γ in F is not sufficient to determine the type of structure we are considering; but most frequently it will be clear from the context what the sheaf $\underline{G}$ is for a given Γ: the sheaf of germs of continuous maps of X into Γ, or of differentiable maps if X is differentiable manifold and Γ a Lie group, etc.) We get thus, in particular, the notions of *fibre bundle with topological structure group* Γ (here $\underline{G}$ is the sheaf of germs of continuous maps of X into Γ), which is exactly the notion of the book of Steenrod; of differentiable fibre bundle of class C^m with Lie structure group Γ (here $\underline{G}$ is the sheaf of germs of maps of class C^m from the manifold X into the Lie group Γ); of analytic fibre bundle (over an analytic manifold X) with structure Lie group Γ ; of complex analytic fibre bundle (over a complex analytic manifold X) with a complex Lie group Γ as structure group; and likewise of algebraic fibre bundle (over an algebraic manifold, with its Zariski topology) with an algebraic group Γ as structure group.

4.3. Definition of a fibre space of structure type Φ by coordinate maps or coordinate transforms. Let again Φ be a fibre space with faithful left sheaf $\underline{G}$ of operators. Let E be a fibre space of structure type Φ ; clearly this structure is entirely determined when we give a family (U_i, f_i) of coordinate maps of E (cf. 4.1.) over open sets U_i which cover X. Given such a family, we have for any two indices i, j such that $U_{ij} = U_i \cap U_j \neq \emptyset$:

$$(4.3.1) \qquad f_j = f_i . f_{ij} \qquad \text{on } U_{ij}$$

where f_{ij} is a well determined section of $\underline{G}$ over U_{ij} (as usual, we simplify the notations by writing f_i and f_j for the restrictions of f_i and f_j to $\Phi \,|U_{ij}$). Conversely, given a fibre space E over X, and a family (f_i, U_i) of isomorphisms f_i: $\Phi \,|U_i \longrightarrow E\,|U_i$, the U_i's covering X, this system is a system of coordinate maps for a suitable structure of type Φ on E if (and only if) the relations 4.1.1. are satisfied or, as we will say, if the maps f_i are compatible with the sheaf $\underline{G}$. Moreover, given a second such family (f'_j, U'_j), this defines on E the same structure of type Φ as the first if and only if the sum of the two families of maps still defines a structure of type Φ on E. The relation thus introduced between certain families of maps is hence an equivalence relation, and the structures of type Φ on E correspond exactly to equivalence classes of such families of maps. Taking sets of maps instead of families with arbitrary sets of indices, it is obvious that a set of maps (f_i, U_i) defining a structure of type Φ is contained in one and only one maximal set of maps compatible with $\underline{G}$ (take the set of all coordinate maps on E provided with the structure of type Φ defined by the f_i's), which is clearly equivalent to the given set

(f_i, U_i). Thus, a structure of type Φ could be equivalently defined as a maximal set of isomorphisms $f_i : \Phi \,|U_i \longrightarrow E\,|U_i$, such that the U_i cover X. We thus rejoin the classical definition of fibre bundles (see Steenrod, The Topology of Fibre Bundles).

Now let (U_i) be an open covering of X, and for each pair (i, j) of indices such that $U_{ij} \neq \emptyset$, let f_{ij} be a section of G on U_{ij}, which can also be interpreted as an automorphism of the fibre space $\Phi\,|U_{ij} = \Phi_{ij}$. We can then consider the fibre space E defined by the system (f_{ij}) of coordinate transformations (cf. definition 1. 5. 1.), provided the usual coherence property

$$(4.3.2.) \qquad f_{ik} = f_{ij} f_{jk} \quad \text{on } U_{ijk}$$

is satisfied for each triple (i, j, k) of indices such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$ Moreover, as seen in section 1. 5., there is for each i a natural isomorphism f_i of $\Phi\,|U_i$ onto $E\,|U_i$, and on any non-empty U_{ij} we have

$$f_{ij} = f_i^{-1} f_j .$$

By what has been said above, this implies that there is on E a unique structure of type Φ such that the f_i be coordinate maps for this structure. Thus we will always consider the fibre space E defined by coordinate transforms (f_{ij}) as above as a fibre space of structure type Φ (hence with structure sheaf $\underline{G}$). Besides, any fibre space E on X of structure type Φ is isomorphic to a fibre space defined in the preceding way; more precisely, if (U_i, f_i) is any family of coordinate maps of E, with (U_i) covering X, then defining sections f_{ij} of $\underline{G}$ over U_{ij} by formula (4. 3. 1.), we get a system of coordinate transformations, and as seen in 1. 5. (cf. proposition 1. 5. 1.) the fibre space E' defined by these is canonically isomorphic to E; as this isomorphism transforms the given coordinate maps of E' into those of E, this is also an isomorphism for the structures of type Φ.

Consider two systems of coordinate transformations (f_{ij}) and (f'_{ij}) relative to a same covering (U_i) of X, let E and E' be the corresponding fibre spaces of structure type Φ. Then there is a one-to-one correspondence between isomorphisms of $\underline{E}$ onto $\underline{E}'$ (isomorphisms in the sense of fibre spaces with structure sheaf) and systems (f_i) of sections of $\underline{G}$ on the U_i's such that

(4.3.3.) $$f'_{ij} = f_i f_{ij} f_j^{-1}.$$

Of course, by proposition 1.5.1., the isomorphisms for the underlying structures of fibre spaces of E and E' are given by such systems, but where the f_i's are arbitrary automorphisms of the restrictions $\Phi | U_i$, satisfying to 4.3.3. But it is trivial that, in order that such an isomorphism be an isomorphism also for the structures of type Φ, it is necessary and sufficient that each f_i be an automorphism of $\Phi | U_i$ considered a fibre space with structure sheaf, which means that f_i is given by a section of $\underline{G}$ on U_i. – In particular, the necessary and sufficient condition for E and E' to be isomorphic is the existence of at least one system (f_i) of sections of $\underline{G}$ on the sets U_i, satisfying the conditions (4.3.3.).

What has been said in 1.5. on the comparison of fibre spaces defined by coordinate transformations relative to two different converings, carries over at once in this more special context. This allows us to determine in principle whether any two fibre spaces given by two arbitrary systems of coordinate transformations (with respect of course to the same Φ, $\underline{G}$) are isomorphic (compare chapter 5 below for the general classification of fibre spaces with structure sheaf $\underline{G}$).

4.4. The associated fibre spaces.

Let again Φ be a fibre space over X with faithful sheaf $\underline{G}$ of left operators, E a fibre space of structure type Φ. Let (U_i, f_i) be the set of all coordinate maps f_i: $\Phi | U_i \longrightarrow E | U_i$ of E, (f_{ij}) the corresponding system of coordinate transformations.

Now consider any other fibre space Ψ with $\underline{G}$ as left group bundle of operators, or what is the same (proposition 3.5.1.) a homomorphism ρ of $\underline{G}$ into the sheaf of germs of automorphisms of the fibre space Ψ. When we wish to recall that Ψ is provided with the supplement of structure by the giving of ρ, we will write $\Psi(\rho)$ instead of merely Ψ. ρ may not be faithful, but in any case ρ determines an injective homomorphism of $\underline{G}/\mathrm{Ker}\,\rho$ into the sheaf of germs of automorphisms of Ψ, (Ker ρ is the Kernel of the homomorphism ρ, therefore is a normal subsheaf of $\underline{G}$), thus $\Psi(\rho)$ can also be considered as a fibre space with faithful sheaf $\underline{G}/\mathrm{Ker}\,\rho$ of left operators. In particular, the notion of fibre space of structure type $\Psi(\rho)$ is therefore determined; the structure sheaf of such a fibre space is $\underline{G}/\mathrm{Ker}\,\rho$.

Definition 4.4.1. Given as above a fibre space E of structure type Φ, structure sheaf $\underline{G}$, and a representation ρ of $\underline{G}$ in the sheaf of germs of automorphisms of a fibre space Ψ, we call fibre space associated to E, and to ρ, the fibre space of structure type $\Psi(\rho)$ (where $\Psi(\rho)$ stands for Ψ with $G/\mathrm{Ker}\,\rho$ as faithful sheaf of operators) defined by the coordinate transformations $\rho(f_{ij})$, where (f_{ij}) is the system of coordinate transformations for E defined by the set (U_i, f_i) of all coordinate maps of E.

This definition of course makes sense, i.e. the conditions of the type (4.3.2.) for coordinate transformations are satisfied by the system $(\rho(f_{ij}))$, as results at once from the fact that they are satisfied for (f_{ij}) itself, and that ρ is a representation.

Functorial behavior. We write $\Psi(\rho)^E$ (or simply Ψ^E if no confusion can arise) for the fibre space of structure type $\Psi(\rho)$ associated to E and the representation ρ. Let $\Psi'(\rho')$ be a second fibre space with $\underline{G}$ as a group bundle of left operators, corresponding to a representation ρ', whence an associated fibre space $\Psi'(\rho')^E$ of structure type $\Psi'(\rho')$. Consider a $\underline{G}$-homomorphism u of $\Psi(\rho)$ into $\Psi'(\rho')$, by which we mean a homomorphism of Ψ into Ψ' which commutes to the operations of $\underline{G}$, more precisely such that for any $x \in X$, $g \in \underline{G}_x$, $z \in \Psi_x$ we have $u(g.z) = g.(u(z))$ or more explicitly

$$u(\rho(g).z) = \rho'(g).(u(z)). \tag{4.4.1.}$$

Let $u_i\colon \Psi|U_i \longrightarrow \Psi|U_i$ be the restriction of u to $\Psi|U_i$, the foregoing formula implies $\rho'(f_{ij})u_j = u_i\,\rho(f_{ij})$, in virtue of proposition 1.5.2. this means that the system (u_i) defines a homomorphism of the fibre space defined by the coordinate transformations $\rho(f_{ij})$ into the one defined by the coordinate transformations $\rho'(f_{ij})$, i.e. of $\Psi(\rho)^E$ into $\Psi'(\rho')^E$.

<u>Definition</u> <u>4.4.2.</u> Given a $\underline{G}$-homomorphism u of a fibre space $\Psi(\rho)$ with $\underline{G}$ as group bundle of left operators into another such $\Psi'(\rho')$, and a fibre space E of structure type Φ (structure sheaf $\underline{G}$), the above constructed homomorphism $u^E : \Psi(\rho)^E \longrightarrow \Psi'(\rho')^E$ of the associated fibre spaces is called <u>associated</u> to E and u.

It is checked easily that the usual functor properties hold: a). <u>If $\Psi'(\rho') = \Psi(\rho)$ and u is the identity, so is u^E</u>. b. <u>If we have two $\underline{G}$-homomorphisms</u> $\Psi(\rho) \xrightarrow{v} \Psi'(\rho') \xrightarrow{v} \Psi''(\rho'')$, <u>then</u> $(vu)^E = v^E u^E$. From this follows: c). <u>If u is an onto-isomorphism</u> $\Psi(\rho) \longrightarrow \Psi'(\rho')$, <u>so is</u> u^E. As <u>locally</u>, for appropriate isomorphisms $\Psi(\rho)^E \approx \Psi(\rho)$ and $\Psi'(\rho')^E \approx \Psi'(\rho')$, u^E can be identified with u, it shares with u any "property of local character", for instance: d) <u>If u is injective (surjective) so is</u> u^E. We now examine, for a fixed $\Psi(\rho)$, how $\Psi(\rho)^E$ varies with E. It is trivial (by "transport de structure") that an isomorphism γ of a fibre space E of structure type Φ onto another E' defines canonically an isomorphism $\gamma_\rho : \Psi(\rho)^E \longrightarrow \Psi(\rho)^{E'}$. Again the two basic functor properties are verified: if E = E' and γ is the identity, so is γ_ρ ; given two consecutive isomorphisms $E \xrightarrow{\gamma} E' \xrightarrow{\gamma'} E''$, we have $(\gamma'\gamma)_\rho = \gamma'_\rho \gamma_\rho$. In particular, we get for E = E' = E'' the following

<u>Proposition</u> <u>4.4.1.</u> Let E be a fibre space of structure type Φ, (structure sheaf $\underline{G}$), Γ the group of automorphisms of E, $\Psi(\rho)$ a fibre space with $\underline{G}$ as group bundle of left operators, $\Psi(\rho)^E$ the fibre space associated to E and ρ (with structure sheaf $\underline{G}/\mathrm{Ker}\,\rho$). Then there is a canonical representation of Γ into the group of automorphisms of $\Psi(\rho)^E$ (considered as fibre space of structure type $\Psi(\rho)$).

The relations between the two ways of considering $\Psi(\rho)^E$ as a functor (either of $\Psi(\rho)$, or of E) are the following.

<u>Proposition</u> <u>4.4.2.</u> Let E, E' be two fibre spaces of structure type Φ, γ an isomorphism of E onto E', $\Psi(\rho)$ and $\Psi'(\rho')$ two fibre spaces with

$\underline{G}$ as group-bundle of left operators, u a $\underline{G}$-homomorphism of $\Psi(\rho)$ into $\Psi'(\rho')$. Then the following diagram is commutative

$$\begin{array}{ccc} \Psi(\rho)^E & \xrightarrow{\gamma_\rho} & \Psi(\rho)^{E'} \\ {\scriptstyle u^E}\downarrow & & \downarrow{\scriptstyle u^{E'}} \\ \Psi'(\rho')^E & \xrightarrow{\gamma_{\rho'}} & \Psi'(\rho')^{E'} \end{array}.$$

(Proof left to the reader.)

Taking in particular E = E', we get the following

Corollary. Let E be a fibre space of structure type Φ, $\Psi(\rho)$ and $\Psi'(\rho')$ two fibre spaces with $\underline{G}$ as group bundle of left operators, u a $\underline{G}$-homomorphism of the first into the second. By proposition 4.4.1., the associated fibre spaces $\Psi(\rho)^E$ and $\Psi'(\rho')^E$ both admit the group Γ of automorphisms of E as a group of operators; then the associated homomorphism u^E (definition 4.4.2.) commutes with the operations of Γ.

The following transitivity property for associated fibre spaces is obviously satisfied: Let ρ be a representation of $\underline{G}$ into the sheaf of germs of automorphisms of a fibre space Ψ, ρ' a representation of $\underline{G}/\mathrm{Ker}\,\rho$ into the sheaf of germs of automorphisms of a fibre space Ψ', $\rho'\rho$ the composition of these representations, E a fibre space with structure of type Φ, F the fibre space associated to E and ρ, F' the fibre space associated to F and ρ', then F' is canonically isomorphic to the fibre space associated to E and $\rho'\rho$.

Functorial characterization of the associated fibre spaces. Consider the category C(Φ) of fibre spaces on X with structure of type Φ, taking as homomorphisms in this category the onto-isomorphisms (in the sense of the structure of type Φ of course!) A (covariant) functor F of this category into the category of all fibre spaces over X is a "function" associating to each $E \in C(\Phi)$ a fibre space F(E) over X, and to an onto-isomorphism u: $E \longrightarrow E'$ (where E, E' are in C(Φ)) a homomorphism F(u): $F(E) \longrightarrow F(E')$, in such a way that (i) if E = E' and u is the identity map, so is $F(u)$; (ii) if $u: E \longrightarrow E'$ and $v: E' \longrightarrow E''$ are given onto-isomorphisms (E, E' and E'' being in $C(\Phi)$) then $F(vu) = F(v)F(u)$. This implies (iii) For $u: E \longrightarrow E'$ as above, $F(u)$ is an onto-isomorphism, and $F(u)^{-1} = F(u^{-1})$.

Definition 4.4.3. A local functor on $C(\Phi)$ is a law associating to each open non-empty subset $U \subset X$ a functor F_U from the category $C(\Phi|U)$ of fibre spaces on U with structure of type $\Phi|U$, into the category of all fibre spaces over U, and to any open non-empty $V \subset U$ an onto-isomorphism $F_U(E)|V \longrightarrow F_V(E|V)$, in such a way that the last law defines a "homomorphism" of the functor $E \longrightarrow F_U(E)|V$ into the functor $E \longrightarrow F_V(E|V)$ (both functors being defined on $C(\Phi|U)$ with values in the category of all fibre spaces on V), i.e. such that for any onto-isomorphism $E \longrightarrow E'$ (E and E' in $C(\Phi|U)$), the diagram

(4.4.3.)
$$\begin{array}{ccc} F_U(E)|V & \longrightarrow & F_U(E')|V \\ \downarrow & & \downarrow \\ F_V(E|V) & \longrightarrow & F_V(E'|V) \end{array}$$

is commutative.

