Preliminary version of the article appeared in *Regular and Chaotic Dynamics* **18**, No. 6 (2013) dedicated to Alain Chenciner, 742–775

Abstract

Two weakly hyperbolic smooth $\mathbf{Z}^k \times \mathbf{R}^m$-action germs are smoothly conjugate if and only if they are formally conjugate, and such.

AMS Subject Classification: 34A26; 34C20; 34C45; 34M35; 37C15; 37C85; 37D10; 37D15; 37D30; 37F75; 37G05; 01-02; 01A80.

Keywords: Hyperbolic, action germ, conjugacy, invariant manifold, normal form, flow, Lyapunov function, Cauchy problem, holomorphic vector field, first integral.

A forgotten theorem on $\mathbf{Z}^k \times \mathbf{R}^m$-action germs and related questions

Marc Chaperon*

- Qu'as tu fait de ta vie, pitance de roi ?

- J'ai vu l'homme.

Je n'ai pas vu l'homme comme la mouette, vague au ventre, qui file rapide sur la mer indéfinie.
J'ai vu l'homme à la torche faible, ployé et qui cherchait. Il avait le sérieux de la puce qui saute, mais son saut était rare et réglementé.
Sa cathédrale avait la flèche molle. Il était préoccupé.

Henri Michaux

How I met Alain Chenciner and what I did under his supervision

In my first year at the École Normale Supérieure, I was disappointed by mathematicians, who got excited about futile problems instead of sticking to meaningful ones[1]. Hence, I studied computer programming during the second year and mathematical economics during the third. The ensuing boredom[2] took me back to mathematics, which after all I loved or at least loved doing.

Then, René Thom's *Stabilité structurelle et morphogénèse* appeared. Even though I did not understand half of it[3], I felt strong affinities with the underlying vision of the world and chose to go that way.

After one year of very well paid purgatory as the mathematics teacher of a *classe préparatoire*[4], I got—thanks to Thom—a temporary research position at the Centre de mathématiques de l'École Polytechnique, directed by its founder Laurent Schwartz. There, in a nice stimulating atmosphere, I met other members of my mathematical family: Michael R. Herman, who had not yet proven *the* Arnold conjecture [16, 25], François Laudenbach, Bernard Teissier and Alain Chenciner; closest to Thom, Alain was to become my doctoral supervisor[5].

*Institut de Mathématiques de Jussieu, Université Paris 7

[1] *Les problèmes qui se posent*, as opposed to *les problèmes qu'on se pose* (Poincaré).

[2] With the notable exception of beautiful lectures by Ivar Ekeland on game theory.

[3] Not an isolated case, I gathered.

[4] Where selected pupils train for the competitions leading to the notorious *grandes écoles*.

[5] For my *thèse d'État*, a kind of habilitation thesis.

Thom having advised me to start with a thesis in pure mathematics ("much easier than applied mathematics"), my first work was to study, complete and extend his wonderful little paper [24] on implicit differential equations, outstanding propaganda for contact geometry and singularity theory. My contribution concerned the singularities of the 1-dimensional characteristic foliation of the submanifold defined by the equation in 1-jet space[6]. Most of it remained unpublished[7] due to work by Takens on constrained systems [23], not the same question but quite the same structure. However, thanks to Herman, I had learned much about the analytic or smooth local classification of dynamical systems, widely ignored at the time because of Thom's structural stability program.

A more permanent position at the CNRS enabled me, after an excursion in partial differential equations [3, 4], to work on a conjecture of Camacho, Kuiper and Palis [1] on the topological classification of the singular 1-dimensional complex foliations defined by holomorphic vector fields near their zeros.

Dumortier and Roussarie had just proven it to be "almost always" true[8] as a consequence of their smooth simultaneous linearisation result for pairs of commuting smooth vector fields near a common zero [15]. Since this was close to my brand new competence domain, my program was to simplify their proof, extend it from formally linearisable germs of $\mathbf{R}^2$-actions to more general germs of $\mathbf{Z}^k \times \mathbf{R}^m$-actions[9] and use the extension to prove the conjecture.

Two years later, I had fulfilled this program and defended my thesis [7]. Alain's help had been unvaluable: though not a specialist, he could understand very quickly what I was doing and give amazingly good advice.

After the excitement of discovery came the much duller task of making my work known. This took me nearly five years[10] and resulted in two publications:

- the book [10], which contains much background material, many unpublished novelties (probably ignored up to now) and my generalisation of the Dumortier-Roussarie linearisation theorem to germs of $\mathbf{Z}^k \times \mathbf{R}^m$-actions
- the article [12], where—among other things—the Camacho-Kuiper-Palis conjecture is proven.

One of the main results of [7] was not included[11], hence the present article. As the chief reason for publishing it lies in current work on complete integrability, theorems on first integrals are stated in the end and their proofs are sketched.

[6]The projected leaves in 0-jet space contain the graphs of solutions but have singularities.
[7]Apart from [2], written under Alain's supervision.
[8]Which already resulted from Siegel's holomorphic linearisation theorem [20].
[9]A problem that no one cared about in such generality. Things are a bit different today.
[10]Partly because in the meantime I contributed to the birth of symplectic topology [8, 9].
[11]In the late eighties, I thought I should publish it but Alain told me: "Everyone knows you have proven it"—an optimistic statement, obviously not true today.

1 Statement of the main theorem

1.1 Germs of $\mathbf{Z}^k \times \mathbf{R}^m$-actions and their linear part

Given a finite dimensional smooth manifold M and $a \in M$, a *smooth germ at a of $\mathbf{Z}^k \times \mathbf{R}^m$-action on M* is defined by the datum of k smooth diffeomorphism germs $g_1, \ldots, g_k : (M, a) \to (M, a)$ and m germs $X_1, \ldots, X_m$ at a of smooth (i.e. C^∞) vector fields vanishing at a, each of which commutes with the others in the sense that

$$\begin{cases} g_i \circ g_j = g_j \circ g_i & \text{for } 1 \le i < j \le k, \\ g_i^* X_j = X_j & \text{for } 1 \le i \le k \text{ and } 1 \le j \le m, \\ [X_i, X_j] = 0 & \text{for } 1 \le i < j \le m. \end{cases} \tag{1}$$

Setting $r = k + m$ and denoting by g_{k+j}^t the flow of X_j, this defines the homomorphism $g : t \mapsto g^t$ of $\mathbf{Z}^k \times \mathbf{R}^m$ into the group of smooth diffeomorphism germs $(M, a) \to (M, a)$ given by

$$g^t = g_1^{t_1} \circ \cdots \circ g_r^{t_r} \text{ for } t = (t_1, \ldots, t_r) \in \mathbf{Z}^k \times \mathbf{R}^m.$$

The endomorphisms $L_i = dg_i(a)$ and $\Lambda_j = dX_j(a)$ of the tangent space

$$E = T_a M$$

all commute; they define the *linear part* of the action germ. Setting $L_{k+j}^s = e^{s\Lambda_j}$, this linear part can be viewed as the linear representation $L : t \mapsto L^t$ of $\mathbf{Z}^k \times \mathbf{R}^m$ on E defined by

$$L^t = L_1^{t_1} \circ \cdots \circ L_r^{t_r} \text{ for } t = (t_1, \ldots, t_r) \in \mathbf{Z}^k \times \mathbf{R}^m.$$

The following proposition will be made precise in subsection 2.1:

Proposition 1.1 *There exist continuous group homomorphisms $a_1, \ldots, a_n$ of $\mathbf{Z}^k \times \mathbf{R}^m$ into $\mathbf{C}^*$ and a decomposition of E as the direct sum of subspaces $E_1, \ldots, E_n$, the* characteristic subspaces *of L, with the following properties:*

i) For $1 \le \ell \le n$, all the automorphisms L^t preserve E_ℓ and

- *either a_ℓ is real-valued, in which case each L^t induces an automorphism of E_ℓ having $a_\ell(t)$ as its single eigenvalue,*
- *or a_ℓ is not real, and then each L^t induces an automorphism of E_ℓ with the sole eigenvalues $a_\ell(t), \overline{a_\ell(t)}$ (which can be real for some t's).*

ii) One has $\{a_\ell, \overline{a_\ell}\} \ne \{a_{\ell'}, \overline{a_{\ell'}}\}$ for $1 \le \ell < \ell' \le n$.

The set $\{E_1, \ldots, E_n\}$ is determined by L in a unique way.

Note. If $(\delta_1, \ldots, \delta_r)$ denotes the canonical basis of $\mathbf{R}^r$, then:

- for $1 \le j \le k$, the numbers $a_\ell(\delta_j), \overline{a_\ell(\delta_j)}$ are the eigenvalues of L_j;
- for $k < j \le r$, one has $a_\ell(s\delta_j) = e^{\alpha_{\ell,j} s}$ for all $s \in \mathbf{R}$, where the numbers $\alpha_{\ell,j}, \overline{\alpha_{\ell,j}}$ are the eigenvalues of Λ_{j-k}.

1.2 Hyperbolicity, conjugacy, the main theorem

Under the same hypotheses and with the same notation, the group homomorphisms $t \mapsto \ln|a_\ell(t)|$ of $\mathbf{Z}^k \times \mathbf{R}^m$ into $\mathbf{R}$, intrinsically associated to L, can be extended to linear forms $c_1, \ldots, c_n$ on $\mathbf{R}^r = \mathbf{R}^{k+m}$; following [5, 6], we call the linear action L (and the action germ g)

- *weakly hyperbolic* if, for $1 \le s \le r$ and $\ell_1, \ldots, \ell_s \in \{1, \ldots, n\}$, the convex hull $\operatorname{conv}\{c_{\ell_1}, \ldots, c_{\ell_s}\}$ does not contain the origin;
- *hyperbolic* if, for $1 \le s \le r$ and $1 \le \ell_1 < \cdots < \ell_s \le n$, the linear forms $c_{\ell_1}, \ldots, c_{\ell_s}$ are linearly independent;
- *strongly hyperbolic* if it is hyperbolic and, moreover, the a_ℓ's are simple in the sense that E_ℓ is a line when a_ℓ is real, a 2-plane otherwise.

Still denoting by $(\delta_1, \ldots, \delta_r)$ the canonical basis of $\mathbf{R}^r$, the numbers $c_\ell(\delta_j)$ are the logarithms of the moduli of the eigenvalues of L_j for $1 \le j \le k$, the real parts of the eigenvalues of Λ_{j-k} for $k < j \le r$.

Examples. For $r = 1$, weak hyperbolicity is just hyperbolicity in the usual sense: if $(k, m) = (1, 0)$ (resp. $(0, 1)$) the automorphism L_1 (resp. the endomorphism Λ_1) has no eigenvalue on the unit circle (resp. the imaginary axis).

A holomorphic vector field germ vanishing at a defines a holomorphic $\mathbf{C}$-action germ, which is the $\mathbf{R}^2$-action germ defined by $X_1 = X$ and $X_2 = iX$ viewed as real vector fields; if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the complex endomorphism $L = dX(a)$, then $E_1, \ldots, E_n$ are the characteristic subspaces of L and the isomorphism $\lambda \mapsto \big((t_1, t_2) \mapsto \Re\mathrm{e}\, \lambda(t_1 + it_2)\big)$ of $\mathbf{C}$ onto $L(\mathbf{R}^2, \mathbf{R})$ identifies the λ_ℓ's to the c_ℓ's; hence, weak hyperbolicity means that no segment $[\lambda_\ell, \lambda_{\ell'}]$ contains the origin, whereas hyperbolicity means that the eigenvalues λ_ℓ are two by two $\mathbf{R}$-independent.

Conjugacy. Smooth $\mathbf{Z}^k \times \mathbf{R}^m$-action germs g at $a \in M$ and g' at $a' \in M'$, defined by $g_1, \ldots, g_k, X_1, \ldots, X_m$ and $g'_1, \ldots, g'_k, X'_1, \ldots, X'_m$ respectively, are C^α*-conjugate*, $0 \le \alpha \le \infty$, when there exists a C^α diffeomorphism germ $h : (M, a) \to (M', a')$ such that $g_j = h^* g'_j := h^{-1} \circ g'_j \circ h$ for $1 \le j \le k$ and $X_j = h^* X'_j$ for $1 \le j \le m$ (if $r = 0$, the relation $X_j = h^* X'_j$ means that, for suitable representatives, h maps the integral curves of X_j to those of X'_j). Hence $h^* g^t = (g')^t$ for all $t \in \mathbf{Z}^k \times \mathbf{R}^\ell$.

The two action germs are *formally conjugate* when there exists a smooth diffeomorphism germ $h : (M, a) \to (M', a')$ such that g_j and $h^* g'_j$ have infinite contact at a for $1 \le j \le k$ and so have X_j and $h^* X'_j$ for $1 \le j \le m$. In other words, $h^* g^t$ and $(g')^t$ have infinite contact at a for all $t \in \mathbf{Z}^k \times \mathbf{R}^\ell$.

Theorem 1.2 *Two weakly hyperbolic smooth $\mathbf{Z}^k \times \mathbf{R}^\ell$-action germs are smoothly* (i.e. C^∞-) *conjugate if and only if they are formally conjugate.*

If $r = 1$, this is the Sternberg-Chen theorem [22, 14]. When L is not weakly hyperbolic, g can be formally but not C^0-conjugate to L ([10], 6.3, Théorème).

2 Proof of Theorem 1.2

The "only if" is obvious. To establish the "if", we can fortunately rely on [10, 12].

Hypotheses and notation Same as in section 1. Weak hyperbolicity is *not* assumed in 2.1 and 2.2, extracted from [10] (to which we refer for details).

2.1 Algebraic part of the proof: normal forms

The following result is not so widely known even for one endomorphism:

Proposition 2.1 ([10], 4.3.1, Proposition 5) *There exist positive integers n, $d_1, \ldots, d_n$, a nonnegative integer $c \leq n$ and an isomorphism of real vector spaces*

$$x = (x_1, \ldots, x_n) = \big((x_{1,p})_{1 \leq p \leq d_1}, \ldots, (x_{n,p})_{1 \leq p \leq d_n}\big) : E \to \prod_{1 \leq \ell \leq c} \mathbf{C}^{d_\ell} \times \prod_{c < \ell \leq n} \mathbf{R}^{d_\ell}$$

such that, for every $t \in \mathbf{Z}^k \times \mathbf{R}^m$, the real automorphism $x_* L^t = x \circ L^t \circ x^{-1}$ of $\prod_{1 \leq \ell \leq c} \mathbf{C}^{d_\ell} \times \prod_{c < \ell \leq n} \mathbf{R}^{d_\ell}$ is given by a block-diagonal matrix

$$\begin{pmatrix} T_1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_n(t) \end{pmatrix}$$

and the following properties are satisfied:

i) Each diagonal block $T_\ell(t)$ is a lower triangular $d_\ell \times d_\ell$ matrix, with coefficients in $\mathbf{C}$ for $\ell \leq c$, in $\mathbf{R}$ otherwise.

ii) For $1 \leq \ell \leq n$ and $t \in \mathbf{Z}^k \times \mathbf{R}^m$, the diagonal elements of $T_\ell(t)$ all equal the same number $a_\ell(t) \in \mathbf{C}^*$.

iii) The continuous group homomorphisms $a_\ell : \mathbf{Z}^k \times \mathbf{R}^m \to \mathbf{C}^*$ so defined take their values in $\mathbf{R}^*$ only for $\ell > c$.

iv) For $1 \leq \ell < \ell' \leq n$, one has $a_\ell \neq a_{\ell'} \neq \overline{a_\ell}$.

v) For $1 \leq \ell \leq n$, one has $T_\ell(t) = a_\ell(t) \exp \varepsilon_\ell(t)$, where ε_ℓ is a linear map of $\mathbf{R}^r$ into the space of lower triangular $d_\ell \times d_\ell$ nilpotent complex matrices, with real coefficients for $\ell > c$.

In particular, Proposition 1.1 holds with $E_\ell = \bigcap_{\ell' \neq \ell} \ker x_{\ell'}$.

A proof. Setting $A_j = L_j$ for $1 \le j \le c$ and $A_j = \Lambda_{j-k}$ for $c < j \le r$, the complex vector space $E^*_{\mathbf{C}} := L(E, \mathbf{C})$ is the direct sum of the characteristic subspaces $F_{1,\ell}$ of the $\mathbf{C}$-linear map $A^*_1 : u \mapsto u \circ A_1$, each of which is stable under every A^*_j because $A_j A_1 = A_1 A_j$; hence, each $F_{1,\ell}$ is the direct sum of the characteristic spaces of the endomorphism induced by A^*_2, yielding a decomposition of $E^*_{\mathbf{C}}$ as the direct sum of A^*_j-invariant subspaces $F_{2,\ell}$ on which the endomorphisms induced by A^*_1 and A^*_2 have a single eigenvalue. Iterating this procedure, one gets a decomposition of $E^*_{\mathbf{C}}$ as the direct sum of A^*_j-invariant subspaces $F_\ell = F_{r,\ell}$ on which the endomorphism $B_{j,\ell}$ induced by every A^*_j has a single eigenvalue $\lambda_{j,\ell}$ and, moreover, to each ℓ corresponds a different (co)vector $(\lambda_{1,\ell}, \dots, \lambda_{r,\ell})$.

For each ℓ, if $d_\ell = \dim F_\ell$, the commuting endomorphisms $B_{j,\ell}$ have a common eigenvector $x_{\ell,1}$, which (replacing it by its real or imaginary part) can be chosen *real* (i.e. in $E^* := L(E, \mathbf{R})$) when every $\lambda_{j,\ell}$ is real. For $d_\ell > 1$ the same argument, applied in $F_\ell / \mathbf{C} x_{\ell,1}$, shows that there exists $x_{\ell,2} \in F_\ell$, real if every $\lambda_{j,\ell}$ is, such that each $(A^*_j - \lambda_{j,\ell}) x_{\ell,2}$ lies in $\mathbf{C} x_{\ell,1}$, and in $\mathbf{R} x_{\ell,1}$ if every $\lambda_{j,\ell}$ is real, etc. This standard triangulation procedure yields Proposition 2.1, except assertion (v), as follows:

- number the F_ℓ's so that every $\lambda_{j,\ell}$ is real for $c < \ell \le n$ and $\overline{\lambda_{j,\ell}} = \lambda_{j,n+\ell}$, hence $\overline{F_\ell} = F_{n+\ell}$, for $1 \le \ell \le c$
- notice that one can take $x_{n+\ell,p} = \overline{x_{\ell,p}}$ for $1 \le \ell \le c$
- remark that $x = \Big((x_{1,p})_{1 \le p \le d_1}, \dots, (x_{n,p})_{1 \le p \le d_n} \Big)$ must be an isomorphism since $\ker x = \{0\}$ and $\dim_{\mathbf{C}} E^*_{\mathbf{C}} = \dim_{\mathbf{R}} E = 2 \sum_{\ell \le c} d_\ell + \sum_{c < \ell \le n} d_\ell$.

Finally, Proposition 2.1 v) follows from the fact ([10], 4.3.1, Lemme 2) that for each real vector space F, the map $\nu \mapsto -\mathrm{Id} + \exp \nu$ is a bijection of the set of nilpotent endomorphisms of F onto itself, whose inverse map is $N \mapsto \ln(\mathrm{Id} + N)$. □

Comments. What is not so widely known is the complex part of the normal form, for which one has to choose one element in each pair $(a_\ell, \overline{a_\ell})$.

The notation ε_ℓ indicates that this is the "small" part of L: indeed, one can make it arbitrarily small by multiplying the $x_{\ell,p}$'s by positive constants.

Notation. Recall that each L^t has a unique *Jordan decomposition* as the sum of a semi-simple endomorphism S^t (meaning that the complexified endomorphism is diagonalisable) and a nilpotent one, commuting with each other. Proposition 2.1 states in particular that the *semi-simple parts* S^t commute with each other and that the isomorphism x diagonalizes them simultaneously. This is the viewpoint we now adopt on normal forms, first in a somewhat cryptic way:

Proposition 2.2 ([10], 4.3.2, Théorème 4) *The action germ g can be put formally into normal form in the following sense: there exists a smooth diffeomorphism germ $h : (M, a) \to (E, 0)$, tangent to the identity at a, such that, for all $t', t \in \mathbf{Z}^k \times \mathbf{R}^m$, the map germs $S^{t'} \circ (h_* g^t)$ and $(h_* g^t) \circ S^{t'}$ have infinite contact at* 0.

