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Boekbespreking

Correspondance Grothendieck-Serre

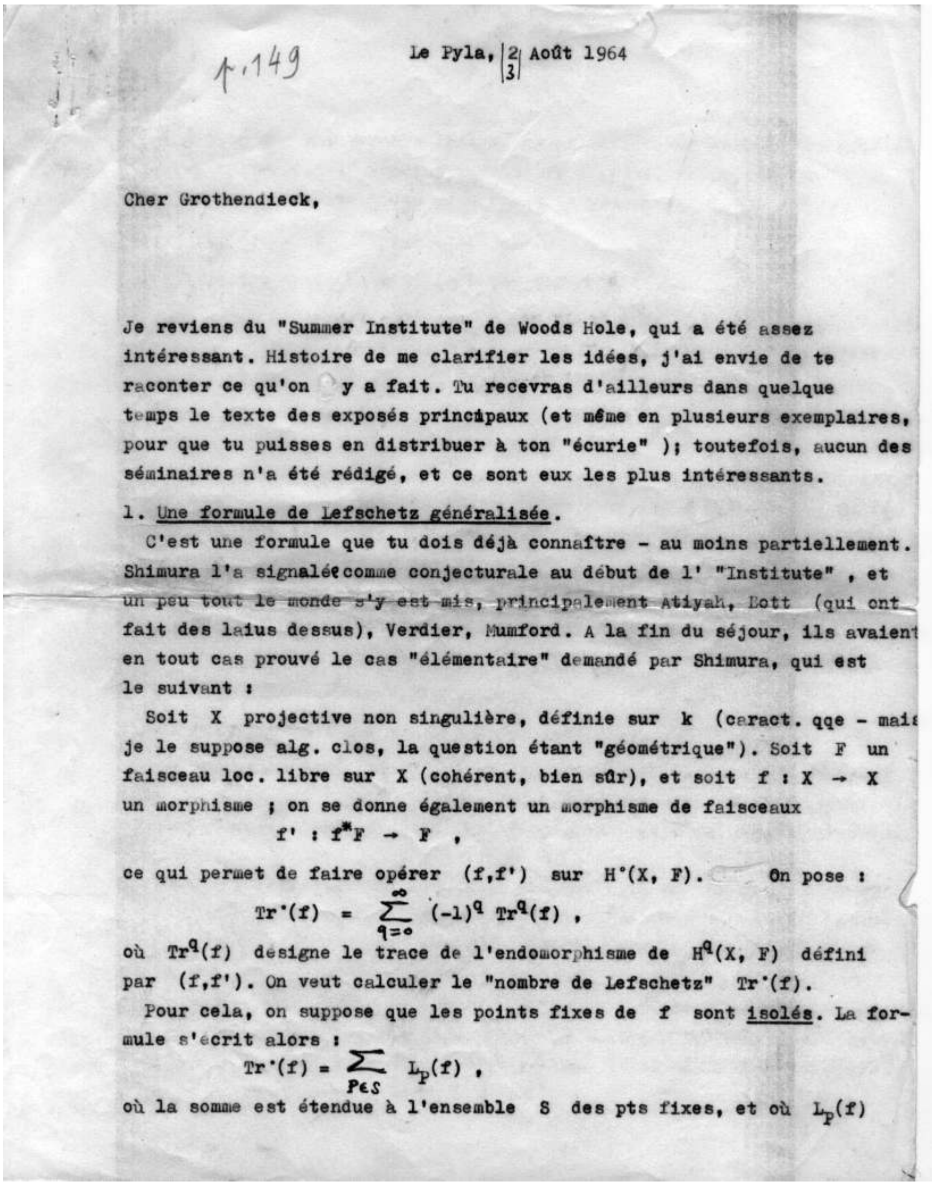
In januari 2004 verscheen bij de American Mathematical Society een Engelse vertaling van de briefwisseling tussen twee van de grootste wiskundigen van de tweede helft van de twintigste eeuw: Jean-Pierre Serre en Alexander Grothendieck. Deze briefwisseling verscheen eerder, in 2001, als publicatie van de Société Mathématique de France. De gerenommeerde wiskundige John Tate heeft, evenals Serre en Grothendieck, baanbrekend werk verricht op het gebied van de algebraïsche meetkunde en getaltheorie. Hij is hoogleraar aan de University of Texas at Austin. Hij is lid van de National Academy of Sciences in de Verenigde Staten, buitenlands lid van de Franse Académie des sciences en erelid van de London Mathematical Society. Hier geeft hij zijn commentaar op de Franse editie.

