Basic facts and naive questions

Marc Chaperon

To Claude Viterbo for his sixtieth birthday

Abstract. We try to find geometric reasons for KAM theorems and such.

Mathematics Subject Classification (2010). Primary 99Z99; Secondary 00A00.

Keywords. V.I. Arnold. Unfoldings. Implicit functions.

I first met Claude at the seminar on contact and symplectic geometry organised from 1982 on by Daniel Bennequin at the École normale supérieure. It was much oriented towards the beautiful conjectures V.I. Arnold had stated in the mid-sixties, inspired by Poincaré's "last geometric theorem." What made the seminar seminal[1] is that its beginning coincided with the first breakthrough in that direction: at the end of 1982, Charles Conley and Eduard Zehnder proved [20] the conjecture on fixed points of Hamiltonian transformations of the standard symplectic $2n$-torus stated in [2, Appendix 9].[2]

It so happened that, the summer before, I had thought about this conjecture, seen how to deduce it from another statement about exact Lagrangian isotopies of the zero section in $T^*\mathbb{T}^n$ and proved a symplectic isotopy extension lemma [6] implying that such an isotopy extends to a compactly supported Hamiltonian isotopy of the ambient space. Almost immediately after reading the preprint of [20], I adapted the Conley-Zehnder proof to get [6] the more general statement, of which I had just learned that a slightly less precise form had also been conjectured by Arnold [1, 3, Appendice 33].

The two weeks spent on the proof of this Arnold conjecture brought me more recognition than the two years of very hard work on my 1980 *thèse d'État* [9, 10, 14]: soon after the Bourbaki seminar [6], I lectured on the

[1] Besides its audience, that comprised Michèle Audin, Abbas Bahri, Alain Chenciner, Nicole Desolneux-Moulis, Ivar Ekeland, Albert Fathi, Michel Herman, Misha Gromov, François Laudenbach, Jean-Claude Sikorav—plus the author and, soon, Claude Viterbo...

[2] Their simple, functional-analytic proof did not make our seminar unanimously happy: not only had the topologists been dreaming of something more geometric, but Conley and Zehnder had proved the conjecture without knowing that it existed: when telling John Mather about their recent work, they had mentioned tori as a side remark, and Mather had informed them that they had solved a famous problem.

Arnold conjectures in Marseille-Luminy and published with Edi Zehnder an expanded version [17] of these lectures. Having felt ill at ease when teaching the fact (established by Amann and Zehnder) that the *non* C^2 infinite dimensional action functional, once reduced to finite dimensions, is as smooth as the Hamiltonian of the isotopy, I found it urgent to design a purely finite dimensional proof of "my" Arnold conjecture; this again took me two weeks [7, 8] and brought me much more recognition.

François Laudenbach liked this new proof; he had an extremely bright (and nice) student of his, Jean-Claude Sikorav, work on its generalisations and consequences. Jean-Claude first proved with François [26] what had been the true aim of [7, 8], namely, the extension of my result from the cotangent bundle of the torus to that of an arbitrary closed manifold.[3] He then noticed that formula (7) in [7] means that my discretised action is a generating phase for the deformed Lagrangian submanifold, and extended this to arbitrary closed manifolds [35]. In [36], he generalised the result to Lagrangian immersions and gave an easy proof of the Arnold conjecture on fixed points in situations including surfaces, first obtained more painfully in [34].

At about the same time, the 25 years old Claude, who had been the student of Laudenbach and Ekeland, solved [38] a big problem: the Weinstein conjecture in $\mathbb{R}^{2n}$. Oddly enough, he did not use Jean Claude's generating phases, with which he would soon do wonders [39].

The last of my favourite Arnold conjectures had been proved [24] via holomorphic disks, and Floer theory[4] had taken off [23], leaving me with my fear of flying. My belief that some room was left for earthlier methods[5] now rested mostly on Claude's shoulders. He did not disappoint me.

1. Subharmonic bifurcations in real or complex dimension one

We first recall the simplest case of the most basic fact.