Idea of the proof. To each smooth diffeomorphism germ $g : (M, a) \to (M, a)$ is associated the automorphism $g_* : f \mapsto f \circ g^{-1}$ of the algebra of smooth germs $f : (M, a) \to \mathbf{R}$. The map $g \mapsto g_*$ induces for every positive integer s an *isomorphism* $\check{g} \mapsto \check{g}_*$ of the group $\check{\mathcal{D}}^s$ of s^{th} order jets (Taylor expansions) at 0 of diffeomorphism germs $(M,a) \to (M,a)$ onto the group of automorphisms of the real algebra $\check{\mathcal{E}}^s$ of s^{th} order jets at 0 of germs $(M,a) \to \mathbf{R}$. Elementary arguments ([10], 4.3.2, proof of Theorème 3) or the Jordan-Chevalley theorem [17] show that, for each $\check{g} \in \check{\mathcal{D}}^s$, the semi-simple part of the automorphism $\check{g}_*$ is itself an automorphism of the algebra $\check{\mathcal{E}}^s$ and therefore of the form $\check{\sigma}_*$ for a unique $\check{\sigma} \in \check{\mathcal{D}}^s$, the *semi-simple part* of $\check{g}$. Now, the very definition of a semi-simple automorphism of $\check{\mathcal{E}}^s$ implies almost immediately ([10], 4.3.2, Théorème 2) that there exists a germ $h : (M,a) \to (E,0)$, tangent to the identity at a, whose s-jet $\check{h}$ linearises $\check{\sigma}$: denoting by S the (semi-simple) differential (1-jet) of $\check{\sigma}$ at a, identified to its s-jet, one has that $\check{h}_*\check{\sigma} = S$.

As the semi-simple parts of commuting endomorphisms commute, so do the semi-simple parts $\check{\sigma}^t$ of the s-jets $\check{g}^t$ at a of the germs g^t; hence, as in Proposition 2.1, one can take the same h for all t, so that $\check{h}_*\check{\sigma}^t = S^t$ (identified to its s-jet) for all t.

This passes to the projective limit when $s \to \infty$ and yields (via the Borel extension theorem) the required diffeomorphism germ h: indeed, the infinite jets at 0 satisfy $\check{h}_*\check{\sigma}^{t'} = S^{t'}$ for all t', hence $S^{t'} \circ (\check{h}_*\check{g}^t) = \check{h}_*(\check{\sigma}^{t'} \circ \check{g}^t) = \check{h}_*(\check{g}^t \circ \check{\sigma}^{t'}) = (h_*g^t) \circ S^{t'}$. $\square$

We can now see what this means in the coordinate system x of Proposition 2.1:

Corollary 2.3 ([10], 4.3.2, Corollaire 3) *With the notation of Propositions 2.1 and 2.2, if we set*

$$\begin{aligned} a_{n+\ell} &:= \overline{a_\ell} : t \mapsto \overline{a_\ell(t)}\,, \quad 1 \le \ell \le c\,, \quad 1 \le p \le d_\ell =: d_{n+\ell} \\ x_{n+\ell,p} &:= \overline{x_{\ell,p}}\,, \quad 1 \le \ell \le c\,, \quad 1 \le p \le d_\ell =: d_{n+\ell} \\ x^\alpha = x_1^{\alpha_1} \cdots x_{n+c}^{\alpha_{n+c}} &:= \prod_{\ell,p} x_{\ell,p}^{\alpha_{\ell,p}}\,,\ \alpha = (\alpha_\ell) = \big((\alpha_{\ell,p})_{1 \le p \le d_\ell}\big)_{1 \le \ell \le n+c} \in \prod_\ell \mathbf{N}^{d_\ell} \end{aligned}$$

and, for $1 \le \ell \le n$,

$$\begin{aligned} \Pi_\ell &:= \big\{\beta \in \mathbf{N}^{n+c} : a_\ell = a^\beta := a_1^{\beta_1} \cdots a_{n+c}^{\beta_{n+c}}, \beta_1 + \cdots + \beta_{n+c} \ge 2\big\} \\ P_\ell &:= \Big\{\alpha \in \prod_{1 \le \ell \le n+c} \mathbf{N}^{d_\ell} : \mu(\alpha) := (|\alpha_\ell|_1)_{1 \le \ell \le n+c} := \big(\sum_p \alpha_{\ell,p}\big)_{1 \le \ell \le n+c} \in \Pi_\ell\Big\}\,, \end{aligned}$$

then, for $1 \le \ell \le n$, $1 \le p \le d_\ell$ and $t \in \mathbf{Z}^k \times \mathbf{R}^m$, the Taylor series of $x_{\ell,p} \circ S^{-t} \circ h_*g^t$ at 0 is of the form

$$x_{\ell,p} + \sum_{1 \le q < p} e_{\ell,p,q}(t) x_{\ell,q} + \sum_{\alpha \in P_\ell} f_{\ell,p,\alpha}(t) x^\alpha$$

where the $e_{\ell,p,q}$'s and $f_{\ell,p,\alpha}$'s are polynomial functions on $\mathbf{R}^r$.

Proof. As h is tangent to the identity, the differential of h_*g^t at 0 is L^t, whose semi-simple part is S^t; hence, the linear terms $x_{\ell,p} + \sum_{q<p} e_{\ell,p,q}(t) x_{\ell,q}$ come from $\exp \varepsilon_\ell(t)$ and are of the required form. The rest of the Taylor series reads $\sum_\alpha f_{\ell,p,\alpha}(t) x^\alpha$ with $\alpha \in \prod_{1 \le s \le n+c} \mathbf{N}^{d_s}$ and $\sum_s |\alpha_s|_1 \ge 2$; for every $t' \in \mathbf{Z}^k \times \mathbf{R}^m$, as $x_{s,p} \circ S^{t'} = a_s(t') x_{s,p}$, the fact that $x_{\ell,p} \circ S^{t'} \circ S^{-t} \circ h_*g^t$ and $x_{\ell,p} \circ S^{-t} \circ h_*g^t \circ S^{t'}$ have infinite contact at 0 writes

$$a_\ell(t') \sum_\alpha f_{\ell,p,\alpha}(t) x^\alpha = \sum_\alpha f_{\ell,p,\alpha}(t) a^{\mu(\alpha)}(t') x^\alpha,$$

hence $f_{\ell,p,\alpha}(t)(a_\ell - a^{\mu(\alpha)}) = 0$ for all α, proving that $f_{\ell,p,\alpha}(t) = 0$ for $\alpha \notin P_\ell$.

Finally, every $f_{\ell,p,\alpha}$ is polynomial for the following reason: for each positive integer s, as S^t is the semi-simple part of the s-jet $\check{h}_*\check{g}^t$ of h_*g^t at 0 for all t, the automorphism $(S^{-t} \circ \check{h}_*\check{g}^t)_*$ of the algebra $\check{\mathcal{F}}^s$ of s-jets of germs $(E,0) \to \mathbf{R}$ is of the form $\exp \nu(t)$, where ν is a linear map of $\mathbf{R}^r$ into the set of nilpotent derivations of $\check{\mathcal{F}}^s$ ([10], Théorème 4 (iii)); in particular, the coefficients of the polynomial $(S^{-t} \circ \check{h}_* g^t)_* x_{\ell,p}$ of the variables x^α are polynomial functions of t. $\square$

Remarks. For a dense open set of linear representations L, the multiplicities d_ℓ all equal 1, hence $\Pi_\ell = P_\ell$; moreover, $P_\ell = \emptyset$ for almost every L (not an open condition in general) and then what we get is a formal linearisation result.

The Jordan decomposition is not the only way to obtain normal forms, but it is probably the most understandable for $r > 1$.

Proposition 2.4 ([10], 4.3.2, Corollaire 4) *Let*

$$\Pi_0 := \{\beta \in \mathbf{N}^{n+c} \smallsetminus \{0\} : a^\beta = 1\};$$

then, if $\mathbf{N}^{n+c}$ is endowed with the ordering "$\beta \leq \beta'$ if and only if $\beta_\ell \leq \beta'_\ell$ for all ℓ ", one has the following:

i) For $0 \leq \ell \leq n$, the set $\min \Pi_\ell$ of minimal elements of Π_ℓ is finite and every element of Π_ℓ is comparable to an element of $\min \Pi_\ell$.

ii) Hence, for each $\beta \in \Pi_\ell$, either β lies in $\min \Pi_\ell$, or it is the sum of an element of $\min \Pi_\ell$ and an element of Π_0.

iii) Thus, if $\Pi_0 = \emptyset$ (which means that L has no non-constant formal first integral), the subset $\Pi_\ell = \min \Pi_\ell$ is finite for $1 \leq \ell \leq n$. In that case, with the notation of Corollary 2.3, the formulae

$$x_{\ell,p} \circ u^t = x_{\ell,p} + \sum_{1 \leq q < p} e_{\ell,p,q}(t) x_{\ell,q} + \sum_{\alpha \in P_\ell} f_{\ell,p,\alpha}(t) x^\alpha\ ,\ t \in \mathbf{R}^r \tag{2}$$

define an algebraic action u of $\mathbf{R}^r$ on E satisfying $u^t \circ S^{t'} = S^{t'} \circ u^t$ for all $t \in \mathbf{R}^r$ and $t' \in \mathbf{Z}^k \times \mathbf{R}^m$, and Corollary 2.3 states that the action germ g is formally conjugate to the (germ at 0 of the) analytic action g_0 of $\mathbf{Z}^k \times \mathbf{R}^m$ on E defined by

$$g_0^t := S^t \circ u^t = u^t \circ S^t,\ \textit{i.e.}\ x_{\ell,p} \circ g_0^t = a_\ell(t)\, x_{\ell,p} \circ u^t. \tag{3}$$

Unfortunately, for $r > 1$ and $\Pi_0 \neq \emptyset$, formal normal forms do not define model action germs in such an obvious way. However, they do provide something interesting along germs at 0 of special submanifolds of E, to which we now turn.

2.2 Strongly invariant manifolds. A preparation lemma

A *strongly invariant manifold* [6] (abbreviated SIM in the sequel) of the action germ g is a germ W at a of C^1 submanifold which is the (strong) unstable manifold of g^t for some $t \in \mathbf{Z}^k \times \mathbf{R}^m$. Now, this unstable manifold W is smooth, its tangent space T_aW is the unstable subspace $\bigoplus_{c_\ell(t)>0} E_\ell$ of L^t, and W is the only g^t-invariant submanifold germ having this tangent space.

It follows that every SIM of g is invariant by all the germs g^t and that the SIM's of g form a finite set of smooth submanifold germs.

The Poincaré domain. The following two properties, expressed by saying that L (or g) is in the *Poincaré domain*, are equivalent:

a) the germ at a of the ambient manifold M is a SIM of g

b) the convex hull $\operatorname{conv}\{c_1, \ldots, c_n\}$ in $\mathbf{R}^{r*}$ does not contain the origin (in particular, g is weakly hyperbolic).

Indeed, (b) means that there exists $t \in \mathbf{R}^r$ such that every $c_\ell(t)$ is positive and, as $\mathbf{Q}^r$ is dense in $\mathbf{R}^r$, one can take $t \in \mathbf{Z}^r$.

Theorem 2.5 ([10], 4.4.2b, Théorème 2) *If g is in the Poincaré domain then, with the notation of Proposition 2.4, the set Π_0 is empty and g is smoothly conjugate to the analytic action germ g_0.*

Idea of the proof. For $\alpha \in \mathbf{N}^{n+c} \setminus \{0\}$, the relation $a^\alpha = 1$ expressing that α lies in Π_0 yields $\sum_{j\le c}(\alpha_j + \alpha_{n+j})c_j + \sum_{j>c} \alpha_j c_j = 0$ and therefore $0 \in \operatorname{conv}\{c_1, \ldots, c_n\}$, hence $\Pi_0 = \emptyset$ in the Poincaré domain. The rest follows from fixed point arguments:

- if E is the unstable subspace of L^{t_0} then there exists an integer q_0 determined by L^{t_0} such that, for every integer $q > q_0$, the map $f \mapsto h_* g^{t_0} \circ f \circ g_0^{-t_0}$ has a unique fixed point h_1 in the space of C^q germs $f : (E, 0) \to (E, 0)$ having q_0^{th} order contact with the identity at 0; this h_1 is smooth, has infinite contact with the identity at 0 and clearly satisfies $(h_1^{-1} \circ h)_* g^{t_0} = g_0^{t_0}$;
- for all $t \in \mathbf{Z}^k \times \mathbf{R}^m$, the two germs $(h_1^{-1} \circ h)_* g^t$ and g_0^t have infinite contact at 0 and commute with $g_0^{t_0}$; hence, they coincide, for the germ $(h_1^{-1} \circ h)_* g^t \circ g_0^{-t}$ is the unique fixed point of the map $f \mapsto g_0^{t_0} \circ f \circ g_0^{-t_0}$ in the previous space, namely the germ of the identity;
- similarly, for $1 \le k \le m$, the germ $(h_1^{-1} \circ h)_* X_j$ and the germ at 0 of $\frac{d}{dt} g_0^{t\delta_{k+j}}|_{t=0}$ have infinite contact at 0 and are $g_0^{t_0}$-invariant; hence, they coincide.

This shows that $h_1^{-1} \circ h$ is a conjugacy between g and g_0. $\square$

As two formally conjugate action germs have the same normal form, this yields

Corollary 2.6 *Theorem 1.2 is true in the Poincaré domain.*

Corollary 2.7 *If g is hyperbolic and satisfies $n \le r$, then it is C^∞-linearisable: there exists a C^∞-conjugacy between g and the germ of L at 0.*

Proof. The linear forms $c_1, \ldots, c_n$ are independent, hence $0 \notin \operatorname{conv}\{c_1, \ldots, c_n\}$ and $P_1 = \cdots = P_n = \emptyset$. $\square$

Notes. The idea of this proof of Theorem 2.5 is essentially due to Sternberg [21]. A key point is that infinite contact between $h_* g^t$ and g_0^t can be replaced by q_0^{th} order contact; for holomorphic germs of vector fields, this h can be taken holomorphic, hence h_1 is, yielding the holomorphic linearisation theorem in Poincaré's thesis. The same fact, applied to the complexified maps of $h_* g^t$ and g_0^t with h analytic, implies that the conjugacy in Theorem 2.5 and Corollaries 2.6-2.7 is analytic when g is. The method also provides a C^q conjugacy between g and g_0 when g is C^q with $q > q_0$. All this follows at once from a very simple invariant manifold theorem obtained later [13].

We refer to [5] for a rather explicit proof of the following more difficult result:

Corollary 2.8 *If g is in the Poincaré domain and strongly hyperbolic, it is C^0-linearisable: there exists a C^0-conjugacy between g and the germ of L at 0.*

Hypothesis. ***From now on, L is assumed to be in the*** **Siegel domain*****, i.e. not in the Poincaré domain*** [12]***.***

If $\Pi_0 = \emptyset$, Corollary 2.3 states that formulae (2) and (3) define respectively an algebraic action u of $\mathbf{R}^r$ and an analytic action g_0 of $\mathbf{Z}^k \times \mathbf{R}^m$ on E. Otherwise, they define *formal actions*, i.e. homomorphisms $\mathbf{R}^r \ni t \mapsto u^t$ and $\mathbf{Z}^k \times \mathbf{R}^m \ni t \mapsto g_0^t$ into the group of infinite jets at 0 of diffeomorphisms $(E,0) \to (E,0)$.

Proposition 2.9 ([10], 4.4.2b, Proposition 1) *For each strongly invariant manifold W of L, the formal actions u and g_0 define* formal actions u_W and g_W of $\mathbf{Z}^k \times \mathbf{R}^m$ along W leaving W invariant, i.e. homomorphisms $\mathbf{R}^r \ni t \mapsto u_W^t$ and $\mathbf{Z}^k \times \mathbf{R}^m \ni t \mapsto g_W^t$ into the group of infinite jets at W of diffeomorphisms $(E,W) \to (E,W)$, *as follows:*

i) The subspace W of E is the unstable subspace of L^{t_0} for some t_0; hence, $W = \bigoplus_{c_\ell(t_0)>0} E_\ell$; if $\hat W = \bigoplus_{c_\ell(t_0)\le 0} E_\ell$, then $E = W \oplus \hat W \cong W \times \hat W$ so we can write the elements of E under the form $(y,z) \in W \times \hat W$.

ii) For all $s \in \mathbf{N}$, the s^{th} partial derivative $\partial_z^s u^t(y,0)$ is a polynomial function of $(t,y) \in \mathbf{R}^r \times W$, hence [13]

$$u^t = \sum_{s\in\mathbf{N}} \frac{1}{s!}\partial_z^s u^t(y,0) z^s \quad \textit{and} \quad g_0^t = S^t \sum_{s\in\mathbf{N}} \frac{1}{s!}\partial_z^s u^t(y,0) z^s$$

do define infinite jets along W.

iii) The $\hat W$-component of $u^t(y,0)$ is zero for all $(t,y) \in \mathbf{R}^r \times W$, implying the rest of our statement since every S^t preserves W and $\hat W$.

Proof. ii) If Π_0 is empty, this is obvious. Otherwise,

[12] The terminology, due to Arnold, refers to Poincaré's holomorphic linearisation theorem and its analogue in the Siegel domain [20], a celebrated triumph over "small denominators".

[13] With the usual convention $z^s = (z, \dots, z)$ repeated s times.

- for $s = 0$, as $0 \notin \operatorname{conv}\{c_\ell : c_\ell(t_0) > 0\}$, every x^α with $\alpha \in P_0$ vanishes on W; hence, by Proposition 2.4 i)-ii), every $x_{\ell,p} \circ u^t$ coincides with the polynomial $x_{\ell,p} + \sum_{1 \le q < p} e_{\ell,p,q}(t) x_{\ell,q} + \sum_{\mu(\alpha) \in \min \Pi_\ell} f_{\ell,p,\alpha}(t) x^\alpha$ on W.
- for arbitrary s, again by Proposition 2.4 i)-ii), for $1 \le \ell \le n$ and $\alpha \in P_\ell$, one has that $\mu(\alpha) = \beta + \gamma_1 + \cdots + \gamma_{s'}$ with $\beta \in \min \Pi_\ell$ and $\gamma_1, \ldots, \gamma_{s'} \in \min \Pi_0$ (this decomposition is not unique in general but there are finitely many of them and we can choose s' maximal); it follows that $\partial_y^s x^\alpha = 0$ on W for $s' > s$, which agains leaves only a finite number of multi-indices $\alpha \in P_\ell$ with $\partial_y^s x^\alpha \neq 0$.

iii) For $c_\ell(t_0) \le 0$, $\alpha \in P_\ell$ and $\beta = \mu(\alpha)$, as $c_\ell = \sum_{\ell' \le c} (\beta_{\ell'} + \beta_{n+\ell'}) c_{\ell'} + \sum_{c < \ell' \le n} \beta_{\ell'} c_{\ell'}$, some ℓ' with $c_{\ell'}(t_0) \le 0$ must satisfy $(\beta_{\ell'} + \beta_{n+\ell'}) > 0$ or $\beta_{\ell'} > 0$, hence $x^\alpha|_W = 0$; as the $x_{\ell,q}$'s also vanish on W for $c_\ell(t_0) \le 0$, this proves what we wanted. $\square$

Preparation Lemma 2.10 ([10], 4.4.2b, Théorème 1) *The action germ g can be put into normal form along the union $\mathcal{V}$ of the SIM's of L: there exists a smooth diffeomorphism germ $h_\infty : (E, 0) \to (E, 0)$ having infinite contact with the identity at 0 such that, for every SIM W of L, the infinite jet of $(h_\infty^{-1} \circ h)_* g^t$ along W is the germ at 0 of g_W^t for all $t \in \mathbf{Z}^k \times \mathbf{R}^m$; it follows that the infinite jet of $(h_\infty^{-1} \circ h)_* X_j$ along W is the germ at 0 of $\frac{d}{dt} g_W^{t\delta_{k+j}}|_{t=0}$ for $1 \le k \le m$.*

In particular, the SIM's of $(h_\infty^{-1} \circ h)_* g$ are the germs at 0 of the SIM's of L.

