Alexander Grothendieck: Enthusiasm and Creativity

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C'est à celui en toi qui sait être seul, à l'enfant, que je voudrai parler et à personne d'autre.[1]

Alexander Grothendieck was born in Berlin on 28 March 1928. His father, Sascha Shapiro, an anarchist originally from Russia, took an active part in the revolutionary movements first in Russia, and then in Germany, during the 1920s, where he met Hanka Grothendieck, Alexander's mother. After the Nazis came to power in Germany it was too dangerous for a Jewish revolutionary to stay there, and the couple moved to France, leaving Alexander in the care of a family near Hamburg. In 1936, during the Spanish Civil War, Sascha joined the anarchists in the resistance against Franco. In 1939 Alexander joined his parents in France, but Sascha was arrested and – partly as a consequence of the race laws enacted by the Vichy government in 1940 – sent to Auschwitz, where he died in 1942. Hanka and Alexander Grothendieck were also deported, but they escaped the holocaust. Alexander, separated from his mother, was able to attend high school at the Collège Cévenol in Chambon-sur-Lignon, lodging at the Secours Suisse, a hostel for refugee children, but he had to flee into the woods every time there was a Gestapo raid. He then enrolled at the University of Montpellier and in autumn 1948 he arrived in Paris with a letter of introduction to Élie Cartan. This led to his being accepted at the École Normale Supérieure as an *auditeur libre* for the 1948–1949 academic year, where he assisted in the debut of algebraic topology in the seminar taught by Henri Cartan (Élie's son). His earliest interests, however, were in functional analysis, and following Cartan's advice, he moved to Nancy. Under the guidance of J. Dieudonné and L. Schwartz, he earned his doctorate in 1953.

During his years in high school and university, Grothendieck never much enjoyed the courses and programs he attended, nor can it be said that he was a model student. His curiosity, coupled with a sense of dissatisfaction, drove him, not quite 20-years old, to develop on his own a theory of measurement and integration.

[1]It is to the one inside you who knows he is alone, to the child, that I wish to speak and to nobody else. *Récoltes et Semailles*, "Promenade, à travers une œuvre", p. 7.

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When he arrived in Paris, he found it had already been written by Lebesgue. He said, "I learned then in solitude the thing that is essential in the art of mathematics – that which no master can really teach".[2] The "official" productive period of Grothendieck's life, as testified by an impressive mass of writings, is 1950 to 1970. While the research topics of the early 1950s were those of functional analysis, the great themes of algebraic geometry, its foundations, such as the redefinition of the concept of space itself, occupied the years 1957–1970.

In 1959, by now a professor at the newly created Institut des Hautes Études Scientifiques (IHES) in Bures, near Paris, Grothendieck taught a lively seminar in which he – in a magnificent display of generosity – shared and gave away his research ideas to students and colleagues, developing them with boundless enthusiasm and creativity. In these early years his frequent and intense contacts with Jean-Pierre Serre, traces of which are left to us in their correspondence, were a source of inspiration and mutual exchange of ideas. In the decade between 1959 and 1969 Grothendieck's ideas were mainly spread, on the one hand, through publications such as *Éléments de Géometrie Algébrique* (*EGA*) – edited in collaboration with Dieudonné – and with the help of the participants in the *Séminaire de Géométrie Algébrique* (*SGA*), and on the other hand, through *Exposés* at the Bourbaki seminars. According to Grothendieck's original idea, the *Séminaire* was considered as a preliminary form of the *Éléments* and was destined to be incorporated into it. The *Éléments* were initially published by the IHES in various weighty tomes. In 1966, Grothendieck was awarded the Fields Medal (the highest recognition a mathematician can receive).

In 1970 Grothendieck, then 42-years old, officially abandoned the scene. There were many reasons that led him to withdraw from the academic world, but certainly his radical anti-war stance was one that he declared openly. It had come to his attention that the IHES received funding from the defence ministry – and had received it for more than 30 years without his being aware of it – and his response was to desert the *Institut*, also taking away with him the publication of the *EGA* and the *SGA*, signing a contract for the new edition with Springer-Verlag. Knowing what it is like to live as a refugee, with a United Nations passport – his own original documents disappeared during the Nazi holocaust – he gave life to the pacifist and environmental movement named *Survivre*. Seen against the background of the major issues of those years, the Vietnam war and the proliferation of nuclear weapons – war and the stockpiling of weapons of mass destruction are still issues still quite pertinent today – Grothendieck's pacifism shows a significant shouldering of responsibility, not the kind that the institutions involved could ignore (even though these still today receive the same kind of funding). Following this decision, Grothendieck spent a couple of years at the Collège de France and then in Orsay before finally returning to the University of Montpellier in 1973. He refused the Crafoord Prize in 1988, the year of his retirement. In these last years, retiring to private life in the country near Mormoiron, having given up travelling, he dedicated

