

Chapter 22. The Mathematical Meditations

22.1. *La longue marche à travers la théorie de Galois and Esquisse d'un programme*

It seems appropriate to discuss these two texts together, because in a way the *Esquisse d'un programme* is summary of *La Longue marche à travers la théorie de Galois*. About five hundred pages of *La longue marche à travers la théorie de Galois* have been transcribed and can be found at the Mathematical Institute of the University of Montpellier II. Grothendieck used *Esquisse d'un programme* in his application for a position at the CNRS. The text contains a summary of his main mathematical reflections from the beginning of the seventies. Since then it has been published together with an English translation¹. The central objects of study are the moduli spaces $M(g,n)$ for Riemann surfaces of genus g with n "distinguished points", objects that had already been investigated by Deligne and Mumford. Grothendieck establishes a relationship between the moduli spaces and certain arithmetic objects, in particular the absolute Galois groups of the field of rational numbers \mathbf{Q} . A theorem of Belyi giving a geometric characterization of algebraic curves defined over algebraic number fields plays a significant role. Grothendieck conceived his theory of *dessins d'enfants* in order to deal with the elementary combinatoric and geometric aspects of these questions. Apart from this theme, he also speculates about what he calls "anabelian geometry", a theory whose goal is to determine those arithmetic varieties that are completely determined by their algebraic fundamental groups. Amongst all of Grothendieck's mathematical "meditations", these are the ones that undoubtedly received the most attention. A number of papers dealing with these questions have been published, some workshops on these topics were organized in the nineties. It has been observed that for the first time in Grothendieck's mathematical development, his work on *dessins d'enfant* allowed him to really recognize the value of explicit examples.

In numerous letters written to mathematician colleagues in the eighties Grothendieck complains that he had had almost no feedback with regard to his research program. Whether these complaints were justified or not is an open question; he lived in almost total isolation and probably was only partially aware of what was going on in the mathematical world.

¹ On the theme of *La Longue marche a travers la Théorie de Galois and Esquisse d'un programme*, the following conference proceedings are available: they include in particular the letter to Faltings and the text of *Esquisse d'un programme*. *The Grothendieck Theory of Dessins d'Enfants*, L. Schneps (ed.), London Math. Soc. Lecture Notes 200 (1994); *Geometric Galois Actions 1 and 2*, P. Lochak and L. Schneps (eds.), London Math. Soc. Lecture Notes 242 and 243 (1997).

22.2. Pursuing Stacks and the Correspondence with Ronald Brown and Tim Porter

In 1982-83, Grothendieck had a long correspondence in English with Ronnie Brown. In a letter dated February 15, 1983, he wrote the following about his mathematical projects:

One reason to my poor answering, is that I feel somewhat "out of the game", and I am keen at not getting caught in any big technical machinery - the machine-building time is over for me now, and I want to be careful not to do more than occasionally throwing a very casual glance at the machine-building others pursue, and possibly making a comment or two, without really getting involved.

Four days later Grothendieck began writing a text in English about homotopic algebra with the title *Pursuing Stacks*, to which he also gave a French heading: *A la Poursuite des Champs*. Nine months later he had written six hundred and fifty pages and completed the editing, although he had not arrived at a definite result or conclusion. (This is what would typically happen whenever Grothendieck said he wanted to just do a little work on something on the side.) The text is accompanied by the extensive correspondence with Ronnie Brown as well as a few letters to his colleague Tim Porter, letters also comprise a total of several hundred pages.

Pursuing Stacks begins as a six-page letter to Quillen, and ends like an ordinary letter with the words "Very cordially yours". Over the next few days, however, Grothendieck added eighteen more pages, although this part of the text seems more like a conversation with himself than with another mathematician. A few months later Grothendieck complained that Quillen had never responded to his letter.

But in any case Grothendieck was not waiting for an answer to continue the work. On 27 February 1983 he started on a new text, called *Reflections on homotopical algebra*. He worked with breathtaking speed right into the month of July, and when assessing his output one must take into account not only on the text but also the accompanying correspondence.

During the first half of that August, Grothendieck took a holiday trip to the Pyrenées, something he had not done since taking a trip there with his family in the early sixties. On August 11, 1983, he noted in *Pursuing Stacks*:

It has been over three weeks now I haven't been working on the notes. Most part of the time was spent wandering in the Pyrenées with some friends (a kind of thing I hadn't been doing since I was a boy), and touring some other friends living the simple life around there, in the mountans. I was glad to meet them and happy to wander and breeze the fresher air of the mountains - and very happy too after two weeks to be back in the familiar surroundings of my home amidst the gentle hills covered with vineyards [...]

