COPY OF DRAFT, July 1969

Dear Professor Grothendieck,

Thank you for your letter of June 12. The argument you give to prove that $\hat{A}[[T]]$ factorial $\Rightarrow A$ rational should extend to the mixed characteristic case because of the following remark: If $f\colon X \to \mathrm{Spec}(A)$ is a desingularization such that $\mathfrak{m}\mathcal{O}_X$ is invertible ($\mathfrak{m} = \max(A)$), say $\mathfrak{m}\mathcal{O}_X = \mathcal{O}_X(-C)$ (with C a curve on X proper over the field $A/\mathfrak{m}$), then A rational $\Leftrightarrow H^1(C, \mathcal{O}_C) = 0$. This follows from Artin's result that A rational $\Leftrightarrow H^1(F, \mathcal{O}_F) = 0$, where F is the "fundamental cycle", which is a closed subscheme of C. Thus the question involves only the Picard scheme of the curve C, rather than the local Pic of A. On the other hand, I could not see why a non–constant formal arc on $\mathrm{Pic}(C)$ should define a non–trivial divisor class (of $C \otimes_{A/\mathfrak{m}} A/\mathfrak{m}((T))$?) in case C has no $A/\mathfrak{m}$-rational point.

You also stated that A rational $\Leftrightarrow$ local Pic of A is zero–dimensional. This is straightforward enough when $A/\mathfrak{m}$ is perfect, but how does it fit in with the following example?

- k a field of characteristic 2 (could be separably closed);
- $a \in k,\ \sqrt{a} \notin k$;
- then $R\colon = k[[X, Y, Z]]/x^2 + aY^2 + Y^3 + Z^7$ is rational and factorial,
- but $R \otimes_k k(\sqrt{a}) = k(C)[[X', Y, Z]]/(X')^2 + Y^3 + Z^7$ ($X' = X + \sqrt{a}Y$) which is not rational, and in fact has local Pic $\cong$ additive group over the residue field.

Your interpretation of $H^1(Z, \mathcal{O}_Z)$ of the Zariski-Riemann space was certainly the correct one. I still have no way of showing that this is a finite module, although the feeling remains that there ought to be a reasonably short proof. Probably a theory of local Pic which did not use resolution of singularities would point the way. I have no information on the question you raised concerning the construction of the local Pic for higher dimensional rings.

I have thought about trying out the following question:

If R is a three dimensional local complete intersection with an isolated singularity, is the local Pic of R zero-dimensional? I think Brieskorn has a topological proof in the classical case. (The corresponding global theorem for surfaces was proved by Hartshorne). Also, under which conditions is this local Pic trivial (local Noether theorem)? Do you know of any results along these lines? I would value any remarks you might make.