[2] *Récoltes et Semailles*, p. 5.

himself to correspondence and to the *Récoltes et Semailles*, a long diary-like narrative about his past as a mathematician, or as he says, a long meditation on "the internal adventure that was and is my life".[3]

Alexander Grothendieck

I received portions of the *Récoltes et Semailles* in 1991, along with a letter from Grothendieck in which he told me that Aldo Andreotti was "a good friend and a truly valuable person: I came to appreciate his peculiar qualities much more now than he has passed away than I did during the 1950s and 1960s when he was still alive". I don't know which Italian mathematicians worked with Grothendieck in those years; the Italian schools were slow to assimilate his methods in algebraic geometry, even though these were partly rooted in the work of Italians such as Severi and Barsotti.

The *Présentation des Thèmes* of the *Récoltes et Semailles* provided the valuable information – along with the letter I have just mentioned – for the sketch of his life given up to now, and for an outline for an overview of his mathematical thinking, to which we will now turn.

Grothendieck's excellence, his mathematical genius, is quite evident in his innate tendency to bring to the fore themes that are obviously crucial but which no one before him had made evident or acknowledged. His productivity had deep roots and expressed itself by means of language that was ever new, emerging like a flowing river of new *notions–abstractions* and *statements–formulations*. Quite frequently statements that sprang perfectly formulated from his fervid and

[3]*Récoltes et Semailles*, p. 8.

implacable imagination turned out to be the foundation of an entire theory that Grothendieck himself outlined, developed and followed through with; in other cases they are only sketched out.

If by mathematical dexterity we mean man's capacity to solve problems, then this tendency of his not just to find solutions to mathematical problems but to *create* mathematics, makes Grothendieck an extremely special and extravagant mathematician. The layman who approaches Grothendieck's mathematical work has to get past the usual concept of a mathematician as a problem solver and try instead to see mathematics as an art and the mathematician as an artist. Of course, mathematics is a very special kind of art, one in which *inventions* borrow from the proofs, that is, imagination has to harmonise with reason. The mathematician' works are theories in a weave, a design, that always make it possible to grasp a oneness in multiplicity. As Grothendieck himself wrote, "it is in this act of going beyond, not in remaining closed within a mandatory circle that we ourselves create, it is above all else in this solitary act that creation is found".[4]

For Grothendieck, mathematical theories are also opportunities for reflection in a lateral sense, and meditative exercises, a kind of contemplation that accompanies us on our internal adventure. Mathematics is thus a *yoga* that diversifies and multiplies into different theories but whose foundations are firmly united. The differentiation of these old and new themes is also interwoven with the history of the ideas that inspired them. According to Grothendieck, there are traditionally three aspects of things that are the objects of mathematical reflections: number, or the arithmetic aspect; measure, or the metric (or analytic) aspect; and shape, or the geometric aspect. "In the most part of the cases studied in mathematics, these three aspects are either present simultaneously or in intimate interaction".[5]

Let's look at some of the topics that algebraic geometry involves from Grothendieck's point of view. His was a perspective that favoured shape and structure and thus the geometric and arithmetic aspects, in a unifying vision that gave birth to a new geometry: *arithmetic geometry*.

> We can state that number is aimed at grasping the structure of the disparate or discrete parts: the systems, sometimes finite, formed of elements or objects that are isolated, if you will, in relation to each other, without any principle of continuous passage from one to the other. Magnitude, on the other hand, is the quality *par excellence*, susceptible to continuous variation; for this reason it is aimed at grasping structure and continuous phenomena: motions, spaces, variations of all kinds, force fields, etc. Thus arithmetic appears (more or less) as the science of discrete structures, and analysis as the science of continuous structures.
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> As far as geometry is concerned, we can state that after more than 2,000 years it exists as a form of science in the modern sense of that term, it straddles the two kinds of structure, discrete and continuous. On the other hand, for a long time there was no real "divorce" between the two different kinds of geometries, one discrete and the other continuous. Instead, there were two different points of view about the investigation of the same geometric figures: one placed an emphasis on the discrete properties … the other on the continuous properties….

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[4] *Récoltes et Semailles*, p. 6.

[5] *Récoltes et Semailles*, p. 26.