1.1. The period doubling bifurcation

Let $h : (u, x) \mapsto h_u(x)$ be a C^k local map $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $k \geq 2$, such that the derivative $h_0'(0) = \partial_x h(0, 0)$ equals -1.

The fixed point 0 of h_0 is *robust*, meaning that every h_u with u small enough has a fixed point $\varphi(u)$ nearby, depending C^k on the parameter u:

Proposition 1.1. *The fixed points of the* unfolding $\tilde{h} : (u, x) \mapsto \big(u, h_u(x)\big)$ *form near* 0 *the graph of a* C^k *function* $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.

Proof. This follows from the implicit function theorem applied to the C^k equation $F(u, x) := x - h(u, x) = 0$, as $F(0, 0) = 0$ and $\partial_x F(0, 0) = 2 \neq 0$. □

Of course 0 is a 2*-periodic point of* h_0, i.e., a fixed point of $h_0^2 := h_0 \circ h_0$.

---

[3] A little sooner, Helmut Hofer had done this [25] in an infinite dimensional framework. Whatever its proof, I thought the result would be more central than it turned out to be.
[4] Whose idea owes as much to Charlie Conley as to Edi Zehnder, see the foreword of [11].
[5] A similar belief is at the origin of the present article.

Proposition 1.2. *Assume that* $\alpha(u) := {h_u}'\big(\varphi(u)\big)$ *satisfies* $\alpha'(0) \neq 0$. *Then, near* 0*, the* 2*-periodic points of* $\tilde{h}$ (solutions of $h_u^2(x) = x$) *form the union of two curves intersecting transversally at* 0*: of course* graph φ*, and a* C^{k-1} *curve* W *of which* $\tilde{h}|_W$ *is an involution, implying that* T_0W *is the* x*-axis.*

Proof. Conjugating $\tilde{h}$ by the local diffeomorphism $(u,x) \mapsto \big(u, x - \varphi(u)\big)$, we may assume $\varphi = 0$—the new ${h_u}'(0)$ is the old ${h_u}'\big(\varphi(u)\big)$.

By Taylor's formula, $h_u(x) = xg_u(x)$ near 0, where $g_u(x) = \int_0^1 {h_u}'(tx)\,dt$, hence $g_u(0) = \alpha(u)$ and therefore $g_0(0) = -1$; the map $g : (u,x) \mapsto g_u(x)$ is C^{k-1} and the equation $h_u^2(x) = x$ writes $xg_u(x)g_u\big(h_u(x)\big) = x$, which means $x = 0$ (the fixed points) or $G(u,x) := g_u(x)g_u\big(h_u(x)\big) - 1 = 0$. As $G(0,0) = 0$ and $\partial_u G(0,0) = \frac{d}{du}\big|_{u=0}\, g_u(0)^2 = -2\alpha'(0) \neq 0$, there exist open neighbourhoods U, V of 0 in $\mathbb{R}$ such that the zeros of $G|_{U\times V}$ form the "graph" $W = \{u = \psi(x)\}$ of a C^{k-1} implicit function $\psi : V \to U$.

The map $\tilde{h}$ is by definition an involution of its set of 2-periodic points, of which $W\setminus\{0\}$ is an open subset, which becomes $\tilde{h}$-invariant if W is replaced by $W \cap \tilde{h}(W)$ (this means restricting conveniently the open subset V). Invariance writes $\psi(x) = \psi\big(h_{\psi(x)}(x)\big)$, hence $\psi'(0) = \lim_{x\to 0} \dfrac{\psi\big(h_{\psi(x)}(x)\big) - \psi(x)}{h_{\psi(x)}(x) - x} = 0$ since the only fixed point of $\tilde{h}$ lying in W is 0; thus, T_0W is the x-axis. $\square$

Examples. The curve W can have various positions with respect to T_0W:

- If $h_u(x) = \alpha(u)x$, where $\alpha : (\mathbb{R}, 0) \to (\mathbb{R}, -1)$ is a C^k function with $\pm\alpha'(0) > 0$, then W is the x-axis; the fixed point 0 of h_u is attracting for $\pm u < 0$, repulsing for $\pm u > 0$, and this cannot be called a bifurcation.
- If $h_u(x) = -(1 + u - x^2)x$, then the fixed point 0 of h_u is attracting for $u < 0$, repulsing for $u > 0$, and W is the parabola $u = x^2$; for $u > 0$, the attracting 2-periodic orbit $\{-\sqrt{u}, \sqrt{u}\}$, born for $u = 0$, takes the place of 0 as an attractor of h_u, a genuine *bifurcation.*
- If $h_u(x) = -(1 + u + x^2)x$ then, for $u < 0$, the repulsing 2-periodic orbit $\{-\sqrt{-u}, \sqrt{-u}\}$ gradually "throttles" the attracting fixed point 0, so that for $u \geq 0$ no attractor of h_u persists near 0, a true *catastrophe.*

The generic situations look like the last two examples (Figure 1).

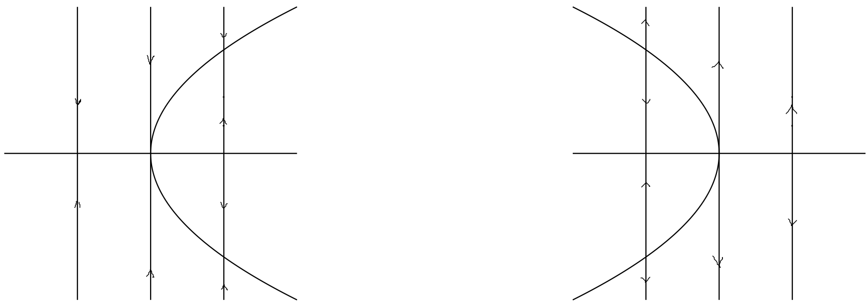

Figure 1. Bifurcation and catastrophe.

1.2. Subharmonic bifurcations, holomorphic case

Proposition 1.3. *Let $h : (u,z) \mapsto h_u(z)$ be a local map $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, holomorphic and such that $h_0'(0) = \partial_z h(0,0)$ is a q^{th} root of unity $\rho = e^{2\pi i \frac{p}{q}}$, $0 < p < q$. Then:*

i) *The fixed points of the unfolding $\tilde{h} : (u,z) \mapsto \big(u, h_u(z)\big)$ form near 0 the graph of a holomorphic function $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$.*
ii) *Assume that $\alpha(u) := {h_u}'\big(\varphi(u)\big)$ satisfies $\alpha'(0) \neq 0$. Then, near 0, the q-periodic points of $\tilde{h}$* (solutions (u,z) of the equation $h_u^q(z) = z$) *form the union of two holomorphic curves intersecting transversally at 0: the fixed point set* graph φ *and a curve W on which $\tilde{h}$ induces a $\mathbb{Z}/q\mathbb{Z}$-action.*
iii) *When ρ is a* primitive *q^{th} root of unity, the curve W and the z-plane are tangent to order $q-1$ at 0.*

Proof. i) As $h_0'(0) = \rho \neq 1$, just apply the holomorphic implicit function theorem to the equation $z - h(u,z) = 0$.

ii) As in the proof of Proposition 1.2, one can assume $\varphi = 0$, hence $h_u(z) = z\, g(u,z)$ with g holomorphic this time and $g(u,0) = \alpha(u)$. The equation $h_u^q(z) = z$ writes $z g_u(z) g_u\big(h_u(z)\big) \cdots g_u\big(h_u^{q-1}(z)\big) = z$, which means either $z = 0$ (the fixed points) or

$$G(u,z) := g_u(z) g_u\big(h_u(z)\big) \cdots g_u\big(h_u^{q-1}(z)\big) - 1 = 0.$$

As $G(u,0) = \alpha(u)^q - 1$ and $\alpha(0) = \rho$, one has $\partial_u G(0) = q\rho^{q-1}\alpha'(0) \neq 0$ and $G(0) = 0$, hence there exist open neighbourhoods U, V of 0 in $\mathbb{C}$ such that the zeros of $G|_{U\times V}$ form the "graph" $W = \{u = \psi(z)\}$ of a holomorphic implicit function $\psi : V \to U$.