Idea of the proof. If W is the unstable subspace of L^{t_0}, let $\tilde{g}_W^{t_0} : (E, W) \to (E, W)$ be a smooth extension of $g_W^{t_0}$ (its only role is to simplify notation). For every integer s,

- taking representatives, the sequence of jets $\left(j^s\left((h_* g^{t_0})^p \circ (\tilde{g}_W^{t_0})^{-p}\right)\big|_W\right)_{p \in \mathbf{N}}$ converges in the C^∞ sense near 0 to the s-jet along W of a local diffeomorphism $h_{s,W} : (E, 0) \to (E, 0)$, obviously such that $j^s(h_{s,W}^{-1} \circ h)_* g^{t_0} = j^s \tilde{g}_W^{t_0}$ along W: this follows from [10], 4.2.2, Théorème 1 (see the proof of Corollary 2.12);
- for all $t \in \mathbf{Z}^k \times \mathbf{R}^m$, the two local maps $\left(j^s(h_{s,W}^{-1} \circ h)_* g^t\right)|_W$ and $(j^s \tilde{g}_W^t)|_W$ coincide near 0 and so do, for $1 \le k \le m$, the germ $\left(j^s(h_{s,W}^{-1} \circ h)_* X_j\right)|_W$ and the germ at 0 of $(j^s(\frac{d}{dt} \tilde{g}_W^{t\delta_{k+j}}|_{t=0}))|_W$: this follows from [10], 4.2.2, Théorème 2.

When $s \to \infty$, the definition domain of $(j^s h_{s,W})|_W$ does not shrink, hence in the limit the infinite jet along W of a smooth local diffeomorphism $h_{\infty,W} : (E, 0) \to (E, 0)$, which has infinite contact with the identity at 0.

One can take the same $h_{\infty,W} = h_\infty$ for every SIM W of L: indeed, the jets $(j^\infty h_{\infty,W})|_W$ corresponding to the various SIM's W of L coincide on their intersections because the previous argument applies when W is the intersection of two SIM's; hence, a mild version of Whitney's extension theorem yields our result. $\square$

Notes. For finite s and fixed W, this proof is essentially that of Theorem 2.5 and [13] could be used. In [11] we show that, for finite s, an analytic action germ can be put into normal form to order s along $\mathcal{V}$ by an analytic diffeomorphism germ (this simplifies the exposition in [12]). Of course, infinite contact is where the analytic and C^∞ theories split apart completely. If $r = 1$, the result is essentially Sternberg's [22]; the case $r > 1$ was a novelty with respect to [15]. The technology of 2.3.2–2.3.3 hereafter makes it possible to obtain normal forms that are the product of S and an action germ commuting with it as in [14], but they are not strikingly good in general for $r > 1$ and $P_0 \neq \emptyset$.

2.3 End of the proof

Hypothesis We now assume g weakly hyperbolic, still in the Siegel domain.

2.3.1 An extension lemma. Proof of Theorem 1.2 for Z-action germs

Extension lemma 2.11 ([10], 4.2.3, théorème 2)[14] *Let N be a compact manifold without boundary, E^+, E^- two nontrivial Euclidean spaces, and let W^+, W^-, Σ be the submanifolds of $Q := N \times E^+ \times E^-$ defined by*

$$W^+ := N \times E^+ \times \{0\}, W^- := N \times \{0\} \times E^-, \Sigma := W^+ \cap W^- = N \times \{(0,0)\}.$$

Writing (x_0, x_+, x_-) the points of $Q := N \times E^+ \times E^-$, assume that the smooth diffeomorphism germ $\varphi = (\varphi_0, \varphi_+, \varphi_-) : (Q, \Sigma) \to (Q, \Sigma)$ leaves invariant the germs of W^+ and W^- at Σ and satisfies the weak normal hyperbolicity condition

$$\max\left\{\max_{x\in\Sigma}\left|\frac{\partial\varphi_+}{\partial x_+}(x)\right|, \max_{x\in\Sigma}\left|\frac{\partial\varphi_-}{\partial x_-}(x)^{-1}\right|\right\} < 1.$$

If $\psi : (Q, \Sigma) \to (Q, \Sigma)$ has infinite contact with φ along $W^+ \cup W^-$, then every smooth germ[15] *$h : (Q \smallsetminus W^-, \Sigma) \to Q \smallsetminus W^-$ having infinite contact with the identity along $W^+ \smallsetminus \Sigma$ and conjugating $\psi|_{Q\smallsetminus W^-}$ to $\varphi|_{Q\smallsetminus W^-}$ extends to a smooth germ $(Q, \Sigma) \to (Q, \Sigma)$ having infinite contact with the identity along $W^+ \cup W^-$ and conjugating ψ to φ.*

Corollary 2.12 ([10], 4.2.3, théorème 2) *If φ is as in the extension lemma then, for every smooth germ $\varphi_1 : (Q, \Sigma) \to (Q, \Sigma)$ having infinite contact with φ along Σ,*

i) there exists a smooth conjugacy $H : (Q, \Sigma) \to (Q, \Sigma)$ of φ_1 to φ having infinite contact with the identity along Σ

ii) for each smooth diffeomorphism germ $H_0 : (Q, \Sigma) \to (Q, \Sigma)$, if $H_0 \circ \varphi$ and $\varphi \circ H_0$ have infinite contact along Σ, then the smooth germs $H : (Q, \Sigma) \to (Q, \Sigma)$ conjugating φ_1 to φ and having infinite contact with H_0 along Σ form an infinite dimensional space—in particular, so do the conjugacies H in (i).

Proof. i) There exists a smooth diffeomorphism germ $h_1 : (Q, \Sigma) \to (Q, \Sigma)$ having infinite contact with the identity along Σ and such that $\psi := h_1^*\varphi_1$ satisfies the hypotheses of the extension lemma: taking representatives of our germs, the jet $j^\infty_{W^\pm} h_1$ is the limit when $p \to \pm\infty$ of $j^\infty_{W^\pm}(\varphi_1^{-p} \circ \varphi^p)$ ([10], 4.2.2, théorème 1).

Let us find h as in the extension lemma: taking representatives, we may assume φ, ψ defined for some $\eta > 0$ on $B := \{x \in Q : \max\{|x_+|, |x_-|\} \leq \eta\}$ and such that

$$\sup_{B\smallsetminus W^-} \max\left\{\frac{|\varphi_+(x)|}{|x_+|}, \frac{|\psi_+(x)|}{|x_+|}\right\} < 1 < \inf_{B\smallsetminus W^+} \min\left\{\frac{|\varphi_-(x)|}{|x_-|}, \frac{|\psi_-(x)|}{|x_-|}\right\} =: c^{-1}. \quad (4)$$

[14] We state only the C^∞ version of this key result, of which two proofs are given in [10].

[15] A germ $(Q \smallsetminus W^-, \Sigma) \to Q \smallsetminus W^-$ is an equivalence class for the relation "there is an open subset $U \supset \Sigma$ in $U_0 \cap U_1$ such that $f_0 = f_1$ on $U \smallsetminus W^-$" between maps $f_j : U_j \smallsetminus W^- \to Q \smallsetminus W^-$ with $U_j \supset \Sigma$ open in Q.

taking a smaller η, we may assume that φ, ψ are embeddings and define h by a "Cauchy problem"[16], i.e. by its restriction $\bar{h}_0$ to the closure $\bar{D}_0$ of $D_0 := B \smallsetminus \varphi(B)$, which must have infinite contact with the identity along $W^+ \smallsetminus \varphi(W^+)$ and have infinite contact with $\psi \circ \bar{h}_0 \circ \varphi^{-1}$ along the inner boundary $\varphi(\partial^+ B) \cap B$, where $\partial^+ B$ is the outer boundary $\{x \in B : |x_+| = \eta\}$; thus, if we look for an $\bar{h}_0$ having infinite contact with the identity along $\partial^+ B$, it must have infinite contact with $\psi \circ \varphi^{-1}$ along $\varphi(\partial^+ B) \cap B$; this defines a Whitney-extendable jet along $\partial^+ B \cup \big(W^+ \smallsetminus \varphi(W^+)\big) \cup \big(\varphi(\partial^+ B) \cap B\big)$, which is indeed the jet of a smooth map $\bar{h}_0 : \bar{D}_0 \to Q$.

We now extend $\bar{h}_0$ to a smooth conjugacy h of ψ to φ defined in $B \smallsetminus W^- \cap \{|x_-| \le \rho\}$ for some small enough positive $\rho \le \eta$:

- by (4), the sets D_0, $D_1 := B \cap \varphi(D_0)$, $D_2 := B \cap \varphi(D_1), \ldots$ form a partition of $B \smallsetminus W^-$, and so do the corresponding sets $\Delta_0, \Delta_1, \ldots$ for ψ;
- if $D_{n,\rho} := D_n \cap \{|x_-| \le \rho\}$, the map $h_n := h|_{D_{n,\rho}}$ should satisfy

$$\begin{aligned} h_n(x) &= \psi^n \circ \bar{h}_0 \circ \varphi^{-n}(x) \\ &= \psi \circ h_{n-1} \circ \varphi^{-1}(x) \text{ for } n > 0; \end{aligned} \tag{5}$$

- we claim that this does define the required h for small enough ρ.

Indeed, if we equip N with a riemannian metric and B with the product metric, given

$$\alpha > 1 \quad \text{such that} \quad c^\alpha \operatorname{Lip} \psi < 1,$$

the number

$$R := \sup_{x \in B_0 \smallsetminus W^+} |x_-|^{-\alpha} d\left(\bar{h}_0(x), x\right)$$

is finite since $\bar{h}_0$ has infinite contact with the identity along W^+; similarly, as φ^{-1}, ψ^{-1} have infinite contact along W^+, the number

$$b_\rho := \sup_{0 < |x_-| \le \rho} |x_-|^{-\alpha} d\left(\varphi^{-1}(x), \psi^{-1}(x)\right)$$

tends to 0 when $\rho \to 0$. Hence, for small enough $\rho \le \eta$, the map $\bar{h}_0|_{\bar{D}_{0,\rho}}$ is an embedding and one has the following:

$$c + R c^\alpha \rho^{\alpha - 1} \le 1 \tag{6}$$

$$\left(c^\alpha + R^{-1} b_\rho\right) \operatorname{Lip} \psi \le 1 \tag{7}$$

$$\rho + R\rho^\alpha \le \eta. \tag{8}$$

If all this holds, let us prove that each h_n is a well-defined map of $D_{n,\rho}$ into Δ_n and

$$\sup_{x \in D_{n,\rho} \smallsetminus W^+} |x_-|^{-\alpha} d\left(h_n(x), x\right) \le R : \tag{9}$$

- this is the case if $n = 0$: indeed, (9) follows from the definition of R; moreover, $\bar{h}_0$ preserves the outer boundary $\partial^+ B$ and maps the inner boundary $\varphi(\partial^+ B) \cap B$ onto $\psi(\partial^+ B) \cap \psi \circ \varphi^{-1}(B)$; now, (8) and the definition of R yield

  $$|\bar{h}_0(x)_-| \le |\bar{h}_0(x)_- - x_-| + |x_-| \le R|x_-|^\alpha + |x_-| \le R\rho^\alpha + \rho \le \eta$$

  for all $x \in \bar{D}_{0,\rho}$; hence, the embedding $\bar{h}_0|_{\bar{D}_{0,\rho}}$ takes its values in $|x_-| \le \eta$ and maps $\big(\partial^+ B \cup \varphi(\partial^+ B)\big) \cap \{|x_-| \le \rho\}$ into $\partial^+ B \cup \psi(\partial^+ B) \cap B$, yielding $h_0(D_{0,\rho}) \subset \Delta_0$;

[16] See the proof of Corollary 2.14 and the subsequent notes.

let us now prove the same for $n > 1$, assuming it true for $n-1$,

- h_n is well-defined by (5) since (4), (6) and (9) for $n-1$ yield, for $x \in D_{n,\rho}$,

$$\begin{aligned} |h_{n-1} \circ \varphi^{-1}(x)_-| &\leq |h_{n-1} \circ \varphi^{-1}(x)_- - \varphi^{-1}(x)_-| + |\varphi^{-1}(x)_-| \\ &\leq R|\varphi^{-1}(x)_-|^\alpha + |\varphi^{-1}(x)_-| \leq Rc^\alpha |x_-|^\alpha + c|x_-| \\ &\leq Rc^\alpha \rho^\alpha + c\rho = \rho\left(Rc^\alpha \rho^{\alpha-1} + c\right) \leq \rho \leq \eta; \end{aligned}$$

- it does satisfy (9) since (4), (6) and (9) for $n-1$ yield, for $x \in D_{n,\rho}$,

$$\begin{aligned} |x_-|^{-\alpha} d\left(h_n(x), x\right) &= |x_-|^{-\alpha} d\left(\psi \circ h_{n-1} \circ \varphi^{-1}(x), \psi \circ \psi^{-1}(x)\right) \\ &\leq \frac{\operatorname{Lip}\psi}{|x_-|^\alpha} d\left(h_{n-1} \circ \varphi^{-1}(x), \psi^{-1}(x)\right) \\ &\leq \frac{\operatorname{Lip}\psi}{|x_-|^\alpha} \left(d\left(h_{n-1} \circ \varphi^{-1}(x), \varphi^{-1}(x)\right) + d\left(\varphi^{-1}(x), \psi^{-1}(x)\right)\right) \\ &\leq \left(R|x_-|^{-\alpha}|\varphi^{-1}(x)_-|^\alpha + b_\rho\right) \operatorname{Lip}\psi \leq R\left(c^\alpha + R^{-1} b_\rho\right) \operatorname{Lip}\psi \\ &\leq R; \end{aligned}$$

- it does map $D_{n,\rho}$ into Δ_n: indeed, as this is assumed true for $n-1$, it takes its values in $\psi(\Delta_{n-1})$ by (5), and (8)-(9) yield

$$|h_n(x)_-| \leq |h_n(x)_- - x_-| + |x_-| \leq R|x_-|^\alpha + |x_-| \leq R\rho^\alpha + \rho \leq \eta$$

  for all $x \in D_{n,\rho}$, hence our result since $\Delta_n = \psi(\Delta_{n-1}) \cap B$.

As the definition of $\bar{h}_0$ implies that each h_n has infinite contact with the identity along $D_n \cap W^+$ and that h_{n-1} and h_n fit together smoothly along the common boundary of $D_{n-1,\rho}$ and $D_{n,\rho}$, this does define the required h.

ii) Since the germs along W^+ of the solutions $\bar{h}_0$ of our extension procedure form an infinite dimensional space, the result is true when H_0 is the germ of the identity. Otherwise, notice that H conjugates φ_1 to φ if and only if $H_0^{-1} \circ H$ conjugates φ_1 to $H_0^{-1} \circ \varphi \circ H_0$ and apply (i) with $\varphi := H_0^{-1} \circ \varphi \circ H_0$. □

Notes. The ingredients of this proof are extracted from that of 4.2.2, Théorème 1 in [10], where the result is not proven (nor used) in such generality. It contrasts sharply with the analytic case: if φ, φ_1, H_0 are analytic, then H_0 is the only analytic H in our infinite dimensional space: the problem is whether it exists.

Corollary 2.13 (Sternberg-Chen) *Theorem 1.2 is true for* $\mathbf{Z}$*-action germs.*

Proof. With the notation of the preparation lemma 2.10, identifying each c_ℓ to $c_\ell(1)$ as usual, we can apply Corollary 2.12 to $\varphi := (h_\infty^{-1} \circ h)_* g_1$ with $\Sigma = \{0\}$ and $E^\pm = \bigoplus_{\pm c_\ell < 0} E_\ell$, equipped with the Euclidean norm $|\cdot|_\delta^2 := \sum_{\pm c_\ell < 0} \sum_p \delta^{2p} |x_{\ell,p}|^2$: indeed, φ satisfies the hypothesis of the extension lemma for small enough $\delta > 0$ since

$$\lim_{\delta \to 0} \max\left\{\left|\partial_{x_+}\varphi_+(0)\right|_\delta, \left|\partial_{x_-}\varphi_-(0)^{-1}\right|_\delta\right\} = \max_{1 \leq \ell \leq n} e^{-|c_\ell|} < 1;$$

thus, if the action germ g' is formally conjugate to g, then g'_1 is smoothly conjugate to a germ $\varphi_1 : (E, 0) \to (E, 0)$ to which Corollary 2.12 applies. □

The case of flows, though accessible to this local approach, will now be treated in a somewhat different spirit.

2.3.2 The Lyapunov map

Let
$$F := \tfrac{1}{2}\big(c_1|x_1|^2 + \cdots + c_n|x_n|^2\big) : E \to \mathbf{R}^{r*},$$
where $|x_\ell|^2 := \sum_p |x_{\ell,p}|^2$ and the coordinates $x_{\ell,p}$ are as in Proposition 2.1.

The case of real flows If $(k,m) = (0,1)$ then F is a real-valued Lyapunov function for the linear flow S^t, as[17]

$$F \circ S^t = \frac{1}{2}\big(c_1|x_1|^2 e^{2c_1 t} + \cdots + c_n|x_n|^2 e^{2c_n t}\big) \tag{10}$$

and therefore $\frac{d}{dt}F \circ S^t\big|_{t=0} = c_1^2|x_1|^2 + \cdots + c_n^2|x_n|^2$.

Weak hyperbolicity means that none of the c_ℓ's is zero, hence the only critical point of F is the origin. As S is in the Siegel domain, the c_ℓ's do not all have the same sign; therefore, (10) shows that, along every orbit of the flow, F tends to $+\infty$ (resp. $-\infty$) when $t \to +\infty$ (resp. $-\infty$), except when the orbit lies in the stable (resp. unstable) manifold $E^+ := \bigoplus_{c_\ell<0} E_\ell$ (resp. $E^- := \bigoplus_{c_\ell>0} E_\ell$) of the flow S at 0, in which case the limit of F is 0.

Thus, if b is a regular value of F (i.e. $b \neq 0$) and $Q_b := F^{-1}(b)$, then

- either b is positive and Q_b is a quotient by the flow S of the invariant open subset $E_b := E \smallsetminus E^+$, meaning that every orbit of S contained in E_b intersects Q_b transversally and exactly once,

- or b is negative and Q_b is a quotient by S of the invariant open subset $E_b := E \smallsetminus E^-$.

If ρ is another smooth $\mathbf{R}$-action on E leaving E_b invariant and for which Q_b is a quotient of E_b, the two flows on E_b induced by ρ and S are smoothly conjugate.

Indeed, there is a unique smooth conjugacy $H : E_b \to E_b$ of ρ to S equal to the identity on Q_b: if the smooth functions $T, \tau : E_b \to \mathbf{R}$ are defined by $S^{-T(P)}(P) \in Q_b$ and $\rho^{-\tau(P)}(P) \in Q_b$, then $H(P) = \rho^{T(P)}(S^{-T(P)}(P))$ and $H^{-1}(P) = S^{\tau(P)}(\rho^{-\tau(P)}(P))$, since $T \circ S^t = t + T$ and $\tau \circ \rho^t = t + \tau$ in E_b.

In general, this conjugacy H between S and ρ does *not* extend continuously to the missing invariant subspace $E^{\pm} = E \smallsetminus E_b$ (dramatic example: $\rho^t = S^{-t}$), but the same idea and the extension lemma yield

Corollary 2.14 (Sternberg-Chen) *Theorem 1.2 is true for* $\mathbf{R}$*-action germs.*

Proof. If we replace each $x_{\ell,p}$ by $\delta^{p-1}x_{\ell,p}$ with $\delta > 0$ small enough, F is a Lyapounov function for the linear flow L and not only for S.

Given an $\mathbf{R}$-action germ g' formally conjugate to g, it is formally conjugate to the same normal form g_0 as g and therefore, by the preparation lemma, smoothly conjugate to an action germ on E whose infinitesimal generator Y_1 has infinite contact with the infinitesimal generator Y_0 of $(h_\infty^{-1} \circ h)_* g$ along $E^+ \cup E^-$. Denote again by

[17] When all the eigenvalues of $dX_1(0)$ are real, S^t is the gradient flow of F with respect to the standard Euclidean metric $\sum dx_{\ell,p}^2$.

Y_j, $j = 0, 1$, a smooth local vector field with that germ at 0, by Λ the infinitesimal generator of L and by $\theta : E \to [0,1]$ a compactly supported smooth function equal to 1 near 0. For small enough positive η, the formulae

$$\tilde{Y}_j(y) = \begin{cases} \Lambda(y) + \theta(\eta^{-1}y)\Big(Y_j(y) - \Lambda(y)\Big) & \text{for } \eta^{-1}y \in \operatorname{supp}\theta \\ \Lambda(y) & \text{otherwise} \end{cases}$$

define two smooth vector fields $\tilde{Y}_0, \tilde{Y}_1$ on E with the following properties:

- they have infinite contact along $E^+ \cup E^-$;
- their flows $\tilde{\varphi}_0^t, \tilde{\varphi}_1^t$ have E^+, E^- as stable and unstable manifolds at 0;
- the function F is a Lyapunov function for both.