For instance, let $\Psi(\rho)$ be a fibre space over X with $\underline{G}$ as group-bundle of left operators, then for any open non-empty $U \subset X$, $\Psi(\rho)|U$ is a fibre space over U with $\underline{G}|U$ as group-bundle of left operators, and therefore we can consider the functor $F_U(E) = (\Psi(\rho)|U)^E$ from the category $C(\Phi|U)$ into the category of all fibre spaces over U; moreover, canonical isomorphisms $F_U(E)|V \longrightarrow F_V(E|V)$ are defined at once, and the commutativity of (4.4.3.) readily checked. Thus $\Psi(\rho)$ defines in a canonical way a local functor on $C(\Phi)$. Conversely, let F be any local functor on $C(\Phi)$, let $\Psi = F(\Phi)$, we will define a representation of $\underline{G}$ into the sheaf of germs of automorphisms of Ψ. Therefore, we must for any non-empty open $U \subset X$, define a homomorphism of the group $H^0(U, \underline{G})$ into the group of automorphisms of $\Psi|U$, in a way to satisfy the usual commutativity requirement with respect to restriction maps relative to $V \subset U$. Now $H^0(U, \underline{G})$ is the group of automorphisms of $\Phi|U$ (4.2.a.), hence a representation of this group into the group of automorphisms of $F_U(\Phi|U)$, but the latter is canonically isomorphic (in virtue of commutativity in (4.4.3.), where U is replaced by X and V by U) to $F_X(\Phi)|U = \Psi|U$, hence the desired homomorphism of $H^o(U, \underline{G})$ into the group of automorphisms of $\Psi|U$; the commutativity relation referred to above results easily from commutativity of (4.4.3.). Thus we have defined a representation ρ of $\underline{G}$ into the sheaf of germs of automorphisms of Ψ, so that Ψ can be considered as a fibre space $\Psi(\rho)$ with $\underline{G}$ as group bundle of left operators. If now we suppose that the functor F had already been obtained as explained above from a fibre space $\Psi'(\rho')$ with $\underline{G}$ as group bundle of left operators, then Ψ is the fibre space associated to the trivial fibre space Φ and the representation ρ', and therefore canonically isomorphic to Ψ'; the definition of this isomorphism is trivial from the definitions, moreover it is easily checked that this isomorphism is even a $\underline{G}$-isomorphism; thus a fibre space $\Psi(\rho)$ with $\underline{G}$ as group bundle of left operators is determined, up to canonical isomorphism, by the local functor it defines. Conversely, let F be any local functor, we consider the fibre space $\Psi(\rho)$ constructed above, let F' be the local functor associated to the latter, we will define a canonical isomorphism of F' onto F, i.e. for each U and each fibre space E in $C(\Phi|U)$, an isomorphism of $F'_U(E)$ onto $F_U(E)$, compatible with the functor maps associated to homomorphisms $E \longrightarrow E'$ and with the maps associated to open sets $V \subset U$. We just give the definition, letting the verification of the compatibilities to the reader. Let (U_i, f_i) be the system of all coordinate maps defining the structure of E; (U_i) is an open covering of U, and f_i an isomorphism of $\Phi|U_i$ onto $E|U_i$. This defines an isomorphism $F(f_i)$ of $F_{U_i}(\Phi|U_i)$ onto $F_{U_i}(E|U_i)$, or what amounts to the same of $F_X(\Phi)|U_i$ onto $F_U(E)|U_i$, i.e. of $\Psi|U_i$ onto $F_U(E)|U_i$. Let as usual $f_{ij} = f_i^{-1}f_j$ (where f_i^{-1} and f_j shortly stand for their restrictions to $E|U_{ij}$ respectively to $\Phi|U_{ij}$), then $F(f_{ij}) = F(f_i)^{-1}F(f_j)$ so that the system $F(f_{ij})$ is the system of coordinate transforms of $F_U(E)$ corresponding to the coordinate maps $F(f_i)$. But by the definition of the associate fibre space, $(F(f_{ij}))$ is also a system of coordinate transforms for the fibre space $F'(E)$ associated to E and the representation ρ of $\underline{G}$ by germs of automorphisms of Ψ, hence the canonical isomorphism $F'(E) \longrightarrow F(E)$. Putting together the results obtained, we get:

Proposition 4.4.3. Let Φ be a fibre space over X with faithful sheaf $\underline{G}$ of left operators. There is a natural one-to-one correspondence between the fibre spaces $\Psi(\rho)$ over X with $\underline{G}$ as sheaf of left operators, and the local functors (definition 4.4.3.) defined on the category $C(\Phi)$ of fibre spaces with structure of type Φ. To $\Psi(\rho)$ corresponds the functor F which, for any non-empty open $U \subset X$ and any fibre space E over U with structure of type $\Phi|U$, associates the fibre space associated to E and $\Psi(\rho)|U$; and to a given local functor F corresponds the fibre space $\Psi = F(\Phi)$, where $\underline{G}$ operates as defined above.

It should be noted also that the notion of $\underline{G}$-homomorphism of a space $\Psi(\rho)$ into a space $\Psi'(\rho')$ can be interpreted, in the above correspondence, by an evident notion of homomorphism of local functor into another (definitions left to the reader!).

Remark. In fact, a more abstract and general formulation of these results should be given, by taking the values for F(E) in a more general category than the specified category of all fibre spaces are open sets of X. For instance, we could take functors with values in the category of group bundles, or of principal sheaves under $\underline{G}$ etc., obtaining a specific result for each given category.

4.5 Particular cases of associated fibre spaces.

a. Associated principal sheaf. Let E be a fibre space with structure of type Φ. Take on the other hand $\Psi = \underline{G}$, $\underline{G}$ operating on Ψ by left regular representation (which is faithful), so that we may write $\Psi(\rho) = \underline{G}_\ell$. Then the fibre space associated to E and ρ is a fibre space with structure of type $\underline{G}_\ell$, or what is the same (4.2., example b.) a principal sheaf under $\underline{G}$ (operating on the right). Using for instance proposition 4.4.3. we get a canonical isomorphism of $\underline{G}_\ell^E$ with principal sheaf associated to E in the sense of definition 4.1.2. More generally, let $\underline{F}$ be a sub-sheaf of groups of $\underline{G}$, and consider the fibre space $\underline{G}/\underline{F}$ with $\underline{G}$ as a sheaf of left operators, (see end of 3.4.). The fibre space of type $\underline{G}/\underline{F}$ associated to E is canonically isomorphic to the quotient sheaf $P/\underline{F}$ where P is the principal sheaf associated to E (in which sheaf $\underline{G}$ and hence $\underline{F}$ operates on the right, so that the quotient $P/\underline{F}$ is of course defined). Let again P be the principal sheaf associated to a fibre space E of structure type Φ; it is readily verified that conversely, the fibre space associated to P and the given representation of $\underline{G}$ into $G(\Phi)$ is canonically isomorphic to E. And conversely, starting with an arbitrary principal sheaf P under $\underline{G}$, let E be the fibre space associated to P and the given representation of $\underline{G}$ by germs of automorphisms of E; then P is canonically isomorphic to the principal sheaf associated with E. (These statements follow at once when considering the coordinate transforms.)

b. Case when Ψ is a fibre space with composition law. As, with the general notations of 4.4., $\Psi(\rho)^E$ is locally isomorphic to Ψ, the isomorphisms of $\Psi|U$ onto $\Psi(\rho)^E|U$ being well determined up to an isomorphism of Ψ defined by a section of $\underline{G}$ on U, it follows that, loosely speaking, each supplementary structure "of local type" on Ψ, which is "invariant under $\underline{G}$", carries over canonically into an analogous structure on $\Psi(\rho)^E$. In order to give a clean statement of sufficient generality to cover all cases effectively to be encountered, it would be necessary to give a treatment of the notion of associated fibre spaces to a given E in a more general context, as was alluded to in the final remark of 4.4. In order not to make this already long report still longer, we will not make this point of view explicit, and will restrict to the case where Ψ is a fibre space with composition law (cf. 3.1.). By saying that the composition law is invariant under $\underline{G}$, we mean that for each $x \in X$, the composition law in Ψ_x is invariant under the operations of G_x, or what is the same, that ρ is a representation of $\underline{G}$ into the sheaf of germs of automorphisms of Ψ considered as fibre space with composition law. Then it is obvious that $\Psi(\rho)^E$ itself is a fibre space with composition law, locally isomorphic to Ψ. Hence if Ψ is a group bundle (respectively an abelian group bundle) so is $\Psi(\rho)^E$. As anyhow $\Psi(\rho)^E$ is a sheaf if and only if Ψ is a sheaf (this notion being a local one) it follows for instance that if Ψ is a sheaf of groups (respectively of abelian groups) so is $\Psi(\rho)^E$. Let now $\Psi'(\rho')$ be another fibre space with composition law and $\underline{G}$ a group bundle of left operators compatible with the composition law, let $u: \Psi \longrightarrow \Psi'$ be a $\underline{G}$-representation which is also a homomorphism for the composition laws;

then the associated u^E: $\Psi(\rho)^E \longrightarrow \Psi'(\rho')^E$ is also a homomorphism for the composition laws.

c. The sheaf of germs of automorphisms of E as associated fibre space. Let $\underline{G}$ be any group bundle on X. For each $x \in X$, $\underline{G}_x$ operates on $\underline{G}_x$ by interior automorphisms, to $g \in \underline{G}_x$ corresponding the permutation $\sigma(g)$ of $\underline{G}_x$ defined by $\sigma(g).g' = gg'g^{-1}$. The map $(g, g') \longrightarrow \sigma(g).g'$ of $\underline{G} \times_X \underline{G}$ into $\underline{G}$ thus obtained is obviously continuous, so that σ defines on $\underline{G}$ a structure of fibre space with group bundle $\underline{G}$ of (left) operators; and more precisely even of a group bundle with $\underline{G}$ as group bundle of automorphisms (automorphisms being understood with respect to the structure of group bundle). Suppose now that $\underline{G}$ is a sheaf of groups, and let Φ as above be a fibre space with $\underline{G}$ as faithful group-bundle of left operators, E a fibre space of structure type Φ, consider the fibre space $\underline{G}(\sigma)^E$ associated to E and the representation σ. By what has been said in b., $G(\sigma)^E$ is a sheaf of groups. Proposition 4.4.3 yields naturally:

Proposition 4.5.1. Let E be a fibre space of structure type Φ, (structure sheaf $\underline{G}$) consider the representation σ of $\underline{G}$ into the sheaf of germs of automorphisms of $\underline{G}$ by interior automorphisms, and the fibre space $\underline{G}(\sigma)^E$ associated to E and σ. As a sheaf of groups, $\underline{G}(\sigma)^E$ is canonically isomorphic to the sheaf of germs of automorphisms of E (considered of course as fibre space of structure type Φ).

In virtue of proposition 4.5.1., we see that, just as Φ admits $\underline{G}$ as faithful sheaf of left operators, so does E admit $\underline{G}(\sigma)^E$ as faithful sheaf of left operators; $\underline{G}$ can be considered as the sheaf of germs of automorphisms of Φ, when we consider Φ as a fibre space with structure sheaf $\underline{G}$ (namely as the trivial fibre space with structure of type Φ, cf. 4.2. example a), and just so may $\underline{G}(\sigma)^E$ be considered as the sheaf of germs of automorphisms of E considered as a fibre space with structure sheaf. Moreover:

Proposition 4.5.2. Let Φ be a fibre space with $\underline{G}$ as faithful sheaf of left operators, and E a fibre space of structure type Φ; consider E as admitting $\underline{G}(\sigma)^E$ (defined in proposition 4.5.1.) as faithful sheaf of left operators. Let F be any fibre space on X. Then there is a canonical correspondence between the structures of type Φ (structure sheaf $\underline{G}$) on F for which the underlying fibre space is F, and the structures of type E (structure sheaf $\underline{G}(\sigma)^E$) is F for which the underlying fibre space is F. In order for F to be trivial for a given structure of the latter type, it is necessary and sufficient that F be isomorphic to E for the corresponding structure of the former type.

In chapter 5 we can restate this in a more condensed form by the formula $H^1(X, \underline{G}) \approx H^1(X, \underline{G}(\sigma)^E)$, a canonical isomorphism transforming the class c of E in the first set, into the unit element e of the second).

We can suppose F to be locally isomorphic to Φ. The structures of the former type on F are the sections of the sheaf $G(\Phi, F) / \underline{G}$, where $G(\Phi, F)$ is the sheaf of germs of isomorphisms of Φ onto F, on which $\underline{G}$ operates on the right as explained in 4.1., and likewise the structures of the second type are the sections of the sheaf $G(E, F) / \underline{G}'$, where for abbreviation we put $\underline{G}' = G(\sigma)^E$. Thus we only need exhibit a natural isomorphism of the sheaves $G(\Phi, F)/\underline{G}$ and $G(E, F)/\underline{G}'$. Now an element of the second quotient comes from a germ v of isomorphism of E onto F at a point $x \in X$, let u be any germ of isomorphism (i.e. of a coordinate map) of Φ onto E at the point x, and let w = vu, which is a germ of isomorphism of Φ onto F at x. The class of w in the quotient $G(\Phi, F)/\underline{G}$ does not change if u is replaced by another germ of isomorphism u', for we have u' = ug ($g \in G_x$, considered as a germ of automorphism of Φ) and hence vu' = (vu)g has the same class as vu; neither does the class of vu change if we replace v by another representative v' of the considered element of $G(E, F) / \underline{G}'$, for we have v' = vg' ($g' \in \underline{G}'_x$ considered as a germ of automorphism of E) hence v'u = vg'u, but of course g'u being still a germ of an isomorphism of Φ onto E is of the form ug ($g \in \underline{G}_x$) hence v'u = v(ug) = (vu)g has the same class as vu. We have thus defined a natural map $G(E, F) / \underline{G}' \longrightarrow G(\Phi, F) / \underline{G}$, and it is seen at once that this is an isomorphism of the first sheaf onto the second. This concludes the proof of the first statement, and the second is as easily checked.

Of course, if F and F' are two fibre spaces with structure of type Φ, then the sheaf of germs of isomorphisms of F onto F' for these structures, is identic to the sheaf of germs of isomorphisms for the corresponding structures of type E(structure sheaf $\underline{G}' = \underline{G}(\sigma)^E$), in particular an isomorphism $F \longrightarrow F'$ for the underlying structures of fibre spaces, is an isomorphism in the sense of the structures of type Φ involved if and only if it is one for the structures of type E. In particular, the classification up to isomorphism of fibre spaces is the same for the structure sheaf $\underline{G}$, or the structure sheaf $\underline{G}'$. It should, however, not be thought that this is true whenever $\underline{G}'$ is a sheaf of groups locally isomorphic to $\underline{G}$; it is quite essential that $\underline{G}'$ be obtained as $\underline{G}' = \underline{G}(\sigma)^E$ for a suitable fibre space E with structure sheaf $\underline{G}$.

d. The sheaf of germs of sections as associated sheaf. Admissible sections.

If E is any fibre space over X, let $\overline{E}$ be the sheaf of germs of sections of E. This can of course be considered as a functor of E, and even a local functor in the sense that we may identify, for $V \subset U \subset X$ (U and V non-empty open sets), and a fibre space E over U, the fibre spaces $\overline{E}\,|V$ and $\overline{(E|V)}$. If now we restrict E to be a fibre space with structure of type $\Phi\,|U$ (defined on a non-specified open $U \subset X$) we have a fortiori a local functor in the precise sense of definition 4.4.3. Using proposition 4.4.3., we get that $\overline{E}$ can be considered as the fibre space associated to E and the sheaf $\overline{\Phi}$ of germs of sections of Φ (on which $\underline{G}$ operates in a natural way). The natural homomorphism $\overline{E} \longrightarrow E$ is associated to the homomorphism $\overline{\Phi} \longrightarrow \Phi$. Moreover, for every subsheaf A of the sheaf $\overline{\Phi}$, stable under $\underline{G}$, the associated sheaf A^E can therefore be considered as a subsheaf of $\overline{E}$, which will behave as a local functor under isomorphisms and restrictions of fibre spaces E (with structure of type Φ).

4.6 Extension and restriction of the structure sheaf.

Let again Φ be a fibre space with a faithful sheaf $\underline{G}$ of left operators. Let $\underline{F}$ be a subsheaf of groups of $\underline{G}$, then $\underline{F}$ operates also faithfully on Φ; in order to distinguish between Φ provided with either the sheaf $\underline{F}$, or $\underline{G}$, as sheaf of operators, we write respectively $\Phi_{\underline{F}}$ and $\Phi_{\underline{G}}$. Thus the notions of fibre space E with structure of type $\Phi_{\underline{F}}$, or with structure of type $\Phi_{\underline{G}}$, are well defined and of course distinct; for the first the structure sheaf is $\underline{F}$, for the second it is $\underline{G}$. Let E be any fibre space on X locally isomorphic to Φ, G(Φ , E) the sheaf of germs of isomorphisms of Φ onto E. $\underline{F}$ and $\underline{G}$ operate faithfully on the right on G(Φ, E), the representation of $\underline{F}$ being induced by the one of $\underline{G}$. Therefore we have a natural onto-homomorphism

$$G(\Phi, E)/\underline{F} \longrightarrow G(\Phi, E)/\underline{G}. \tag{4.6.1.}$$

Definition 4.6.1 Let Φ be a fibre space with $\underline{G}$ as faithful sheaf of left operators, $\underline{F}$ a subsheaf of groups of $\underline{G}$, E a fibre space with structure of type $\Phi_{\underline{F}}$ i.e. a fibre space E locally isomorphic to Φ, provided with a section of the sheaf on the left side of (4.6.1.). The canonical image of this section yields a section of G(Φ , E) / $\underline{G}$, i.e. on E a structure of type $\Phi_{\underline{G}}$, which is said obtained from the given structure (with structure sheaf $\underline{G}$) by extension of the structure sheaf to $\underline{G}$.

By this definition, if the structure of type $\Phi_{\underline{F}}$ of E is defined by coordinate maps maps $f_i : \Phi | U_i \longrightarrow E | U_i$, (subject to the condition that the corresponding coordinate transforms f_{ij} be given by sections of $\underline{F}$ on U_{ij}, see 4.3.), then its structure of type $\Phi_{\underline{G}}$ is defined by the same coordinate maps (and hence corresponding to the same coordinate transformations). It follows at once that if E is defined by a priori given coordinate transformations (f_{ij}) (the f_{ij} being sections of $\underline{F}$ on U_{ij} satisfying the conditions 4.3.2.), then the fibre space obtained by extension of the structure sheaf to $\underline{G}$ is defined by the same coordinate transformations, but now considered as sections of $\underline{G}$ rather than of $\underline{F}$. This shows that the notion of associated bundle (4.4.) and of extension of the structure sheaf, are both particular cases of the following more general construction (where the notations are slightly changed with respect to 4.4.). Let Φ be a fibre space with $\underline{F}$ as a faithful sheaf of left operators, Ψ a fibre space with $\underline{G}$ as a faithful sheaf of left operators, E a fibre space with structure of type Φ, ρ a homomorphism of $\underline{F}$ into $\underline{G}$, then let $\Psi(\rho)^E$ stand for the fibre space of structure type $\underline{G}$ obtained by taking the set (U_i, f_i) of all coordinate maps $f_i : \Phi | U_i \longrightarrow E | U_i$ for E and the corresponding family (f_{ij}) of coordinate transforms, and considering the fibre space of structure type Ψ defined by the coordinate transforms $\rho(f_{ij})$ (which obviously satisfy $\rho(f_{ik}) = \rho(f_{ij}) \rho(f_{jk})$, ρ being a homomorphism). Of course, in the case ρ is an onto-homomorphism, we find the fibre space of definition 4.4 1., and in the case $\Phi = \Psi$, $\underline{F} \subset \underline{G}$ and ρ being the injection homomorphism, we have said that the space just constructed is canonically isomorphic to the one of definition 4.6.1. We may thus call in the general case $\Psi(\rho)^E$ the <u>fibre space with structure sheaf G associated to the homomorphism</u> ρ, or <u>obtained by extension of the structure sheaf to G with respect to the homomorphism</u> ρ.