The map $\tilde{h}$ induces by definition an action of $\mathbb{Z}/q\mathbb{Z}$ on its set of q-periodic points, of which $W \setminus \{0\}$ is an open subset, that becomes $\tilde{h}$-invariant if W is replaced by $W \cap \tilde{h}^{-1}(W) \cap \cdots \cap \tilde{h}^{1-q}(W)$ (as before, this means restricting conveniently the open subset V). Invariance writes

$$\psi(z) = \psi\Big(h\big(\psi(z), z\big)\Big). \tag{1.1}$$

iii) Still assuming $\varphi = 0$, if we derivate (1.1), we get

$$\psi'(z) = \psi'\Big(h\big(\psi(z), z\big)\Big)\Big(\partial_1 h\big(\psi(z), z\big)\psi'(z) + {h_{\psi(z)}}'(z)\Big).$$

For $z = 0$, as the identity $h(u,0) = 0$ implies that $\partial_1 h(u,0) = 0$, this reads

$$\psi'(0) = \psi'(0)\, h_0'(0), \text{ that is, } (\rho - 1)\psi'(0) = 0, \text{ hence } \psi'(0) = 0,$$

which proves our result if $q = 2$. Otherwise assuming inductively that ψ vanishes to order $k-1$ at 0 for some $k \in \{2, \ldots, q-1\}$ and derivating k times (1.1) at 0, the Faà di Bruno formula and the identity $\partial_1 h(u,0) = 0$ yield

$$\psi^{(k)}(0) = \psi^{(k)}(0)\, h_0'(0)^k, \quad \text{that is,} \quad (\rho^k - 1)\, \psi^{(k)}(0) = 0,$$

hence $\psi^{(k)}(0) = 0$ as ρ is a *primitive* q^{th} root of unity. □

Examples. If $h_u(z) = \alpha(u)z$, where $\alpha : (\mathbb{C}, 0) \to (\mathbb{C}, \rho)$ is a holomorphic function such that $\alpha'(0) \neq 0$, then W is the z-plane.

If $h_u(z) = (\rho + u - z^q)z$, then W is the curve $u = z^q$.

1.3. Opening Pandora's box

Under the hypotheses of Proposition 1.3 ii)-iii), α is a holomorphic local diffeomorphism $(\mathbb{C}, 0) \to (\mathbb{C}, \rho)$. Viewing it as a local parameter change and performing the variable changes in the proof of Proposition 1.3, the following hypotheses are verified with $u_0 = \rho$:

Hypotheses. Given $u_0 \in \mathbb{S}^1$, set $\tilde{u}_0 := (u_0, 0) \in \mathbb{C}^2$ and let $h : (u, z) \mapsto h_u(z)$ be a holomorphic local map $(\mathbb{C}^2, \tilde{u}_0) \to (\mathbb{C}, 0)$ such that $h_u(0) = 0$ and ${h_u}'(0) = u$. Proposition 1.3 now reads as follows:

Proposition 1.4. *If $u_0 = e^{2\pi i \frac{p}{q}}$, $0 < p < q$, $\gcd(p, q) = 1$, then the q-periodic points of $\tilde{h}$ near $\tilde{u}_0$ form the union of $\{z = 0\}$ and the $\tilde{h}$-invariant "graph" $W_{p/q} = \{u = \psi_{p/q}(z)\}$ of a holomorphic $\psi_{p/q} : (\mathbb{C}, 0) \to (\mathbb{C}, u_0)$ such that ${\psi_{p/q}}^{(j)}(0) = 0$ for $1 \leq j < q$. The function $\psi := \psi_{p/q}$ verifies* (1.1)*, and $\tilde{h}$ generates a $\mathbb{Z}/q\mathbb{Z}$-action on $W_{p/q}$, namely*

$$\big(m, \big(\psi(z), z\big)\big) \longmapsto \tilde{h}^m\big(\psi(z), z\big) = \big(\psi(z), {h_{\psi(z)}}^m(z)\big)\,,$$

induced by the $\mathbb{Z}/q\mathbb{Z}$-action $(m, z) \mapsto {h_{\psi_{p/q}(z)}}^m(z)$ on $\operatorname{Dom}\psi_{p/q}$. □