For negative b, it follows as before that there is exactly one smooth conjugacy $H : E_b = E \smallsetminus E^- \to E_b$ of $\tilde{Y}_1$ to $\tilde{Y}_0$ equal to the identity on Q_b; as it has infinite contact with the identity along $E^+ \smallsetminus \{0\}$, the extension lemma applies to the germs h, φ, ψ of $H, \tilde{\varphi}_0^1, \tilde{\varphi}_1^1$ respectively at 0 since φ has the required properties, hence our corollary. □

Notes. Via the conjugacy relation, this local extension can then be globalised, a remark that will prove useful in the sequel.

If b is negative enough, then $\tilde{\varphi}_0^t(y) = \tilde{\varphi}_1^t(y) = L^t(y)$ for all $y \in Q_b$ and $t \leq 0$, implying that the extended H is the limit of $\tilde{\varphi}_0^t \circ \tilde{\varphi}_1^{-t}$ when $t \to +\infty$ (of course, t can be restricted to the integers): this is Nelson's approach [19].

Off W^-, the Lyapunov function F and the "curved" hypersurface Q_b can be replaced respectively by $x \mapsto |x_+|^2$ and by the cylinder $\{|x_+| = 1\}$.

Basic facts about the $\mathbf{R}^r$-action S when $k = 0$

Notation. As in [10], for $y \in E$, we let $J(y) := \{\ell : x_\ell(y) \neq 0\}$ and, for $I \subset \{1, \ldots, n\}$, $E_I := \bigoplus_{\ell \in I} E_\ell$, $V_I := \sum_{\ell \in I} \mathbf{R} c_\ell$ and $C_I := \sum_{\ell \in I} \mathbf{R}_+ c_\ell$.

Proposition 2.15 *For $I \subset \{1, \ldots, n\}$, a point $y \in E_I$ is a critical point of $F|_{E_I}$ if and only if it is a critical point of F; hence, every critical value of $F|_{E_I}$ is a critical value of F, and every regular value of F is a regular value of $F|_{E_I}$.*

Proof. An occasion to recall definitions. A *critical point* of F is an $y \in E$ such that $DF(y) : E \to \mathbf{R}^{r*}$ is onto. As $DF(y)v = \sum c_\ell \, x_\ell(y) \cdot x_\ell(v)$ (scalar product) for all $y, v \in E$, it is clear that $DF(y)E = DF(y)E_I = D(F|_{E_I})(y)E_I$ for $y \in E_I$, implying the first assertion. The rest is pure terminology: if y is a critical point of F, then $F(y)$ is called a *critical value* of F, hence the second assertion. A *regular value* is a point in the target space which is not a critical value, hence the last assertion. □

Formula (10) still holds for $r > 1$ and has about the same consequences as before:

Proposition 2.16 *The map F is a "Lyapunov map" for S: for each $y \in E$,* *i) the map* $\Psi_y : t \mapsto F \circ S^t(y)$ *of* $\mathbf{R}^r$ *into* $\mathbf{R}^{r*}$ *is the derivative of the real function* $f_y : t \mapsto \frac{1}{4}\sum_\ell |x_\ell(y)|^2 e^{2c_\ell t}$*, which is convex since its second derivative* $D^2 f_y(t) = \sum_\ell |x_\ell(y)|^2 e^{2c_\ell t} c_\ell \otimes c_\ell$ *is nonnegative for all* t*;*

ii) the bilinear form $D^2 f_y(t)$ *is positive definite for all* t *if and only if* y *does not belong to the* S*-invariant set*

$$\operatorname{Crit} F = \{y \in E : V_{J(y)} \neq \mathbf{R}^{r*}\},$$

which is indeed the critical set of F *since* $DF(y)\mathbf{R}^r = V_{J(y)}$*;*

iii) for $y \notin \operatorname{Crit} F$*, it follows that* $\Psi_y = Df_y$ *is a diffeomorphism of* $\mathbf{R}^r$ *onto an open subset of* $\mathbf{R}^{r*}$*, which turns out to be the interior* $\mathring{C}_{J(y)}$ *of the closed convex cone* $C_{J(y)}$*;*

iv) either y *lies in the union* $\mathcal{V}$ *of the SIM's of* S *(which contains* $\operatorname{Crit} F$*), or* Ψ_y *is a diffeomorphism of* $\mathbf{R}^r$ *onto* $\mathbf{R}^{r*}$*;*

v) the restriction of F *to each SIM of* S *and therefore to* $\mathcal{V}$ *is a proper map.*

Proof. i) is obvious.

ii) The nonnegative bilinear form $D^2 f_y(t)$ is degenerate if and only if there exists $s \in \mathbf{R}^r \setminus \{0\}$ such that $D^2 f_y(t)s^2 = 0$, i.e. $c_\ell(s) = 0$ for $x_\ell(y) \neq 0$; this does mean that the c_ℓ's with $x_\ell(y) \neq 0$ belong to the hyperplane $\{c \in \mathbf{R}^{r*} : c(s) = 0\}$ for some nonzero $s \in \mathbf{R}^r$, i.e. that they do not span $\mathbf{R}^{r*}$. To see that $DF(y) : \delta y \mapsto \sum_\ell x_\ell(\delta y) \cdot x_\ell(y) c_\ell$ maps $\mathbf{R}^r$ onto $V_{J(y)}$, notice that $DF(y)\mathbf{R}^r \subset V_{J(y)}$ and that every linear combination $\sum_{\ell \in J(y)} \lambda_\ell c_\ell$ is of the form $DF(y)\delta y$ with $x_\ell(\delta y) = \lambda_\ell x_\ell(y)/|x_\ell(y)|^2$ for $\ell \in J(y)$.

iii) The first assertion is classical and the inclusion $\Psi_y(\mathbf{R}^r) \subset \mathring{C}_{J(y)}$ clear; for the whole, see [10], 5.1, proof of Théorème 1.

iv) One shoud prove that $y \in \mathcal{V}$ if and only if $\mathring{C}_{J(y)} \neq \mathbf{R}^{r*}$. The "only if" is clear: if y is in the unstable manifold of S^t, then either $t = 0$, yielding $y = 0$ and $\mathring{C}_{J(y)} = \emptyset$, or one has $c_\ell(t) > 0$ for all $\ell \in J(y)$, hence $C_{J(y)}$ is contained in $\{c : c(t) \geq 0\}$; to prove the "if", notice that $\mathring{C}_{J(y)}$ is then contained in the half-space $\{c : c(t) > 0\}$ for some nonzero t; now, by weak hyperbolicity, the convex hull of the c_ℓ's lying in the hyperplane $\{c : c(t) = 0\}$ does not contain the origin, hence t can be changed a bit so that they satisfy $c_\ell(t) > 0$, which will in particular be the case for all $\ell \in J(y)$.

v) If W is the unstable manifold of S^t, then $W \ni y \mapsto F(y)t$ is a positive definite quadratic form. □

Notes. Proposition 2.16 holds in the Poincaré domain. In the Siegel domain, (iv) implies that F is onto, hence every regular value of F is a value of F.

By weak hyperbolicity, the orbits of S contained in $\mathcal{V}$ are precisely those adherent to 0, called the *Poincaré leaves* of the singular foliation defined by S.

Proposition 2.17 *Let* $b \in \mathbf{R}^{r*}$*.*

i) It is a critical value of F *if and only if it lies in some* C_I *with* $\dim V_I < r$*.*

ii) Otherwise, the union $\hat{\mathcal{V}}_b$ *of the SIM's of* S *on which* F *does not take the value* b *consists of all* $y \in E$ *with* $b \notin \mathring{C}_{J(y)}$ *and therefore contains* $\operatorname{Crit} F$*.*

iii) Still assuming that b is a regular value of F, the submanifold $Q_b := F^{-1}(b)$ is a quotient by S of the dense S-invariant open subset $E_b := E \smallsetminus \hat{\mathcal{V}}_b$ in the same sense as for flows; in other words, the map $\varphi_b : Q_b \times \mathbf{R}^r \to E_b$ defined by $\varphi_b(y,t) := S^t(y)$ is a diffeomorphism[18]*.*

Proof. i) If $y \in F^{-1}(b)$ is a critical point of F, one can take $I = J(y)$ since $V_{J(y)} \neq \mathbf{R}^{r*}$ and $b = F(y) \in C_{J(y)}$. If $b \in C_I$ with $V_I \neq \mathbf{R}^{r*}$, there clearly exists $y \in F^{-1}(b)$ with $J(y) \subset I$, hence $y \in \operatorname{Crit} F$.

ii) Given $I \subset \{1,\ldots,n\}$, one clearly has $F(E_I) \subset C_I$ hence, by (i), $b \notin F(E_I)$ for $\dim V_I < r$; if now $V_I = \mathbf{R}^{r*}$, then either $b \notin F(E_I)$, or

If $y \in \hat{\mathcal{V}}_b$ then of course b does not lie in $\Psi_y(\mathbf{R}^r)$ which, by Proposition 2.16 ii)-iii), equals $\mathring{C}_{J(y)}$ unless $\mathring{C}_{J(y)} = \emptyset$; conversely, if $b \notin \mathring{C}_{J(y)}$ then $b \notin C_{J(y)}$ by (i), hence there exists $t \in \mathbf{R}^r$ such that $bt < 0$ and $c_\ell t \geq 0$ for all $\ell \in J(y)$, implying that F does not take the value b on $W := \bigoplus_{c_\ell t \geq 0} E_\ell$; now W contains y and is a SIM of S since, by weak hyperbolicity, the inequalities $c_\ell t \geq 0$ can be made strict.

iii) By (ii), one has $y \in E_b$ if and only if b lies in $\mathring{C}_{J(y)}$, which is the image of the diffeomorphism Ψ_y by Proposition 2.16 iii). □

2.3.3 Untangling the $\mathbf{R}^r$-action S when $k = 0$

Notation. For $0 \leq j \leq r$, we identify $\mathbf{R}^j$ to the subgroup $\mathbf{R}^j \times \{0\}$ of $\mathbf{R}^r$ and denote by $\pi_j : \mathbf{R}^{r*} \to \mathbf{R}^{j*}$ the restriction map $c \mapsto c|_{\mathbf{R}^j}$. The canonical basis of $\mathbf{R}^r$ is still denoted by $(\delta_1, \ldots, \delta_r)$.

Proposition 2.18 ([10], 5.2, Propositions 1 and 2) *Replacing S^t by S^{At} for a suitable automorphism A of $\mathbf{R}^r$, one may assume that, for all $j \in \{0,\ldots,r\}$ and $I \subset \{1,\ldots,n\}$, the restricted projection $\pi_j|_{V_I}$ has maximal rank, which implies the following:*

i) For $0 \leq j \leq r$ the action $S_j := S|_{\mathbf{R}^j \times E}$ is weakly hyperbolic and the previous properties of S, F hold for S_j and $F_j := \pi_j \circ F$.

ii) For $0 \leq j < r$, if I belongs to

$$\mathcal{K}_j := \{I \subset \{1,\ldots,n\} : \dim V_I = j \text{ and } c_\ell \in V_I \Rightarrow \ell \in I\},$$

(when S is hyperbolic, $I \in \mathcal{K}_j$ if and only if I has j elements), then there exists one $\mathbf{g}_I \in \mathbf{R}^j$ such that

$$\pi_{j+1} V_I = \{c \in \mathbf{R}^{j+1*} : c(\delta_{j+1} - \mathbf{g}_I) = 0\}.$$

iii) For almost all $b \in \mathbf{R}^r$, every F_j admits $b_j := \pi_j(b)$ as a regular value. Hence, if $\hat{\mathcal{V}}_{b_j}$ denotes the union of those SIM's of S_j on which F_j does not take the value b_j, the submanifold $Q_{b_j} := F_j^{-1}(b_j)$ is a quotient by S_j of the dense S_j-invariant open subset $E_{b_j} := E \smallsetminus \hat{\mathcal{V}}_{b_j}$; in other words, the map $\varphi_{b_j} : Q_{b_j} \times \mathbf{R}^j \to E_{b_j}$ defined by $\varphi_{b_j}(y,t) := S_j^t(y)$ is a diffeomorphism.

Idea of the proof. This relies on simple general position arguments: see [10], where the subgroups $H_j = A\mathbf{R}^j$ are considered rather than A itself. □

[18] When S is a linear holomorphic $\mathbf{C}^s$-action in the Poincaré domain, this provides examples of compact holomorphic manifolds with no real symplectic structure [18].

Hypothesis and notation We assume that S and b have the properties stated in Proposition 2.18. As in [10], for $0 \leq j < r$, we let

$$\mathcal{I}_{b_j} := \left\{ I \in \mathcal{K}_j : Q_{b_j} \cap E_I \neq \emptyset \right\}.$$

We denote by $\xi : \mathbf{R}^r \to L(E, E)$ the homomorphism of Lie algebras which to $u \in \mathbf{R}^r$ associates the infinitesimal generator of the linear flow $(t, y) \mapsto S^{tu} y$.

Proposition 2.19 ([10], 5.2, Proposition 3) *For $0 \leq j < r$, the action $S_{j+1}|_{E_{b_j} \times \mathbf{R}^{j+1}}$ reads as follows via the diffeomorphism φ_{b_j}: for all $y \in Q_{b_j}$, $s, s' \in \mathbf{R}^j$ and $t \in \mathbf{R}$,*

$$\varphi_{b_j}^* S_{j+1}^{(s,t)}(y, s') = \Big(\Phi_{b_j}^t(y), s + s' + \int_0^t \mathbf{g}_{b_j} \circ \Phi_{b_j}^\tau(y) \, d\tau \Big),$$

where $\Phi_{b_j}^t : Q_{b_j} \to Q_{b_j}$ and $\mathbf{g}_{b_j} : Q_{b_j} \to \mathbf{R}^j$ are defined in the following way:

- *$\Phi_{b_j}^t$ is the flow of the vector field on Q_{b_j} whose value at $y \in Q_{b_j}$ is the image of $\xi(\delta_{j+1}) y$ by the projection of E onto $T_y Q_{b_j}$ along the tangent space $\xi(\mathbf{R}^j) y$ of the orbit of y by S_j;*
- *this vector field is of the form $y \mapsto \xi\big(\delta_{j+1} - \mathbf{g}_{b_j}(y)\big) y$ with $\mathbf{g}_{b_j}(y) \in \mathbf{R}^j$, which defines $\mathbf{g}_{b_j}$.*

It follows that the flow $\Phi_{b_j}^t$ is complete and that its orbits are the intersections of Q_{b_j} with the orbits of S_{j+1}.

Proof. Natural. □

We now describe the structure of the flow $\Phi_{b_j}^t$:

Proposition 2.20 ([10], 5.2, Théorèmes 1-2) *For $0 \leq j < r$, the real function $f_{b_j} : y \mapsto F(y) \delta_{j+1}$ on Q_{b_j} and the flow $\Phi_{b_j}^t$ have the following properties:*

i) The critical set of f_{b_j} is $\Sigma_{b_j} = \{ y \in Q_{b_j} : \dim V_{J(y)} = j \}$, disjoint union of the compact $\Phi_{b_j}^t$-invariant submanifolds $\Sigma_{b_j, I} := Q_{b_j} \cap E_I$ with $I \in \mathcal{I}_{b_j}$, which satisfy $\mathbf{g}_{b_j}(\Sigma_{b_j, I}) = \{ \mathbf{g}_I \}$ and $f_{b_j}(\Sigma_{b_j, I}) = \{ b_j(\mathbf{g}_I) \}$.

ii) The restriction of f_{b_j} to $Q_{b_j} \smallsetminus \Sigma_{b_j}$ is a Lyapunov function for $\Phi_{b_j}^t|_{Q_{b_j} \smallsetminus \Sigma_{b_j}}$.

iii) For each $y \in Q_{b_j}$, the function $\psi_y : t \mapsto f_{b_j} \circ \Phi_{b_j}^t(y)$ is bounded from above (resp. below) if and only if y belongs to the stable (resp. unstable) manifold $W_{b_j, I}^+$ (resp. $W_{b_j, I}^-$)[19] of $\Sigma_{b_j, I}$ for some $I \in \mathcal{I}_{b_j}$.

[19] $W_{b_j, I}^{\pm}$ is the set of all $y \in Q_{b_j}$ such that the distance of $\Phi_{b_j}^t(y)$ to $\Sigma_{b_j, I}$ tends to 0 when $t \to \pm\infty$.

iv) For each $I \in \mathcal{I}_{b_j}$, the $\Phi^t_{b_j}$-invariant subset $W^{\pm}_{b_j,I}$ equals $Q_{b_j} \cap E_{b_j,I} \cap E_{I_0^{\pm}}$, where $E_{b_j,I}$ is the S-invariant open subset

$$E_{b_j,I} := \left\{ y \in E : b_j \in \pi_j C_{J(y)\cap I} \right\}$$

and $E_{I_0^{\pm}}$ is the SIM of S_{j+1} defined by

$$I_0^{\pm} := \{\ell \in \{1, \dots, n\} : \pm c_\ell(\delta_{j+1} - \mathbf{g}_I) \leq 0\}.$$

v) The set $\bigcup_{I\in\mathcal{I}_{b_j}} W^+_{b_j,I} \cup W^-_{b_j,I}$ is the intersection of Q_{b_j} with the union $\mathcal{V}_{j+1}$ of the SIM's of S_{j+1}; hence, it is closed and contains $Q_{b_j} \cap \mathcal{V}_j$. Moreover, $f_{b_j}|_{Q_{b_j}\cap\mathcal{V}_{j+1}}$ is proper,

$$Q_{b_j} \cap \hat{\mathcal{V}}_{b_{j+1}} = \Big(\bigcup_{b(\delta_{j+1}-\mathbf{g}_I)>0} W^+_{b_j,I} \Big) \cup \Big(\bigcup_{b(\delta_{j+1}-\mathbf{g}_I)<0} W^-_{b_j,I} \Big) \qquad (11)$$

and $Q_{b_{j+1}}$ is a quotient of $Q_{b_j} \setminus \hat{\mathcal{V}}_{b_{j+1}}$ by the flow $\Phi^t_{b_j}$.

vi) The set of those $y \in Q_{b_j}$ for which $\psi_y : t \mapsto f_{b_j} \circ \Phi^t_{b_j}(y)$ is bounded (bounded orbits of the flow) is $Q_{b_j} \cap \mathcal{V}_j$; it is compact, contains Σ_{b_j} and

$$\min f_{b_j}(Q_{b_j} \cap \mathcal{V}_j) = \min_{I\in\mathcal{I}_{b_j}} b(\mathbf{g}_I)\,,\ \max f_{b_j}(Q_{b_j} \cap \mathcal{V}_j) = \max_{I\in\mathcal{I}_{b_j}} b(\mathbf{g}_I)$$

Proof. i) As b_j is a regular value of F_j, every $y \in Q_{b_j}$ satisfies $\dim(\pi_j V_{J(y)}) = j$, and the critical set of f_{b_j} is $Q_{b_j} \cap \operatorname{Crit} F_{j+1} = \{y \in Q_{b_j} : \dim(\pi_{j+1} V_{J(y)}) \leq j\}$; by the general position hypothesis stated in Proposition 2.18, this is Σ_{b_j}. Of course, every $y \in \Sigma_{b_j}$ lies in only one $\Sigma_{b_j,I}$, defined by $I = \{\ell : c_\ell \in V_{J(y)}\}$; in that case, the formula

$$DF(y)\,(\xi(u)y) = \sum_{\ell\in J(y)} |x_\ell(y)|^2 c_\ell(u) c_\ell,\ u \in \mathbf{R}^r, \qquad (12)$$

shows that $DF(y)\,(\xi(\delta_{j+1} - \mathbf{g}_I)y) = 0$, hence $\xi(\delta_{j+1} - \mathbf{g}_I) \in \ker DF_j(y) = T_y Q_{b_j}$ and therefore $\mathbf{g}_{b_j}(\Sigma_{b_j,I}) = \{\mathbf{g}_I\}$. For $y \in \Sigma_{b_j,I}$, one does have

$$f_{b_j}(y) = \frac{1}{2}\sum_{\ell\in I} |x_\ell(y)|^2 c_\ell(\delta_{j+1}) = \frac{1}{2}\sum_{\ell\in I} |x_\ell(y)|^2 c_\ell(\mathbf{g}_I) = F_j(y)\mathbf{g}_I = b_j(\mathbf{g}_I).$$