This book gives us intimate views of the collaboration which took place between A. Grothendieck and J-P. Serre during the period 1955–1969, when they played such key roles in revolutionary developments in algebraic geometry. The volume contains about eighty essentially unedited letters from that period plus a few from the mid eighties. It exists thanks to Serre who preserved the letters, to Pierre Colmez who obtained Grothendieck's permission to publish them through an intermediary Jean Malgoire and organized an army of students to TEX the letters, to the efforts of those students, and to the French Mathematical Society which created a new book series, *Documents Mathématiques* to accomodate the publication of the letters. The book is attractively and readably printed, with reproductions of two of Grothendieck's letters, one handwritten, one typed, which show that TEXing the letters was not always a routine job. Serre has added very helpful notes explaining unclear references, noting assertions which turned out to be false, and indicating as far as possible the subsequent fate of conjectures made in the letters.

Unfortunately the historical record provided by the letters is full of gaps because the letters were written for the most part only when Serre and Grothendieck were not both in the Paris region. When together there they spent hours on the phone. One can only regret that those conversations were not taped. But if not furnishing a complete historical record, the letters do afford a real understanding of how these men thought and worked together. Their collaboration was especially fruitful because the two think so differently: Grothendieck rushes ahead on his own, full of optimism, guided by the big picture, putting everything in the most general context, sometimes overlooking details in his hurry; Serre, with an amazing knowledge of the literature and special cases, very careful with details, working with a good balance between special and general, and always ready to construct an example or counterexample to keep Grothendieck on track.

Faisceaux Algébriques Cohérents

The collaboration by letter begins in January 1955. Serre has seen that sheaf theory which was so useful in the Cartan seminars on complex analytic varieties serves also in abstract algebraic geometry despite the weirdness of the Zariski topology. He is finishing his paper 'Faisceaux Algébriques Cohérents' (FAC) on the subject. Grothendieck has decided to abandon topological vector spaces for algebraic geometry. In the early letters he bombards Serre with questions, some naive,



f.149 Le Pyla, 2/3 Août 1964

Cher Grothendieck,

Je reviens du "Summer Institute" de Woods Hole, qui a été assez intéressant. Histoire de me clarifier les idées, j'ai envie de te raconter ce qu'on y a fait. Tu recevras d'ailleurs dans quelque temps le texte des exposés principaux (et même en plusieurs exemplaires, pour que tu puisses en distribuer à ton "écurie"); toutefois, aucun des séminaires n'a été rédigé, et ce sont eux les plus intéressants.

1. Une formule de Lefschetz généralisée.

C'est une formule que tu dois déjà connaître – au moins partiellement. Shimura l'a signalée comme conjecturale au début de l' "Institute" , et un peu tout le monde s'y est mis, principalement Atiyah, Bott (qui ont fait des laius dessus), Verdier, Mumford. A la fin du séjour, ils avaient en tout cas prouvé le cas "élémentaire" demandé par Shimura, qui est le suivant :

Soit X projective non singulière, définie sur k (caract. qqe – mais je le suppose alg. clos, la question étant "géométrique"). Soit F un faisceau loc. libre sur X (cohérent, bien sûr), et soit $f : X \to X$ un morphisme ; on se donne également un morphisme de faisceaux

$$f' : f^*F \to F ,$$

ce qui permet de faire opérer (f,f') sur $H^\bullet(X, F)$. On pose :

$$Tr^\bullet(f) = \sum_{q=0}^{\infty} (-1)^q \, Tr^q(f) ,$$

où $Tr^q(f)$ désigne le trace de l'endomorphisme de $H^q(X, F)$ défini par (f,f'). On veut calculer le "nombre de Lefschetz" $Tr^\bullet(f)$.

Pour cela, on suppose que les points fixes de f sont isolés. La formule s'écrit alors :

$$Tr^\bullet(f) = \sum_{P \in S} L_P(f) ,$$

où la somme est étendue à l'ensemble S des pts fixes, et où $L_P(f)$

First page of a letter from Serre to Grothendieck, dated August 2–3, 1964. Serre tells of his mathematical experiences at the *Summer Institute* in Woods Hole. Grothendieck's answer is printed on the next page.

some not so, which Serre answers patiently. Much of the discussion concerns sheaves.

Grothendieck is developing his own view of abelian categories, derived functors, the various theories of sheaf cohomology, spectral sequences,...as only he could. His summary was later published as 'Sur quelques points d'algèbre homologique' in *Tohoku Math. J. (1957)*. By the end of 1955 Grothendieck explains to Serre a general formulation of Serre duality, saying it is almost evident, and implicit in FAC. Serre is excited by this formulation, saying it is *the* good way to state duality in both the algebraic case and the analytic case. FAC finished, Serre is following it with GAGA, 'Géométrie Algébrique et Géométrie Analytique', proving that for projective varieties, the analytic coherent sheaves are the 'same' as algebraic ones.