Let us come back to the general conditions of the beginning of this section, suppose now that we are given on E a structure $S(\underline{G})$ of type $\Phi_{\underline{G}}$, and we want to find those structures $S(\underline{F})$ of type $\Phi_{\underline{F}}$ on E such that $S(\underline{G})$ be obtained from $S(\underline{F})$ by extension of the structure sheaf (we then say that $S(\underline{F})$ is obtained from $S(\underline{G})$ by <u>restriction of the structure sheaf</u>; of course such a $S(\underline{F})$ may not exist, and if it exists, not be unique). If $P_{\underline{G}}$ is the principal sheaf associated to $S(\underline{G})$, i.e. the inverse image in the sheaf $G(\Phi, E)$ of the section $S(\underline{G})$ of $G(\Phi, E)/\underline{G}$ (which is a principal sheaf under the operations of G), then of course $P_{\underline{G}}/\underline{F}$ is the subsheaf of $G(\Phi, E)/\underline{F}$ inverse image of $S(\underline{G})$ under the homomorphism (4.6.1.), and therefore the sections $S(\underline{F})$ we are looking for are exactly the sections of $P_{\underline{G}}/\underline{F}$. $P_{\underline{G}}/\underline{F}$ can also be described as the fibre space of structure type $\underline{G}/\underline{F}$ associated to E and the natural representation of $\underline{G}$ as sheaf of operators in $\underline{G}/\underline{F}$ (see 4.5. example a.). Thus we finally get:

<u>Proposition 4.6.1.</u> Let E be a fibre space with structure sheaf $\underline{G}$, $\underline{F}$ a subsheaf of groups of $\underline{G}$. Then there is a canonical one-to-one correspondence between the structures on E obtained by restriction of

the structure sheaf to $\underline{F}$, and the sections of the sheaf $P/\underline{F}$ (P the principal sheaf associated to E), i.e. the sections of the sheaf associated to E and the natural representation of $\underline{G}$ by operators in $\underline{G}/\underline{F}$.

If we remark that if $\underline{F}$ is reduced to the unit sheaf, a structure of type $\Phi_{\underline{F}}$ on E is identic with the giving of an isomorphism of Φ onto E, we see

Corollary. In order that a fibre space of structure type Φ be trivial, it is necessary and sufficient that its principal sheaf $\underline{P}$ admits of a section.

This follows also trivially from the fact that $\underline{P}$ is also the sheaf of germs of isomorphisms of Φ onto E (4.2., example a.).

4.7 *Case of fibre spaces with a structure group.*

Let G be a topological group operating faithfully (at left) in a topological space F, and $\underline{G}$ a sheaf of germs of continuous maps of X into G, containing the germs of constant maps, so that $\underline{G}$ can be considered as a sheaf of groups operating faithfully (at left) on the trivial fibre space $X >< F$, and admitting G as a structure group (definition 3.5.1.), and we say that a fibre space of structure type $X >< F$ (structure sheaf $\underline{G}$) admits G as a *structure group* (see 4.2., examples d). Now let in the same way F' be a topological space with a faithful topological group G' of operators, $\underline{G}'$ a subsheaf of the sheaf of germs of continuous maps of X into G'. Let ρ be a representation of the topological group G into G', such that for each germ $f \in \underline{G}$, the composed germ $\rho . f$ be in $\underline{G}'$; we will say that ρ is a *representation compatible with the sheaves $\underline{G}$ and $\underline{G}'$*. Then $f \longrightarrow \rho . f$ is a homomorphism of $\underline{G}$ into $\underline{G}'$, which we will denote by $\underline{\rho}$. Now let E be a fibre space of structure type $X >< F$ (fibre F, structure sheaf $\underline{G}$, hence structure group G), we can consider the fibre space of structure type $X >< F'$, structure sheaf $\underline{G}'$ associated to E and the representation $\underline{\rho}$ (cf. 4.6.); in practice, when the sheaves $\underline{G}$ and $\underline{G}'$ are supposed fixed once for all, this new fibre space is simply called the *fibre space associated to E* (given fibre space with structure group G) *and the homomorphism $\underline{\rho}$*

of $\underline{G}$ into $\underline{G}'$ (operating on F'). We may simply be given a representation ρ of G by homeomorphisms of F' onto itself (without a group G' or a sheaf $\underline{G}'$), such that the corresponding map $G \times F' \longrightarrow F'$ be continuous. We then take $G' = G / \mathrm{Ker}\ \rho$, $\underline{G}'$ = image of G under ρ, and get (for any E with structure sheaf $\underline{G}$) an associated fibre space with fibre F', structure group G' (and even structure sheaf $\underline{G}'$). For instance, taking $F' = G$, and for ρ the left regular representation, we get an associated fibre space P with fibre G, structure group G (operating by left regular representation) and more precisely, with structure sheaf $\underline{G}$: this is called the associated principal bundle of the fibre space E with structure group G. It can be given an interpretation analogous to the associated principal sheaf (see definition 4.1.2. and example 4.5.a.): it is isomorphic, as a fibre space, to the space of isomorphisms of F onto (variable) fibres E_x of E, which are "compatible with the structure group $\underline{G}$" i.e. which are induced by a germ at x of a coordinate map from $X \times F$ to E; this space being topologized in the obvious way (we suppose, to fix the ideas, that $G \times F \longrightarrow F$ is continuous for the product topology and that we take on $X \times F$ the product topology): for each coordinate map $U \times F \longrightarrow E|U$, defining in an evident way a bijective map $U \times G \dashrightarrow P|U$, we take on $P|U$ the topology deduced from the topology of $U \times G$, and then on P the unique topology inducing on each $P|U$ the given one. From this alternative definition, we see easily that G operates on the right on P (the map $P \times G \longrightarrow P$ being continuous) and that X is the quotient space P/G, (this is the second well known aspect of the notion of principal fibre bundle. All of these facts, which depend only on the giving of G operating on F and not on $\underline{G}$ itself, are quite classical and it is therefore useless to give more details.) Let $\widetilde{P}$ be the principal sheaf associated to the fibre space E. There is a natural injective homomorphism of $\widetilde{P}$ into the sheaf $\underline{P}$ of germs of sections of P, since each $f \in H^0(U, \widetilde{P})$, being an isomorphism of $U \times F$ onto $E|U$, can be considered as a map $x \longrightarrow f(x)$ which to each $x \in U$ associates an admissible isomorphism of $(U \times F)_x = F$ onto the fibre E_x of E over x, i.e. as a section of P. The sections of P thus obtained will be called admissible sections (for structure sheaf $\underline{G}$). For any coordinate map $U \times G \dashrightarrow P|U$ (stemming from a coordinate map $U \times F \dashrightarrow E|U$) the sheaf $\underline{G} | U$, considered as a sheaf of sections of $U >< G$, is transformed into the sheaf $P|U$ of admissible sections of $P|U$.

More generally, let F be a topological subgroup of G, and $H = G/H$ the corresponding homogenous space, let $\underline{F}$ be the subsheaf of $\underline{G}$ of germs of maps into F, and $\underline{H}$ the canonical image of $\underline{G}$ in the sheaf of germs of maps of X into H, clearly we have $\underline{H} = \underline{G}/\underline{F}$, thus $\underline{H}$ is stable under the operations of $\underline{G}$ on the sheaf of germs of maps of X into G/H. If now E is a fibre space with structure sheaf $\underline{G}$, we can consider the fibre space associated to E and the operations of G on H, which is a fibre space of type $X >< H$, and structure sheaf $\underline{G}'$ a quotient of $\underline{G}$. As easily seen, this space can be identified to the quotient P/F (if we remember that G and hence F operate on the right on the principal fibre space P). The sheaf $\underline{H}$ of germs of sections of $X >< H$ being stable under the structure sheaf $\underline{G}'$, it follows that we can consider the sheaf associated to P/F and $\underline{H}$, which is a sheaf of sections of P/F, the sections of which are still called <u>admissible</u> <u>sections</u>. The germs of admissible sections of P/F are exactly those which are canonical images of admissible sections of P.

If for example G is a Lie group, F a closed subgroup (and hence itself a Lie group), X a manifold of class C^m (respectively an analytic manifold), $\underline{G}$ the sheaf of germs of maps of class C^m (respectively of analytic maps) of X into G, then $\underline{F}$ and $\underline{H}$ are the sheaves of germs of maps of X into F, respectively of X into H, which are of class C^m(respectively analytic). This is of course trivial for $\underline{F}$, and for $\underline{H}$ it follows from the well known fact that through each point of G (and in fact it is sufficient to know it for the neutre element e) passes an analytic section (G being considered as a fibre space over H, with group F). If E is a fibre space with structure sheaf $\underline{G}$, i.e. a fibre space of class C^m (respectively an analytic fibre space) with structure group G, it is checked at once that the admissible sections of the associated fibre space P/F are exactly the sections which are of class C^m (respectively analytic). The analogous remarks apply when G and F are complex Lie groups, X being a complex analytic variety. Same results in the abstract algebraic case, X being an algebraic variety and G and F algebraic groups, but with the important difference that here we must explicitly include in the hypotheses that G considered as fibered over G/F admits a local regular cross-section, which implies that each germ of a regular map from X into H can be lifted to a germ of regular map of X into G.

CHAPTER V

THE CLASSIFICATION OF FIBRE SPACES WITH STRUCTURE SHEAF

5.1 The functor $H^1(X, \underline{G})$ and its interpretation.

Let $\underline{G}$ be a sheaf of groups (not necessarily abelian) on X. Let $\underline{U} = (U_i)_{i \in I}$ be an open covering of X, as usual we write $U_{i_o i_1 \dots i_p}$ for the intersection $U_{i_o} \cap \dots \cap U_{i_p}$, for any p-tuple $(i_o, \dots, i_p) \in I^{p+1}$. We define the group of p-cochains of the covering $\underline{U}$ with coefficients in the sheaf of groups $\underline{G}$ as the product group

$$(5.1.1.) \qquad C^p(\underline{U}, \underline{G}) = \prod_{I^{p+1}} H^o(U_{i_o \dots i_p}, \underline{G}) .$$

(We recall that for any open $U \subset X$, $H^o(U, G)$ is the group of sections of $\underline{G}$ on U). Thus an element of $C^p(\underline{U}, \underline{G})$ is an arbitrary family $(g_{i_o \dots i_p})$ of sections of $\underline{G}$ on the intersections $U_{i_o \dots i_p}$. It is understood of course that in the product (5.1. 1.) we take for $H^o(U_{i_o \dots i_p}, G)$ the unit group if $U_{i_o \dots i_p} = \emptyset$, or what amounts to the same, that we restrict to systems of indices $(i_o, \dots, i_p)$ for which the intersection $U_{i_o \dots i_p}$ is non-empty. Here we are interested mainly in $C^o(\underline{U}, \underline{G}) = \prod H^o(U_i, \underline{G})$ and $C^1(\underline{U}, \underline{G}) = \prod H^o(U_{ij}, \underline{G})$. A 1-cochain (g_{ij}) is called a 1-cocycle if it satisfies the conditions

$$g_{ik} = g_{ij}g_{jk} \quad \text{on } U_{ijk} \tag{5.1.2.}$$

whenever $U_{ijk} \neq \emptyset$. The set of 1-cocycles in $C^1(\underline{U}, \underline{G})$ (which set in general is not a subgroup however) is denoted by $Z^1(\underline{U}, \underline{G})$. It should be noted that the relations (5.1.2.) imply

$$g_{ji} = g_{ij}^{-1}, \qquad g_{ii} = e \text{ (unit section)} \tag{5.1.3.}$$

(the second relation is obtained in taking $i = j = k$, and the first relation then follows in taking $k = i$ in (5.1.2)). Let $g^o = (g_i) \in C^o(\underline{U}, \underline{G})$, $g^1 = (g_{ij}) \in C^1(\underline{U}, \underline{G})$, we define an element $D(g^o).g^1$ of $C^1(\underline{U}, \underline{G})$ by

$$(D(g^o).g^1)_{ij} = g_i g_{ij} g_j^{-1}. \tag{5.1.4.}$$

With this definition, D is a representation of the group $C^o(\underline{U}, \underline{G})$ by permutations of the set $C^1(\underline{U}, \underline{G})$; it is moreover trivial that $Z^1(\underline{U}, \underline{G})$ is stable under the operations of $C^o(\underline{U}, \underline{G})$. We can therefore take the quotient set $Z^1(\underline{U}, \underline{G}) / D(C^o(\underline{U}, \underline{G}))$, which we denote by $H^1(\underline{U}, \underline{G})$:

$$H^1(\underline{U}, \underline{G}) = Z^1(\underline{U}, \underline{G}) / D(C^o(\underline{U}, \underline{G})). \tag{5.1.5.}$$

Let now $\underline{V} = (V_j)_{j \in J}$ be another open covering of X, finer than $\underline{U}$. We will define a canonical map

$$\varphi_{\underline{V}, \underline{U}} : H^1(\underline{U}, \underline{G}) \longrightarrow H^1(\underline{V}, \underline{G}). \tag{5.1.6}$$

Let $\tau : J \longrightarrow I$ be such that for each $j \in J$, $V_j \subset U_{\tau j}$. Then for each natural integer p, in particular for $p = 0$ and $p = 1$, we have a corresponding natural homomorphism

$$\varphi^p_\tau : C^p(\underline{U}, \underline{G}) \longrightarrow C^p(\underline{V}, \underline{G})$$

transforming the cochain $g^p = (g_{i_o \ldots i_p}) \in C^p(\underline{U}, \underline{G})$ into

$\varphi^p_\tau g^p \in C^p(\underline{V}, \underline{G})$ given by

(5.1.7.) $\quad (\varphi^p_\tau g^p)_{j_o \ldots j_p}$ = restriction to $V_{j_o \ldots j_p}$ of $g_{j_o \ldots j_p}$.

It is obvious that for, $g^o \in C^o(\underline{U}, \underline{G})$ $g^1 \in C^1(\underline{U}, \underline{G})$, we have

(5.1.8.) $$\varphi^1_\tau(D(g^o).g^1) = D(\varphi^o_\tau g^o).(\varphi^1_\tau g^1)$$

and on the other hand that $\varphi^1_\tau(Z^1(\underline{U}, \underline{G})) \subset Z^1(\underline{V}, \underline{G})$, so that φ^1_τ defines a map of $Z^1(\underline{U}, \underline{G})$ into $Z^1(\underline{V}, \underline{G})$ compatible with the equivalence relations defined respectively by the groups of permutations $D(C^o(\underline{U}, \underline{G}))$ and $D(\underline{C}^o(\underline{V}, \underline{G}))$, hence a well determined homomorphism (5.1.6.). We have only to show that this does not depend on the choice of τ. This is quite easy directly, we prefer, however, to give a geometric argument, which at the same time will show us the meaning of our constructions. Let Φ be a fibre space admitting $\underline{G}$ as faithful sheaf of left operators: we can always find such a Φ, e.g. $\underline{G}$ itself operating by the left regular representation. By what has been said in 4.3.3., we see that there is a one-to-one natural correspondence between those classes (for isomorphism relation) of fibre spaces E of structure type Φ such that for each i, $E|U_i$ be trivial, i.e. isomorphic (in the sense of the structures of type Φ) to $\Phi|U_i$ (i.e. such that for each U_i there is a coordinate map for E defined on $\Phi|U_i$), and classes of systems (g_{ij}) of coordinate transformations (relative to the covering $\underline{U}$ and the sheaf $\underline{G}$) modulo the equivalence relation found in section 4.3. (cf. (4.3.3.)); now a system of coordinate transformations relative to $\underline{U}$ is nothing but an element of $Z^1(\underline{U}, \underline{G})$, and the equivalence relation in question is the one defined by the group $D(C^o(\underline{U}, \underline{G}))$ operating as defined by formula (5.1.4.), therefore the classes of such systems are exactly the elements of $H^1(\underline{U}, \underline{G})$.

On the other hand, as remarked in 4.3. as a consequence of the general considerations of 1.5., the map φ^1_{τ}: $Z^1(\underline{U}, \underline{G}) \longrightarrow Z^1(\underline{V}, \underline{G})$ transforms a system of coordinate transformations for a fibre space **E** into a system of coordinate transformations for a fibre space isomorphic to E; thus by passing to the quotient, we obtain a map $H^1(\underline{U}, \underline{G}) \longrightarrow H^1(\underline{V}, \underline{G})$ which, interpreted as a map on sets of classes of fibre spaces of type Φ, is nothing but the inclusion map, hence canonically defined, and moreover <u>injective.</u> Let now $\underline{W} = (W_k)_{k \in K}$ be a third open covering of X, finer than $\underline{V}$, then we have

$$\varphi_{\underline{W}, \underline{U}} = \varphi_{\underline{W}, \underline{V}} \ \varphi_{\underline{V}, \underline{U}} \tag{5.1.9.}$$

as is seen at once either directly, or as a trivial consequence of the geometric interpretation. Let $\mathcal{R}$ be the set of all trivially indexed open coverings of X, preordered by the refinement relation which turns it into a preordered filtering set. The maps (5.1.6.) satisfying (5.1.9.) define the system of sets $(H^1(\underline{U}, \underline{G}))_{\underline{U} \in \mathcal{R}}$ as an <u>inductive system</u> on the preordered set $\mathcal{R}$. Therefore, we can take the inductive limit of this <u>inductive system</u>, which only depends on $\underline{G}$ and is denoted by $H^1(X, \underline{G})$:

$$H^1(X, \underline{G}) = \varinjlim_{u} H^1(\underline{U}, \underline{G}). \tag{5.1.10.}$$

<u>Definition 5.1.1.</u> Let $\underline{G}$ be a bundle of groups on X. We denote by $H^1(X, \underline{G})$ (and call first cohomology set of X with coefficients in $\underline{G}$), the inductive limit (5.1.9.) of the sets $H^1(U, \underline{G})$ defined by (5.1.5.), corresponding to all open coverings of X, under the homomorphisms defined above.