> At the end of the 1800s there was a divorce, with the birth and the development of what was sometimes known as abstract (algebraic) geometry. Roughly speaking, its aim was to introduce, for every prime number p, an (algebraic) geometry of characteristic p, based on the (continuous) model of the (algebraic) geometry inherited from earlier centuries, but in a context that appeared, however, to be irreducibly discontinuous, discrete. These new geometrical objects became increasingly important at the beginning of the 1900s, and this in particular, given the close connection with arithmetic ... would seem to be one of the guiding ideas in the work of André Weil. . . . It is in this spirit that he formulated, in 1949, his celebrated Weil conjectures. Conjectures that are absolutely astounding, if truth be told, making it possible for us to see, by means of these new discrete kinds of varieties (or spaces), the possibility of certain kinds of constructions and topics that up to that time had seems conceivable only in the context of those spaces that the analysts deemed worthy of being called by that name. . . .
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> It is possible to believe that the new geometry is above all a synthesis of these two worlds ... the arithmetic world ... and the world of continuous magnitudes. In this new vision, the two worlds that were once separate, now form a single world.[6]

This unifying vision is embodied in the concepts of *scheme* and *topos*, revealing hidden structures: the geometrical richness of the discrete world is brought to light in all of its beauty and detail, thus making possible for Grothendieck himself and his student, Pierre Deligne, to prove the so-called Weil conjectures.

The concept of scheme constitutes an enlargement or generalisation of the concept of algebraic variety as it had been studied by the Italian and German schools in the early years of the 1900s. Grothendieck's idea of scheme and the basic ideas of a scheme theory, by means of the concept of maps, that is, by a suitable transformation (or morphism) of schemes, goes back to the years 1957–1958 and were briefly illustrated at the International Congress of Mathematicians in Edinburgh in 1958. It was precisely the concept of *sheaf* – already introduced and studied by Leray, Henri Cartan and Serre – that turned out to be essential because it made it possible to reconstruct a global datum starting from an open set of locally defined data, and thus making it possible to apply continuous reasoning in a discrete context.

While algebraic geometry is the study of polynomial equations and the geometric loci that they define, sheaf theory and scheme theory are the language for expressing it faithfully, a language that is easy to use and natural, and aimed at explicitly describing the details of the inner structure of these geometric entities.

Classically, each affine variety has a corresponding coordinate ring that describes it algebraically by means of polynomial equations in an ambient space:

$$\text{affine variety} \Leftrightarrow \text{coordinate ring}$$

The fundamental idea of scheme theory is that this correspondence can be extended by associating each ring A with its *spectrum* Spec(A). We can see that the set of all the primes p of A gives rise to a collection of local rings A_p (germs). Vice versa, we want it to be possible to reconstruct A from this collection taken as a

[6] *Récoltes et Semailles*, pp. 28–30.

whole. This collection is actually the local reflection of an object – a sheaf – that is also topological in nature, so that A embodies the global aspect. The affine scheme Spec(A) results precisely from the synergy of topology (called Zariski topology) and the set of primes and the ranges of the corresponding local rings.

A scheme will thus admit a covering by affine schemes, that is, it is a topological space X and a structure sheaf O_x such that for every point of X there exists an open neighbourhood of the type Spec(A). The range now embodied by the structure sheaf follows and faithfully reflects the shape of the space underlying the scheme.

One advantage of this definition of *shape* consists mainly in the fact that it intrinsically describes geometric entities, schematically speaking, as a network of primary entities, omitting any reference to an ambient space. A further advantage of the scheme concept is its relative versatility, which makes it possible to conceive a scheme defined by a morphism based on what can even be a *family* of schemes.

A morphism of schemes X $\rightarrow$ S is simply a continuous application of the underlying spaces compatible with the structure sheafs. If S=Spec(A), such a scheme on S is equivalent to the fact that the structure sheaf O_X is a sheaf of A-algebras. For example, every scheme X can be considered as a scheme over S=Spec(*Z*).

Further, there exists a fibered product $X \times_s S' \rightarrow S'$ for schemes X $\rightarrow$ S and S′ $\rightarrow$ S that effects the base change from S to S′. This product corresponds to the operation of extension or reduction of the scalars of the hypothetical equations for X. For example, every scheme is reduced modulo a prime number $p \in \mathbf{Z}$ by means of the product with S′=Spec($\mathbf{Z}/p$), producing in this way a family of schemes corresponding to the reduction modulo p of its hypothetical equations. Further, the product of X with S=Spec(**C**) produces a scheme in zero characteristic (an analytical space corresponding to the prime $p=\infty$). By isolating properties of "good behaviour" of the family by means of the concept of flat morphism, and rediscovering the concept of compactness by means of proper morphism, it is also possible to develop concepts of a differential nature in a purely algebraic context via the concept of smooth morphism.

