Feb. 15, 1970.

Dear Professor Grothendieck.

I appreciate very much your letter of last September with its program for studying the local Picard functor. I have been working on that part of the problem which involves representing, over the perfect residue field k of an Artin local ring R, the Pic of an X which is proper over R. So far, progress is limited, mainly, I suppose, for lack of a feeling about $W(A)$ (W = Witt vectors) when A is a k-algebra with nilpotents. What I can prove is representability when one restricts to the category of perfect schemes over k. (a property which might be referred to as "quasi-representability" in analogy with Serre's quasi-algebraic ($\doteqdot$ perfect) group-schemes).

The approach which I found most useful so far is to imitate the "dévissage de Oort". Let me indicate the main steps.

R = artin local ring with algebraically closed (for now) residue field k of characteristic $p>0$

W = infinite Witt vectors, coefficients in k

X = scheme proper over R and (for simplicity) connected

You have suggested to consider the functor of k-algebras

$$A \rightsquigarrow \mathrm{Pic}(X_A) = \mathrm{Pic}(X \otimes_R R(A))$$

The first observation is that the canonical homomorphism $W \to R$ gives rise to a map, functorial in A,

save till end (page 5)

$$\mathrm{Pic}(X \otimes_W W(A)) \to \mathrm{Pic}(X \otimes_R R(A))$$

which induces an <u>isomorphism</u> of associated f.p.q.c. sheafs. To see this one notes that for any A there exists a faithfully flat A-algebra A_1 such that the Frobenius of A_1 is surjective, $A_1^p = A_1$, (it suffices to show this for A = polynomial ring over k...); hence if $F \to G$ is a morphism of functors such that $F(A) \xrightarrow{\approx} G(A)$ whenever $A^p = A$, then $\tilde{F} \xrightarrow{\approx} \tilde{G}$ is an isomorphism of the associated f.p.q.c. sheafs. To apply this to our situation use the fact that if $A^p = A$ then

$$R \otimes_W W(A) \xrightarrow{\approx} R(A).$$

Thus we may assume $R = W$. (and $p^M \mathcal{O}_X = 0$ for ~~some~~ M)

Next, if X is <u>reduced</u> (or, more generally, if $p\mathcal{O}_X = 0$) then our functor becomes

$$P_1(A) = \mathrm{Pic}\left(X \otimes_k \frac{W(A)}{pW(A)}\right)$$

whose associated sheaf is the <u>representable</u> functor

$$\mathrm{Pic}'(A) \doteq \mathrm{Pic}(X \otimes_k A)/(\text{image of})\, \mathrm{Pic}(A)$$

(This is so because $W(A)/pW(A) = A$ when $A^p = A$ so that if $P(A) = \mathrm{Pic}(X \otimes_k A)$ we have $\tilde{P}_1 \xrightarrow{\approx} \tilde{P} = \mathrm{Pic}'$).

In the general case, let $\mathcal{N}$ = sheaf of nilpotents of X, $X_n = (|X|, \mathcal{O}_X/\mathcal{N}^n)$, $\mathcal{I}_n = \mathcal{N}^{n-1}/\mathcal{N}^n$ $(n \geq 1)$ and define, for $n \geq 2$, the functors

$$E_n(A) = \text{cokernel of } H^0(X_A, \mathcal{O}_{X_{n-1,A}}) \to H^1(X_A, \mathcal{I}_n \mathcal{O}_{X_n,A})$$

$$F_n(A) = H^2(X_A, \mathcal{I}_n \mathcal{O}_{X_n,A})$$

$$P_n(A) = \mathrm{Pic}(X_{n,A}) \qquad (\text{for } n=1,\ P_1 \text{ is as above})$$

By Oort's method, using the (truncated) exponential, we get <u>exact sequences</u>

$$0 \to E_n(A) \to P_n(A) \to P_{n-1}(A) \to F_n(A)$$

and so (since $\tilde{P}_1$ is representable) we could finish up by "dévissage" if we could show that the f.p.q.c. sheafs $\tilde{E}_n$, $\tilde{F}_n$ are representable. ($\tilde{E}_n$ by an <u>affine</u> scheme).