After this, Grothendieck's work progressed more slowly; the last pages were written on November 12, 1983, without reaching a definitive conclusion. The final lines, in Grothendieck's inimitable style, read:

I daresay life has been generous with me for the last three months, while I haven't even taken the trouble to stop with the mathematical nonsense for any more than a week or two. This week too I still did some mathematical scratchwork, still along the lines of abelianization, which keeps showing a lot richer than suspected.

Grothendieck generally wrote his meditations with the intention of publishing them. Indeed, at this time he was envisioning a work of several volumes comprising all of his mathematical meditations. Apparently he had even contacted a publisher. On September 26, 1983, he noted on page 500: "Also I more or less promised the publisher, Pierre Berès, that a first volume would be ready for the printer by the end of this calendar year, and I would like to keep promise."²

At the turn of the year 1983/84, following the completion of *Pursuing Stacks*, Grothendieck corresponded with H.J. Baues from Bonn on questions of abstract algebraic homotopy theory. Baues invited Grothendieck to a small conference in Bonn, in which Bénabou, Brown, Porter and Kamps were to participate as well. Grothendieck naturally refused the invitation:

I thank you warmly for your invitation to a short meeting of mathematicians in Bonn. Ronnie Brown and J. L. Loday made similar suggestions, as both of them are interested in a meeting on the subject of the basis of homotopy theory. There is no doubt that participating in such a meeting would be beneficial to my crass lack of culture in homotopy theory - over the last ten years however I have developed a sort of allergy to "learned encounters" of all kinds, which I wish to allow for, and for which I request your sympathetic understanding!

As already mentioned in Chapter 21, the meditation *Pursuing Stacks* contains two unusual inserts. On August 22, 1983, Grothendieck wrote of the death of his nine-year-old granddaughter Ella, the daughter of Serge, and on October 22, he vividly described the birth of his grandson Suleyman, his daughter Johanna's second child.

In the summer of 1989, Grothendieck received a visit from Bill Lawvere in Les Aumettes. Grothendieck had just made a vow of silence, so that apart from the single word of greeting ("Bill!"), the two of them communicated only in writing. Grothendieck felt that Lawvere was the right person to publish *Pursuing Stacks*. But in turn, Lawvere felt that the text needed to be thoroughly edited, and numerous errors corrected. Grothendieck would not hear of it: students should see that even a great mathematician makes mistakes. The argument that even mathematical texts without

² Pierre Berès, 1913-2008, was an important antiquarian and collector of autographs and books, who was acquainted with many notable personalities of his time. In 1956 he acquired the publishing house Editions Hermann, which among other things published the Bourbaki volumes. He was not the only person with whom Grothendieck negotiated about the publication of his meditations. The details of these negotiations have not been unraveled.

errors were already quite difficult to understand failed to impress Grothendieck in the least, so that in the end the whole project fell through.

Around 2002, the *Société Mathématique de France* asked George Maltsiniotis to be in charge of the publication of *Pursuing Stacks*. At the end of 2009, after Maltsiniotis, with the help of Matthias Kunzer, had already devoted more than two thousand hours of work to the project (by his own estimation), Grothendieck distributed a message to Luc Illusie in which he expressly forbade the publication or reproduction of any of his texts whatsoever, whether already published or not. Nevertheless, the editing work of Maltsiniotis continued.

22.3. A manuscript on Topology.

In the middle of 1986 Grothendieck wrote down with frantic speed his long-held thoughts on the refoundations of topology. It was not a typed manuscript, but consisted of handwritten notes, which seem rather tentative. It would be very difficult, if not impossible, to create a readable text on the basis of these notes. In a letter to Jun-Ichi Yamashita dated July 9, 1986, Grothendieck reported on his work:

I have been intensely busy for about a month now, with writing down some altogether different foundations of "topology", starting with the "geometrical objects" or "figures", rather than with a set of "points" and some kind of notion of "limit" or (equivalently) "neighborhoods". Like the language of topoi (and unlike the so-called "moderate space" theory foreshadowed in the *Esquisse*, still waiting for someone to take hold of the work in store...), it is a kind of topology "without points" - a direct approach to "shape". I do hope the language I have started developing will be appropriate for dealing with finite spaces, which come off very poorly in "general topology" (even when working with non-Hausdorff spaces). After all, presumably the space-time space we are living in is finite - at any rate there is no philosophical evidence whatever that it isn't, and still less, that it is adequately represented as a mathematical "continuum" (more specifically, as a topological or differentiable or Riemannian or pseudo-Riemannian "manifold") - and as for physical evidence; it is clear there cannot be any by the very nature of things, as measurements never yield anything else but approximate locations of the would-be (ideal) "points". These "points" however do not have any empirical existence whatever. As Riemann pointed out, I believe, the mathematical continuum is a convenient fiction for dealing with physical phenomena, and the mathematics of infinity are just a way of approximating (by simplification through "idealization") an understanding of finite aggregates, whose structures seem too elusive or too hopelessly intricate for a more direct understanding (at least it has been so up to now).

The letter then goes on to explain in greater depth Grothendieck's thoughts on physics and on the basis of topology (and also analysis and geometry). A thought he frequently reiterates is that he considers it possible that physical space has a discrete structure.

22.4. *Réflexions Mathématiques*

At some point around the middle of the eighties, Grothendieck decided to publish his *Réflexions* in several volumes. He wrote a foreword of slightly more than one page to which he added several handwritten corrections. The author possesses only a poor

quality and partly illegible photocopy of this, so that the following text is really just an approximation of what he actually wrote or intended to write. It is clear that he soon gave up on this project. In fact he never made a truly serious effort to have his works published.

After a twelve years' silence, the time has come for the author's works of maturity, with his vision and style renewed. Here is a day-by-day account of an explorer's travel, going on through the very concept of writing - and this travel gives rise to occasional reflections on the travel (as well as on the traveller and on the manifold world around) thus recapturing its genuine nature: of an impassioned adventure, rooted in life.

In the two forthcoming volumes, under the common title "Pursuing Stacks" (the first volume of which is subtitled "The Modelizing Story"), the author sketches some main themes of a vast synthesis of homotopical algebra, (co)homological algebra, and topos theory - ripe now for more than fifteen years and however never yet begun. The leading thread, tenacious and omnipresent (while often remaining implicit) comes from algebraic geometry, and from intuitions (rich, precise and provocatingly fragmentary) around the notions of sheaves, stacks, 2-stacks "etc." on a topos.

These two volumes are the first in a planned considerably wider series of "*Réflexions Mathématiques*" where the author intends to present, among other things, some "naive Zideas" about two "new continents", eager to be discovered and explored. One takes birth with a suitable notion of a "tame" topological space [...] The other one appears as the confluence of a multitude of streams of thought and insights, coming from topology, conformal geometry, algebraic geometry, discrete and profinite groups, algebraic groups over number fields, regular polyhedra, arithmetics, "motives" ... bound for joining in the exploration of the action of profinite "absolute" Galois groups [...] on certain (so-called "anabelian") profinite geometrical fundamental groups [...]

In truth, probably not many mathematicians would describe these texts as Grothendieck's "work of maturity".

On November 14, 1985, Grothendieck wrote to his German friend N. W., whom he apparently knew only through correspondence, with regard to the same project:

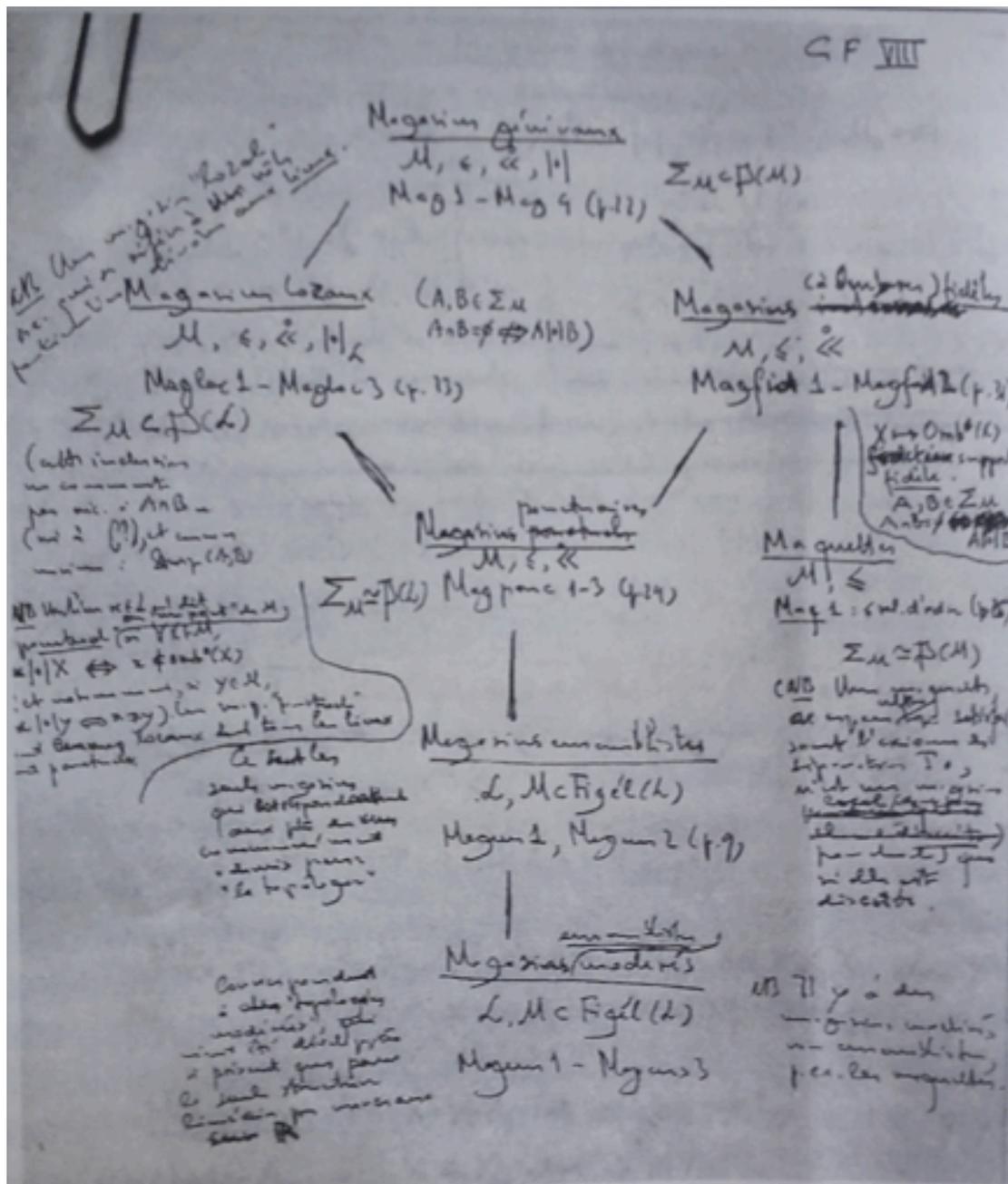
It is certainly not my intention, however, to get back onto the mathematical treadmill or to compete with gravediggers, but only to set on paper, in broad strokes, a vision which has been torn up, trampled on and buried, and thus to make it available to all those who might one day be interested in it. That should probably take about three years of work (if not interrupted too much by other things, such as meditation). It is possible, and I actually hope for this, that I will then be able to call it quits as far as mathematics is concerned.

22.5. *Les Dérivateurs*

Les Dérivateurs was written after Grothendieck had already survived some terrible psychological crises. Once again, he wrote a text of over two thousand handwritten pages over the course of just a few months, this time situated between September 1990 and May 1991. Contrary to his usual habit, only a few of the pages have dates, namely those from November 1990. Georges Maltsiniotis and Matthias Künzer undertook to decipher these pages and typeset them using TeX. Large sections of the

text are available online. The definitive publication of *Les Dérivateurs* was delayed and finally dropped, primarily because work on *Pursuing Stacks* seemed more urgent.

The idea of "derivators" was already implicit in SGA IV, and was further elaborated by Grothendieck in *Pursuing Stacks* (Section 69 and after). Alex Heller, Bernhard Keller and Jens Franke developed similar ideas independently. It appears that Grothendieck wrote *Les Dérivateurs* because the representation of derived categories by Verdier and also the ideas of Illusie and Deligne on this subject did not satisfy him. He certainly was unaware of later developments. According to Maltsiniotis, *Les Dérivateurs* is a fully structured and nearly definitive text, in contrast to the journalistic style of *Pursuing Stacks*.



Mathematical notes by Grothendieck from around 1986