When u_0 is not a root of unity, the following result can apply to $f = h_{u_0}$:

Theorem 1.5 (Brjuno [5], **Yoccoz** [40]**).** *If $u_0 = e^{2\pi i\omega}$ with $\omega \in [0, 1] \setminus \mathbb{Q}$, the following two conditions are equivalent:*

i) *ω is a* Brjuno number*, meaning that the convergents $\frac{p_n}{q_n}$ of its continued fraction expansion verify $\sum \frac{\log q_{n+1}}{q_n} < \infty$.*

ii) *Every holomorphic germ $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $f'(0) = u_0$ is holomorphically linearisable.* □

Notes. The implication i)⇒ii) is Brjuno's. In 1942, Siegel [32] had proved ii) under the stronger condition $\sup \frac{\log q_{n+1}}{\log q_n} < \infty$. This already defines a full measure set of numbers $u_0 \in \mathbb{S}^1$, but Theorem 1.5 provides the *optimal* set.

Back to families, in the trivial case $h_u(z) = uz$, every h_u is linear(isable). However, in general, h_{u_0} is linearisable if $u_0 = e^{2\pi i\omega}$ with ω Brjuno.

In that case, linearisability means that there exists a holomorphic local coordinate (conjugacy) $Z_\omega : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $Z_\omega \circ h_{u_0} = u_0 Z_\omega$; as the rotation $z \mapsto u_0 z$ preserves each circle $S_r = \{|z| = r\}$, every closed curve $C_r = Z_\omega^{-1}(S_r)$ with $r > 0$ small enough is h_{u_0}-invariant and, of course, $Z_\omega|_{C_r}$ conjugates $h_{u_0}|_{C_r}$ to the rotation $z \mapsto e^{2\pi i\omega}z$ restricted to S_r.

Question 1.6. Is this the limit of what happens near $u = e^{2\pi i p_n/q_n}$? Do the holomorphic functions ψ_{p_n/q_n} tend to the constant $\psi_\omega = u_0 = e^{2\pi i\omega}$ in some uniform neighbourhood of 0 and, for $z \in \mathbb{C}$ close to 0, do the periodic orbits $\Big\{\Big(\psi_{p_n/q_n}(z), {h_{\psi_{p_n/q_n(z)}}}^k(z)\Big) : 0 \leq k < q_n\Big\}$ of $\tilde{h}$ tend to the closed $\tilde{h}$-invariant curve $\{u_0\} \times C_r$ such that $z \in C_r$?

More precisely, does the (holomorphic) standard linearisation[6]

$$Z_{p_n/q_n}(z) = \frac{1}{q_n}\sum_{k=0}^{q_n-1} e^{-2\pi i k p_n/q_n} {h_{\psi_{p_n/q_n}(z)}}^k(z)$$

of the $\mathbb{Z}/q_n\mathbb{Z}$-action $(m,z)\mapsto {h_{\psi_{p_n/q_n}(z)}}^m(z)$ tend to Z_ω when $n\to\infty$?

Notes. If $h_u(z) = uz$, the answer is trivially positive even when ω is not Brjuno. The question is whether this holds for arbitrary families h.

My hope would be to deduce the Siegel-Brjuno theorem from the uniform convergence of ψ_{p_n/q_n} and maybe Z_{p_n/q_n} in a uniform neighbourhood of 0, at least for some well-chosen family h. One might get invariant fractals at the limit when ω is irrational but not Brjuno, as in [19] – the Pérez-Marco hedgehogs [28], independent of any arithmetic conditions, might be obtained in this fashion.

2. Subharmonic bifurcations, Arnold tongues and KAM circles

Here, *smooth* means real analytic or C^∞.

2.1. Subharmonic bifurcations in real dimension two

Let $h:(u,z)\mapsto h_u(z)$ be a smooth local map $(\mathbb{R}^2\times\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ such that the eigenvalues of the derivative $Dh_0(0) = \partial_z h(0,0)$ are primitive q^{th} roots of unity $\rho = e^{2\pi i\frac{p}{q}}, \bar\rho = e^{-2\pi i\frac{p}{q}}$, $1\le p<q$, $q\ge 3$.