At present I cannot do this. But something can be said in case A is <u>reduced</u>, because then $W(A)$ is <u>flat</u> over W, and hence the natural morphism of functors

$$E_n^*(A) = E_n(k) \otimes_W W(A) = E_n(k) \otimes_k \frac{W(A)}{pW(A)} \longrightarrow E_n(A)$$

(defined for all A, reduced or not) is an <u>isomorphism</u> on the category of reduced A. Moreover, if $A^p = A$,

$$E_n^*(A) \xrightarrow{\approx} E_n(k) \otimes_k A .$$

Thus $E_n(A)$ is <u>quasi-representable</u>: it is represented on the category of perfect schemes by the perfect closure of $\mathbb{V}(\check{E}_n(k))$. Similarly for F_n. Also, the Zariski sheaf associated to $P_1(A)$ takes the value $\mathrm{Pic}(X_1 \otimes_k A)/\mathrm{Pic}(A)$ for perfect A and so it is (quasi-) represented by the perfect closure of the usual group scheme $\underline{\mathrm{Pic}}(X_1/k)$. So we can conclude by standard arguments that the Zariski sheaf associated with P_n is quasi-representable for all n.

The next remark shows that we are actually quasi-representing $\tilde{P}_n$:

If F is any functor of k-algebras, with associated f.p.q.c. sheaf $\tilde{F}$, and if F is quasi-representable, then for all <u>perfect</u> A, $F(A) \xrightarrow{\approx} \tilde{F}(A)$.

[This follows from two facts:

(i) if $B^p = B$ and B is faithfully flat over a perfect A, then B_{red} (which is perfect) is also faithfully flat over A.

(ii) if A is perfect and B and C are reduced <u>flat</u> A-algebras, then $B \otimes_A C$ is also reduced (so that $B \otimes_A C$ is perfect if both B and C are).]

Finally, let us show that, for perfect A,

$$\tilde{P}_n(A) = P_n^*(A) = \mathrm{Pic}(X_{n,A}) \Big/ \overset{\text{(image of)}}{\mathrm{Pic}(W_M(A))} \qquad \text{whenever } p^M \mathcal{O}_X = 0.$$

(And, furthermore, $\mathrm{Pic}(W_M(A)) \xrightarrow{\approx} \mathrm{Pic}(A)$) so that $\tilde{P}_n(A) = \mathrm{Pic}(X_{n,A})$ if $\mathrm{Pic}(A)$ is trivial.

This goes by induction, using the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E_n(A) \arrow[r] \arrow[d, "\approx"] & P_n^*(A) \arrow[r] \arrow[d] & P_{n-1}^*(A) \arrow[r] \arrow[d] & F_n(A) \arrow[d, "\approx"] \\ 0 \arrow[r] & \tilde{E}_n(A) \arrow[r] & \tilde{P}_n(A) \arrow[r, "\alpha"] & \tilde{P}_{n-1}(A) \arrow[r, "\beta"] & \tilde{F}_n(A) \end{tikzcd}$$

(Note that for f in A, $W_M(A_f) = W_M(A)_{(f,0,0,0,\dots,0)}$ so that each element of $\mathrm{Pic}(W_M(A))$ is ~~also~~ trivial locally on A, so the diagram makes sense, and, moreover the first row is exact. The second row is also exact <u>except</u> possibly at $\tilde{P}_{n-1}(A)$; but all we will need is $\beta \circ \alpha = 0$)

To start the induction, recall that

$$\tilde{P}_1(A) = \mathrm{Pic}(X_1 \otimes_R A)/\mathrm{Pic}(A) = P_1^*(A).$$

(since $\mathrm{Pic}(W_M(A)) \xrightarrow{\sim} \mathrm{Pic}(A)$). The rest is just diagram chasing (five-lemma).

I hope it will be possible to obtain better results in the near future (at least to be able to represent the functor on the category of schemes <u>smooth</u> over k). I will write when there is something further.

Thanks again for your help.

Sincerely,
J. Lipman

P.S. I would be most grateful if I could be put on a mailing list for exposés of current developments (S.G.A., etc.), since previously it has taken several years for such material to reach here through "regular" channels, and I would like very much to keep abreast of your work.