An element of this inductive limit is a class of elements in the union of the sets $H^1(\underline{U}, \underline{G})$, an element in $H^1(\underline{U}, \underline{G})$ being equivalent to an element in $H^1(\underline{V}, \underline{G})$ if and only if for a suitable common refinement $\underline{W}$ of $\underline{U}$ and $\underline{V}$, the canonical images of these elements in $H^1(\underline{W}, \underline{G})$ are the same. If we now interpret the sets $H^1(\underline{U}, \underline{G})$ as sets of classes of fibre spaces, we see at once that $H^1(X, \underline{G})$ can be canonically identified with the set of all isomorphism classes of fibre spaces of type Φ (structure sheaf $\underline{G}$) on X. Thus we have obtained:

Proposition 5.1.1. Let $\underline{U} = (U_i)$ be an open covering of X. Then $H^1(\underline{U}, \underline{G})$ is in canonical one-to-one correspondence with the set of isomorphism classes of fibre spaces with structure of type Φ (structure sheaf $\underline{G}$) the restriction of which on each U_i is trivial. $H^1(X, \underline{G})$ is in canonical one-to-one correspondence with the set of all isomorphism classes of fibre spaces with structure of type Φ. With these identifications, the maps (5.1.6.) and the natural maps of the sets of $H^1(\underline{U}, \underline{G})$ into their inductive limit, are the inclusion maps for sets of isomorphism classes of fibre spaces. (In these statements, Φ is any fibre space admitting $\underline{G}$ as sheaf of left operators.)

In particular, we see that in the classification problem, Φ itself does not play any part; only the structure sheaf $\underline{G}$ is significant.

Let now $\underline{U}$ be a fixed open covering, and let u: $\underline{G} \longrightarrow \underline{G}'$ be a homomorphism of a sheaf of groups into another. This defines corresponding homomorphisms $u^p: C^p(\underline{U}, \underline{G}) \longrightarrow C^p(\underline{U}, \underline{G}')$ by putting

$$(u^p(g^p))_{i_o \dots i_p} = u_* (g_{i_o \dots i_p}) \tag{5.1.11.}$$

where u_* is the homomorphism on the sets of sections deduced from u (cf. 1.7.). Of course

$$u^1(D(g^o).g^1) = D(u^o g^o).(u^1 g^1)$$

and moreover u^1 transforms cocycles into cocycles, so that by passing to the quotient we obtain a map

$$u^1 : H^1(\underline{U}, \underline{G}) \longrightarrow H^1(\underline{U}, \underline{G}'). \tag{5.1.12.}$$

This map clearly depends on u in a way to satisfy the usual two functor requirements: a) if $\underline{G} = \underline{G}'$ and u the identity, the corresponding $u^1 : H^1(\underline{U}, \underline{G}) \longrightarrow H^1(\underline{U}, \underline{G}')$ is the identity; b) is we have two homomorphisms of sheaves of groups: $\underline{G} \xrightarrow{u} \underline{G}' \xrightarrow{v} \underline{G}''$, then $(vu)^1 = v^1 u^1$. On the other hand, these maps are compatible with the restriction maps corresponding to a pair $(\underline{U}, \underline{V})$ of a covering $\underline{U}$ and a refinement $\underline{V}$. More precisely, given such a pair, and a homomorphism of sheaves u: $\underline{G} \longrightarrow \underline{G}'$, the following diagram of maps

$$\begin{array}{ccc} H^1(\underline{U}, \underline{G}) & \xrightarrow{u^1} & H^1(\underline{U}, \underline{G}') \\ \downarrow & & \downarrow \\ H^1(\underline{V}, \underline{G}) & \xrightarrow{u^1} & H^1(\underline{V}, \underline{G}') \end{array}$$

(where the vertical arrows are the "restriction maps" $\varphi_{\underline{V}, \underline{U}}$) is commutative. From this follows that for a fixed u: $\underline{G} \longrightarrow \underline{G}'$, the maps $H^1(\underline{U}, \underline{G}) \longrightarrow H^1(\underline{U}, \underline{G}')$ corresponding to all open coverings of X define a natural map of the inductive limit $H^1(X, \underline{G})$ of the sets $H^1(\underline{U}, \underline{G})$ into the inductive limit $H^1(X, \underline{G}')$ of the sets $H^1(\underline{U}, \underline{G}')$, map which we will still denote by u^1 :

(5.1.13.) $$u^1 \ : \ H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{G}');$$

and that these maps, for varying u, satisfy again the functor properties:

(5.1.14.) $$u = \text{identity implies } u^1 = \text{identity}; \ (vu)^1 = v^1 u^1.$$

Thus $H^1(X, \underline{G})$, for variable $\underline{G}$, is defined as a functor of $\underline{G}$.

Let Φ be a fibre space with $\underline{G}$ as faithful sheaf of left operators, Φ' a fibre space with $\underline{G}'$ as faithful sheaf of left operators, u a homomorphism $\underline{G} \longrightarrow \underline{G}'$, then we have defined in section 4.6. for each fibre space E with structure of type Φ the fibre space $\Phi'(u)^E$ associated to E and the homomorphism u, and we have seen that if (g_{ij}) is a system of coordinate transformations for E, then (ug_{ij}) is a system of coordinate transformations of the associated fibre space. Hence, if the isomorphism class of E is defined by an element $c \in H^1(\underline{X}, \underline{G})$, then the isomorphism class of the associated fibre space is defined by $u^1 c \in H^1(\underline{X}, \underline{G}')$.

Connection with the classical notion of cohomology with coefficients in an abelian sheaf. As well known, if $\underline{G}$ is an abelian sheaf, one defines for every integer $p \geq 0$ the p.th cohomology group of X with coefficients in $\underline{G}$, $H^p(X, \underline{G})$. There are different axiomatic or constructive approaches to this notion, see in particular H. Cartan, Seminaire de Topologie Algébrique, 1950-51. The most convenient approach in the present context is the definition by the Čech covering method, cf. J. P. Serre, Faisceaux algébriques cohérents, 1, Annals of Mathematics, vol. 61, No. 2, 1955. It is obvious that the definitions given here for $H^o(X, \underline{G})$ and $H^1(X, \underline{G})$ for any sheaf of groups $\underline{G}$, coincide with those given there when $\underline{G}$ is abelian. In this case, $H^1(X, \underline{G})$ is even an (abelian) group; this will be generalized below (5. 5.) to a somewhat more general situation. Now we only remark that if $\underline{G}$ is not commutative, there is no natural way of defining a group structure in $H^1(X, \underline{G})$. However, we can define in this set a privileged element, the trivial or neutre or unit element of $H^1(X, \underline{G})$, which can either be defined as the class defined by a cocycle (g_{ij}) such that $g_{ij} = e$ for each (i, j) (e being a unit section), or as the class of the trivial fibre space (4. 2. example a).

The remainder of this chapter consists in the step by step development, according to the stringency of the hypotheses implied, of the definition and properties of a generalized exact cohomology sequence.

5. 2 The first coboundary map.

The following terminology will be useful:

Definition 5. 2. 1. A sequence of two consecutive maps

(5. 2. 1.)
$$A \xrightarrow{u} B \xrightarrow{v} C$$

where A, B, C are sets and C is provided with a "neutre element" e, is called exact if $v^{-1}(e) = u(A)$. If B is also provided with a "neutre element" (again denoted e by "abus de langage"), if A is a group (hence also a "neutre element" in A) and if we are given a left or right representation ρ of A by permutations of B, the system (A, B, C, u, v, ρ) (where A is understood with its group structure, and B and C with their neutre elements) is called exact if (i) either $u x = \rho(x). e$ for any x, or

$ux = \rho(x^{-1}).e$ for any x; (ii) two elements of B have the same image in C if and only if they are congruent mod the group A of permutations, (iii) $v(e) = e$.

Exactness of such a system implies obviously exactness in the sequences of maps (5.2.1.), in the sense of the first part of the definition. If A, B, C are groups, the neutre elements of B and C their unit elements, and u and v homomorphisms, we can let A operate on B by composition of u and either the left or the right regular representation of B; then the two notions of exactness are the same, and mean that the image of u is the kernel (in the usual sense) of v. Suppose now given a sequence of arbitrary length of maps between sets A_i each of which is provided with a neutre element:

$$A_o \longrightarrow A_1 \longrightarrow \dots A_{n-1} \longrightarrow A_n \longrightarrow \dots .$$

(this sequence may of course be infinite); some of the A_i may be given group structures (we then assume that the given neutre element is precisely the unit element), and may operate moreover on the following set A_{i+1} (on the left or on the right). We say that the system thus defined is an <u>exact sequence,</u> if each sub-system $A_o \longrightarrow A_1 \longrightarrow A_2$, $A_1 \longrightarrow A_2 \longrightarrow A_3$ etc, is an exact sequence in the sense of definition 5.2.1.

Let $\underline{G}$ be any sheaf (not necessarily a sheaf of groups) on X, $\underline{F}$ a <u>faithful</u> sheaf of <u>right</u> operators on $\underline{G}$. We have a natural homomorphism

(5.2.2.) $$H^o(X, \underline{G}) \longrightarrow H^o(X, \underline{H}) \quad \text{where } \underline{H} = \underline{G}/\underline{F}$$

on the other hand the group $H^o(X, \underline{F})$ operates on the left-hand side of (5.2.2.) and two elements are congruent if and only if they have same right-hand image (proposition 3.4.1.). We will now define a natural map

(5.2.3.) $$\delta : H^o(X, \underline{H}) \longrightarrow H^1(X, \underline{F})$$

the <u>coboundary map</u> δ. Let $h \in H^o(X, \underline{H})$ be a section of $\underline{H}$, let (U_i, g_i) be the set of all systems (U_i, g_i) of open set U_i and a $g_i \in H^o(X, \underline{G})$ the image of which under the map corresponding to (5.2.2.) is $h|U_i$. Then (U_i) is an open covering of X. If $U_{ij} \neq 0$, then $g_i|U_{ij}$ and $g_j|U_{ij}$ define the same element of $H^o(U_{ij}, \underline{H})$, hence by what has been said above (applied to U_{ij} instead of X), we see that there exists a unique $f_{ij} \in H^o(U_{ij}, \underline{F})$ such that

$$(5.2.4.) \qquad g_j = g_i f_{ij} \quad \text{on } U_{ij}.$$

The system (f_{ij}) thus obtained obviously satisfies to the condition $f_{ik} = f_{ij} f_{jk}$, and can therefore be considered as a system of coordinate transforms for a fibre space with structure sheaf $\underline{F}$ (which space is well determined for every choice of a faithful representation of $\underline{F}$ by germs of automorphisms of a fibre space Φ). The class of this fibre space, i.e. the element of $H^1(X, \underline{F})$ defined by the cocycle (f_{ij}), is by definition δh. It should be noted that in this definition, we can clearly replace (U_i, f_i) by a subset, provided the U_i's considered still cover X; the corresponding fibre space is indeed canonically isomorphic to the first, a fortiori its class is the same.

Proposition 5.2.1. Let $\underline{F}$ be a sheaf of groups operating faithfully on the right on a sheaf $\underline{G}$, let $\underline{H}$ be the quotient sheaf. Then

$$H^o(X, \underline{G}) \xrightarrow{j_o} H^o(X, \underline{H}) \xrightarrow{\delta} H^1(X, \underline{F})$$

is an exact sequence (definition 5.2.1.) i.e. the image of the first map is the "kernel" of the second.

Let indeed g be a section of $\underline{G}$ the image of which is h, then on each U_i we must have $g = g_i f_i$ with a well determined $f_i \in H^o(U_i, \underline{F})$; writing $g_i f_i = g_j f_j$ on U_{ij} and using (5.2.4.), we get $f_i = f_{ij} f_j$ i.e. $f_{ij} = f_i . e. f_j^{-1}$, which means that the element of $H^1(X, \underline{F})$ defined by the cocycle (f_{ij}) is the same as the one defined by the unit cocycle, i.e. the neutre element of $H^1(X, \underline{F})$. Conversely, if this is so, then we see at once that the sections $g_i f_i$ are the restrictions of a single section g of $\underline{G}$, because $g_i f_i$ and $g_j f_j$ have same restriction to U_{ij}; this g of course has image h.

Corollary Suppose given a "neutre element" e of $H^o(X, \underline{G})$ and take its image in $H^o(X, \underline{H})$ as neutre element in the latter set. Define $H^o(X, \underline{F}) \longrightarrow H^o(X, \underline{G})$ by $f \longrightarrow e.f$. Let also e be the unit group, $e \longrightarrow H^o(X, \underline{F})$ the trivial homomorphism. Then, with respect to the natural operations of e in $H^o(X, \underline{F})$ and of the group $H^o(X; \underline{F})$ on the set $H^o(X, \underline{G})$, and the given neutre elements, the sequence

$$(5.2.6.) \qquad e \longrightarrow H^o(X, \underline{F}) \longrightarrow H^o(X, \underline{G}) \longrightarrow H^o(X, \underline{H}) \longrightarrow H^1(X, \underline{F})$$

is exact (definition 5.2.1.)

We have only to show exactness in $H^o(X, \underline{G})$ and in $H^o(X, \underline{F})$. For the former, this results from the definition of the map $H^o(X, \underline{F}) \longrightarrow H^o(X, \underline{G})$, and that we already noticed that two elements of $H^o(X, \underline{G})$ have the same image in $H^o(X, \underline{H})$ if and only if they are congruent under the group $H^o(X, \underline{F})$; for the latter, this means nothing else but that $H^o(X, \underline{F}) \longrightarrow H^o(X, \underline{G})$ is injective, which is trivial because of the faithfulness of $\underline{F}$.

Suppose now that $\underline{G}$ is itself a sheaf of groups, $\underline{F}$ a subsheaf of groups operating on $\underline{G}$ by the right regular representation, then $\underline{H}$ is the usual quotient $\underline{G}/\underline{F}$ (cf. 3.4.). Here $H^o(X, \underline{G})$ will be considered as a group; As such, first, it has a neutre element, so that corollary 1 applies; on the other hand, it operates on the left on $H^o(X, \underline{H})$ because $\underline{G}$ operates on the left on $\underline{H}$ (cf. 3.4.), and when now considering the sequence (5.2.6.) this additional structure will be kept in mind.

Corollary 2. If $\underline{G}$ is itself a sheaf of groups and $\underline{F}$ a subsheaf of groups (operating on $\underline{G}$ by the right regular representation), the sequence (5.2.6.) is still exact if we take into account the operations of $H^o(X, \underline{G})$ on $H^o(X, \underline{H})$ and if moreover we complete it by the map $H^1(X, \underline{F}) \longrightarrow H^1(X, \underline{G})$ arising from the injection $\underline{F} \longrightarrow \underline{G}$:

$$(5.2.7.) \qquad e \longrightarrow H^o(X, \underline{F}) \xrightarrow{i_o} H^o(X, \underline{G}) \xrightarrow{j_o} H^o(X, \underline{H}) \xrightarrow{\delta} H^1(X, \underline{F}) \xrightarrow{i_1} H^1(X, \underline{G}).$$

We have to show again exactness in the more restricted sense in $H^o(X, \underline{H})$, and moreover exactness in $H^1(X, \underline{F})$. For the first, it is obvious that the image of a $g \in H^o(X, \underline{G})$ is $g.e$ (e being the neûtre element of $H^o(X, \underline{H})$), so we only must prove that two elements h, h' of $H^o(X, \underline{H})$ have same image under δ if and only if they are congruent under $\underline{G}$. Now let (U_i) be an open covering of X fine enough that for each i we can find $g_i \in H^o(U_i, \underline{G})$ the image of which in $H^o(U_i, \underline{H})$ be $h|U_i$, then 5.2.4. can be written as

$$(5.2.8.) \qquad f_{ij} = g_i^{-1} g_j \quad \text{in } U_{ij}$$

and δh is defined by the cocycle (f_{ij}) thus defined. Taking (U_i) fine enough, we can suppose that in the same way we can find liftings $g_i' \in H^o(U_i, \underline{G})$ for the $h'|U_i$, so that the cocycle $(f_{ij}') = (g_i'^{-1} g_j')$ defines $\delta h'$. Suppose $h' = g.h$ with $g \in H^o(X, \underline{G})$, then we may take $g_i' = gg_i$ hence $f_{ij}' = f_{ij}$ and a fortiori $\delta h = \delta h'$; conversely, suppose $\delta h = \delta h'$, hence (as $H^1(\underline{U}, \underline{F}) \longrightarrow H^1(X, \underline{F})$ is injective) there exists a O-cochain (f_i) ($f_i \in H^o(U_i, \underline{F})$) such that $f_{ij}' = f_i f_{ij} f_j^{-1}$, i.e. $g_i'^{-1} g_j' = f_i g_i^{-1} g_j f_j^{-1}$, which can be written also $g_i f_i^{-1} g_i'^{-1} = g_j f_j^{-1} g_j'^{-1}$ on U_{ij}, and means that there is a unique section g of $\underline{G}$ such that $g|U_i = g_i f_i^{-1} g_i'^{-1}$, i.e. $g_i f_i^{-1} = gg_i'$ on U_i, which implies (by taking the images in $H^o(U_i, \underline{H})$ that $h = gh'$. - We now only have to prove exactness in $H^1(X, \underline{F})$. First we note that any element δh ($h \in H^o(X, \underline{H})$) has as image in $H^o(X, \underline{G})$ the trivial element, this resulting from the very form (5.2.8.) of the cocycle defining δh (and thus defining at the same time the image of δh in $H^1(X, \underline{G})$). Conversely, consider an element c of $H^1(X, \underline{F})$ having as image in $H^1(X, \underline{G})$ the trivial element, we prove that c is of the form δh. c is defined by a cocycle (f_{ij}) for a suitable open covering $\underline{U} = (U_i)$ of X, by hypothesis this defines the trivial element of $H^1(X, \underline{G})$ i.e. can be written $f_{ij} = g_i^{-1} g_j$, where $g_i \in H^o(U_i, \underline{G})$. Let h_i be the image of g_i in $H^o(U_i, \underline{H})$, then the preceding relation implies that $h_i = h_j$ in U_{ij}, thus there is a unique $h \in H^o(X, \underline{H})$ such that $h|U_i = h_i$, and by its definition we see that $\delta h = c$. This ends the proof of corollary 2.