These considerations led Grothendieck to develop systematically an algebraic geometry *relative* to the basis that makes it possible to "join together the various geometries associated with the various prime numbers".[7]

Seen in this way, a point of a scheme over a base will be simply a morphism of the base towards the scheme, and scheme may fail to have any point, that is, that it has points only when its base is changed.

An S-point of a scheme X→S is a morphism S→X that leaves S fixed. If k is a field S=Spec(k) it reduces topologically to a true point and the schemes of finite type over k, with their relative points, play the role of the new algebraic varieties, making it possible to visualise infinitesimal concepts by means of nilpoint elements. For example, the morphisms from Spec(k[$\varepsilon/\varepsilon^2$]) to a scheme X correspond to S-points of X over S=Spec(k) together with their tangent vectors.

[7] *Récoltes et Semailles*, p. 33.

In this sense the scheme X→S itself can be seen as a collection of fibres (X_s) as the points s ∈ S of the base vary, but also as the collection of all of its points relative to the base, that is, as all the schemes T→S and morphisms T→X that leave S fixed. This vision of a scheme leads to the concept of representability that makes it possible to construct schemes by representing them by means of their hypothetical (relative) points.

Just as the concept of scheme constitutes a broadening of the concept of algebraic variety, the concept of topos constitutes a metamorphosis of the concept of topological space.[8] The *étale* topos and the *crystalline* topos associated with a scheme constitute the fundamental step for visualising the structure, that is, for the constructing the *cohomology invariants* of the scheme. With the concept of site already developed in 1958 – "the most fertile year of my whole life"[9] – Grothendieck similarly developed a relative topology in which some morphisms serve the role of open sets. The topos corresponding to such a site makes the arithmetic nature of the schemes completely clear. To put it briefly,

$$\text{scheme} \Rightarrow \text{topos} \Rightarrow \text{cohomology.}$$

> . . .consider the set of all sheaves on a given topological space or, if you like, the prodigious arsenal of all the "meter sticks" that measure it. We consider this "set" or "arsenal" as equipped with its most evident structure, the way it appears so to speak "right in front of your nose"; that is what we call the structure of a "category". . . From here on, this kind of "measuring superstructure" called the "category of sheaves" will be taken as "incarnating" what is most essential to that space. . . .We can by now "forget" the initial space, keep and use the category (or arsenal) associated to it, which will be considered to be the most adequate incarnation of the topological (or spatial) structure that we intend to express.
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> As often happens in mathematics, we have succeeded here (thanks to the crucial idea of sheafs and cohomological measuring stick) to express a given notion (that of a certain space) in terms of another (that of category). As always, the discovery of this kind of translation of one notion (which expresses a certain kind of situation) into the terms of another (corresponding to another kind of situation) enriches our understanding of both of them through the unexpected confluence of specific intuitions in relationship to each other. Thus, a "topological" situation (incarnated in the given space) or, if you will, the incarnated "continuum" of the space is translated or expressed by the structure of the category, which is "algebraic".[10]

According to Grothendieck, a *cohomology theory* naturally follows from the six operations associated to the category derived from the topos.

Grothendieck's *six operations* are functors between derived categories. They are the derived tensor product $\overset{L}{\otimes}$, the ***RH***om (which produces the values of Ext^i) and, for any scheme morphism f: X→S, the direct image functors Rf_* and $Rf_!$ and the

[8]*Récoltes et Semailles*, p. 40.

[9]*Récoltes et Semailles*, p. 24.

[10]*Récoltes et Semailles*, pp. 38–39.

inverse image functors Lf_* and $Rf^!$. A theory of relative duality is expressed here by the adjunction between $Rf^!$ and $Rf_!$.

Grothendieck associates each geometry of characteristic p to an ℓ-adic cohomology corresponding to every prime $\ell \neq p$ by means of the étale topos, and a crystalline cohomology by means of the crystalline topos.

This arsenal of structures and operations is supposed to arrive at the same result. "It is in order to arrive and express this intuition of kinship between different cohomology theories that I have formulated the notion of motive associated to an algebraic variety".[11] This theme suggests that there is a common *motive* underlying the multitude of possible cohomology theories.