Finally, each $\Sigma_{b_j,I}$ is a submanifold since b_j is a regular value of $F_j|_{E_I}$ by Propositions 2.15 and 2.18; it is compact because it lies in $Q_{b_j} \cap \mathcal{V}_j$, see the proof of (vi).

ii) For $y \in Q_{b_j}$, the relation $\xi\left(\delta_{j+1} - \mathbf{g}_{b_j}(y)\right) y \in T_y Q_{b_j}$ and (12) yield

$$\begin{aligned} 0 &= DF_j(y)\left(\xi\left(\delta_{j+1} - \mathbf{g}_{b_j}(y)\right) y\right) \mathbf{g}_{b_j}(y) \\ &= \sum_{\ell\in J(y)} |x_\ell(y)|^2 c_\ell\left(\delta_{j+1} - \mathbf{g}_{b_j}(y) c_\ell\right) c_\ell\left(\mathbf{g}_{b_j}(y)\right), \end{aligned}$$

hence, again by (12),

$$\begin{aligned} Df_{b_j}(y)\left(\xi\left(\delta_{j+1} - \mathbf{g}_{b_j}(y)\right) y\right) &= \sum_{\ell\in J(y)} |x_\ell(y)|^2 c_\ell\left(\delta_{j+1} - \mathbf{g}_{b_j}(y)\right) c_\ell(\delta_{j+1}) \\ &= \sum_{\ell\in J(y)} |x_\ell(y)|^2 c_\ell\left(\delta_{j+1} - \mathbf{g}_{b_j}(y)\right)^2; \end{aligned}$$

this is positive unless $\pi_{j+1}c_\ell$ vanishes on $\delta_{j+1}-\mathbf{g}_{b_j}(y)$ for all $\ell\in J(y)$, in which case $\dim(\pi_{j+1}V_{J(y)})\le j$, that is, $y\in Q_{b_j}\cap \mathrm{Crit}\,F_{j+1}=\Sigma_{b_j}$.

iii) The "only if" being clear, we prove the "if". If $\psi_y(t)=f_{b_j}\circ\Phi^t_{b_j}(y)$ is bounded from above (resp. below) then, by Proposition 2.19, the map $\mathbf{R}^{j+1}\ni s\mapsto F_{j+1}(S^s y)$ is not onto (it does not take every value $c\in\mathbf{R}^{j+1*}$ with $c|_{\mathbf{R}^j}=b_j$) and therefore $y\in Q_{b_j}\cap\mathcal{V}_{j+1}$ by Proposition 2.16 iv). As $F_{j+1}|_{\mathcal{V}_{j+1}}$ is proper by Proposition 2.16 v), so is $f_{b_j}|_{Q_{b_j}\cap\mathcal{V}_{j+1}}$; hence, there exists a real sequence (t_p) tending to $+\infty$ (resp. $-\infty$) such that $\Phi^{t_p}_{b_j}(y)$ tends to a limit $z\in Q_{b_j}$. We now use the following obvious result:

Lemma 2.21 *One has $E_{b_j}=\bigcup_{I\in\mathcal{I}_{b_j}}E_{b_j,I}$. For all $I\in\mathcal{I}_{b_j}$, if $I^\pm:=I_0^\pm\setminus I$,*

i) a quotient of the S-invariant subset $E_{b_j,I}$ by S_j is the submanifold $Q_{b_j,I}$ that reads $\Sigma_{b_j,I}\times E_{I^+}\times E_{I^-}$ in the identification of $E=E_I\oplus E_{I^+}\oplus E_{I^-}$ to $E_I\times E_{I^+}\times E_{I^-}$;

ii) the image of $\Phi^t_{b_j}|_{Q_{b_j}\cap E_{b_j,I}}$ by the diffeomorphism $\mathcal{C}_{b_j,I}:Q_{b_j}\cap E_{b_j,I}\to Q_{b_j,I}$ obtained by following the orbits of S_j is

$$\Phi^t_{b_j,I}:=S^{t(\delta_{j+1}-\mathbf{g}_I)}|_{Q_{b_j,I}};$$

iii) In the identification (i), the flow $\Phi^t_{b_j,I}$ splits:

$$\Phi^t_{b_j,I}(y_0,y_+,y_-)=\left(\Phi^t_{b_j}(y_0),S^{t(\delta_{j+1}-\mathbf{g}_I)}y_+,S^{t(\delta_{j+1}-\mathbf{g}_I)}y_-\right),$$

and the linear endomorphism $y_\pm\mapsto S^{t(\delta_{j+1}-\mathbf{g}_I)}y_\pm$ of $E_{I^\pm}$ is a strict contraction for $\pm t>0$; in particular, the stable and unstable manifolds $Y^+_{b_j,I}$ and $Y^-_{b_j,I}$ of $\Sigma_{b_j,I}$ for the flow $\Phi^t_{b_j,I}$ are

$$Y^\pm_{b_j,I}:=Q_{b_j,I}\cap E_{I_0^\pm}$$

and the germ of $\Phi^t_{b_j,I}$ at $\Sigma_{b_j,I}$ satisfies the hypotheses of the extension lemma 2.11. ■

Back to the proof of Proposition 2.20 iii). The first assertion of the lemma shows that $z\in E_{b_j,I}$ for some $I\in\mathcal{I}_{b_j}$, hence $\Phi^{t_p}_{b_j}(y)\in Q_{b_j}\cap E_{b_j,I}$ for large enough p and in fact $y\in Q_{b_j}\cap E_{b_j,I}$; as $\mathcal{C}_{b_j,I}\circ\Phi^{t_p}_{b_j}(y)=\Phi^{t_p}_{b_j,I}\circ\mathcal{C}_{b_j,I}(y)$ tends to $\mathcal{C}_{b_j,I}(z)$, the form of the flow $\Phi^{t_p}_{b_j,I}$ implies $\mathcal{C}_{b_j,I}(y)\in Y^\pm_{b_j,I}$ and $\mathcal{C}_{b_j,I}(z)\in\Sigma_{b_j,I}$, hence $y\in W^\pm_{b_j,I}$ and $z\in\Sigma_{b_j,I}$.

iv) What we have just done shows that $W^\pm_{b_j,I}=\mathcal{C}^{-1}_{b_j,I}(Y^\pm_{b_j,I})=Q_{b_j}\cap E_{b_j,I}\cap E_{I_0^\pm}$.

v) We have just proven the inclusion $\bigcup_{I\in\mathcal{I}_{b_j}}W^+_{b_j,I}\cup W^-_{b_j,I}\subset Q_{b_j}\cap\mathcal{V}_{j+1}$; to establish equality, remember that if $y\in Q_{b_j}$ belongs to some v.f.i. of S_{j+1}, then the interior of $\pi_{j+1}C_{J(y)}$ differs from $\mathbf{R}^{j+1*}$ by Proposition 2.16 iv). If this interior is empty, i.e. $\dim\pi_{j+1}V_{J(y)}\le j$, then $y\in\Sigma_{b_j}$. Otherwise, as $F_j(y)=b_j$ and $y\notin\mathrm{Crit}\,F_j$, the interior of $\pi_jC_{J(y)}$ contains b_j, hence one of the j-dimensional faces $\pi_{j+1}C_K$, $K\subset J(y)$, of the closed convex cone $\pi_{j+1}C_{J(y)}$ must satisfy $b_j\in\pi_jC_K$; if I is the element of $\mathcal{I}_{b_j}$

such that $V_I = V_K$ then, as $\pi_{j+1} C_{J(y)}$ lies in one of the closed half-spaces bounded by $\pi_{j+1} V_I$, we have $y \in Q_{b_j} \cap E_{b_j,I} \cap E_{I_0^\pm} = W^\pm_{b_j,I}$.

The function $f_{b_j}|_{Q_{b_j} \cap \mathcal{V}_{j+1}}$ is proper because $F_{j+1}|_{\mathcal{V}_{j+1}}$ is. Finally, for $y \in Q_{b_j}$, it follows from Proposition 2.19 that $y \in \hat{\mathcal{V}}_{b_{j+1}}$ if and only if $t \mapsto f_{b_j} \circ \Phi^t_{b_j}(y)$ does not take the value $b_{j+1}(\delta_{j+1}) = b(\delta_{j+1})$; now, by (iii), this means that either $y \in W^+_{b_j,I}$ for some I such that the value $b_j(\mathbf{g}_I) = b(\mathbf{g}_I)$ of f_{b_j} on $\Sigma_{b_j,I}$ is less than $b(\delta_{j+1})$, or $y \in W^-_{b_j,I}$ for some I with $b(\mathbf{g}_I) > b(\delta_{j+1})$, hence (11).

vi) As $F_j|_{\mathcal{V}_j}$ is proper, $Q_{b_j} \cap \mathcal{V}_j$ is compact; since $\mathcal{V}_j$ is S-invariant, this compact subset is invariant by the flow $\Phi^t_{b_j}$ and therefore contained in the set of $y \in Q_{b_j}$ for which ψ_y is bounded. Conversely, given such an y, it follows from (iii) that it belongs to $W^-_{b_j,I} \cap W^+_{b_j,J}$ for uniquely determined $I, J \in \mathcal{I}_{b_j}$ which, by (i)-(ii), satisfy either $I = J$, or $b(\mathbf{g}_I) < b(\mathbf{g}_J)$; in the second case, by (iv), one has $y \in E_{I_0^-} \cap E_{J_0^+}$, i.e. $c_\ell(\delta_{j+1} - \mathbf{g}_J) \le 0 \le c_\ell(\delta_{j+1} - \mathbf{g}_I)$ for all $\ell \in J(y)$, implying that y belongs to the SIM $\bigoplus_{c_\ell(\mathbf{g}_J - \mathbf{g}_I) \ge 0}$ of S_j; in the first case $I = J$, one has $y \in \Sigma_{b_j,I}$ and therefore the c_ℓ's with $\ell \in J(y)$ lie in the j-dimensional subspace V_I where, by weak hyperbolicity, they are contained in an open half-space H with $0 \in \partial H$; as $\pi_j|_{V_I}$ has maximal rank, every $\pi_j c_\ell$ with $\ell \in J(y)$ lies in the open half-space $\pi_j H$ of $\mathbf{R}^{j*}$, proving that $y \in \mathcal{V}_j$. The bounds of $f_{b_j}(Q_{b_j} \cap \mathcal{V}_j)$ follow at once from (iii). $\square$

2.3.4 Proof of Theorem 1.2 for $\mathbf{R}^r$-action germs

Even though we could remain at the local level as in [12] or in the proof of Corollary 2.12, it will be more comfortable to extend our action germs into genuine actions with properties very close to those of S; the required conjugacy will then be the solution of a Cauchy problem as in the proof of Corollary 2.14.

Hypothesis and notation. We still assume that S and b have the properties stated in Proposition 2.18. For $0 \le j < r$, we let

$$m_j := \min\left\{b(\delta_{j+1}), \min_{I \in \mathcal{I}_{b_j}} b(\mathbf{g}_I)\right\} \quad \text{and} \quad M_j := \max\left\{b(\delta_{j+1}), \max_{I \in \mathcal{I}_{b_j}} b(\mathbf{g}_I)\right\},$$

hence $Q_{b_j} \cap \mathcal{V}_j \subset f^{-1}_{b_j}([m_j, M_j])$ by Proposition 2.20 vi).

Lemma 2.22 *Given a smooth function $\theta_r : Q_{b_r} = Q_b \to [0,1]$ with compact support, equal to 1 near $Q_b \cap \mathcal{V}$, one defines inductively a compactly supported smooth function $\theta = \theta_0 : E \to [0,1]$ as follows:*

i) Its restriction to Q_{b_r} is θ_r.

ii) For $0 \le j < r$, its restriction θ_j to Q_{b_j} is determined from θ_{j+1} by the formula

$$\forall y \in Q_{b_j+1}\ \forall t \in \mathbf{R} \quad \theta_j\left(\Phi^t_{b_j}(y)\right) = \kappa_j\left(f_{b_j}\left(\Phi^t_{b_j}(y)\right)\right)\theta_{j+1}(y)$$

where $\kappa_j \in \mathcal{C}^\infty(\mathbf{R}, [0,1])$ has compact, connected support and equals 1 on a compact interval K_j containing $[m_j, M_j]$ in its interior.

For $0 \leq j < r$, the function θ_j is equal to 1 in a neighbourhood of $Q_{b_j} \cap \mathcal{V}_j$; in particular, $\theta = 1$ near 0.

Proof. We use the following

Fact *For $0 \leq j < r$, if N is a compact neighbourhood of $Q_{b_{j+1}} \cap \mathcal{V}_{j+1}$ in $Q_{b_{j+1}}$ and K a compact interval with $b_{j+1} \in \mathring{K}$, then the closure $f_{b_j}^{-1}(K) \cap \left(\bigcup_t \Phi_{b_j}^t(N) \cup \hat{\mathcal{V}}_{b_{j+1}}\right)$ of $f_{b_j}^{-1}(K) \cap \bigcup_t \Phi_{b_j}^t(N)$ is a compact neighbourhood of $f_{b_j}^{-1}(K') \cap \mathcal{V}_{j+1}$ in Q_{b_j} for every compact interval $K' \subset \mathring{K}$; in particular, for $[m_j, M_j] \subset \mathring{K}$, it is a compact neighbourhood in Q_{b_j} of $Q_{b_j} \cap \mathcal{V}_j$.*

This fact implies our lemma: indeed, for $N = \operatorname{supp} \theta_{j+1}$ and $K = \operatorname{supp} \kappa_j$, it shows that $\operatorname{supp} \theta_j$ is compact; for $N = \theta_{j+1}^{-1}(1)$ and $\operatorname{supp} \kappa_j \subset \mathring{K}$, it shows that θ_j is smooth, being equal to $\kappa_j \circ f_{b_j}$ near the only litigious part, namely $\mathcal{V}_{j+1} \cap Q_{b_j}$; for $N = \theta_{j+1}^{-1}(1)$ and $K' = K_j$, one gets that $\theta_j = 1$ near $Q_{b_j} \cap \mathcal{V}_j$.

The previous fact follows immediately from Proposition 2.20 ii-iii) if K does not contain any critical value of f_{b_j}, but it remains true otherwise because $\Phi_{b_j}^t$ is (normally) hyperbolic at each $\Sigma_{b_j,I}$ (see step 3 in the proof of the globalisation lemma 2.24). $\square$

Notation

- We fix θ as in Lemma 2.22 and let B be a bounded open subset of E containing $\operatorname{supp} \theta$, hence $0 \in B$.
- With the notation of the preparation lemma 2.10, we let $Y_1, \dots, Y_r$ be commuting smooth vector fields on an open neighbourhood of 0 in E, whose germs at 0 are $(h_\infty^{-1} \circ h)_* X_1, \dots, (h_\infty^{-1} \circ h)_* X_r$; if ξ is as in the previous paragraph, the semi-simple part of $\Lambda_j = DY_j(0)$ is $\xi_j := \xi(\delta_j)$.
- For $\eta > 0$, we define the weighted zooming map $\zeta_\eta : E \to E$ by

$$x_{\ell,p} \circ \zeta_\eta := \eta^{p-2d-1} x_{\ell,p}\,, \text{ where } d := \max_\ell d_\ell.$$

Lemma 2.23 *When η tends to 0, the vector fields $\zeta_{\eta*} Y_j - \xi_j$ tend uniformly to 0 on B and so do their derivatives at all orders.*

Proof. Since $\zeta_{\eta*}\xi_j = \xi_j$ and $Y_j = \Lambda_j + R_j$ with $j^1 R_j(0) = 0$, it is enough to check that

a) $\zeta_{\eta*} R_j$ and its derivatives tend uniformly to 0 on B and

b) each linear vector field $\zeta_{\eta*}(\Lambda_j - \xi_j)$ tends to 0 in $L(E, E)$.

As $\Lambda_j - \xi_j$ is in triangular form $x_{\ell,p} \circ (\Lambda_j - \xi_j) = \sum_{1 \leq q < p} \varepsilon_{\ell,p,q} x_{\ell,q}$,

$$x_{\ell,p} \circ \zeta_\eta \circ (\Lambda_j - \xi_j) = \eta^{p-2d-1} \sum_{1 \leq q < p} \varepsilon_{\ell,p,q} x_{\ell,q} = \sum_{1 \leq q < p} \eta^{p-q} \varepsilon_{\ell,p,q} x_{\ell,q} \circ \zeta_\eta,$$

hence $x_{\ell,p} \circ \zeta_{\eta*}(\Lambda_j - \xi_j) = \sum_{1 \leq q < p} \eta^{p-q} \varepsilon_{\ell,p,q} x_{\ell,q}$, proving b).
By Taylor's formula, $R_j = \sum_{m,q,m',q'} x_{m,q} x_{m',q'} a_{m,q,m',q'}$ near $0 \in E$ with $a_{m,q,m',q'}$

smooth, hence

$$\begin{aligned} x_{\ell,p}\circ\zeta_{\eta*}R_j\,(y) &= \eta^{p-2d-1}\sum_{m,q,m',q'}\eta^{4d+2-q-q'}x_{m,q}\,(y)\,x_{m',q'}\,(y)\,a_{m,q,m',q'}\left(\zeta_\eta^{-1}(y)\right)\\ &= \sum_{m,q,m',q'}\eta^{2d+1+p-q-q'}x_{m,q}\,(y)\,x_{m',q'}\,(y)\,a_{m,q,m',q'}\left(\zeta_\eta^{-1}(y)\right), \end{aligned}$$

which tends uniformly to 0 on B as well as its derivatives, since $\zeta_\eta^{-1}(y)$ does and $q+q'-p\le d_m+d_{m'}-0\le 2d$, in other words $2d+1+p-q-q'\ge 1$. □

The following result shows in particular that every weakly hyperbolic smooth $\mathbf{R}^r$-action germ on E is the germ of a smooth $\mathbf{R}^r$-action.

Globalisation lemma 2.24 *For small enough $\eta>0$, the formulae*

$$\forall y\in Q_{b_{j-1}}\quad \tilde{Y}_j(y)=\begin{cases}\xi_j(y)+\theta(y)(\zeta_{\eta*}Y_j-\xi_j)(y) & \text{for } y\in\operatorname{supp}\theta\\ \xi_j(y) & \text{otherwise}\end{cases}\tag{13}$$

define the generators $\tilde{Y}_j:=\frac{d}{d\tau}\tilde{g}^{\tau\delta_j}|_{\tau=0}$ of a unique smooth action $\tilde{g}$ of $\mathbf{R}^r$ on E, whose germ at 0 is $(\zeta_\eta\circ h_\infty^{-1}\circ h)_*g$: for each j, the germ of $\tilde{Y}_j$ at 0 is the germ $(\zeta_\eta\circ h_\infty^{-1}\circ h)_*X_j$ of $\zeta_{\eta*}Y_j$. This action possesses (for our fixed b and small enough η) *all the properties of S stated in Propositions 2.18 iii), 2.19 and 2.20—which will be detailed in the proof.*

Proof. For small enough η, every vector field $\zeta_{\eta*}Y_j$ is well-defined on B, hence (13) does define a vector field $\tilde{Y}_j$ on E over $Q_{b_{j-1}}$.

The expression "For small enough $\eta>0$ and $0\le j<r$," is implicit at each of the following steps.