Of course, in (5. 2. 7.), the map $H^o(X, \underline{F}) \longrightarrow H^o(X, \underline{G})$ is also the map which corresponds to the inclusion map $\underline{F} \longrightarrow \underline{G}$. The exact sequences (5. 2. 6.) and (5. 2. 7.), and their extensions to be defined below, will be referred to as the <u>exact cohomology sequences corresponding to the exact sequence of sheaves</u> $e \longrightarrow \underline{F} \xrightarrow{i} \underline{G} \xrightarrow{j} \underline{H} \longrightarrow e$. The reader may have remarked that information of the exactness type in our cohomology sequences was partially incomplete, for in (5. 2. 6.) we did not give a characterization of the image of the last homomorphism $H^o(X, \underline{H}) \longrightarrow H^1(X, \underline{F})$, and in (5. 2. 7.) we did not give a criterion whether two elements of $H^1(X, \underline{F})$ have same image in $H^1(X, \underline{G})$, neither a characterization of the image of $H^1(X, \underline{F}) \longrightarrow H^1(X, \underline{G})$. Now for the latter such a characterization has already been given in corollary to proposition 4. 6. 1. . The two former questions will be answered in section 5. 6.

<u>Functorial character of the coboundary operator and of the exact cohomology sequences.</u> Consider two exact sequences of same set of indices $A_o \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots$ and $A'_o \longrightarrow A'_1 \longrightarrow A'_2 \longrightarrow \ldots$ where the A_i and A'_i are, as explained above, either sets with a neutre element, or groups, which may then operate on the left or on the right on the consecutive set. We suppose that the structures involved in the two sequences are similar (i. e. if A_i is a group so is A'_i, and if A_i operates on the left respectively on the right on A_{i+1}, so does A'_i on A'_{i+1}). We then call <u>homomorphism</u> of the first exact sequence into the second a system of maps $u_i = A_i \longrightarrow A'_i$ such that (i) u_i transforms unit into unit (ii) if A_i and hence A'_i is a group, u_i is a homomorphism (iii) if moreover A_i operates on A_{i+1}, on the left for instance, then

$u_{i+1}(\rho_i(x_i). x_{i+1}) = \rho'_i(u_i x_i). (u_{i+1} x_{i+1})$ for any $x_i \in A_i$ and

$x_{i+1} \in A_{i+1}$ (where ρ_i and ρ'_i denote the given representations of

A_i respectively A'_i) (iv) commutativity holds in each of the diagrams

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \downarrow u_i & & \downarrow u_{i+1} \\ A'_i & \longrightarrow & A'_{i+1} \end{array} .$$

(This notion could be developed without assuming the given sequences to satisfy the exactness requirements.) When all sets A_i are groups and all mappings involved are homomorphisms, we find of course the usual notion of exact sequences of groups, and homomorphisms between such.

Proposition 5.2.2. Let $\underline{G}$ be a sheaf with faithful sheaf $\underline{F}$ of right operators, $\underline{G}'$ a sheaf with faithful sheaf $\underline{F}'$ of right operators, ρ a homomorphism of $\underline{F}$ into $\underline{F}'$ and u a homomorphism of the fibre space $\underline{G}$ into the fibre space $\underline{G}'$ compatible with ρ. Then the diagram

$$\begin{array}{ccc} H^0(X, \underline{G}/\underline{F}) & \xrightarrow{\delta} & H^1(X, \underline{F}) \\ \downarrow & & \downarrow \\ H^0(X, \underline{G}'/\underline{F}') & \xrightarrow{\delta} & H^1(X, \underline{F}') \end{array}$$

(where the vertical arrows are the functor maps stemming from the map $\underline{G}/\underline{F} \longrightarrow \underline{G}'/\underline{F}'$ obtained from u) is commutative.

The straightforward verification is left to the reader.

Corollary. Let $\underline{G}$ be a sheaf of groups, $\underline{F}$ a subsheaf of groups, and same for $\underline{G}'$ and $\underline{F}'$, let $\underline{H} = \underline{G}/\underline{F}$, $\underline{H}' = \underline{G}'/\underline{F}'$. Let u be a homomorphism of $\underline{G}$ into $\underline{G}'$ mapping $\underline{F}$ into $\underline{F}'$, and hence defining also a homomorphism of fibre spaces $\underline{G}/\underline{F} \longrightarrow \underline{G}'/\underline{F}'$, hence corresponding maps from the terms of the exact cohomology sequence of $\underline{F}$, $\underline{G}$ into those for $\underline{F}'$, $\underline{G}'$:

$$\begin{array}{ccccccccccc} e \longrightarrow & H^0(X, \underline{F}) & \longrightarrow & H^0(X, \underline{G}) & \longrightarrow & H^0(X, \underline{H}) & \longrightarrow & H^1(X, \underline{F}) & \longrightarrow & H^1(X, \underline{G}) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ e \longrightarrow & H^0(X, \underline{F}') & \longrightarrow & H^0(X, \underline{G}') & \longrightarrow & H^0(X, \underline{H}') & \longrightarrow & H^1(X, \underline{F}') & \longrightarrow & H^1(X, \underline{G}'). \end{array}$$

These maps define a homomorphism of the first exact sequence into the second.

In the following sections, for each new structure in the cohomology sequence a statement of the type of the preceding corollary on the "functorality" of the new construction should be made, and of course is indispensable for applications. But as these statements and their proof are entirely mechanical, we let them once for all to the reader.

5.3 Case when $\underline{F}$ is normal in $\underline{G}$.

Let $e \to \underline{F} \xrightarrow{i} \underline{G} \xrightarrow{j} \underline{H} \to e$ be an exact sequence of sheaves as in 5.2., where $\underline{G}$ is a sheaf of groups and $\underline{F}$ a subsheaf of groups, but suppose now $\underline{F}$ normal, so that $\underline{H} = \underline{G}/\underline{F}$ is itself a sheaf of groups, the map $\underline{G} \to \underline{H}$ being a homomorphism of sheaves of groups, and therefore defining a map $H^1(X, \underline{G}) \to H^1(X, \underline{H})$ which can be added after the last term of sequence (2.5.7.) so we get:

$$(5.3.1.) \qquad e \to H^o(X, \underline{F}) \xrightarrow{i_o} H^o(X, \underline{G}) \xrightarrow{j_o} H^o(X, \underline{H}) \xrightarrow{\delta} H^1(X, \underline{F})$$

$$\xrightarrow{i_1} H^1(X, \underline{G}) \xrightarrow{j_1} H^1(X, \underline{H}).$$

On the other hand, $H^o(X, \underline{H})$ is now also a group, and the map $j_o: H^o(X, \underline{G}) \to H^o(X, \underline{H})$ a group homomorphism, the operations of the first group on the set $H^o(X, \underline{H})$ is obtained by composing this homomorphism with the left regular representation of $H^o(X, \underline{H})$. We will now define a representation of the group $H^o(X, \underline{H})$ by permutations of $H^1(X, \underline{F})$. Let $h \in H^o(X, \underline{H})$, let $\underline{U} = (U_i)$ a covering fine enough that for each i there exists a $g_i \in H^o(X, \underline{G})$ the image of which in $H^o(U_i, \underline{H})$ be $h|U_i$, let $g = (g_i) \in C^o(\underline{U}, \underline{G})$, and define the operation $\rho(g)$ of g on $C^1(\underline{U}, \underline{F})$ by $\rho(g).(f_{ij}) = (g_i f_{ij} g_j^{-1})$; the second member is a cochain with coefficients in $\underline{F}$, and not only in $\underline{G}$, for $g_i f_{ij} g_j^{-1} = (g_i f_{ij} g_i^{-1})(g_i g_j^{-1})$, and the two right-hand factors are sections of $\underline{F}$, the first because $\underline{F}$ is normal in $\underline{G}$, the second because the image $g_i g_j^{-1}$ in $H^o(U_{ij}, \underline{H})$ is the unit section. It is obvious that $\rho(g)$ transforms a cocycle into a cocycle, and on the other hand transforms a cocycle (f'_{ij}) equivalent to (f_{ij}), i.e. of the form $(f_i f_{ij} f_j^{-1})$ (where $(f_i) \in C^o(\underline{U}, \underline{F})$) into a cocycle equivalent to $\rho(g).(f_{ij})$, for $\rho(g).(f'_{ij}) = (g_i f_i f_{ij} f_j^{-1} g_j^{-1}) = (f'_i g_i f_{ij} g_j^{-1} f'_j{}^{-1})$ which is equivalent to $\rho(g).(f_{ij}) = (g_i f_{ij} g_j^{-1})$ (we put $f'_i = g_i f_i g_i^{-1}$, which is a section of $\underline{F}$ on U_{ij}

because $\underline{F}$ is invariant). Therefore $\rho(g)$ defines an operation in the quotient $H^1(\underline{U}, \underline{F})$, which will also be denoted by $\rho(g)$. Moreover, if for the same h we take another system of lifting sections $g' = (g_i')$, then $\rho(g)$ and $\rho(g')$ in $H^1(\underline{U}, \underline{F})$ are the same, for we must have $g_i' = f_i g_i$ with $f_i \in H^o(U_i, \underline{F})$, hence $\rho(g').(f_{ij}) = (f_i g_i f_{ij} g_j^{-1} f_j^{-1})$ defines the same element of $H^1(\underline{U}, \underline{F})$ then $\rho(g).(f_{ij}) = (g_i f_{ij} g_j^{-1})$. Therefore, we may write $\rho(h)$ for the operation in $H^1(\underline{U}, \underline{F})$ thus defined. Obviously $\rho(e)$ is the identity, and $\rho(hh') = \rho(h)\rho(h')$, whenever h and h' are such that they can both be lifted on each U_i. Moreover, if $\underline{V}$ is a covering finer than $\underline{U}$, it is still obvious that the operations $\rho(h)$ on $H^1(\underline{U}, \underline{F})$ and on $H^1(\underline{V}, \underline{F})$ agree. We have therefore defined a representation of $H^o(X, \underline{H})$ by permutations of the set $H^1(X, \underline{F})$, and this supplementary structure will now be kept in mind when speaking of the sequence (5. 3. 1.).

Proposition 5. 3. 1. Two elements of $H^1(X, \underline{F})$ have same image in $H^1(X, \underline{G})$ if and only if they are congruent under the group of permutations $\rho(H^o(X, \underline{H}))$ defined above; an element of $H^1(X, \underline{G})$ is in the image of $H^1(X, \underline{F})$ if and only if it is in the kernel of the map j_i: $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$. The sequence (5. 3. 1.), provided with the different structures explained above, is still exact, (definition 5. 2. 1.).

Taking into account proposition 5. 2. 1. corollary 2, exactness is equivalent to the first two statements of the proposition, together with the formula

$$(5.3.2.) \qquad \delta h = \rho(h^{-1}).e.$$

The latter results trivially from the definitions. Consider now elements $c \in H^1(X, \underline{F})$ and $h \in H^o(X, \underline{H})$, then $\rho(h).c$ is defined, for a sufficient fine covering $\underline{U} = (U_i)$ of X, by a cochain $(g_i f_{ij} g_j^{-1})$, where (f_{ij}) is a cocycle defining c, and the $g_i \in H^o(U_i, \underline{G})$ lift $h|U_i$; the image of c respectively $\rho(h).c$ in $H^1(X, \underline{G})$ are defined by the same cocycles (f_{ij}) and $(g_i f_{ij} g_j^{-1})$ as c and $\rho(h).c$, and thus obviously define the same element of

$H^1(X, \underline{G})$. Conversely, suppose $c, c' \in H^1(X, \underline{F})$ have same image in $H^1(X, \underline{G})$; for a sufficiently fine covering $\underline{U} = (U_i)$, c and c' are defined respectively by cocycles (f_{ij}) and (f'_{ij}) on $\underline{U}$, and as these cocycles define the same element of $H^1(X, \underline{G})$ we must have $(f'_{ij}) = (g_i f_{ij} g_j^{-1})$ for a suitable $(g_i) \in C^o(\underline{U}, \underline{G})$. But $f'_{ij} g_j = g_i f_{ij}$ implies, if h_i is the image of g_i in $C^o(U_i, \underline{H})$, that h_i and h_j are the same on U_{ij}, hence there exists a unique $h \in H^o(X, \underline{H})$ such that $h_i = h|U_i$ for each i, and by definition of $\rho(h)$ we see that $c' = \rho(h).c$. - We now only have to prove that the image of $H^1(X, \underline{F})$ in $H^1(X, \underline{G})$ is the kernel of $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$; this could be done very easily directly, but we can also remark that for a fibre space E with structure sheaf $\underline{G}$, the associated fibre space of type $\underline{G}/\underline{F}$ is a principal sheaf with structure sheaf $\underline{H}$ the class of which is the image c' of the class c of E under the map $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$; in order that the structure sheaf of E may be <u>restricted</u> to $\underline{F}$, it is necessary and sufficient that the associated bundle admits of a section (proposition 4.6.1.) but this is also the condition for this associated bundle to be trivial (proposition 4.6.1., corollary). This completes the proof.

$\underline{G}$ operates in itself by interior automorphisms (4.5., example c.), in particular every section $g \in H^o(X, \underline{G})$ defines an automorphism of the sheaf of groups $\underline{G}$, which will be called the <u>interior automorphism</u> defined by g, and denoted by $\sigma(g)$. As $\underline{F}$ is invariant, $\sigma(g)$ defines also an automorphism of the sheaf of groups $\underline{F}$, and hence induces a bijective map of $H^1(X, \underline{F})$ into itself. We thus get a representation of $H^o(X, \underline{G})$ by permutations of the set $H^1(X, \underline{F})$, which will still be denoted by $g \longrightarrow \sigma(g)$.

<u>Proposition 5.3.2</u>. Let σ be the representation of $H^o(X, \underline{G})$ by permutations of $H^1(X, \underline{F})$ deduced from the representation by interior automorphisms of $\underline{G}$. Let $g \in H^o(X, \underline{G})$, and $h = j_o g$ its image in $H^o(X, \underline{H})$, then

(5.3.3.) $$\rho(j_o g) = \sigma(g).$$

This results trivially from the definitions. An important particular case is

<u>Corollary</u>. Suppose that j_o: $H^o(X, \underline{G}) \longrightarrow H^o(X, \underline{H})$ is surjective, and that $H^o(X, \underline{G})$ is the product of $H^o(X, \underline{F})$ and the centralisator of $H^o(X, \underline{F})$. Then $H^o(X, \underline{H})$ operates trivially on $H^1(X, \underline{F})$, i.e. the map i_1: $H^1(X, \underline{F}) \longrightarrow H^1(X, \underline{G})$ is injective.

In section 5.6. we will give a criterion for two elements of $H^1(X, \underline{G})$ to have the same image in $H^1(X, \underline{H})$, and in section 5.7. we will (at least when $\underline{F}$ is abelian) characterize the image of the map j_1: $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$.

5.4. <u>Case when F is normal and abelian.</u>

In this case, as recalled in 5.1., $H^1(X, \underline{F})$ is itself an abelian group, the product of course being defined in terms of the product of two cocycles $(f_{ij}).(f'_{ij}) = (f_{ij}f'_{ij})$ (which is still a cocycle because $\underline{F}$ is abelian). We will see however that the map δ: $H^o(X, \underline{H}) \longrightarrow H^1(X, \underline{F})$ is in general <u>not</u> a homomorphism. Moreover, $H^1(X, \underline{F})$ does <u>not</u> operate on $H^1(X, \underline{G})$ as could be expected from the foregoing, and neither can we define at this stage a second coboundary map $H^1(X, \underline{H}) \longrightarrow H^2(X, \underline{F})$; for this $\underline{F}$ should be supposed in the center of $\underline{G}$ as in the next section. In section 5.8. however, we will define in the present context a suitable substitute for the second coboundary map. Here we will develop a convenient interpretation of the representation ρ of $H^o(X, \underline{H})$ defined in 5.3.

$\underline{F}$ being abelian, the germ of automorphism of $\underline{F}$ induced by the germ of interior automorphism defined by a $g \in \underline{G}$ which is in $\underline{F}$, is trivial, therefore for general $g \in \underline{G}$ the germ of automorphism of $\underline{F}$ defined by g depends only on the image of g in $\underline{H}$, so that <u>we have thus defined a representation σ of the sheaf H into the sheaf of automorphisms of the abelian sheaf F</u>; from this results <u>a representation, still denoted σ, of the group $H^o(X, H)$ into the group of automorphisms of the abelian sheaf F</u>, which again defines a <u>representation σ of $H^o(X, \underline{H})$ by automorphisms of the abelian group $H^1(X, \underline{F})$</u>. Let now, for any $c \in H^1(X, \underline{F})$, T(c) stand for the operation $c' \longrightarrow c' + c$ of translation by c.

Proposition 5.4.1. The coboundary operator δ, the represenation ρ of $H^o(X, \underline{H})$ by permutations of the set $H^1(X, \underline{F})$ considered in 5.3., and the representation σ of $H^o(X, \underline{H})$ by automorphisms of the abelian group $H^1(X, \underline{F})$ defined just before, are related by

$$(5.4.1.) \qquad \rho(h) = T(\delta(h^{-1}))\sigma(h) \qquad (h \in H^o(X, \underline{H})).$$

This formula results at once from the definitions and the formula $(g_i f_{ij} g_j^{-1}) = (g_i f_{ij} g_i^{-1})(g_i g_j^{-1})$. It shows that ρ is representation of the group $H^o(X, \underline{H})$ by affine transformation in the abelian group $H^1(X, \underline{F})$, the "linear representation" associated with it being σ. In order to simplify the notations, let $\delta'(h)$ for $\delta(h^{-1})$, and write the composition law in the abelian group $H^1(X, \underline{F})$ additively as usual, then using formula (5.4.1.) and expressing that ρ is a representation, we get

$$(5.4.2.) \qquad \delta'(hh') = \delta' h + \sigma(h).\ \delta' h' \qquad \delta' e = 0$$

$$(h, h' \in H^o(X, \underline{H}), \ \delta'(h) = \delta(h^{-1})).$$

Using classical notions of the cohomology of groups (see for instance the forthcoming book "Homological Algebra" by Cartan-Eilenberg), these formulae mean that δ' is a normalized 1 cochain of the group $H^o(X, \underline{H})$, with coefficients in the abelian group $H^1(X, \underline{F})$ where $H^o(X, \underline{H})$ operates by σ. - We see on (5.4.2.) that δ is a homomorphism if and only if σ is the identity on the image of δ, which in general of course is not the case.