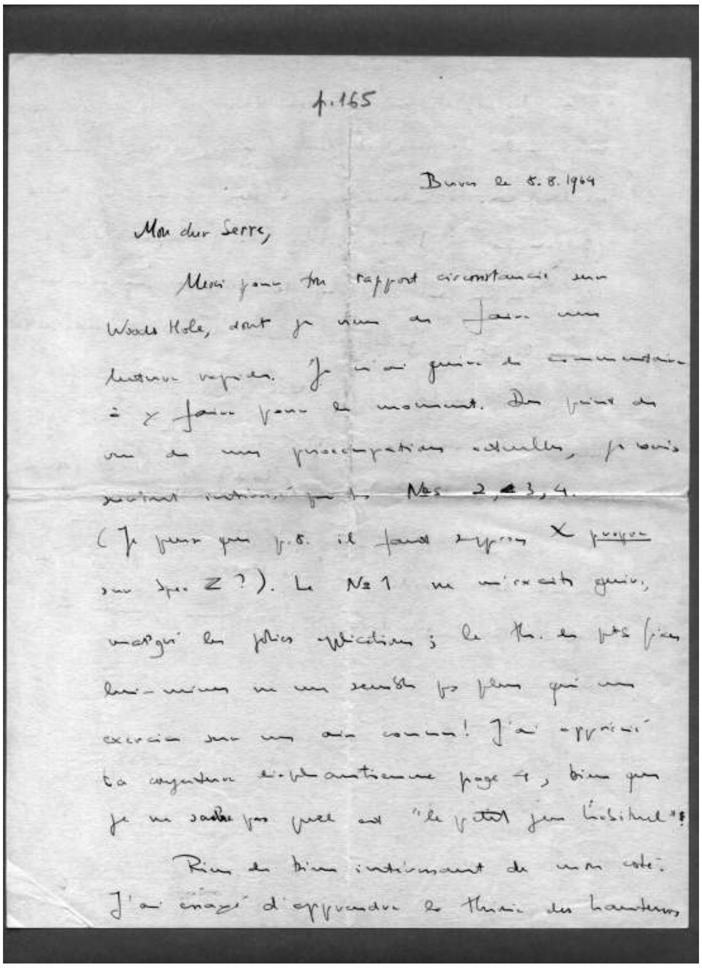
The best category of commutative rings

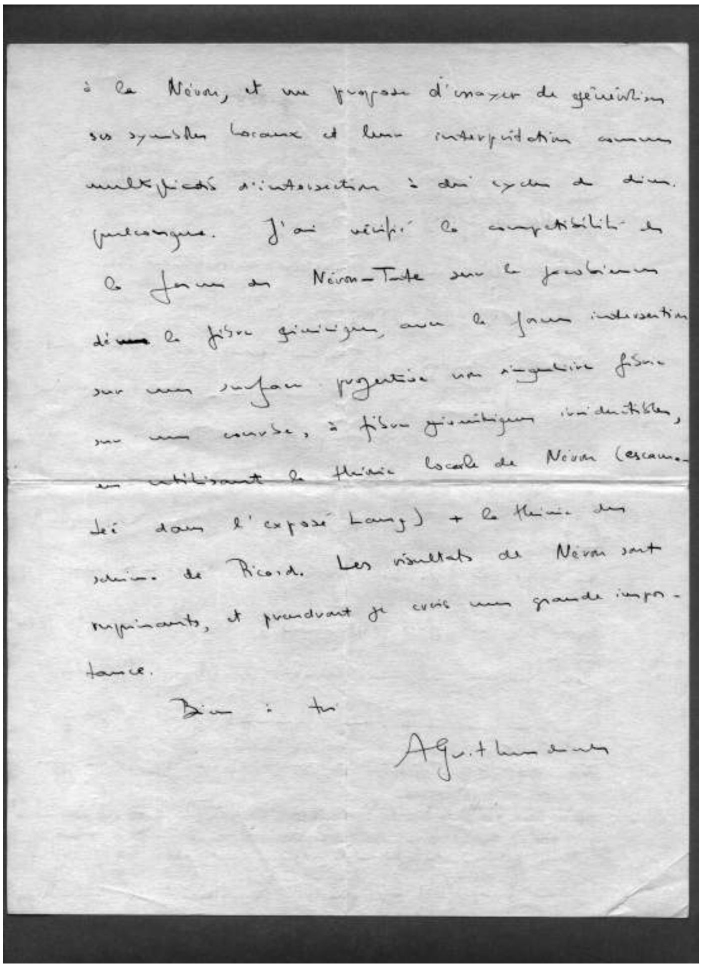
Schemes were already in the air, though always with restrictions on the rings involved. In February 1955 Serre mentions that the theory of coherent sheaves works on the spectrum of commutative rings in which every prime ideal is an intersection of maximal ideals. A year later, Grothendieck tells of Cartier's introducing the technically useful notion of quasi-coherent sheaf on arithmetic varieties made by gluing together the spectra of noetherian rings. But it would take two more years before Grothendieck realized that the noetherian condition should be dropped and one should include all prime ideals in the spectrum, that in the end, as Serre puts it in the notes, the best category of commutative rings is the category of *all* commutative rings.