Step 1 *One defines a smooth vector field Z_{b_j} on Q_{b_j} as follows: each $Z_{b_j}(y)$ is the projection of $\tilde{Y}_{j+1}(y)$ in $T_yQ_{b_j}$ along $\bigoplus_{1\le i\le j}\mathbf{R}\tilde{Y}_i(y)$. The flow $\Psi^t_{b_j}$ of Z_{b_j} is complete and preserves the intersection of Q_{b_j} with every SIM W of S.*

Indeed, with the notation of Lemma 2.22, recall that $\operatorname{supp}\theta_j:=Q_{b_j}\cap\operatorname{supp}\theta$; for $y\in Q_{b_j}\smallsetminus\operatorname{supp}\theta_j$, one has $\tilde{Y}_i(y)=\xi_j(y)$ for $1\le i\le j+1$, hence $Z_{b_j}(y)=\xi\left(\delta_{j+1}-\mathbf{g}_{b_j}(y)\right)$; on $\operatorname{supp}\theta_j$, every $\tilde{Y}_i$ with $1\le i\le j$ converges uniformly to ξ_i when $\eta\to 0$; now, as Q_{b_j} is a quotient of E_{b_j} by S_j, the vectors fields ξ_i with $1\le i\le j$ are linearly independent at every point of $\operatorname{supp}\theta_j$, hence so are, for small enough η, the vectors fields $\tilde{Y}_i$ with $1\le i\le j$, implying that Z_{b_j} is well-defined and smooth. The flow $\Psi^t_{b_j}$, having the same generator as the complete flow $\Phi^t_{b_j}$ off the compact subset $\operatorname{supp}\theta_j$, is complete. It preserves $Q_{b_j}\cap W$ for each SIM W of S because every ξ_i and every $\zeta_{\eta*}Y_i|_{\operatorname{supp}\theta_j}$ with $i\le j+1$ is tangent to W.

Step 2 *The restriction of f_{b_j} to the $\Psi^t_{b_j}$-invariant subset $Q_{b_j}\smallsetminus\Sigma_{b_j}$ is a Lyapunov function for the flow $\Psi^t_{b_j}|_{Q_{b_j}\smallsetminus\Sigma_{b_j}}$.*

Indeed, by Step 1, every $\Sigma_{b_j,I}=Q_{b_j}\cap E_{I_0^+}\cap E_{I_0^-}$ with $I\in\mathcal{I}_{b_j}$ is $\Psi^t_{b_j}$-invariant since $E_{I_0^+},E_{I_0^-}$ are SIM's of S. We should verify that $\mathcal{L}_{Z_{b_j}}f_{b_j}(y)=df_{b_j}(y)Z_{b_j}(y)$

is positive off Σ_{b_j}. By Proposition 2.20 ii), this is true in $Q_{b_j} \smallsetminus \operatorname{supp}\theta_j$, where Z_{b_j} generates $\Phi^t_{b_j}$; by Lemma 2.23 and Proposition 2.20 ii), for every open subset $V \supset \Sigma_{b_j}$ of Q_{b_j}, the function $\mathcal{L}_{Z_{b_j}} f_{b_j}$ is positive on $\operatorname{supp}\theta_j \smallsetminus V$ for small enough η; hence, the problem is to find an open subset $V \supset \Sigma_{b_j}$ of Q_{b_j} such that $\mathcal{L}_{Z_{b_j}} f_{b_j}$ is positive on $V \smallsetminus \Sigma_{b_j}$ for small enough η. This follows essentially from the fact that *each $\Sigma_{b_j,I}$ is a critical submanifold of $\mathcal{L}_{Z_{b_j}} f_{b_j}$ along which the Hessian of $\mathcal{L}_{Z_{b_j}} f_{b_j}$ is positive definite in the normal direction $E_{I^+} \oplus E_{I^-}$ for small enough η.*

Let us prove this: for $y \in Q_{b_j}$, since $F_j(y) = b_j$ and $c_\ell(\delta_{j+1} - \mathbf{g}_I) = 0$ for $\ell \in I$,

$$f_{b_j}(y) = b_j(\mathbf{g}_I) - F_j(y)\mathbf{g}_I + f_{b_j}(y) = b_j(\mathbf{g}_I) + \frac{1}{2}\sum_{\ell \notin I} c_\ell(\delta_{j+1} - \mathbf{g}_I)|x_\ell(y)|^2 \qquad (14)$$

hence, denoting the standard Euclidean scalar product by a dot,

$$\mathcal{L}_{Z_{b_j}} f_{b_j}(y) = \sum_{\ell \notin I} c_\ell(\delta_{j+1} - \mathbf{g}_I)\, x_\ell(y) \cdot x_\ell\left(Z_{b_j}(y)\right).$$

For $y \in \Sigma_{b_j,I}$, as $Z_{b_j}(y)$ is tangent to $\Sigma_{b_j,I}$, every $x_\ell\left(Z_{b_j}(y)\right)$ (and of course $x_\ell(y)$) with $\ell \notin I$ vanishes, yielding

$$d\,\mathcal{L}_{Z_{b_j}} f_{b_j}(y) = \sum_{\ell \notin I} c_\ell(\delta_{j+1} - \mathbf{g}_I)\left(x_\ell(dy) \cdot x_\ell\left(Z_{b_j}(y)\right) + x_\ell(y) \cdot d\left(x_\ell \circ Z_{b_j}\right)(y)\right) = 0$$

and implying that the Hessian $D^2(\mathcal{L}_{Z_{b_j}} f_{b_j})(y)$ is the quadratic form

$$v \mapsto 2\sum_{\ell \notin I} c_\ell(\delta_{j+1} - \mathbf{g}_I)\, x_\ell(v) \cdot d\left(x_\ell \circ Z_{b_j}\right)(y)v;$$

when $\eta \to 0$, this quadratic form tends uniformly on $\Sigma_{b_j,I}$ to its analogue for the flow $\Phi^t_{b_j}$, namely $v \mapsto 2\sum_{\ell \notin I} c_\ell(\delta_{j+1} - \mathbf{g}_I)^2\,|x_\ell(v)|^2$; as this is positive definite on $E_{I^+} \oplus E_{I^-}$, so is $D^2(\mathcal{L}_{Z_{b_j}} f_{b_j})(y)$ for all $y \in \Sigma_{b_j,I}$ when η is small enough.

To conclude, writing the points of $E_{b_j,I}$ under the form (y,z) with $y \in \Sigma_{b_j,I}$ and $z \in E_{I^+} \oplus E_{I^-}$ and remembering that $\mathcal{L}_{Z_{b_j}} f_{b_j}$ vanishes to order 1 along $\Sigma_{b_j,I}$, Taylor's formula writes $\mathcal{C}_{b_j,I*}\mathcal{L}_{Z_{b_j}} f_{b_j}(y,z) = \chi_{y,z}(z)$ where $\chi_{y,z}$ is the quadratic form $\int_0^1 (1-t)D^2\mathcal{C}_{b_j,I*}\mathcal{L}_{Z_{b_j}} f_{b_j}(y,tz)dt$; when $\eta \to 0$, this quadratic form tends uniformly near $\Sigma_{b_j,I}$ to its analogue for S, which is positive definite if the neighbourhood is small enough since, by what we have just done, it is positive definite on $\Sigma_{b_j,I}$.

Step 3 *With the notation of Lemma 2.21, the germ of $\mathcal{C}_{b_j,I*}\Psi^t_{b_j}$ at $\Sigma_{b_j,I}$ satisfies the hypotheses of the extension lemma 2.11 for all $I \in \mathcal{I}_{b_j}$ and $t \neq 0$, with $W^+ = Y^{\pm}_{b_j,I}$ and $W^- = Y^{\mp}_{b_j,I}$ for $\pm t > 0$. Hence, a neighbourhood basis of $\Sigma_{b_j,I}$ in Q_{b_j} consists of the compact subsets*

$$B_{\varepsilon,V} := \left(f_{b_j}^{-1}([b(\mathbf{g}_I) - \varepsilon, b(\mathbf{g}_I) + \varepsilon]) \cap \bigcup_t \Psi^t_{b_j}(V)\right) \cup \left(f_{b_j}^{-1}([b(\mathbf{g}_I), b(\mathbf{g}_I) + \varepsilon]) \cap W^-_{b_j,I}\right)$$

with $\varepsilon > 0$ small and V a small compact neighbourhood of $W^+_{b_j,I} \cap f_{b_j}^{-1}(b(\mathbf{g}_I) - \varepsilon)$ in $f_{b_j}^{-1}(b(\mathbf{g}_I) - \varepsilon)$. For $y \in B_{\varepsilon,V}$, the function $t \mapsto f_{b_j} \circ \Psi^t_{b_j}(y)$ takes the value $b(\mathbf{g}_I) \pm \varepsilon$ if and only if $y \notin W^{\pm}_{b_j,I}$.

Indeed, since $\Psi^t_{b_j}$ preserves the intersections of Q_{b_j} with the SIM's $E_{I_0^\pm}$ of S, the germs at $\Sigma_{b_j,I}$ of their images by $\mathcal{C}_{b_j,I}$ are preserved by $\mathcal{C}_{b_j,I*}\Psi^t_{b_j}$. Hence, as $\Psi^t_{b_j}$ tends in the C^1 sense to $\Phi^t_{b_j}$ near $\Sigma_{b_j,I}$ when $\eta \to 0$, the first part of our statement follows from the analogous fact for $\Phi^t_{b_j}$ (last assertion of Lemma 2.21).

We now deduce the second part in a more or less standard way: the boxes

$$B_\rho = \mathcal{C}^{-1}_{b_j,I}\left(\left\{y \in Q_{b_j,I} : \max\left\{\sum_{\ell\in I^+} |x_\ell(y)|^2, \sum_{\ell\in I^-} |x_\ell(y)|^2\right\} \le \rho^2\right\}\right)$$

with $\rho > 0$ small enough form a neighbourhood basis of $\Sigma_{b_j,I}$ in Q_{b_j} and, for all $y \in B_\rho$, the orbit $\Psi^t_{b_j}(y)$ leaves B_ρ for some t with $\pm t \ge 0$ through

$$\partial^\pm B_\rho = \left\{y \in B_\rho : \sum_{\ell\in I^\mp} |x_\ell \circ \mathcal{C}_{b_j,I}(y)|^2 = \rho^2\right\}$$

unless $y \in W^\pm_{b_j,I}$, in which case $\Psi^t_{b_j}(y)$ remains in B_ρ for $\pm t \ge 0$ and tends to $\Sigma_{b_j,I}$. Moreover[20], *the closures $C_\rho(N)$ of the subsets $B_\rho \cap \bigcup_t \Psi^t_{b_j}(N)$ with N a neighbourhood of $\partial^- B_\rho \cap W^+_{b_j,I}$ in $\partial^- B_\rho$ form a neighbourhood basis of* $(W^+_{b_j,I} \cup W^-_{b_j,I}) \cap B_\rho$ in B_ρ; thus, when N is small enough, one has $\max f_{b_j}(N) < b(\mathbf{g}_I) < \min f_{b_j}\big(C_\rho(N) \cap \partial^+ B_\rho\big)$, since $\max f_{b_j}(W^+_{b_j} \cap \partial^- B_\rho) < b(\mathbf{g}_I) < \min f_{b_j}(W^-_{b_j} \cap \partial^+ B_\rho)$ by, e.g., (14); it follows that, for $\max f_{b_j}(N) < b(\mathbf{g}_I) - \varepsilon < b(\mathbf{g}_I) + \varepsilon < \min f_{b_j}\big(C_\rho(N) \cap \partial^+ B_\rho\big)$ the subset $C_\rho(N) \cap f^{-1}_{b_j}([b(\mathbf{g}_I) - \varepsilon, b(\mathbf{g}_I) + \varepsilon])$ of B_ρ is a neighbourhood of $\Sigma_{b_j,I}$ of the form $B_{\varepsilon,V}$ with $V = C_\rho(N) \cap f^{-1}_{b_j}(b(\mathbf{g}_I) - \varepsilon)$.

Step 4 *For each $y \in Q_{b_j}$, the function $\chi_y : t \mapsto f_{b_j} \circ \Psi^t_{b_j}(y)$ is bounded from above (resp. below) if and only if y belongs to the stable (resp. unstable) manifold of $\Psi^t_{b_j}$ at $\Sigma_{b_j,I}$ for some $I \in \mathcal{I}_{b_j}$. These global stable and unstable manifolds are the same as for $\Phi^t_{b_j}$, namely $W^+_{b_j,I} = Q_{b_j} \cap E_{b_j,I} \cap E_{I_0^+}$ and $W^-_{b_j,I} = Q_{b_j} \cap E_{b_j,I} \cap E_{I_0^-}$. Hence, $Q_{b_{j+1}}$ is a quotient of $Q_{b_j} \smallsetminus \hat{\mathcal{V}}_{b_{j+1}}$ by the flow $\Psi^t_{b_j}$.*

Indeed, if χ_y is bounded, say, from above, then $\Psi^{t_0}_{b_j}(y) \in \operatorname{supp}\theta_j$ for some t_0 since, otherwise, $\Psi^t_{b_j}(y) = \Phi^t_{b_j}(y)$ could not tend to any $\Sigma_{b_j,I}$, which lies inside $\operatorname{supp}\theta_j$, yielding the contradiction $\chi_y(t) \to +\infty$ by Proposition 2.20 iii); we claim that $\Psi^t_{b_j}(y)$ remains in $\operatorname{supp}\theta_j$ for all $t \ge t_0$: it cannot escape through the "lateral" part of the boundary of $\operatorname{supp}\theta_j$ that consists of segments of orbits of $\Phi^t_{b_j}$ coming from the boundary of $\operatorname{supp}\theta_{j-1}$, which are segments of orbits of $\Psi^t_{b_j}$ because there $\theta = 0$; the only remaining escape, through the part of the boundary where f_{b_j} is maximal, is also excluded because $\Psi^t_{b_j}(y)$ would never come back and therefore $\chi_y(t) \to +\infty$ as before.

Now $\chi'_y(t) = \mathcal{L}_{Z_{b_j}} f_{b_j}\Big(\Psi^t_{b_j}(y)\Big)$ accumulates for $t \ge t_0$ to 0 hence, by Proposition 2.20 i), $\Psi^t_{b_j}(y)$ accumulates to $\Sigma_{b_j,I}$ and therefore $\lim_{t\to+\infty} \chi_y(t) = b(\mathbf{g}_I)$ for some $I \in \mathcal{I}_{b_j}$; it does follow that $\Psi^t_{b_j}(y) \in W^+_{b_j,I}$ for all large enough t, since it must enter some neighbourhood $B_{\varepsilon,V}$ as in Step 3 and $\chi_y(t)$ cannot take the value $b(\mathbf{g}_I) + \varepsilon$.

As $\Psi^t_{b_j}(y) \in W^+_{b_j,I}$ for large t and $W^+_{b_j,I}$ is included in the $\Psi^t_{b_j}$-invariant manifold $E_{I_0^+} \cap Q_{b_j}$, we have $y \in E_{I_0^+} \cap Q_{b_j}$ and there remains to show that $y \in E_{b_j,I}$; now, by

[20]See for example [12], (2.2), Isolating Block Lemma (iii).

Proposition 2.20 v), the closed subset $E_{I_0^+} \cap Q_{b_j} \smallsetminus E_{b_j,I} = E_{I_0^+} \cap Q_{b_j} \smallsetminus W^+_{b_j,I}$ equals $E_{I_0^+} \cap \Big(\Big(\bigcup_{J\neq I} W^+_{b_j,J}\Big) \cup \Big(\bigcup_J W^-_{b_j,J} \smallsetminus W^+_{b_j,I}\Big)\Big)$; as it is closed, this yields

$$E_{I_0^+} \cap Q_{b_j} \smallsetminus W^+_{b_j,I} = E_{I_0^+} \cap \Big(\Big(\bigcup_{J\neq I} Q_{b_j} \cap E_{J_0^+}\Big) \cup \Big(\bigcup_J Q_{b_j} \cap E_{J_0^-}\Big)\Big)$$

(since $W^+_{b_j,J}$ is dense in $Q_{b_j} \cap E_{J_0^+}$ and $W^-_{b_j,J} \smallsetminus W^+_{b_j,I}$ dense in $Q_{b_j} \cap E_{J_0^-}$ for all $J \in \mathcal{I}_{b_j}$), implying that $E_{I_0^+} \cap Q_{b_j} \smallsetminus W^+_{b_j,I}$ is $\Psi^t_{b_j}$-invariant and therefore $y \in W^+_{b_j,I}$.

Step 5 *For small η, if N_r is a compact neighbourhood of $Q_b \cap \mathcal{V}$ in Q_b* (which we choose to be a submanifold with boundary)*, one defines inductively a compact neighbourhood N_j of $Q_{b_j} \cap \mathcal{V}_j$ in Q_{b_j} for $0 \le j < r$, which is a submanifolds with corners, as follows: N_j is the closure $f_{b_j}^{-1}(K_j) \cap \Big(\bigcup_t \Psi^t_{b_j}(N_{j+1}) \cup \hat{\mathcal{V}}_{b_{j+1}}\Big)$ of $f_{b_j}^{-1}(K_j) \cap \bigcup \Psi^t_{b_j}(N_{j+1})$. When η and N_r are small enough, the compact neighbourhood N_0 of 0 in E is contained in $\theta^{-1}(1)$ and therefore so is $N_j = N_0 \cap Q_{b_j}$ for $0 \le j \le r$.*

Indeed, the first part of the statement is the analogue for $\Psi^t_{b_j}$ of the fact used in the proof of Lemma 2.22 and its proof is the same. The second part follows from the fact that every N_j tends to its analogue for $\Phi^t_{b_j}$ when $\eta \to 0$.

Hypothesis. We fix η, N_r as in step 5 so that $N_0 \subset \theta^{-1}(1)$ and conclude by induction:

Step 6 *For $1 \le j \le r$, the vector fields $\tilde{Y}_i$ defined by* (13) *on $Q_{b_{i-1}}$ for $1 \le i \le j$ extend in a unique fashion to commuting smooth vector fields on E, whose germs at 0 are $(\zeta_\eta \circ h_\infty^{-1} \circ h)_* X_1, \ldots, (\zeta_\eta \circ h_\infty^{-1} \circ h)_* X_j$. More precisely, $\tilde{Y}_i = \zeta_{\eta *} Y_i$ on N_0.*

The vector fields $\tilde{Y}_1, \ldots, \tilde{Y}_j$ define a smooth action $\tilde{g}_j$ of $\mathbf{R}^j$ on E preserving every SIM of S and possessing (for our fixed b) *all the properties of S_j stated in Propositions 2.18 iii), 2.19 and 2.20. In particular, the map $\psi_{b_j} : Q_{b_j} \times \mathbf{R}^j \to E$ defined by $\psi_{b_j}(y,t) := \tilde{g}^t_j(y)$ is a diffeomorphism onto E_{b_j}.*

This is obvious if $j = 1$, as (13) defines a vector field $Z_{b_0} := \tilde{Y}_1$ on $Q_{b_0} = E$ equal to $\zeta_{\eta *} Y_1$ on $\theta^{-1}(1) \supset N_0$ and for which we have just proven our statement.

Assuming the result established for some $j < r$, let us prove it for $j+1$. The action $\tilde{g}_{j+1}$ must read as follows via ψ_{b_j}: for all $y \in Q_{b_j}$, $s, s' \in \mathbf{R}^j$ and $t \in \mathbf{R}$,

$$\psi^*_{b_j} \tilde{g}^{(s,t)}_{j+1}(y, s') = \Big(\Psi^t_{b_j}(y), s + s' + \int_0^t \mathbf{s}_{b_j} \circ \Psi^\tau_{b_j}(y)\, d\tau\Big) \tag{15}$$

where $\Psi^t_{b_j}$ is the flow of Z_{b_j} and $\mathbf{s}_{b_j}(y)$ is the element $(\sigma_1, \ldots \sigma_j)$ of $\mathbf{R}^j$ such that $Z_{b_j}(y) = \tilde{Y}_{j+1}(y) - \sum_{1 \le i \le j} \sigma_i \tilde{Y}_i(y)$. The formula (15) defines an action $\tilde{g}_{j+1}$ of $\mathbf{R}^{j+1}$ on the dense open subset E_{b_j} (its extension to $\mathbf{R}^{j+1} \times E$ will therefore be unique), whose generators are $\tilde{Y}_1, \ldots \tilde{Y}_j$ (already extended to E) and the vector field $\tilde{Y}_{j+1}$ on E_{b_j} given by $\psi^*_{b_j} \tilde{Y}_{j+1}(y, s') = \big(Z_{b_j}(y), \mathbf{s}_{b_j}(y)\big)$, i.e.

$$\forall y \in Q_{b_j}\ \forall s \in \mathbf{R}^j \quad \tilde{g}^{s*}_j \tilde{Y}_{j+1}(y) = \tilde{Y}_{j+1}(y). \tag{16}$$

we should show that it extends smoothly to E so that $\tilde{Y}_{j+1} = \zeta_{\eta *} Y_{j+1}$ on N_0. We claim that (16) defines a vector field $\tilde{Y}_{j+1}$ on E_{b_j} equal to $\zeta_{\eta *} Y_{j+1}$ on $N_0 \cap E_{b_j} = N_0 \smallsetminus \hat{\mathcal{V}}_{b_j}$;

this will prove our result since $\tilde{Y}_{j+1}$ will extend by density to a smooth vector field $\hat{Y}_{j+1}$ equal to $\zeta_{\eta *}Y_{j+1}$ on N_0, which will extend smoothly to the rest of $\hat{\mathcal{V}}_{b_j}$ because it is invariant by $\tilde{g}_j$.