5.5 Case when $\underline{F}$ is in the center of $\underline{G}$.

Then of course $\underline{F}$ is invariant and abelian, so that 5.4. applies. Moreover here the representation σ of $\underline{H}$ into the sheaf of germs of automorphisms of $\underline{F}$ is trivial, a fortiori the same is true for the corresponding representation of $H^o(X, \underline{H})$ by automorphisms of $H^1(X, \underline{F})$. Hence by formula (5.4.2.) and (5.4.1.) we get:

Proposition 5.5.1. If $\underline{F}$ is in the center of $\underline{G}$, the coboundary map δ

is a homomorphism and the representation ρ of $H^o(X, \underline{H})$ by permutations of $H^1(X, \underline{F})$ is obtained by composing the representation $\delta' = -\delta$ with the regular representation of the abelian group $H^1(X, F)$.

Corollary. The kernel of the map j_1: $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$ is canonically isomorphic with the quotient group $H^1(X, \underline{F}) / \delta(H^o(X, \underline{H}))$.

This results at once from the foregoing, and exactness of the sequence (5. 3. 1.)

Let $\underline{U} = (U_i)$ be an open covering of X, $f = (f_{ij})$ a cocycle of $\underline{F}$ and $g = (g_{ij})$ a cocycle of $\underline{G}$ relative to this covering, then we see at once that $fg = (f_{ij}g_{ij})$ is again a cocycle, and the class of the latter in $H^1(\underline{U}, \underline{G})$ depends only of the classes c and c' of f and g in $H^1(\underline{U}, \underline{F})$ respectively $H^1(\underline{U}, \underline{G})$, and can therefore be denoted by $\rho(c).c'$. It is obvious then that $\rho(e)$ is the identity operation on $H^1(\underline{U}, \underline{G})$ and $\rho(c_1c_2) = \rho(c_1)\,\rho(c_2)$, so that we get a representation of $H^1(\underline{U}, \underline{F})$ by permutations of the set $H^1(\underline{U}, \underline{G})$. If $\underline{V}$ is an open covering of X finer than $\underline{U}$, and if we identify $H^1(X, \underline{U})$ with a subset of $H^1(X, \underline{V})$, we see also at once that for $c \in H^1(\underline{U}, \underline{F})$ the operations $\rho(c)$ defined on $H^1(\underline{U}, \underline{G})$ and $H^1(\underline{V}, \underline{G})$ agree, so that we get a in a natural way a representation of the abelian group $H^1(X, \underline{F})$ by permutations of the set $H^1(X, \underline{G})$.

Proposition 5. 5. 2. Suppose $\underline{F}$ in the center of $\underline{G}$. In order that two elements of $H^1(X, \underline{G})$ have same image in $H^1(X, \underline{H})$, it is necessary and sufficient that they be congruent under the group of permutations $\rho(H^1(X, \underline{F}))$ defined above; in other words, the cohomology sequence (5. 3. 1.) still remains exact (definition 5. 2. 1.) if we add to its structures the operations of $H^1(X, \underline{F})$ on $H^1(X, \underline{G})$.

Of course, both statements are equivalent, for we have of course

(5. 5. 1.) $\qquad i_1 c = \rho(c). e \qquad (c \in H^1(X, \underline{F}))$.

The proof of proposition 5. 5. 2. is of the same entirely mechanical character as in the preceding sections, and therefore left to the reader.

As stated already, we can complete here the exact sequence $H^1(X, \underline{H})$ (at least in general cases as: X paracompact) by a <u>second coboundary map</u> $H^1(X, \underline{H}) \longrightarrow H^2(X, \underline{F})$; as we will define an analogous map under more general conditions in section 5.7. below, the reader is referred to that section.

5.6. <u>Transformation of the exact sequence of sheaves</u>.

Let again

$$(5.6.1.) \qquad e \longrightarrow \underline{F} \xrightarrow{i} \underline{G} \xrightarrow{j} \underline{H} \longrightarrow e$$

be an exact sequence of sheaves, where $\underline{F}$ and $\underline{G}$ are sheaves of groups and i is a homomorphism of sheaves of groups, so that $\underline{F}$ can be identified with a subsheaf of groups of $\underline{G}$ (not necessarily normal, i.e. $\underline{H}$ itself is not assumed to be a sheaf of groups). Now $\underline{G}$ operates on itself by interior automorphisms, and this representation of $\underline{G}$ induces a representation of $\underline{F}$ by germs of interior automorphisms of $\underline{G}$; let $\underline{G}(\sigma)$ be $\underline{G}$ considered as a group bundle with sheaf of left operators $\underline{F}$. $\underline{F}$ as a subsheaf is stable under the operations of $\sigma(\underline{F})$, and therefore we can consider $\underline{F} = \underline{F}(\sigma)$ as group bundle with sheaf of left operators $\underline{F}$ (the representation of $\underline{F}$ thus obtained being of course the "adjoint representation" by interior automorphisms). Of course, the operations of $\underline{F}$ on $\underline{H}$ obtained by passing to the quotient are trivial, we write however $\underline{H}(\sigma)$ for $\underline{H}$ provided with $\underline{F}$ acting (trivially) as sheaf of left operators. (5.6.1.) can now be written as an exact sequence of sheaves with sheaf of left operators $\underline{F}$ (the homomorphisms i, j being compatible with these structures):

$$(5.6.2.) \qquad e \longrightarrow \underline{F}(\sigma) \xrightarrow{i} \underline{G}(\sigma) \xrightarrow{j} \underline{H}(\sigma) \longrightarrow e.$$

Let now E be any fibre space with structure sheaf $\underline{F}$, consider the fibre spaces associated to E and the different terms of (5.6.2.), (cf. 4.4.), then the latter sequence is transformed into an exact sequence:

$$(5.6.3.) \qquad e \longrightarrow \underline{F}(\sigma)^E \xrightarrow{i} \underline{G}(\sigma)^E \xrightarrow{j} \underline{H}(\sigma)^E \longrightarrow e$$

where of course the first two terms are still sheaves of groups and the first homomorphism a homomorphism of sheaves of groups (cf. 4.5.b.), the last term being here canonically isomorphic to $\underline{H}$ (because $\underline{F}$ operates trivially on $\underline{H}$). Let us now write the exact cohomology sequence relative to (5.6.3.):

$$\text{(5.6.4.)} \qquad e \longrightarrow H^o(X, \underline{F}(\sigma\!\!-)^E) \longrightarrow H^o(X, \underline{G}(\sigma\!\!-)^E) \longrightarrow H^o(X, \underline{H}(\sigma\!\!-)^E)$$

$$\begin{array}{ccc} \longrightarrow H^1(X, \underline{F}(\sigma\!\!-)^E) & \longrightarrow & H^1(X, \underline{G}(\sigma\!\!-)^E) \\ \downarrow & & \downarrow \\ H^1(X, \underline{F}) & \longrightarrow & H^1(X, \underline{G}). \end{array}$$

The vertical arrows are the canonical isomorphisms established in proposition 4.5.2. This of course applies without further comment to the first arrow; for the second, we simply notice that, in virtue of the transitivity principle explained at the end of section 4.4., $\underline{G}(\sigma\!\!-)^E$ is canonically isomorphic to $\underline{G}(\sigma\!\!-)^{E'}$, where now $\underline{G}(\sigma\!\!-)$ stands for $\underline{G}$ with $\underline{G}$ as sheaf of left operators (acting by interior automorphisms) and where E' is the fibre space with structure sheaf $\underline{G}$ associated to E. Thus the second vertical arrow is defined in virtue of the same proposition 4.5.2., and moreover it is easily checked that the square of homomorphisms thus obtained is commutative (this being contained in an evident functor property of the isomorphism of proposition 4.5.2.; for obvious reasons, one never states <u>all</u> functor properties which are actually currently needed, relying on the somewhat vague but intuitive fact that all maps defined in a reasonably "canonic" way have the obvious functor properties, i.e. that all diagrams constructed with such maps are commutative; which truth, once admitted, will save a considerable amount of ink in many an expository work). Therefore, in the exact cohomology sequence, we may replace the last two terms by $H^1(X, \underline{F})$ and $H^1(X, \underline{G})$, provided we take now, as new unit elements, the class c of E in $H^1(X, \underline{F})$ respectively its image c' in $H^1(X, \underline{G})$, according to proposition 4.5.2. Interpretation of the first three terms of the cohomology sequence is readily obtained: the first two are the groups Aut(E) and Aut(E') of automorphisms of E respectively the associated E' with structure sheaf $\underline{G}$

(in virtue of proposition 4.5.1.), the third is identic to $H^o(X, \underline{H})$, so that we finally get the

Proposition 5.6.1. Given an exact sequence of sheaves (5.6.1.), where i is a homomorphism of sheaves of groups, and a fibre space E with structure sheaf $\underline{F}$, let E' be the associated fibre space with structure sheaf $\underline{G}$. Then we have a (canonical) exact sequence:

$$(5.6.5.) \qquad e \longrightarrow \mathrm{Aut}(E) \xrightarrow{i_o} \mathrm{Aut}(E') \xrightarrow{j_o} H^o(X, \underline{H}) \xrightarrow{\delta^E} H^1(X, \underline{F}) \xrightarrow{i_1} H^1(X, \underline{G})$$

where we take as unit elements in $H^1(X, \underline{F})$ and $H^1(X, \underline{G})$ the class c of E respectively the class c' of E'.

Of course, "exact sequence" involves also the operations of Aut(E') on $H^o(X, \underline{H})$ defined in 5.2., and moreover, if $\underline{F}$ is normal (and hence $\underline{F}(\sigma)^E$ of course normal in $\underline{G}(\sigma)^E$), the operations of $H^o(X, \underline{H})$ in $H^1(X, \underline{F})$ as defined in 5.3. We let the interpretation of the first homomorphisms of (5.6.5.) to the reader; it should be noted that the coboundary operator in (5.6.5.) is relative to (5.6.3.) and not to (5.6.1.), and therefore has been denoted by δ^E rather than δ. As a consequence of exactness in $H^1(X, \underline{F})$, we get a result promised in 5.2.:

Corollary. Under the conditions of proposition 5.6.1., the elements of $H^1(X, \underline{F})$ which have the same image c' in $H^1(X, \underline{G})$ as c are exactly those which are in the image of $H^o(X, \underline{H})$ by the coboundary operators δ^E associated with E. Therefore, the set of these elements is in one-to-one correspondence with the set of classes of intransitivity of $H^o(X, \underline{H})$ under the operations of the group Aut (E').

(The last statement of course results from exactness of (5.6.5.) in $H^o(X, \underline{H})$).

Suppose now that $\underline{F}$ is normal in $\underline{G}$. Let now $\underline{G}(\sigma)$ stand for $\underline{G}$ provided with $\underline{G}$ as sheaf of left automorphisms (by interior automorphisms), then $\underline{F}$ being normal is stable under the operations of $\underline{G}$ and we can consider $\underline{F}$ as group bundle with $\underline{G}$ as a sheaf of left automorphisms, denoted by $\underline{F}(\sigma)$. $\underline{H}$ is now itself a sheaf of groups, and the operations of $\underline{G}$ on $\underline{G}$ pass to the quotient, so that $\underline{H}$ can be considered as group bundle with $\underline{G}$ as sheaf of left automorphisms, denoted by $\underline{H}(\sigma)$. With these new notations, the exact sequence (5.6.2.) still holds. Let now E' be a fibre space with structure sheaf $\underline{G}$, then (5.6.2.) gives rise to an exact sequence of associated sheaves of groups:

$$(5.6.6.) \qquad e \longrightarrow F(\sigma)^{E'} \xrightarrow{i} \underline{G}(\sigma)^{E'} \xrightarrow{j} \underline{H}(\sigma)^{E'} \longrightarrow e.$$

If E" is the fibre space with structure sheaf $\underline{H}$ associated to E', the last term in this sequence is canonically isomorphic with $\underline{H}(\sigma)^{E''}$, where now $\underline{H}(\sigma)$ stands for $\underline{H}$ provided with $\underline{H}$ as sheaf of left automorphisms ($\underline{H}$ acting by interior automorphisms). (5.6.6.) now gives rise to an exact sequence of cohomology (proposition 5.3.1.):

$$(5.6.7.)$$

$$\begin{tikzcd}
e \arrow[r] & H^o(X, \underline{F}(\sigma)^{E'}) \arrow[r] & H^o(X, \underline{G}(\sigma)^{E'}) \arrow[r] & H^o(X, \underline{H}(\sigma)^{E'}) \\
\arrow[r] & H^1(X, \underline{F}(\sigma)^{E'}) \arrow[r] & H^1(X, \underline{G}(\sigma)^{E'}) \arrow[r] \arrow[d] & H^1(X, \underline{H}(\sigma)^{E'}) \arrow[d] \\
& & H^1(X, \underline{G}) \arrow[r] & H^1(X, \underline{H})
\end{tikzcd}$$

where the two vertical arrows are defined as above and give rise to the same remarks. If E' is obtained from a fibre space E with structure sheaf $\underline{E}$ by extension of the structure sheaf to $\underline{G}$, then $\underline{H}(\sigma)^{E'}$ is of course canonically isomorphic to $\underline{H}$, and $\underline{F}(\sigma)^{E'}$ to the $\underline{F}(\sigma)^{E}$ considered in the beginning, so that (5.6.7.) is nothing but (5.6.5.) with the supplementary map $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$ added. In the general case however, $H^1(X, F(\sigma)^{E'})$ is in no direct relation with $H^1(X, \underline{F})$, and has no interpretation other than its definition. The second and third term in (5.6.7.) are canonically isomorphic with Aut(E') respectively Aut(E") in virtue of proposition 4.5.1. (making use of the isomorphism $\underline{H}(\sigma)^{E'} = \underline{H}(\sigma)^{E''}$). As to $H^o(X, \underline{F}(\sigma)^{E'})$, it appears as a privileged normal subgroup of Aut(E') corresponding to the given normal subsheaf $\underline{F}$ of the structure sheaf $\underline{G}$ of E'; it can be defined alternatively as the set of those automorphisms of E' which are expressed, when E is given by a system (g_{ij}) of coordinate transforms as explained in 4.3., by a system (g_i) (with $g_{ij} = g_i g_{ij} g_j^{-1}$) where g_i is a section of $\underline{F}$ (and not only $\underline{G}$) on U_i. We will denote this subgroup of Aut(E') by $\mathrm{Aut}_{\underline{F}}(E')$ and call it the group of $\underline{F}$-automorphisms of $\underline{E}$; when the structure sheaf can be restricted to $\underline{F}$, $\mathrm{Aut}_{\underline{F}}(E')$ is nothing else than the group of automorphisms of E' provided with the structure sheaf $\underline{F}$. We thus get the:

Proposition 5.6.2 Consider an exact sequence (5.1.1.) of sheaves of groups $\underline{F}$, $\underline{G}$, $\underline{H}$, let E' be a fibre space with structure sheaf $\underline{G}$, E" the associated fibre space with structure sheaf $\underline{H}$. Then we have a (canonical) exact sequence:

$$(5.6.7.) \qquad e \longrightarrow \mathrm{Aut}_{\underline{F}}(E') \xrightarrow{i_o} \mathrm{Aut}(E') \xrightarrow{j_o} \mathrm{Aut}(E'')$$
$$\xrightarrow{\delta E'} H^1(X, \underline{F}(\sigma)^{E'}) \xrightarrow{i_1} H^1(X, \underline{G}) \xrightarrow{j_1} H^1(X, \underline{H})$$

where $\mathrm{Aut}_{\underline{F}}(E')$ is the group of $\underline{F}$-automorphisms of E, and where we take as unit elements in $H^1(X, \underline{G})$ and $H^1(X, \underline{H})$ the class c' of E' respectively the class c" of E".

Of course the first two maps are group homomorphisms, and the exactness statement is understood with respect to the structures referred to in proposition 5.3.1., in particular takes into account the natural representation, now denoted by $\rho^{E'}$, of Aut(E") by permutations of $H^1(X, \underline{F}(\sigma)^{E'})$. Thus we get a result promised in section 5.3.:

Corollary. Under the conditions of proposition 5.6.2., the elements of $H^1(X, \underline{G})$ which have the same image c" as c' are exactly those which are in the image of $H^1(X, \underline{F}(\sigma)^{E'})$ in (5.6.7.). Therefore, the set of these elements is in one-to-one correspondence with the set of classes of intransitivity in $H^1(X, \underline{F}(\sigma)^{E'})$ under the group of permutations $\rho^{E'}(\mathrm{Aut}(E''))$.

This gives us the characterization of the inverse image of an element of $H^1(X, \underline{H})$ when this inverse image is non-empty. Together with corollary to proposition 5.6.1., it allows to a certain extent to reduce the classification of fibre spaces with structure sheaf $\underline{G}$ to the classification with structure sheaves $\underline{F}$ and $\underline{H} = \underline{G}/\underline{F}$.