Grothendieck went on to suggest new conjectures, enhancing the unifying vision of the new geometry, the so-called "standard conjectures" that point to and predict the laws of a new *yoga* mediating between *form* and *structure*. While the Weil conjectures predicted the existence of a cohomology called, naturally enough, the Weil cohomology, later constructed by Grothendieck by means of the topos étale, that is, a structure associated to the form capable of grasping both the geometric and the arithmetic aspects, in the context of the dawning abstract (algebraic) geometry described above, Grothendieck's standard conjectures predict the existence of a motivic cohomology capable of synthesising in a single "invariant of the form" all of the structures that can be associated to it. Their formulation – obtained independently by Bombieri as well – appeared in a brief paper entitled "Standard conjectures on algebraic cycles" in the proceedings of the 1968 colloquium on algebraic geometry that took place in Bombay (Tata Institute of Fundamental Research, Mumbai).

The geometric construction of Grothendieck's motives is performed through the algebraic cycles that had already been introduced by Severi in the 1930s and then studied by Chow in the 1950s; these cycles are formal linear combinations of subvarieties and the correspondences from X to Y are defined by means of the cycles on the product $X \times Y$.

For a Weil cohomology $X \to H^*_\ell(X)$ there is a cycle map $Z^J(X) \to H_\ell{}^{2J}(X)$ which associates a cohomology class to every *algebraic cycle* of codimension j on X. The algebraic part of $H_\ell{}^{2*}(X)$ is that generated by classes of algebraic cycles. By means of Künneth's formula, we can also consider

$$H^*_\ell(X \times Y) = H^*_\ell(X) \otimes H^*_\ell(Y) = \mathrm{Hom}\,(H^*_\ell\,(X),\, H^*_\ell\,(X))$$

since $H^*_\ell(-)$ are vector spaces of finite dimension. The principle suggested by this identification is that cohomology operators of an algebraic kind have to be defined algebraically by means of a class associated to a cycle on the product, and thus by a correspondence.

[11] *Re´coltes et Semailles*, p. 46.

The two *standard conjectures* can be briefly summarised like this: the first, called the *Lefschetz standard conjecture*, states that a given operator $\Lambda{:}H^*_\ell(X) \to H^*_\ell(X)$ which is the quasi-inverse of the Lefschetz operator L is induced by an algebraic cycle, that is, that the operator induced – by iteration – from the Lefschetz operator restricted to the algebraic part is an isomorphism. The second conjecture, called the *Hodge standard conjecture*, states that a given definite bilinear form on the primitive algebraic cohomology class is positive definite.

One simple consequence of the standard conjectures is the validity of Riemann geometric hypothesis as stated in the famous Weil conjectures, as well as the coincidence of the cohomological and numerical equivalence for algebraic cycles: an open question even in zero characteristic.

This mediating *yoga* based on the concept of *motive* and the corresponding theory of motives should provide the most refined structures associated with forms like invariants:

$$\text{form} \Rightarrow \text{motive} \Rightarrow \text{structure}.$$

Just as a musical motive has various thematic incarnations, so the motive can have various incarnations, or avatars, such that the familial structures of the (cohomological) invariants of the forms will be "simply the faithful reflection of properties and structures internal to the motive".[12]

The first congress entirely dedicated to motives took place in Seattle in 1991. Noteworthy advances in this area were achieved by Vladimir Voevodsky – winner of the Fields Medal in 2002 – who constructed a triangulated category of motives, by using methods from algebraic homotopy that had also been partly presaged by Grothendieck as "motivic homotopy types".[13] Voevodsky's construction makes it possible to obtain an "incarnation" of the motivic cohomology but it does not, however, find a solution to the standard conjectures, which are still today – along with the Hodge conjecture – the fundamental open question in modern algebraic geometry.

To close, Grothendieck and Einstein, through a "mutation of the conception that we have of space, in a mathematical sense on one hand and a physical sense on the other",[14] and an innovation in the way we look at the world via a unifying vision drawn from mathematics on the one hand and physics on the other, have turned out to be the mathematician and the physicist who revolutionised scientific thought through the concept of relativity.

[12] *Récoltes et Semailles*, p. 46.
[13] *Récoltes et Semailles*, p. 47.
[14] *Récoltes et Semailles*, p. 59.

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All biographical information, bibliographies and more are available at the URL http://www.grothendieckcircle.org. A complete bibliography of Grothendieck's writings can also be found in the first volume of the *Grothendieck Festschrift* published by Birkhäuser in 1990. Here we note only a few essential sources used in this present work:

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