To establish our claim, first recall that $\tilde{Y}_{j+1} = \zeta_{\eta *}Y_{j+1}$ on $\theta_j^{-1}(1) \supset N_j$ by (13) with $j := j+1$. Now, by definition of N_0, the subset $N_0 \cap E_{b_j}$ consists of points y_0, each of which is obtained from a unique $y_j \in N_j$ by concatenating the paths $[0, t_i] \ni t \mapsto \Psi^t_{b_{i-1}}(y_i) \in N_i$ from $y_i \in N_i$ to $y_{i-1} := \Psi^{t_i}_{b_{i-1}}(y_i) \in N_{i-1}$ for $j \geq i \geq 1$; as (15) with $j := i-1$ yields

$$\Psi^t_{b_{i-1}}(y_i) = \tilde{g}_i^{(-\int_0^t \mathbf{s}_{b_{i-1}} \circ \Psi^\tau_{b_{i-1}}(y_i)\, d\tau\, ,\, t)}(y_i),$$

it follows that y_0 is the endpoint of a path $t \to \tilde{g}^{\gamma(t)}(y_j)$ in N_0 with $\gamma : [0, T] \to \mathbf{R}^j$ continuous, piecewise smooth and $\gamma(0) = 0$; now, if $\gamma = (\gamma_1, \ldots, \gamma_j)$,

$$\begin{aligned}
\frac{d}{dt}\tilde{g}_j^{\gamma(t)*}(\tilde{Y}_{j+1} - \zeta_{\eta *}Y_{j+1})(y_j) &= \tilde{g}_j^{\gamma(t)*}\Big[\sum_{i=1}^j \gamma_i'(t)\tilde{Y}_i\, ,\, \tilde{Y}_{j+1} - \zeta_{\eta *}Y_{j+1}\Big](y_j) \\
&= \sum_{i=1}^j \gamma_i'(t)\, \tilde{g}_j^{\gamma(t)*}\Big[\tilde{Y}_i\, ,\, \tilde{Y}_{j+1} - \zeta_{\eta *}Y_{j+1}\Big](y_j) \\
&= \sum_{i=1}^j \gamma_i'(t)\, \tilde{g}_j^{\gamma(t)*}\Big([\tilde{Y}_i, \tilde{Y}_{j+1}] - [\zeta_{\eta *}Y_i, \zeta_{\eta *}Y_{j+1}]\Big)(y_j) \\
&= 0
\end{aligned}$$

since $\tilde{Y}_i$ and $\zeta_{\eta *}Y_i$ coincide on the submanifold with corners N_0 and therefore have the same 1-jet at $\tilde{g}_j^{\gamma(t)}(y_j)$ for $1 \leq i \leq j$ and $0 \leq t \leq T$; hence,

$$\begin{aligned}
\tilde{g}_j^{\gamma(T)*}(\tilde{Y}_{j+1} - \zeta_{\eta *}Y_{j+1})(y_j) &= \tilde{g}_j^{\gamma(0)*}(\tilde{Y}_{j+1} - \zeta_{\eta *}Y_{j+1})(y_j) = (\tilde{Y}_{j+1} - \zeta_{\eta *}Y_{j+1})(y_j) \\
&= 0
\end{aligned}$$

i.e. $\tilde{Y}_{j+1}(y_0) = \zeta_{\eta *}Y_{j+1}(y_0)$, as claimed. □

End of the proof

Hypotheses and notation. Given an $\mathbf{R}^r$-action germ g' formally conjugate to g, it is formally conjugate to the same normal form g_0 as g and therefore, by the preparation lemma, smoothly conjugate to an action germ on E whose infinitesimal generators possess representatives $Y_1', \ldots, Y_r'$ having infinite contact with $Y_1, \ldots, Y_r$ along $\mathcal{V}$. If we apply to them the globalisation lemma 2.24 *with the same* η *and* θ as for $Y_1, \ldots, Y_r$ (which is possible if η is chosen small enough), we get an $\mathbf{R}^r$-action ρ on E with the same properties as $\tilde{g}$, having infinite contact with $\tilde{g}$ along $\mathbf{R}^r \times \mathcal{V}$.

Lemma 2.25 *There exists a unique smooth conjugacy $h : E \to E$ of $\tilde{g}$ to ρ equal to Id_E on Q_b, and it has infinite contact with the identity along $\mathcal{V}$. In particular, Theorem 1.2 is true for $\mathbf{R}^r$-action germs.*

Proof. The last assertion is clear since $h^* Y_j' = Y_j$ near 0 for $1 \leq j \leq r$.
If the diffeomorphisms $\psi_b, \chi_b : Q_b \times \mathbf{R}^r \to E_b$ are given by $\psi_b(y, t) = \tilde{g}^t(y)$ and

$\chi_b(y,t) = \rho^t(y)$, the conjugacy relation $h \circ \tilde{g}^t(y) = \rho^t \circ h(y)$ reads $h \circ \tilde{g}^t(y) = \rho^t(y)$ for $y \in Q_b$, i.e. $h \circ \psi_b = \chi_b$, hence $h|_{E_b} = \chi_b \circ \psi_b^{-1}$. As for flows, this defines a smooth diffeomorphism of E_b onto itself, conjugating $\tilde{g}|_{\mathbf{R}^r \times E_b}$ to $\rho|_{\mathbf{R}^r \times E_b}$. Moreover, as $\tilde{g}$ and ρ have infinite contact along $\mathbf{R}^r \times \mathcal{V}$, the diffeomorphism $h|_{E_b}$ has infinite contact with the identity along $\mathcal{V} \cap E_b$. Since E_b is dense, uniqueness follows and all we have to show is that the bijection $h : E \to E$ defined by

$$h(y) = \begin{cases} \chi_b \circ \psi_b^{-1}(y) & \text{for } y \in E_b \\ y & \text{for } y \in \hat{\mathcal{V}}_b \end{cases}$$

is smooth and has infinite contact with the identity along $\mathcal{V}$. We will prove inductively that $h|_{E_{b_j}}$ is smooth, maps E_{b_j} onto itself and has infinite contact with the identity along $\mathcal{V} \cap E_{b_j} = \mathcal{V} \setminus \hat{\mathcal{V}}_{b_j}$ for $0 \le j \le r$, hence Lemma 2.25 for $j = 0$, as $E_{b_0} = E$.

By what we have just done, the result is true if $j = r$; given $j < r$, we now prove it for j assuming it true for $j+1$. First note that, as h is a bijection equal to the identity on $\hat{\mathcal{V}}_b \supset \hat{\mathcal{V}}_{b_j}$, we do have $h(E_{b_j}) = E_{b_j}$.

Let $\psi_{b_j}, \chi_{b_j} : Q_{b_j} \times \mathbf{R}^j \to E_{b_j}$ be the diffeomorphisms given by $\psi_{b_j}(y,t) = \tilde{g}^t(y)$ and $\chi_{b_j}(y,t) = \rho^t(y)$. If we set

$$h_{b_j} := \chi_{b_j}^{-1} \circ h \circ \psi_{b_j}$$

and, for $y \in Q_{b_j}$,

$$\big(h^0(y), h^1(y)\big) := h_{b_j}(y,0) = \chi_{b_j}^{-1} \circ h(y),$$

then for $s \in \mathbf{R}^j$ the relation $h \circ \tilde{g}^s(y) = \rho^s \circ h(y)$ reads

$$h_{b_j}(y,s) = \big(h^0(y), s + h^1(y)\big) \tag{17}$$

and, for $t \in \mathbf{R}$, the relation $h \circ \tilde{g}^{(s,t)}(y) = \rho^{(s,t)} \circ h(y)$ writes

$$h_{b_j} \circ \psi_{b_j}^* \tilde{g}^{(s,t)}(y,0) = \chi_{b_j}^* \rho^{(s,t)} \circ h_{b_j}(y,0).$$

By (15), $\psi_{b_j}^* \tilde{g}^{(s,t)}(y,0) = \psi_{b_j}^* \tilde{g}^{(0,t)}(y,s)$; similarly, $\chi_{b_j}^* \rho^{(s,t)}(y',s') = \chi_{b_j}^* \rho^{(0,t)}(y', s+s')$ hence, by (17), $\chi_{b_j}^* \rho^{(s,t)} \circ h_{b_j}(y,0) = \chi_{b_j}^* \rho^{(0,t)} \circ h_{b_j}(y,s)$; setting

$$\tilde{\Psi}_{b_j}^t := \psi_{b_j}^* \tilde{g}^{(0,t)} \quad \text{and} \quad \tilde{\Xi}_{b_j}^t := \chi_{b_j}^* \rho^{(0,t)}$$

it follows that the relation $h \circ \tilde{g}^{(s,t)}|_{E_{b_j}} = \rho^{(s,t)} \circ h|_{E_{b_j}}$ is equivalent to

$$h_{b_j} \circ \tilde{\Psi}_{b_j}^t = \tilde{\Xi}_{b_j}^t \circ h_{b_j}. \tag{18}$$

Now, by (15),

$$\tilde{\Psi}_{b_j}^t(y,s) = \left(\Psi_{b_j}^t(y), s + \int_0^t \mathbf{s}_{b_j} \circ \Psi_{b_j}^\tau(y)\, d\tau\right) \tag{19}$$

and similarly, if $\Xi_{b_j}^t, \sigma_{b_j}$ denote the analogues of $\Psi_{b_j}^t, \mathbf{s}_{b_j}$ for ρ,

$$\tilde{\Xi}_{b_j}^t(y,s) = \left(\Xi_{b_j}^t(y), s + \int_0^t \sigma_{b_j} \circ \Psi_{b_j}^\tau(y)\, d\tau\right). \tag{20}$$

By (17)-(19)-(20), both h_{b_j} and the flows $\tilde{\Psi}_{b_j}^t, \tilde{\Xi}_{b_j}^t$ commute with the action of $\mathbf{Z}^j$ on $Q_{b_j} \times \mathbf{R}^j$ by translation on the second factor; hence, they induce a bijection $\bar{h}_{b_j}$ of $Q_{b_j} \times \mathbf{T}^j$ onto itself and two flows $\bar{\Psi}_{b_j}^t, \bar{\Xi}_{b_j}^t$ on $Q_{b_j} \times \mathbf{T}^j$.

Our induction hypothesis is that h is smooth on $E_{b_{j+1}}$ and has infinite contact with the identity along $\mathcal{V} \smallsetminus \hat{\mathcal{V}}_{b_{j+1}}$, i.e. that h_{b_j} is smooth on

$$\psi_{b_j}^{-1}(E_{b_{j+1}}) = (Q_{b_j} \cap E_{b_{j+1}}) \times \mathbf{R}^j = (Q_{b_j} \smallsetminus \hat{\mathcal{V}}_{b_{j+1}}) \times \mathbf{R}^j$$

and has infinite contact with the identity along

$$\psi_{b_j}^{-1}(\mathcal{V} \smallsetminus \hat{\mathcal{V}}_{b_{j+1}}) = (Q_{b_j} \cap \mathcal{V} \smallsetminus \hat{\mathcal{V}}_{b_{j+1}}) \times \mathbf{R}^j;$$

we wish to show that h_{b_j} is smooth and has infinite contact with the identity along $(Q_{b_j} \cap \hat{\mathcal{V}}_{b_{j+1}}) \times \mathbf{R}^j$. Using the extension lemma 2.11, we will now prove that $\bar{h}_{b_j}$, which is smooth on $(Q_{b_j} \smallsetminus \hat{\mathcal{V}}_{b_{j+1}}) \times \mathbf{T}^j$, is smooth on the whole of $Q_{b_j} \times \mathbf{T}^j$ and has infinite contact with the identity along $(Q_{b_j} \cap \hat{\mathcal{V}}_{b_{j+1}}) \times \mathbf{T}^j$; this will imply what we want.

The flows $\bar{\Psi}_{b_j}^t, \bar{\Xi}_{b_j}^t$ have all the properties of $\Psi_{b_j}^t$ stated in the proof of the globalisation lemma 2.24 if $Q_{b_j}, \Sigma_{b_j,I}, W_{b_j,I}^{\pm}$ are replaced respectively by their cartesian products $\bar{Q}_{b_j}, \bar{\Sigma}_{b_j,I}, \bar{W}_{b_j,I}^{\pm}$ with $\mathbf{T}^j$ and f_{b_j} by $\bar{f}_{b_j} : (y,s) \mapsto f_{b_j}(y)$. By (11),

$$(Q_{b_j} \cap \hat{\mathcal{V}}_{b_{j+1}}) \times \mathbf{T}^j = \Bigg(\bigcup_{b(\delta_{j+1} - \mathbf{g}_I) > 0} \bar{W}_{b_j,I}^+ \Bigg) \cup \Bigg(\bigcup_{b(\delta_{j+1} - \mathbf{g}_I) < 0} \bar{W}_{b_j,I}^- \Bigg);$$

let us explain how to "fill in the gap" along $\bigcup_{b(\delta_{j+1} - \mathbf{g}_I) < 0} \bar{W}_{b_j,I}^-$ by showing that there, $\bar{h}_{b_j}$ is smooth and has infinite contact with the identity:

- if $b(\delta_{j+1})$ is larger than $b(\mathbf{g}_I)$ for all $I \in \mathcal{I}_{b_j}$, there is no gap to fill;
- otherwise, there is a smallest $b(\mathbf{g}_I) > b(\delta_{j+1})$; if we call it v_1 then, for every[21] $I \in \mathcal{I}_{b_j}$ with $b(\mathbf{g}_I) = v_1$, the hypotheses of the extension lemma 2.11 are satisfied for positive t by the germs at $\bar{\Sigma}_{b_j,I}$ of $\bar{\Psi}_{b_j}^t, \bar{\Xi}_{b_j}^t$ and $\bar{h}_{b_j}|_{\bar{Q}_{b_j} \smallsetminus \bar{W}_{b_j,I}^-}$; hence, the latter extends to a smooth germ having infinite contact with the identity along the germ of $\bar{W}_{b_j,I}^-$, which of course is the germ of $\bar{h}_{b_j}$ at $\bar{\Sigma}_{b_j,I}$;
- now, (18) yields $\bar{h}_{b_j} \circ \bar{\Psi}_{b_j}^t = \bar{\Xi}_{b_j}^t \circ \bar{h}_{b_j}$ for all positive t; as $\bar{\Psi}_{b_j}^t$ and $\bar{\Xi}_{b_j}^t$ have infinite contact along their common unstable manifold $\bar{W}_{b_j,I}^-$ at $\bar{\Sigma}_{b_j,I}$, it follows that $\bar{h}_{b_j}$ is smooth and has infinite contact with the identity along $\bar{W}_{b_j,I}^-$ for all $I \in \mathcal{I}_{b_j}$ with $b(\mathbf{g}_I) = v_1$;
- if there is no $I \in \mathcal{I}_{b_j}$ with $b(\mathbf{g}_I) > v_1$, the gap is filled;
- otherwise, if v_2 denotes the smallest $b(\mathbf{g}_I) > v_1$ then, for all $I \in \mathcal{I}_{b_j}$ with $b(\mathbf{g}_I) = v_2$, as every possible gap along $\bar{W}_{b_j,I}^+$ has just been filled, the same argument shows that $\bar{h}_{b_j}$ is smooth and has infinite contact with the identity along $\bar{W}_{b_j,I}^-$;
- iterating this procedure, the gap is filled.

The gap along $\bigcup_{b(\delta_{j+1} - \mathbf{g}_I) > 0} \bar{W}_{b_j,I}^+$ is filled in the same way by considering first the largest $b(\mathbf{g}_I) < b(\delta_{j+1})$ (if any), calling it v_{-1}, applying the extension lemma for some *negative* t to the germs at $\bar{\Sigma}_{b_j,I}$ of $\bar{\Psi}_{b_j}^t, \bar{\Xi}_{b_j}^t$ and $\bar{h}_{b_j}|_{\bar{Q}_{b_j} \smallsetminus \bar{W}_{b_j,I}^+}$ for all $I \in \mathcal{I}_{b_j}$ with $b(\mathbf{g}_I) = v_{-1}$, concluding that $\bar{h}_{b_j}$ is smooth and has infinite contact with the identity along $\bigcup_{b(\mathbf{g}_I) = v_{-1}} \bar{W}_{b_j,I}^+$, passing to the largest $b(\mathbf{g}_I) < v_{-1}$ (if any), etc. □

[21] One could add to Proposition 2.18 the general position hypothesis that $b(\mathbf{g}_I) \neq b(\mathbf{g}_J)$ for $I \neq J$ but, for $j > 1$, this strict ordering of $\mathcal{I}_{b_j}$ would not mean much dynamically.

2.3.5 Idea of the proof of Theorem 1.2 for $k > 0$

In that case, the dispensable analogue of the globalisation lemma 2.24 [7] is lengthy[22].

The linear action $S|_{2(\mathbf{Z}^k\times\mathbf{R}^m)\times E}$ is the restriction of a linear action σ of $\mathbf{R}^r$ on E with $x_{\ell,p}\circ\sigma^t = u_\ell(t)c_\ell(t)x_{\ell,p}$, $|u_\ell(t)| = 1$ ([10], 6.1.1, Proposition 1), possessing all the properties described in 2.3.2. Moreover, the automorphism A in Proposition 2.18 can be obtained from an automorphism A of $\mathbf{Z}^k \times \mathbf{R}^m$ ([10], 6.1.1, Proposition 2, where this is expressed in terms of subgroups), so that σ has all the properties stated in 2.3.3 once the action germs composed with A, for every b as in Proposition 2.18.

We now explain how to prove that two weakly hyperbolic formally conjugate smooth $\mathbf{Z}^2$-action germs g, g' are C^∞-conjugate: after applying the preparation lemma, we get two pairs g_1, g_2 and g_1', g_2' of commuting local diffeomorphisms $(E, 0) \to (E, 0)$ defining the same formal normal form along $\mathcal{V}$; by our general position hypothesis, g_1 and g_1' are hyperbolic and therefore conjugate by Corollary 2.13; moreover, this conjugacy can be chosen with the same infinite jet as the identity along $\mathcal{V}$: just impose this as part of the Whitney extension problem in the proof of Corollary 2.12 i).

Assuming therefore $g_1' = g_1$, we should now construct a smooth local conjugacy $(E, 0) \to (E, 0)$ of g_2 to g_2' preserving g_1. With the notation of 2.3.4,

- we fix $b \in \mathbf{R}^{2*}$ with $b_1 < 0$ and $b(\delta_2) < \min_{I\in\mathcal{I}_{b_1}} b(\mathbf{g}_I)$ (hence $[m_0, M_0] = [b_1, 0]$, $\hat{\mathcal{V}}_{b_1} = W^-_{b_0,\emptyset}$ and $[m_1, M_1] = [b(\delta_2), \max_{I\in\mathcal{I}_{b_1}} b(\mathbf{g}_I)]$);
- denoting by $\Phi^t_{b_1}$ the flow on Q_{b_1} defined by σ as in 2.3.3, we choose a number $\mu_2 > \max f_{b_1}\circ\Phi^1_{b_1}\left(\mathcal{V}\cap f^{-1}_{b_1}(\max_{I\in\mathcal{I}_{b_1}} b(\mathbf{g}_I))\right)$ and set $K_1 := [b(\delta_2), \mu_2]$;
- we choose a positive number μ_1 so that $S^{2\delta_1}\left(\mathcal{V}\cap f^{-1}_{b_1}(K_1)\right) \subset f^{-1}_{b_0}([b_1, \mu_1])$, and set $K_0 := [b_1, \mu_1]$;
- we fix a bounded open neighbourhood $B \ni 0$ in E of the compact subset

  $$\mathcal{K} := \left(\hat{\mathcal{V}}_{b_1} \cup \bigcup_{t\geq 0}\sigma^{t\delta_1}\left(\mathcal{V}\cap f^{-1}_{b_1}(K_1)\right)\right)\cap f^{-1}_{b_0}(K_0)$$

  (which contains $\mathcal{V}_1 \cap f^{-1}_{b_0}(K_0)$ and $\mathcal{V}\cap f^{-1}_{b_1}(K_1)$).