The case $\underline{F}$ normal *abelian* is particularly simple. As explained in 5.4., the representation of $\underline{G}$ by germs of automorphisms of $\underline{F}$ is then induced by a representation σ of $\underline{H}$, therefore in virtue of the transitivity principle $\underline{F}(\sigma)^{E'}$ is canonically isomorphic to $\underline{F}(\sigma)^{E''}$, where now $\underline{F}(\sigma)$ stands for $\underline{F}$ with $\underline{H}$ as sheaf of germs of left automorphisms, and therefore $\underline{F}(\sigma)^{E''}$ *depends only on the fibre space E'' with structure sheaf $\underline{H}$ associated to E'*. On the other hand, $H^1(X, \underline{F}(\sigma)^{E'})$ is an abelian group, which we will denote by $M(E'')$ or simply M, and we saw in 5.4. that the operations of $\mathrm{Aut}(E'')$ on M are affine and can be expressed in terms of the coboundary map $\delta^{E'}: \mathrm{Aut}(E'') \dashrightarrow M$ and the representation σ of $\mathrm{Aut}(E'')$ by automorphisms of M associated with the exact sequence (5.6.6.), so that the inverse image of the class c'' of E'' appears as the quotient of the abelian group M by the group $\rho^{E'}(\mathrm{Aut}(E''))$ of affine transformations of M. However, though M does depend only on E'', the representation $\rho^{E'}$ depends in general on the fibre space E' (with structure sheaf $\underline{G}$) we started with. We will rapidly give some precisions in this direction. First, in virtue of proposition 4.4.1, there is a canonical representation of Aut (E'') by automorphisms of $\underline{F}(\sigma)^{E''}$, and it is checked at once that this is the same as the representation σ associated with the exact sequence (5.6.6.), where $\underline{F}(\sigma)^{E'} = \underline{F}(\sigma)^{E''}$ is abelian invariant. Therefore *the representation σ of $\mathrm{Aut}(E'')$ by automorphisms of the group $M = H^1(X, \underline{F}(\sigma)^{E''})$ considered above is also the natural representation stemming from the operations of $\mathrm{Aut}(E'')$* on $\underline{F}(\sigma)^{E''}$; it therefore depends only on E''. As we have (5.4.1.):

$$\rho^{E'}(g) = T(\delta^{E'}(g^{-1})).\sigma(g) \qquad (g \in \mathrm{Aut}(E'')) \tag{5.6.8.}$$

(where $T(m)$ is the translation operator in M corresponding to $m \in M$), in order to see how $\rho^{E'}(g)$ depends on the manner in which E'' has been obtained from a fibre space E' with structure sheaf $\underline{G}$, we only need investigate the behavior of the coboundary operator $\delta^{E'}$. More precisely, suppose given a priori a fibre space E" with structure sheaf $\underline{H}$ (to fix the ideas, all fibre spaces will be supposed principal sheaves), the class c" of which is in the image of $H^1(X, \underline{G})$, i.e. such that there is at least one fibre space E' with structure sheaf $\underline{G}$, such that the associated fibre space $\overline{E}'$ with structure sheaf $\underline{H}$ be isomorphic to E". For any such E' and any isomorphism u: $\overline{E}' \longrightarrow E''$, the representation $\rho^{E'}$ of Aut($\overline{E}'$) by automorphisms of M(E') and the map $\delta^{E'}$: Aut(E') $\longrightarrow$ M(E') are transformed into a representation ρ^u of Aut(E") by automorphisms of M(E") and a map δ^u : Aut(E") $\longrightarrow$ M(E") and clearly we have, in virtue of (5. 6. 8.):

$$\text{(5. 6. 9.)} \qquad \rho^u(g) = T(\delta^u(g^{-1})).\sigma(g) \qquad (g \in \mathrm{Aut}(E'')).$$

As already noticed, if we introduce $\rho'^u(g) = \rho^u(g^{-1})$, ρ'^u is a normalized cochain of degree 1 of Aut(E") with coefficients in M(E") considered as a group with Aut(E") as group of operators (under σ). On the other hand, in virtue of proposition 5. 6. 2., and using the isomorphism of M(E') onto M(E") corresponding to u, we have an onto isomorphism:

$$\text{(5. 6. 10.)} \qquad i^u\text{: } M(E'')/\rho^u(\mathrm{Aut}(E'')) \longrightarrow j_1^{-1}(c'')$$

of the set of classes of intransitivity of M(E") under ρ^u onto the set of classes $c' \in H^1(X, \underline{G})$ such that $j_1 c' = c''$. We now investigate how δ'^u (and hence ρ^u) and i^u vary with u (for a fixed $c'' \in H^1(X, \underline{H})$). First, let φ be an isomorphism of E" onto a second fibre space E_1'' with structure sheaf $\underline{H}$, hence a corresponding isomorphism $g \longrightarrow \varphi g \varphi^{-1}$ of Aut(E") onto Aut(E_1'') and an isomorphism $\underline{F}(\sigma)^{E''} \longrightarrow \underline{F}(\sigma)^{E_1''}$ defining an isomorphism $\sigma(\varphi)$: M(E") $\longrightarrow$ M(E_1''). Then it is evident by "transport de structure" that

$$\text{(5. 6. 11.)} \qquad \rho^{\varphi u}(g) = \sigma(\varphi)\rho^u(\varphi^{-1} g \varphi)\sigma(\varphi)^{-1} \qquad (g \in \mathrm{Aut}(E_1''))$$

which implies

(5.6.12.) $\quad \delta' \varphi^u = (\delta'^u)_\varphi$ where $(\delta'^u)_\varphi(g) = \sigma(\varphi)\,\delta'^u(\varphi^{-1} g \varphi)$

$(g \in \mathrm{Aut}(E''_1))$.

In virtue of (5.6.11.) $\sigma(\varphi)$ transforms $\rho^u(\mathrm{Aut}(E''))$ into $\rho^{\sigma u}(\mathrm{Aut}(E''_1))$, and hence defines a map on the sets of classes of intransitivity, and of course the diagram

(5.6.13.)
$$\begin{array}{ccc} M(E'') / \rho^u(\mathrm{Aut}(E'')) & \longrightarrow & j_1^{-1}(c'') \\ \Big\downarrow \sigma(\varphi) & & \| \\ M(E''_1) / \rho^u(\mathrm{Aut}(E''_1)) & \longrightarrow & j_1^{-1}(c'') \end{array}$$

is commutative. Let us take now $E''_1 = E''$, hence $\varphi \in \mathrm{Aut}(E'')$. Then the second part of (5.6.12.) defines quite generally operations of the group $\Gamma = \mathrm{Aut}(E'')$ on the group of normalized cocycles in $M = M(E'')$ (on which Γ operates by σ), $\varphi \in \Gamma$ defining the operation $\delta' \to \delta'_\varphi$ defined by this formula; and it is seen at once that the cocycles homologous to zero form a stable subgroup, so that Γ operates on the quotient $H^1(\Gamma, M)$ - all this being of course defined whenever a group Γ operates in an abelian group M: then Γ operates in a natural way on all groups $H^i(\Gamma, M)$. However, it is easily checked that these operations are always trivial. It follows at once that $H^1(\mathrm{Aut}(E''), M(E''))$ is a set which is intrinsically determined by the class c'', more precisely: if E'' and E''_1 are of the same class c'', then the bijective maps $H^1(\mathrm{Aut}(E''), M(E'')) \to H^1(\mathrm{Aut}(E''_1), M(E''_1))$ defined by the isomorphisms of E'' onto E''_1 are all the same, thus the two quotient-sets above are canonically isomorphic (and these isomorphisms satisfy of course the obvious transitivity property). So we may denote this set symbolically by $H^1(\mathrm{Aut}(c''), M(c''))$. Now formula (5.6.12.) shows that, given E' and E'' and an isomorphism u of E' onto E'', the element in $H^1(\mathrm{Aut}(c''), M(c''))$ defined by the cocycle δ'^u does not depend on E'' or u, but only on E', and of course even only on the class c' of E'. We will now see that it does not even depend on the choice of c', i.e. does not change if we replace E' by another fibre space E'_1 with structure sheaf $\underline{G}$, class c'_1 such that $j_1(c'_1) = c''$. Indeed, in virtue of exactness of (5.6.7.), such a fibre space is isomorphic to the fibre space associated with a fibre space E with structure sheaf $\underline{F}(\sigma)^{E'}$, so that we can suppose that $E'_1 = \underline{G}(\sigma)^E$ is this associated fibre space. Then we have a canonical isomorphism u of $\underline{H}(\sigma)^{E'_1} = E'_1$ onto $E'' = E'$, hence a corresponding cocycle δ'^u: $\mathrm{Aut}(E'') \longrightarrow M(E'') = M$; it is seen at once that this cocycle does only depend on the class $c \in H^1(X, \underline{F}(\sigma)^E) = M$ of E, and may therefore be written δ'^c. This is one of the cocycles which correspond to the different isomorphisms of E'_1 onto E". We now compare $\delta'c$ with the cocycle $\delta'^{E'}$ of (5.6.7.) (i.e. the δ'^v corresponding to the identity isomorphism $E' \longrightarrow E''$). An easy direct computation by coordinate transforms, which being entirely mechanical is left to the reader, shows:

$$(5.6.14.) \qquad (\delta'^c - \delta'^{E'})(g) = \sigma(g).c - c \qquad (g \in \mathrm{Aut}(E''))$$

i.e. $\delta'^c - \delta'^{E'}$ is the "coboundary" of the normalized O-cochain $c \in M$. (It may be remarked that this means also that the affine representation ρ^c corresponding to δ'^c is obtained by transforming $\rho^{E'}$ by the translation T(c) with $c \in M$). So (5.6.12.) together with (5.6.14.) show that for given E", the cocycles δ'^u corresponding to all possible isomorphisms u onto E" of fibre spaces $\overline{E}'_1$ associated with fibre spaces E'_1 with structure sheaf $\underline{G}$, form exactly one class of the quotient $H^1(\mathrm{Aut}(E''), M(E''))$, which class is therefore canonically determined by the class $c'' \in H^1(X, \underline{H})$ alone. Putting together the results obtained, we get:

Proposition 5.6.3. Let $\underline{G}$ be a sheaf of groups, $\underline{F}$ an abelian normal subsheaf, $\underline{H} = \underline{G}/\underline{F}$ the quotient. Then in virtue of the representation σ of $\underline{H}$ by germs of automorphisms of $\underline{F}$ deduced from the interior automorphisms of $\underline{G}$, each fibre space E" with structure sheaf $\underline{H}$ gives rise to an associated abelian sheaf $\underline{F}(\sigma)^{E''}$ on which the group Aut(E") of automorphisms of E" operates as group of automorphisms (representation still denoted σ), hence Aut(E") operates also on the groups $H^i(X, \underline{F}(\sigma)^{E''}) = M^i(E'')$, and we can consider in the usual way the associated cohomology groups $H^j(\mathrm{Aut}(E''), M^i(E''))$

which are intrinsically determined by the class $c'' \in H^1(X, \underline{H})$ of E''. If now E' is a fibre space with structure sheaf $\underline{G}$ and u an isomorphism of the associated fibre space $\bar{E}'$ with structure sheaf $\underline{H}$ onto E'', then to the map $\delta'^{E'}$ of the sequence (5.6.7.) corresponds a map δ'^{u}: $\mathrm{Aut}(E'') \to M^1(E'')$ which is a normalized 1-cocycle. The set of all such cocycles (for all possible choices of E and u) is exactly one class in $H^1(\mathrm{Aut}(E''), M^1(E'')) = H^1(\mathrm{Aut}(c''), M^1(c''))$, which is thus canonically determined by the class $c'' \in H^1(X, \underline{H})$ and the given extension $\underline{G}$ of $\underline{H}$ by $\underline{F}$. Given any u as above, and taking the affine representation ρ^u of Aut(E'') defined by the corresponding cocycle δ'^{u} by formula (5.6.9.), the inverse image of c'' in $H^1(X, \underline{G})$ is canonically isomorphic to the quotient set $M^1(E'') \,/\, \rho^u(\mathrm{Aut}(E''))$.

We intend to give elsewhere a general cohomological interpretation of the invariant just obtained for a fibre space E'' with structure sheaf $\underline{H}$ and an extension of $\underline{H}$ with abelian kernel $\underline{F}$. For the sake of shortness, we will neither give here the special remarks which apply to the case when the representation of Aut(E'') by automorphisms of M(E'') is trivial (which generalizes the case when $\underline{F}$ is in the center of $\underline{G}$) or when the class of $H^1(\mathrm{Aut}(E''), M(E''))$ obtained above is the "zero".

5.7. The second coboundary map ($\underline{F}$ normal abelian in $\underline{G}$).*

Suppose still $\underline{F}$ normal abelian in $\underline{G}$, $\underline{H} = \underline{G}/\underline{F}$, let E'' be any fibre space with structure sheaf $\underline{H}$, consider again the associated abelian sheaf $\underline{F}(\sigma)^{E''}$, which we will write shortly $\underline{F}(E'')$. Under general conditions, for instance if X is paracompact, we will define an element $\delta E'' \in H^2(X, \underline{F}(E''))$, such that this be 0 if and only if E'' can be obtained as fibre space associated with a fibre space E' with structure sheaf $\underline{G}$.

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* Footnote (added for the second edition): A general definition of the second coboundary map, without any paracompactness assumptions, and an application of the latter to a question of algebraic geometry, is given in A. Grothendieck, Sur quelques points d'Algèbre Homologique, Chapter 3, paragraph 4, Tohoku Math. Journal, 1958. It should be noticed that this definition is possible only using the "true" cohomology groups of a space (and not Čech cohomology), as defined by using injective resolutions. See also Roger Godement: Théorie des Faisceaux, Act. Scient. et Ind. Paris 1958, for the general theory of the cohomology of sheaves.

However, in order not to let escape the fibre spaces occuring in algebraic geometry, we will not restrict to the case X paracompact; this will give rise to some mild technical complications. We follow closely the exposition of the coboundary map (for the case of abelian sheaves) given by Serre in the paper cited in 5.1.

We will first define a variant for the set $H^1(X, \underline{H})$. Let $U = (U_i)$ be an open covering of X, let $C^o(\underline{U}, \underline{H})_o$ and $\overline{C}^1(\underline{U}, \underline{H})_o$ be the subsets of $C^o(\underline{U}, \underline{H})$ respectively $C^1(\underline{U}, \underline{H})$ images of the sets $C^o(\underline{U}, \underline{G})$ and $\overline{C}^1(\underline{U}, \underline{G})$. Let $Z^1(\underline{U}, \underline{H})_o = Z(\underline{U}, \underline{H}) \cap C^1(\underline{U}, \underline{H})_o$ be the set of cocycles in $\overline{C}^1(\underline{U}, \underline{H})_o$. Coming back to the definitions of 5.1., we see that $C^o(\underline{U}, \underline{H})_o$ is a group operating on the set $C^1(\underline{U}, \underline{H})_o$ and letting invariant the subset of cocycles, thus we can consider the quotient

$$(5.7.1.) \qquad H^1(\underline{U}, \underline{H})_o = Z^1(\underline{U}, \underline{H})_o / D(C^o(\underline{U}, \underline{H})_o).$$

Of course, we have a canonical isomorphism

$$(5.7.2.) \qquad H^1(\underline{U}, \underline{H})_o \rightarrow H^1(\underline{U}, \underline{H})_o.$$

On the other hand, if $\underline{V}$ is a covering finer than $\underline{U}$, we define as in 5.1. a canonical map $H^1(\underline{U}, \underline{H})_o \rightarrow H^1(\underline{V}, \underline{H})_o$, and the diagram

$$(5.7.3.) \qquad \begin{array}{ccc} H^1(\underline{U}, \underline{H})_o & \longrightarrow & H^1(\underline{U}, \underline{H}) \\ \downarrow & & \downarrow \\ H^1(\underline{V}, \underline{H})_o & \longrightarrow & H^1(\underline{V}, \underline{H}) \end{array}$$

is commutative. These maps define the system of sets $H^1(\underline{U}, \underline{H})_o$ ($\underline{U}$ variable covering) as an inductive system of sets, hence we can take the inductive limit, which we denote by $H^1(X, \underline{H})_o$. The maps (5.7.2.) and commutativity of (5.7.3.) define a canonical map

$$(5.7.4.) \qquad H^1(X, \underline{H})_o \longrightarrow H^1(X, \underline{H}).$$

Lemma 5.7.1. The map (5.7.4.) is injective. It is bijective in each of the following two cases: a) X is paracompact b) X is an algebraic irreductible curve with the Zariski topology.

Consider two elements of $H^1(X, \underline{H})_o$ having same image, we will prove that they are identic. We can find an open covering $\underline{U} = (U_i)_{i \in I}$ so that both elements be defined by cocycles (h_{ij}) and (h'_{ij}) in $Z^1(\underline{U}, \underline{H})_o$, and the hypothesis means that we can find $(h_i) \in C^o(\underline{U}, \underline{H})$ such that $h'_{ij} = h_i h_{ij} h_j^{-1}$. Now it is obvious that we can find a covering $\underline{V} = (V_j)_{j \in J}$ finer than $\underline{U}$, and a map $\tau: J \longrightarrow I$ such that $V_u \subset U_{\tau j}$ for any j, and such that for each j, $h_{\tau j}$ can be lifted into a section of $\underline{G}$ over V_j : it is sufficient to take $J = X$, and to associate with each j an element $\tau j \in I$ such that $j \in U_{\tau j}$, and then to take an open neighborhood V_j of j contained in $U_{\tau j}$ and small enough that $h_{\tau j}$ can be lifted on this set. Then the cochain $\varphi_\tau^1((h'_{ij}))$ in $Z^1(\underline{V}, \underline{H})_o$ is transformed of the cochain $\varphi_\tau^1((h_{ij}))$ by the O-cochain $\varphi_\tau^o((h_i))$, which belongs to $C^o(\underline{V}, \underline{H})_o$, and they therefore define the same element of $H^1(\underline{V}, \underline{H})_o$ and a fortiori of $H^1(X, \underline{H})_o$, which proves the first statement of the lemma. In order for (5.7.4.) to be surjective, it is obviously sufficient to find for any cocycle $(h_{ij}) \in Z^1(\underline{U}, \underline{H})$, a covering $\underline{V}$ finer than $\underline{U}$ and a map $\tau: J \longrightarrow I$ such that $V_j \subset U_{\tau j}$ for each j, and such that for each pair (j, j') of elements of J, the section $h_{\tau j, \tau j'}$ of $\underline{H}$ can be lifted into a section of $\underline{G}$ on $V_{jj'}$. It is easily seen that this condition is satisfied in the two cases considered in the lemma, the details are left to the reader.

For each $\underline{U}$, we have a canonical map $H^1(\underline{U}, \underline{G}) \longrightarrow H^1(\underline{U}, \underline{H})_o$ (derived from the map $Z^1(U, \underline{G}) \longrightarrow Z^1(\underline{U}, \underline{H})_o$). These maps, for variable $\underline{U}$, are compatible with the restriction maps, so that we get a natural map

(5.7.5.bis) $\quad H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})_o$.

Of course, the composition of this map with (5.7.4.) is nothing but the functorial map $H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H})$ defined in 5.1.