So far, we have given a very sketchy and incomplete description of the mathematics involved in the first two years of the collaboration, as revealed by 27 letters covering 55 pages. The problems under discussion were largely foundational ones. The next seven years saw the exchange of 26 letters, about 85 pages. Some of the topics discussed during these years are: Gothendieck's proof of Riemann-Roch, the progress of his writing of *Éléments de Géométrie Algébrique*(EGA), begun in 1958, Serre's geometric class field theory, Grothendieck's theory of the fundamental group (one of the first big successes of the theory of schemes), Weil's restriction of scalars functor, the astonishing news of Dwork's proof of the rationality of the zeta function of a variety over a finite field (1959), generalized Jacobians and local symbols, purity, the (sad?) situation of mathematics in Paris in 1961, p-adic Lie algebras, M. Artin's proof that every non-singular variety is covered by 'good' opens, the local invariants in the formula expressing the Euler characteristic of an étale sheaf on a curve in terms of its singularities in the case of higher ramification which were requested by Grothendieck, and supplied by Serre, the Swan representation.

An outburst of correspondence

Suddenly, in 1964, there are 22 letters in one year. A big theme is algebraic cycles and the topics are becoming more arithmetical. Serre writes to Grothendieck an extensive report on the Woods Hole Summer Institute: Woods Hole fixed point formula, conjectures on l-adic cohomology of a variety over a local field inspired by Ogg's work on elliptic curves, my conjectures on algebraic cycles and poles of zeta functions, formal groups, the Serre-Tate theorem and Serre's canonical lift. Much of the discussion in the following months is stimulated by this report and by a letter from Serre to A. Ogg announcing the criterion of Ogg-Néron-Shafarevich for good reduction of an abelian variety and its agreeable corollaries, together with a question which Serre calls "naive": is it true that after a finite extension of the ground field, the connected component of the special fiber of the Néron model of an abelian variety is an extension of an abelian variety by a torus. Grothendieck has no sentiment at first, but later proves it. There is much discussion of zeta and L-functions.





Grothendieck's answer, dated August 8, 1964.

Grothendieck discusses vanishing cycles and introduces a new concept which he calls a "motive", which is to become one of his greatest contributions. Soon after, we have motivic Galois groups, and in early 1965 he formulates the "standard conjectures".

Farewell to Grothendieck

The collaboration is coming to an end. There are three more letters exchanging mathematical ideas. The last, in January 1969 from Grothendieck contains his thoughts on what Serre has told him about Steinberg's theorem. Not long after that Grothendieck abandoned the writing of EGA and dropped out of the mathematical world.

The rest of the book, six letters from the mid 1980's is sad to read. Grothendieck has isolated himself and begun writing *Récoltes et Semailles*, a rambling account of his bitter somewhat paranoid reflections on his mathematical life and on the behaviour of his former students and his colleagues, which he distributes to them. Serre gets the first chapters and writes Grothendieck his comments. Grothendieck finds that Serre accepts the nice things in his account but rejects the disagreeable ones, and says that since he's known him Serre has always shown a horror of serious self-examination. In his reply, Serre asks why Grothendieck abandoned his mathematical program and hazards two guesses. Had the writing of EGA and the seminars simply become too great a burden? Or was the problem a deeper one, that Grothendieck realized that his method of doing mathematics, building big general theories, which worked so well for topological vector spaces and algebraic geometry was not as effective for the problems of number theory and modular forms which were looming in the late sixties when Grothendieck abandoned his program?

At the end of 1986, Serre writes a last mathematical letter, explaining to Grothendieck his conjectures on modular Galois representations of degree 2, and sending best wishes for 1987. Grothendieck replies, thanking Serre for his trouble, but explaining that it is not worth the effort. He has his own projects and is not interested, but wishes Serre a happy 1987 and all success in his work. Of course the book will be a great resource for future historians of 20th century mathematics, but it is much more than that. It gives today's reader a feel for a very different mathematical era and a unique opportunity to be present at the exchange of mathematical ideas at the highest level, in complete comradeship, between two masters.

Correspondance Grothendieck-Serre, Pierre Colmez–Jean-Pierre Serre (Eds.), Société Mathématique de France, 2001, ISBN 2-85629-104-X.

Grothendieck-Serre Correspondence, Bilingual edition, Pierre Colmez–Jean-Pierre Serre (Eds.), Catriona Maclean–Leila Schneps (translation), American Mathematical Society, Société Mathématique de France, 2004, ISBN 0-8218-3424-X.