When $\eta \to 0$, the map $\zeta_{\eta*}g_1$ tends in the C^∞ sense to S^{δ_1} on B and $\zeta_{\eta*}g_2$, $\zeta_{\eta*}g_2'$, to S^{δ_2}. It follows that, for small enough η,

- on $B \smallsetminus \{0\}$, the function $f_{b_0} = F_1$ is "Lyapunov" for $\zeta_{\eta*}g_1$, i.e. $F_1\circ\zeta_{\eta*}g_1 > F_1$;
- the compact set $\mathcal{K}$ admits a relatively open neighbourhood $\mathcal{N} \subset B$ in $f^{-1}_{b_0}(K_0)$ which is $\zeta_{\eta*}g_1$-*saturated*, meaning that $y \in \mathcal{N}$ satisfies $\zeta_{\eta*}g_1(y) \notin \mathcal{N}$ (resp. $\zeta_{\eta*}g_1^{-1}(y) \notin \mathcal{N}$) if and only if $f_{b_0}(\zeta_{\eta*}g_1(y)) > \mu_1$ (resp. $f_{b_0}\left(\zeta_{\eta*}g_1^{-1}(y)\right) < b_1$;
- hence, $\mathcal{N}\smallsetminus\hat{\mathcal{V}}_{b_1} = \mathcal{N}\smallsetminus W^-_{0,\emptyset}$ is the disjoint union of the sequence $\mathcal{D}_p$ of nonempty subsets defined by $\mathcal{D}_0 := \mathcal{N}\smallsetminus\zeta_{\eta*}g_1(\mathcal{N})$ and $\mathcal{D}_{p+1} := f^{-1}_{b_0}(K_0)\cap\zeta_{\eta*}g_1(\mathcal{D}_p)$;
- one has $\sup f_{b_0}(\mathcal{D}_0) < \mu_1$ (thanks to the choice of μ_1);
- thus, the boundary of $\mathcal{D}_0$ consists of $\mathcal{D}_0\cap Q_{b_1} = \mathcal{N}\cap Q_{b_1}$ and its image by $\zeta_{\eta*}g_1$; if one identifies the two by $\zeta_{\eta*}g_1$, one gets a manifold $\bar{\mathcal{N}}$ (which can be assumed diffeomorphic to a circle bundle over the open subset $\mathcal{D}_0\cap Q_{b_1}$ of Q_{b_1}, as it is a small deformation of the corresponding object for S^{δ_1}).

[22] This is why Theorem 1.2 in the Siegel domain was proven in [10] only for linearisations.

On the "quotient" $\bar{\mathcal{N}}$ of $\mathcal{N} \setminus \hat{\mathcal{V}}_{b_1}$ by $\zeta_{\eta *} g_1$, the two embeddings $\zeta_{\eta *} g_2$ and $\zeta_{\eta *} g_2'$ induce (partially defined) embeddings $\bar{g}_2$ and $\bar{g}_2'$; if we can construct a conjugacy $\bar{h}$ of $\bar{g}_2$ to $\bar{g}_2'$ defined near the quotient $\bar{\mathcal{V}}_1$ of $\mathcal{V}_1 \cap \mathcal{D}_0$ and having infinite contact with the identity along it then, by the extension lemma 2.11, it will extend to a local diffeomorphism h preserving $\zeta_{\eta *} g_1$ and having infinite contact with the identity along $\mathcal{V}_1$ near 0, which will conjugate $\zeta_{\eta *} g_2$ to $\zeta_{\eta *} g_2'$, hence our result.

To obtain $\bar{h}$, we solve the same kind of Cauchy problem as in the proof of Corollary 2.12: using a reasonable presentation of $\bar{\mathcal{N}}$ as a circle bundle (for which the quotient $\bar{W}$ of $W \cap \mathcal{D}_0$ is a union of fibres for every SIM W of S), we extend f_{b_1} to the function $\bar{f}_{b_1}$ on $\bar{\mathcal{N}}$ constant on the fibres. Denoting by $\bar{\mathcal{V}}$ the quotient of $\mathcal{V} \cap \mathcal{D}_0$, we then choose $\bar{h}$ to have infinite contact with the identity along a compact neighbourhood N of $\bar{\mathcal{V}} \cap \{\bar{f}_{b_1} = b(\delta_2)\}$ in $\{\bar{f}_{b_1} = b(\delta_2)\}$, to have the corresponding jet along $\bar{g}_2(N)$ and to have the same jet as the identity in between along $\bar{\mathcal{V}}$; the Whitney extension theorem then yields a "fattened" version of the Cauchy problem in the proof of Lemma 2.25, which can be solved by the same repeated use of the extension lemma.

2.4 Consequences and variants

Corollary 2.26 *With the notation of Proposition 2.4, if g is weakly hyperbolic and Π_0 empty, then g is smoothly conjugate to the analytic action germ g_0.*

Proof. In the Poincaré domain, this is Theorem 2.5. Otherwise, as g and g_0 are formally conjugate, they are smoothly conjugate by Theorem 1.2. □

Theorem 2.27 *Under the hypotheses of Corollary 2.26, if g is strongly hyperbolic, then it is C^0-linearisable, i.e. C^0-conjugate to S.*

Idea of the proof. Similar to that of Theorem 3.1 hereafter but much easier analytically, as the action u of Proposition 2.4 is algebraic. The additional fact needed (because of the commutation relation) is that one can define an ordering on $\{1, \dots, n\}$ as follows: i is strictly less than j if and only if there exists $p \in P_j$ with $p_i \neq 0$; this enables one to "kill" first the monomials x^p with $p \in P_i$ and i maximal, etc. □

Theorem 2.28 *If g is weakly hyperbolic, in the Siegel domain and formally conjugate to a smooth $\mathbf{Z}^k \times \mathbf{R}^m$-action germ g' at $a' \in M'$, meaning that there exists a smooth diffeomorphism germ $h_0 : (M, a) \to (M', a')$ such that $h_0^* g'$ and g have infinite contact at a, then, for each such h_0, the smooth conjugacies of g to g' having infinite contact with h_0 at a form an infinite dimensional space.*

Idea of the proof. As in the proof of Corollary 2.13, we may assume $h_0 = \mathrm{Id}_E$.

If $k = 0$, the result follows from the fact that, in Lemma 2.25, the Cauchy problem $h|_{Q_b}$ can be any C^1-small enough perturbation of $\mathrm{Id}_E|_{Q_b}$, equal to $\mathrm{Id}_E|_{Q_b}$ off a compact subset and having infinite contact with $\mathrm{Id}_E|_{Q_b}$ along $\mathcal{V} \cap Q_b$. The germs at $\mathcal{V} \cap Q_b$ of such perturbations obviously form an infinite dimensional space and identify to the conjugacies between the action germs g and g'.

For $k > 0$, the result is more obvious since the solutions of our Whitney extension problem form an infinite dimensional space, as in the proof of Corollary 2.13 ii). □

Theorem 2.29 *All the previous results remain true if* $\mathbf{Z}^k \times \mathbf{R}^m$ *is replaced by an* elementary Abelian group, *i.e. a Lie group* G *isomorphic to* $G_0 \times \mathbf{Z}^k \times \mathbf{R}^m$, *where* G_0 *is the product of a finite Abelian group and a torus*[23].

Idea of the proof [7, 10]. By a theorem of Bochner ([10], 3.1.4, théorème), the action germ restricted to the maximal compact subgroup G_0 of G can be linearised smoothly and then all the proofs can be made invariant by this linear G_0-action. □

Remark. This might be helpful in the study of smooth completely integrable systems via Nguyen Tien Zung's moto: "Always look for the torus action".

3 Related results and questions

3.1 Germs of holomorphic vector fields

Theorem 3.1 ([12], Theorem 2) *Let* X *be a germ at* $a \in M$ *of holomorphic vector field with* $X(a) = 0$, *generating a weakly hyperbolic* $\mathbf{C}$*-action germ* g, *and let* σ *be the semi-simple part of the endomorphism* $dX(a)$ *of the complex vector space* $E = T_aM$. *For every positive integer* k, *the action germ* g *is* C^k*-conjugate to the holomorphic* $\mathbf{C}$*-action germ generated by a polynomial normal form, i.e. a complex polynomial vector field* $\sigma + \nu$ *on* E *with* $d\nu(0)$ *nilpotent and* $[\sigma, \nu] = 0$.

Proof. In this case, the algebraic part of the proof of Theorem 1.2 reads as follows: for every integer s, there exists a holomorphic conjugacy $(M, a) \to (E, 0)$ of X to a holomorphic vector field germ having s^{th} order contact at 0 with a polynomial normal form $\sigma + \nu_s$ of degree s; passing to the projective limit, one gets a smooth diffeomorphism germ $h : (M, a) \to (E, 0)$ such that h_*g has s^{th} order contact at 0 with the $\mathbf{C}$-action generated $\sigma + \nu_s$ for all s; hence, for every integer q, the smooth action germ $(h_\infty^{-1} \circ h)_*g$ of the preparation lemma 2.10 has q^{th} order contact along $\mathcal{V}$ with the $\mathbf{C}$-action germ $g^{(q)}$ generated by a polynomial normal form $\sigma + \nu_{s_q}$; globalising $(h_\infty^{-1} \circ h)_*g$ and $g^{(q)}$ as in lemma 2.24, one gets two $\mathbf{R}^2$-actions having q^{th} order contact along $\mathcal{V}$. For large enough q, the C^p version of the extension lemma 2.11, used repeatedly as in the proof of lemma 2.25, yields a C^k conjugacy between them. □

Notes. The proof in [12] is much the same but remains holomorphic as long as possible: by [11], for every integer q, there exists a holomorphic conjugacy of X to a holomorphic vector field germ having q^{th} order contact along $\mathcal{V}$ with $\sigma + \nu_{s_q}$; the end of the proof is restricted to something like N_0 instead of globalising the actions.

In the Poincaré domain, it has been known since Poincaré and Dulac that X is holomorphically conjugate to a polynomial normal form. Theorem 3.1 shows that in the weakly hyperbolic case small denominators have no C^k meaning for any k. Finding the best (least) possible degree for ν seems very difficult.

[23] (Weak) hyperbolicity is that of the action germ restricted to $\mathbf{Z}^k \times \mathbf{R}^m$, a notion independent of the isomorphism chosen [10].

For general smooth $\mathbf{Z}^k \times \mathbf{R}^m$-action germs with $k+m>1$, I do not see how to get nice normal forms like $\sigma+\nu$; this prevented me from obtaining the general version of the following result:

Theorem 3.2 ([12], Theorem 1) *Under the hypotheses of Theorem 3.1, if g is strongly hyperbolic, it is C^0-conjugate to the germ of the linear $\mathbf{C}$-action $e^{t\sigma}$.*

Idea of the proof. By Theorem 3.1, it is enough to C^0-conjugate the complex flow generated by the normal form $\sigma+\nu$ to $e^{t\sigma}$. This is done by solving the same Cauchy problem as before, but several times and with much more care: using a first b, one can C^0-conjugate the complex flow of $\sigma+\nu$ to that of the vector field obtained from $\sigma+\nu$ by killing the monomials of ν which vanish on $\hat{\mathcal{V}}_b$; one can then kill successively the other monomials by using different b's. The key remark is that the formal flow defined by a normal form converges in a domain large enough to define its complex flow where needed for the proof. □

Notes. This explicit method yields conjugacies that are Hölder continuous of every exponent less than 1 (but not Lipschitzian in general). Despite superficial analogy, Theorem 3.2 is much more difficult than the Grobman-Hartman theorem, as it is shown in [1] that there are moduli already for topological *equivalence* between germs of complex linear vector fields. The Camacho-Kuiper-Palis conjecture was the weaker version of Theorem 3.2 where C^0-conjugacy is replaced by topological equivalence. It seems that no simpler proof has been found.

3.2 First integrals

Hypotheses, notation and definition. We go back to the hypotheses and notation of section 1. A *first integral* of g is a smooth function germ $\mathcal{J}:(M,a)\to\mathbf{R}$ such that $\mathcal{J}\circ g_j=\mathcal{J}$ for $1\le j\le k$ and $\mathcal{L}_{X_j}\mathcal{J}=0$ for $1\le j\le m$, hence $\mathcal{J}\circ g^t=\mathcal{J}$ for all $t\in\mathbf{Z}^k\times\mathbf{R}^m$. A *formal first integral* of g is a smooth function germ $\mathcal{J}_0:(M,a)\to\mathbf{R}$ such that $\mathcal{J}_0\circ g^t$ and $\mathcal{J}_0$ have infinite contact at a for all $t\in\mathbf{Z}^k\times\mathbf{R}^m$.

Here again, the contrast between the Poincaré and Siegel domains is striking:

Theorem 3.3 *If g is in the Poincaré domain, its only first integrals are the germs at a of constant functions.*

If g is in the Siegel domain and weakly hyperbolic, it possesses the following property: for every formal first integral $\mathcal{J}_0$ of g, the first integrals of g having infinite contact with $\mathcal{J}_0$ at a form an infinite dimensional space.

Idea of the proof. When g is in the Poincaré domain, taking representatives, some g^t satisfies $\lim_{n\to\infty} g^{nt}(x)=a$ for every x close enough to a; if $\mathcal{J}$ is a first integral of g, as $\mathcal{J}(x)=\mathcal{J}\circ g^{nt}(x)$ for all $n\in\mathbf{N}$, it follows that $\mathcal{J}$ is the constant $\mathcal{J}(a)$ (note that this holds assuming only that $\mathcal{J}$ is continuous at a).

In the Siegel domain, with the notation of the preparation lemma 2.10, the germ $(h_\infty^{-1}\circ h)_*\mathcal{J}_0$ defines a formal integral $\mathcal{J}_{\mathcal{V}}$ of $(h_\infty^{-1}\circ h)_*g$ *along* $\mathcal{V}$, whose jet along the unstable manifold W of S^t is (taking representatives) $\lim\limits_{n\to\infty}\left(j^\infty\left((h_\infty^{-1}\circ h)_*(\mathcal{J}_0\circ g^{-nt})\right)\right|_W$.

One can then conclude as in the proof of Theorem 1.2: the infinite jet of $\mathcal{J}_\mathcal{V}$ along $\mathcal{V}$ yields a unique jet along the whole of $\mathcal{V}$ of first integral of $\tilde{g}$; then, if for example $k = 0$, every smooth function $\mathcal{J}_b : Q_b \to \mathbf{R}$ having the induced jet along $Q_b \cap \mathcal{V}$ extends to a unique first integral $\mathcal{J}$ of $\tilde{g}$, having the same jet as the extended $\mathcal{J}_\mathcal{V}$ along $\mathcal{V}$: this follows from the analogue for first integrals of the extension lemma 2.11 ([10], 4.2.4, Théorème 2). The case $k \neq 0$ is similar. $\square$

Notes. "In general" $\Pi_0 = \emptyset$, hence the only formal first integrals of g are the smooth function germs $\mathcal{J}_0$ having infinite contact with a constant at a. However, this proof shows that, in the C^∞ sense, every weakly hyperbolic $\mathbf{R}^r$-action germ g in the Siegel domain is more or less completely integrable, as it is possible to find $\dim(M) - r$ first integrals functionally independent off $\mathcal{V}$.

Theorem 3.3 remains true for elementary Abelian group action germs.

The same methods apply to various problems, for example the solution of "(co)homological equations", all of which (including the conjugacy problem and the problem of first integrals) are particular cases of invariant manifold problems [13], as I plan to show in a forthcoming book. In the Poincaré domain, every formal solution yields a unique smooth solution whereas, in the Siegel domain, there is a huge amount of flexibility.

Acknowledgements As explained in the introduction, this rather heavy birthday cake owes much to Alain Chenciner, whose friendship has illuminated my life in the last thirty-nine years. Whatever the interest of this work, it required concentration and freedom, available because Laurent Schwartz valued results above office hours. Had not Ivan Kupka encouraged me, I might have given up just before finding the missing piece of the puzzle, namely the Lyapunov map F. The initial impulse came from Robert Roussarie's talk about [15] at the Thom seminar; a few years later, he was the courageous referee of [10] and I am very sorry some of his excellent suggestions had to be ignored by fear of a riot among the secretaries in charge of typing the manuscript (those were hard, pre-TEX times). Even though the ideology of this paper comes from [7], many details are new; I am grateful to Alain Albouy and Alexey Borisov for offering me the opportunity for such improvements. Finally, I thank Eva Miranda and Nguyen Tien Zung, whose interest in the domain caused the renewal of mine.

References

[1] C. Camacho, N. Kuiper, J. Palis. The topology of holomorphic flows with singularity. *Publ. Math. I.H.E.S.* **48** (1978), 5–38

[2] M. Chaperon. Sur certains difféomorphismes normalement hyperboliques II. *C. R. Acad. Sci., Paris, Sér. A,* **281** (1975), 731–733

[3] —— Modèles microlocaux pour certains opérateurs pseudodifférentiels fuchsiens, *Séminaire Goulaouic-Schwartz 1977–78*, exposé XI

[4] —— Singularités en géométrie de contact. *Astérisque* **59–60** (1978), 95–118

[5] —— Linéarisation des germes hyperboliques d'actions différentiables de $\mathbf{R}^k \times \mathbf{Z}^m$: le domaine de Poincaré. *C. R. Acad. Sc. Paris* **289***, série A* (1979), 325–328

[6] —— Linéarisation C^∞ des germes d'actions C^∞ de $\mathbf{Z}^k \times \mathbf{R}^m$. *C. R. Acad. Sc. Paris* **290***, série A* (1980), 13–15

[7] —— Propriétés génériques des germes d'actions différentiables de groupes de Lie commutatifs élémentaires. *Thèse d'État*, Université de Paris 7, 1980, 285 pages

[8] —— Quelques questions de géométrie symplectique [d'après, entre autres, Poincaré, Arnold, Conley et Zehnder]. *Séminaire Bourbaki*, Astérisque **105–106** (1983), 231–249

[9] —— Une idée du type "géodésiques brisées" pour les systèmes hamiltoniens. *C. R. Acad. Sc. Paris* **298** (1984) *série A*, 293–296

[10] —— Géométrie différentielle et singularités de systèmes dynamiques. *Astérisque* **138–139** (1986), 434 pages

[11] —— Invariant manifolds and a preparation lemma for local holomorphic flows and actions. *Springer Lecture Notes in mathematics* **1345** (1988), 95–110

[12] —— C^k-conjugacy of holomorphic flows near a singularity. *Publ. Math. I.H.E.S.* **64** (1986), 143–183

[13] —— Variétés stables et formes normales, *C. R. Acad. Sc. Paris* **317***, Série 1* (1993), 87–92

[14] K.-T. Chen. Equivalence and decomposition of vector fields about an elementary critical point. *Amer. J. Math.* **85** (1963), 693–722

[15] F. Dumortier, R. Roussarie. Smooth linearization of germs of $\mathbf{R}^2$–actions and holomorphic vector fields. *Ann. Inst. Fourier, Grenoble,* **30** (1980), 31–64

[16] M.R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. I.H.E.S.* **49** (1979), 5–233

[17] J.E. Humphrey. *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972

[18] S. López de Medrano, A. Verjovsky. A new family of complex, compact, non symplectic manifolds. *Bol. Soc. Mat. Bras.* **28**, 2 (1997), 253–269.

[19] E. Nelson. *Topics in dynamics. I: Flows.* Mathematical Notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1969

[20] C.L. Siegel. Über die Normalformen analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. *Nachr. Akad. Wiss. Göttingen, Math.-phys Kl.* (1952), 21–30

[21] S. Sternberg. Local contractions and a theorem of Poincaré. *Amer. J. of Math.* **79** (1957), 809–824

[22] —— On the structure of local homeomorphisms of euclidean n-space. II. Amer. J. Math. **80** (1958), 623–631

[23] F. Takens. Constrained equations; a study of implicit differential equations and their discontinuous solutions. *Structural stability, the theory of catastrophes, and applications in the sciences (Proc. Conf. Battelle Seattle Res. Center, Seattle, Wash., 1975),* Lecture Notes in Math. **525**, Springer, Berlin (1976), 143–234

[24] R. Thom, *Sur les équations différentielles multiformes et leurs intégrales singulières.* Bol. Soc. Bras. Mat. **3** (1972) no. 1, 1–11

[25] J.-C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Ann. Scient. Éc. Norm. Sup.* 4$^{\text{e}}$ série, t. **18** (1984), 333–359.

Index