Let now E" be a fibre space with structure sheaf $\underline{H}$ the class c" of which belongs to the subset $H^1(X, \underline{H})_o$ of $H^1(X, \underline{H})$, we will define an element $\delta E''$ of $H^2(X, E''(\underline{F}))$. Let (h_{ij}) be a system of coordinate transforms of E", we can suppose $(h_{ij}) \in Z^1(\underline{U}, \underline{H})_o$. We denote by φ_i the natural coordinate maps $\underline{H}|U_i \longrightarrow E''|U_i$ and by $\overline{\varphi}_i$ the coordinate maps $\underline{F}|U_i \longrightarrow \underline{F}(E'')|U_i$. so we have

(5.7.5.) $h_{ij} = \varphi_i^{-1}\varphi_j \qquad \sigma(h_{ij}) = \bar{\varphi}_i^{-1}\bar{\varphi}_\sigma$.

Recall also that $h_{ik} = h_{ij}h_{jk}$, therefore, if we take for every h_{ij} a lifting $g_{ij} \in H^o(U_{ij}, \underline{G})$, we can choose them such that

(5.7.6.) $g_{ij} = g_{ji}^{-1}$

and moreover we must have $g_{ij}g_{jk} = g_{ik}$ mod $\underline{F}$, i.e. (using (5.7.6.)) we have

(5.7.7.) $f_{ijk} = g_{ij}g_{jk}g_{ki} \in H^o(U_{ijk}, \underline{F})$.

We now define

(5.7.8.) $f'_{ijk} = \bar{\varphi}_i(f_{ijk}) = \bar{\varphi}_i(g_{ij}g_{jk}g_{ki}) \in H^o(U_{ijk}, \underline{F}(E''))$.

I claim that (f'_{ijk}) <u>is an alternated cocycle of dimension</u> 2 for $\underline{U}$ and the abelian sheaf $\underline{F}(E'')$ (see the paper of Serre cited in 5.1. for the definitions). Indeed, (5.7.6.) and (5.7.7.) imply $f_{ikj} = f_{ijk}^{-1}$, hence taking the image by $\bar{\varphi}_i$:

$$f'_{ikj} = f'_{ijk}{}^{-1}.$$

Also $f'_{kji} = f'_{ijk}{}^{-1}$, which means (by transforming with $\bar{\varphi}_i^{-1}$):
$\bar{\varphi}_i^{-1}\bar{\varphi}_k(f_{kij}) = f_{ijk}^{-1}$ i.e. $\sigma(g_{ik})f_{kij} = f_{ikj}$ i.e.
$g_{ik}g_{kj}g_{jk}g_{ik}g_{ik}^{-1} = f_{ikj} = g_{ik}g_{kj}g_{ji}$ which is of course true. And again we have $f'_{jik} = f'_{ijk}{}^{-1}$, which means (transforming by $\bar{\varphi}_i^{-1}$)
$\bar{\varphi}_i^{-1}\bar{\varphi}_j f_{jik} = f_{ijk}^{-1}$ i.e. $\sigma(g_{ij})f_{jik} = f_{ikj}$ i.e.
$g_{ij}g_{ji}g_{ik}g_{kj}g_{ij}^{-1} = g_{ik}g_{kj}g_{ji}$ which follows indeed from (5.7.6.). Thus we have proved that (f'_{ijk}) is an alternating 2-cochain, we now prove that it is a cocycle, i.e. that we have

$$f'_{jkl} f'^{-1}_{ikl} f'_{ijl} f'^{-1}_{ijk} = e \text{ in } U_{ijkl}.$$

This is proved as above by transforming with $\bar{\varphi}_i^{-1}$, the verification is only some lines longer (of course, the fact that $\underline{F}$ is abelian and invariant has to be used throughout). Therefore, this cochain defines an element of the cohomology group $H^2(\underline{U}, \underline{F}(E''))$, and hence of $H^1(X, \underline{F}(E''))$. It is easily checked that this does not depend on the choice of the system of coordinate transforms defining E", and therefore may be denoted by $\delta E''$. It should be noted that in this general context, however, it is not possible to define the $\delta c''$ of the class c" of E", for $\underline{F}(E'')$ is not defined by the mere class of E", and neither is $H^1(X, \underline{F}(E''))$. If a map defined on the set $H^1(X, \underline{H})_o$ itself is desired, one can notice that the quotient $M^2(E'')/\mathrm{Aut}(E'')$ of $M^2(E'') = H^2(X, \underline{F}(E''))$ by the group of automorphisms of E" (acting in the usual way on $M(E'')$) is intrinsically determined by the class c", and so is the class of $\delta E''$ in this quotient, so that we may define symbolically a $\delta c'' \in M^2(c'')/\mathrm{Aut}(c'')$, and the <u>second</u> <u>coboundary</u> map is defined on $H^1(X, \underline{H})_o$ and takes its values on a set depending on the argument considered.– We now come to an important exactness property connected with this second coboundary map:

<u>Proposition 5.7.2.</u> Let $\underline{F}$ be an abelian normal subsheaf of the sheaf of groups $\underline{G}$, let $\underline{H} = \underline{G}/\underline{F}$, let E" be a fibre space with structure sheaf $\underline{H}$, $\underline{F}(E'')$ the associated sheaf (corresponding to the natural operation of $\underline{H}$ on $\underline{F}$). Let $c'' \in H^1(X, \underline{H})$ be the class of E". In order that c" be in the image of $H^1(X, \underline{G})$, it is necessary and sufficient that $c'' \in H^1(X, \underline{H})_o$ (as defined in 5.7.1.) and that the $\delta E'' \in H^2(X, \underline{F}(E''))$ defined above be equal to 0.

<u>Necessity</u>. We can assume that E" is the fibre space associated to a fibre space E' with structure sheaf $\underline{G}$, let (g_{ij}) be a system of coordinate transforms of E', let $h_{ij} \in H^o(U_{ij}, \underline{H})$ be the image of g_{ij}, then (h_{ij}) is a system of coordinate transforms for E". In particular this shows $c'' \in H^1(X, \underline{H})_o$, and by definition $\delta E''$ is defined by the 2-cocycle of formula (5.7.8.), which is now the unit cocycle, hence $\delta E'' = 0$.

Sufficiency. We can assume that E" is defined by coordinate transforms (h_{ij}), where h_{ij} can be lifted into g_{ij}, and following the notations above, δ E" defined by (5. 7. 8.) is supposed zero, which means that there exists an alternated 1-cochain $(f'_{ij}) \in C^1(\underline{U}, \underline{F}(E''))$ such that

$$(5.7.9.) \qquad f'_{ijk} = f'_{jk} f'_{ki} f'_{ij}.$$

Let $f_{ij} = \bar{\varphi}_i^{-1}(f'_{ij}) \in H^o(U_{ij}, \underline{F})$, then the relations $f'_{ij} = f'^{-1}_{ji}$ and (5. 7. 9.) yield

$$(5.7.10.) \qquad f_{ij} = \sigma(g_{ij}) f_{ji}^{-1} \text{ and } f_{ijk} = \sigma(g_{ij}) f_{jk} \; \sigma(g_{ik}) f_{ki} f_{ij}$$

i. e., taking into account that the first relation (by substituting k to j) yields $\sigma(g_{ik}) f_{ki} = f_{ik}^{-1}$ and substituting in the second:

$g_{ij} g_{jk} g_{ki} = g_{ij} f_{jk} g_{ij}^{-1} f_{ik}^{-1} f_{ij}$ i. e.

$$(g_{jk}^{-1} f_{jk}) (g_{ij}^{-1} f_{ij}) = g_{ik}^{-1} f_{ik}.$$

Letting $\qquad g'_{ij} = f_{ij}^{-1} g_{ij}$

and taking the inverses of both members of the above formula, we get $g'_{ik} = g'_{ij} g'_{jk}$, which means that (g'_{ij}) is a 1-cocycle $\in Z^1(\underline{U}, \underline{G})$. The element of $Z^1(\underline{U}, \underline{H})$ image of (g'_{ij}) is the same as the image of the cochain (g_{ij}), i. e. (h_{ij}), which completes the proof.

Corollary. Under the conditions of lemma 5. 7. 1., in particular if X is paracompact, the element $\delta E'' \in H^2(X, \underline{F}(E''))$ is defined for any fibre space E" with structure sheaf $\underline{H}$, and in order that E" be isomorphic to a fibre space associated to a fibre space with structure sheaf $\underline{G}$, it is necessary and sufficient that $\delta E'' = 0$.

It should be noted that, $\underline{H}$ being any sheaf of groups and E" a fibre space with structure sheaf $\underline{H}$, an element $\delta E''$ is defined whenever we are given an extension of $\underline{H}$ by an abelian sheaf $\underline{F}$ in which $\underline{H}$ operates (so that $\underline{F}(E'')$ is defined and hence $H^2(X, \underline{F}(E''))$. We intend to give elsewhere the characterization of $\delta E''$ in terms of the cohomological invariant characterizing the extension $\underline{G}$ of $\underline{H}$ by $\underline{F}$. In the case the sheaves involved are defined in terms of structure groups F, G, H, when either these are discrete of H is connected and G is a covering group, E'' can be expressed in terms of classical cohomological notions.

Particular case: $\underline{F}$ is in the center of $\underline{G}$. This means that $\underline{H}$ operates trivially on $\underline{F}$, so that $\underline{F}(E'') = \underline{F}$, therefore the second coboundary map is a map $H^1(X, \underline{H})_o \longrightarrow H^2(X, \underline{F})$, which can be added to the exact cohomology sequence (5.3.1.) to yield a sequence:

(5.7.11.)
$$e \longrightarrow H^o(X, \underline{F}) \longrightarrow H^o(X, \underline{G}) \longrightarrow H^o(X, \underline{H}) \xrightarrow{\delta} H^1(X, \underline{F})$$
$$\longrightarrow H^1(X, \underline{G}) \longrightarrow H^1(X, \underline{H}) \xrightarrow{\delta} H^2(X, \underline{F}).$$

Proposition 5.7.2. here means that this new sequence is still exact. Of course, in the usual cases referred to in lemma 5.7.1., $H^1(X, \underline{H})_o$ can be replaced by $H^1(X, \underline{H})$ itself.

A characterization of the image in $H^2(X, \underline{F})$ of the last homomorphism is lacking. However, if $\underline{G}$ itself is abelian, then $H^2(X, \underline{G})$ and $H^2(X, \underline{H})$ are also defined, and (5.7.11.) fits into the classical exact cohomology sequence for abelian sheaves (see cited paper of Serre):

(5.7.12.)
$$e \longrightarrow H^o(X, \underline{F}) \longrightarrow \ldots \longrightarrow H^1(X, \underline{H})_o \longrightarrow H^2(X, \underline{F})$$
$$H^2(X, \underline{G}) \longrightarrow H^2(X, \underline{H})_o \longrightarrow \ldots .$$

5.8. Geometric interpretation of the first coboundary map. Let G be a topological group, F a topological subgroup, H the quotient space, $\underline{f}$ a subsheaf of groups of the sheaf of germs of continuous maps of H into F containin the constant germs. G can be considered as a fibre space over H, all fibers being homeomorphic to F (the projection map being by definition the canonical map of G onto the quotient G / F). We suppose now that we are given on G a structure of fibre space of type $H \times F$, structure sheaf $\underline{f}$ (F operating on the fibre F by left regular representation). This of course implies that G fibered over H is locally trivial, which here means also that there exists a germ of a section of G (fibered by F over H) passing through the unit element (for then by translation, we see that there is a section over a neighborhood of any point of H, which classically and trivially implies local triviality). If $\underline{f}$ is the sheaf of all germs of continuous maps of H into F, then existence of a local section is also sufficient for a structure with structure sheaf $\underline{f}$ on G to exist, and this is unique. Other examples: Suppose G is a Lie group and F closed (hence itself a Lie group) then taking into account the analytic structure on G and using the well known fact of the existence of analytic local sections of G over H, we see that G can be considered as analytic fibre bundle over H with Lie structure group F (fibre F on which F operates by regular left representation), hence we may take in the foregoing $\underline{f}$ = sheaf of germs of analytic maps of H into F. A fortiori for any sheaf $\underline{f}' \supset \underline{f}$, G is provided with the structure sheaf $\underline{f}'$ by "extension of the structure sheaf $\underline{f}$"; for instance, we can take for f' the sheaf of germs of infinitely differentiable maps of H into F etc. Analogous remarks if G is a complex analytic Lie group and F a complex subgroup, then G is a complex fibre bundle with complex structure group F over the complex manifold H; we need only the well known fact of the existence of complex analytic sections of G over H. As a last example, suppose that G is an algebraic group over an arbitrary ground field k, F an algebraic subgroup, so that H is an algebraic variety without singularities; as usual, we take on algebraic varieties the Zariski topology. Let $\underline{f}$ be the sheaf of germs of regular (i.e. rational and "defined" at the given point) maps from H into F. Then G can be considered as a fibre space of type $H \times F$, structure sheaf $\underline{f}$, provided there exists a regular section of G over a neighborhood of the neutre element of H. As such a section is the restriction of a "rational section" of G over H (i.e. a rational map $H \longrightarrow G$ which, composed with the projection $G \longrightarrow H$, gives the identity), the preceding hypothesis is equivalent to the existence of a rational section of G over H. In case F is discrete or an abelian variety, this appears to be a very strong condition, for as easily seen a rational section is even regular, and thus the fibering should be trivial with respect to the structure sheaf $\underline{f}$, which of course is seldom the case. However, if F is a connected linear algebraic group, it seems very likely that there must exist a rational section of G over H; this can at least be checked directly in the most important cases, and it has been proved by Rosenlicht (forthcoming paper) in case F is moreover solvable.

Let us now come back to the general case of a topological group G fibered by a subgroup F, and provided with a structure of type $H >< F$, structure sheaf $\underline{f}$. Let moreover X be a topological space, suppose given sheaves $\underline{F}$, $\underline{G}$, $\underline{H}$ of germs of continuous maps of X into F respectively G, respectively H. Suppose the maps $i: F \longrightarrow G$ and $j: G \longrightarrow H$ compatible with these sheaves, i.e. the composition of i with a germ in $\underline{F}$ is in $\underline{G}$, and the composition of j with a germ in $\underline{G}$ is in $\underline{H}$. Then we have a sequence of homomorphisms of sheaves

$$(5.8.1.) \qquad e \longrightarrow \underline{F} \longrightarrow \underline{G} \longrightarrow \underline{H} \longrightarrow e$$

where $\underline{F} \longrightarrow \underline{G}$ is a homomorphism also for the structures of sheaves of groups. This sequence is exact if and only if: (i) if a germ in $\underline{G}$ is a germ of maps into F, it belongs to $\underline{F}$ (ii) any germ in $\underline{H}$ can be lifted into a germ in $\underline{G}$. We will assume this, and moreover (iii) the composition of a germ in $\underline{H}$ and a germ in $\underline{f}$ is in $\underline{F}$. From the exactness of the sequence (5.8.1.) follows an exact sequence of cohomology (5.2.7.). From (iii) follows that for any fibre space E on H with structure sheaf $\underline{f}$, and any map h of X into H belonging to $H^0(X, H)$, we can consider on the inverse image of E under h(which is a fibre space on X) in a natural way a structure with structure sheaf $\underline{F}$ (via a general definition of inverse images of fibre spaces with structure sheaf, which was not given in 4.1. and is left to the reader). Now we supposed G itself a fibre space with structure sheaf $\underline{f}$, hence its inverse image is a fibre space with structure sheaf $\underline{F}$ on X, determined canonically by $h \in H^0(X, \underline{H})$. Then the proof of the following statement is straightforward and equally left to the reader:

Proposition 5.8.1. Let G be a topological group, F a subgroup,

$H = G/F$, $\underline{f}$ a sheaf of germs of continuous maps of H into F, we suppose given on G, considered as fibre space over H, (fibre F) a structure of type $H >< F$, structure sheaf $\underline{f}$. Let X be a topological space, $\underline{F}$, $\underline{G}$, $\underline{H}$ three sheaves of germs of continuous maps of X into F respectively G respectively H, satisfying conditions (i) (ii) and (iii) above. Then the sequence (5. 8. 1.) is exact. Let h be a map $X \longrightarrow H$ belonging to $H^o(X, \underline{H})$, and E the fibre space with structure sheaf $\underline{F}$ (fibre F) on X inverse image of the fibre space G over H by the map h. This fibre space is canonically isomorphic with the fibre space defined by means of h in 5. 2. (F operating on $X >< \underline{F}$ in the usual way), and therefore its class in $H^1(X, \underline{F})$ is the image of h under the coboundary operator associated with (5. 8. 1.).

Corollary. In order that a fibre space with structure sheaf $\underline{F}$ be trivial in the sheaf $\underline{G}$, it is necessary and sufficient that the associated principal bundle (cf. 4. 7.) be isomorphic to the inverse image of the fibre space G over H by a suitable map $h \in H^o(X, \underline{H})$ of X into H; this h determined up to an operation of $H^o(X, \underline{G})$.

This follows at once from proposition 5. 8. 1. and the exactness of the cohomology sequence.

It could be remarked that the different supplementary structures involved in the exact cohomology sequence defined in a rather algebraic way in 5. 3., 5. 4., and 5. 5., namely the operations ρ of $H^o(X, \underline{H})$ on $H^1(X, \underline{F})$ if $\underline{F}$ is normal in $\underline{G}$, the operations σ of $H^o(X, \underline{H})$ on $H^1(X, \underline{F})$ when moreover $\underline{F}$ is abelian, and the operations ρ of $H^1(X, F)$ on $H^1(X, \underline{G})$ when $\underline{F}$ is even in the center of $\underline{G}$, could have been dealt with in a more geometric way, by defining in the first instance, for each $h \in H^o(X, \underline{H})$ and each fibre space E with structure sheaf $\underline{F}$, a fibre space $\rho(h).E$ with structure sheaf $\underline{F}$ (via the system of all coordinate maps for E), and proceeding in an analogous way in the other two instances. Then statements like: $\rho(h).c$ and c have same image in $H^1(X, \underline{G})$ (where $c \in H^1(X, \underline{F})$) could be stated in a more geometric and slightly more precise form: there is a canonical isomorphism between the fibre spaces $\rho(h).E$ and E, where E is any fibre space with structure sheaf $\underline{F}$. It does not seem that the exposition would be much heavier if in each possible instance the geometric point of view were thus taken into account.