Proposition 2.1. i) *The fixed points of the unfolding $\tilde h:(u,z)\mapsto \big(u,h_u(z)\big)$ form near 0 the graph of a smooth function $\varphi:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$.*

ii) *There is a smooth local function $\alpha:(\mathbb{R}^2,0)\to(\mathbb{C},\rho)$ such that the eigenvalues of $Dh_u\big(\varphi(u)\big)$ are $\alpha(u),\overline{\alpha(u)}$.*

iii) *If $D\alpha(0):\mathbb{R}^2\to\mathbb{C}$ is bijective then, near 0, the q-periodic points of $\tilde h$ form the union of two surfaces intersecting transversally at 0: of course* graph φ, *plus a C^{q-3} surface W on which $\tilde h|_W$ induces a $\mathbb{Z}/q\mathbb{Z}$-action.*

Proof. i) follows from the implicit mapping theorem applied to the smooth equation $F(u,z) := z - h(u,z) = 0$, as $\partial_z F(0,0):\mathbb{R}^2\to\mathbb{R}^2$ is invertible.

ii) follows from the formula for the eigenvalues of a real 2×2 matrix with no real eigenvalue.

iii) We may assume $\varphi = 0$, and the new $Dh_u(0)$ is the old $Dh_u\big(\varphi(u)\big)$.

Lemma 2.2. a) *An $\mathbb{R}$-linear change of variables $J(u):\mathbb{R}^2\to\mathbb{C}$, depending smoothly on u, yields $h:(\mathbb{R}^2\times\mathbb{C},0)\to(\mathbb{C},0)$ and $Dh_u(0)z = \alpha(u)z$.*

b) *Modulo a change of variables, polynomial of degree $q-1$ with respect to $z,\bar z$, whose coefficients are smooth functions of u, the Taylor polynomial Q_u of h_u to order $q-1$ at 0 for small u is of the form*

$$Q_u(z) = z\Big(\alpha(u) + \sum_{k=1}^{[\frac{q-1}{2}]} b_k(u)|z|^{2k}\Big) + \beta(u)\bar z^{q-1}.$$

[6]The other holomorphic local linearisations Z satisfy $Z\circ Z^{-1}_{p_n/q_n}(z) = za(z^{q_n})$, $a(0)\neq 0$.

Proof of the lemma. a) The isomorphism $J(u) \in L(\mathbb{R}^2, \mathbb{C})$ is an eigenvector of $Dh_u(0)^T : \lambda \mapsto \lambda \circ Dh_u(0)$ associated to the eigenvalue $\alpha(u)$. Under the condition, e.g., $J(u)(1,0) = 1$, it is unique and depends smoothly on u.

b) By normal form theory or direct computation, one can assume that $Q_u(z) - \alpha(u)z$ is a $\mathbb{C}$-linear combination (depending smoothly on u) of monomials $z^j \bar{z}^k$ with $1 < j + k \le q - 1$ and $u_0^j \bar{u}_0^k = u_0$, that is, $e^{2\pi i(j-k-1)p/q} = 1$, which writes $(j - k - 1)p = \ell q$ with $\ell \in \mathbb{Z}$. As $\gcd(p, q) = 1$, one has $\ell = mp$, $m \in \mathbb{Z}$, hence $j - k - 1 = mq$ and either $m = 0$, hence $z^j \bar{z}^k = z|z|^{2k}$, or $m = -1$ and $j = 0$, yielding $z^j \bar{z}^k = \bar{z}^{q-1}$. $\diamond$

By Taylor's formula,

$$
\begin{aligned}
h_u(z) &= Q_u(z) + \sum_{j=0}^{q} z^j \bar{z}^{q-j} \int_0^1 \frac{(1-t)^{q-1}}{(q-1)!} \begin{pmatrix} q \\ j \end{pmatrix} \partial_z^j \partial_{\bar{z}}^{q-j} h_u(tz)dt \\
&= z\Big(a(u,z) + b(u,z)\frac{\bar{z}^{q-1}}{z}\Big)
\end{aligned}
$$

where a, b are smooth, $a(u, 0) = \alpha(u)$ and $b(u, 0) = \beta(u)$. It follows that $h_u(z) = zg_u(z)$ with $g : (u, z) \mapsto g_u(z)$ only C^{q-3} in general and $g_u(0) = \alpha(u)$.

- For $q > 3$, the same arguments as in the proof of Proposition 1.3 yield a C^{q-3} implicit function $\psi : (\mathbb{C}, 0) \to (\mathbb{R}^2, 0)$ whose graph W has the required properties near the origin—in particular, (1.1) holds.
- If $q = 3$, then $h_u(z) = A_u(z)z$ near 0, where $A_u(z) = \int_0^1 Dh_u(tz)\, dt$ (hence $A_u(0)z = \alpha(u)z$), and one can similarly apply the implicit map theorem along $r = 0$ after dividing by r the equation $h_u^3(re^{i\theta}) = re^{i\theta}$.[7]

The details are left to the reader. $\square$

Example. If $h_u(z) = (\rho + u)z - \bar{z}^{q-1}$, $u, z \in \mathbb{C}$, then W is the surface $u = \bar{z}^{q-1}/z$, which is C^{q-3} but not C^{q-2}. Thus, our bound for the differentiability of W is sharp. No such problem arised in the holomorphic case.

2.2. Arnold tongues

Under the hypotheses of Proposition 2.1 iii), α is a smooth local diffeomorphism $(\mathbb{R}^2, 0) \to (\mathbb{C}, \rho)$. Viewing it as a local parameter change, the following hypotheses are verified with $u_0 = \rho$, modulo the variable changes in the proof of Proposition 2.1:

Hypotheses. For $u_0 \in \mathbb{S}^1$, setting $\tilde{u}_0 := (u_0, 0) \in \mathbb{C}^2$, let $h : (u, z) \mapsto h_u(z)$ be a smooth local map $(\mathbb{C}^2, \tilde{u}_0) \to (\mathbb{C}, 0)$ such that $h_u(0) = 0$ and $Dh_u(0)z = uz$. Proposition 2.1 now reads as follows:

Proposition 2.3. *If $u_0 = e^{2\pi i \frac{p}{q}}$, $0 < p < q$, $\gcd(p, q) = 1$, then, near $\tilde{u}_0$, the q-periodic points of $\tilde{h}$ form the union of $\{z = 0\}$ and the $\tilde{h}$-invariant "graph" $W_{p/q} = \{u = \psi_{p/q}(z)\}$ of a C^{q-3} function $\psi_{p/q} : (\mathbb{C}, 0) \to (\mathbb{C}, u_0)$. The function $\psi := \psi_{p/q}$ verifies* (1.1)*, and $\tilde{h}|_{W_{p/q}}$ generates a $\mathbb{Z}/q\mathbb{Z}$-action on $W_{p/q}$ as in Proposition* 1.4. $\square$

[7] The "blown-up" surface $\breve{W} = \operatorname{graph} \breve{\psi}$ in polar coordinates is smooth, see section 3.

The functions $\psi = \psi_{p/q}$ of Proposition 1.4, being holomorphic, are either constant or *open.* Thus the invariant manifold $W_{p/q}$, projected into parameter space, is either $\{e^{2\pi ip/q}\}$ or (in general) an open neighbourhood of $e^{2\pi ip/q}$. The non-holomorphic case is altogether different:

Proposition 2.4. *If u_0 is a primitive q^{th} root of unity $e^{2\pi ip/q}$, $0 < p < q$, $q > 4$, then:*

i) *Up to a smooth local change of variables $(\mathbb{C}^2, \tilde{u}_0) \to (\mathbb{C}^2, \tilde{u}_0)$, of the form $(u, z) \mapsto (u, Z_u(z))$ with Z_u (real) polynomial of degree $q-2$, one has the following: near $\tilde{u}_0$, the unfolding $\tilde{h}$ is tangent to order $q-2$ along $\mathbb{C} \times \{0\}$ to a smooth unfolding $\tilde{P}(u, z) = (u, P_u(z))$ of the form*

$$P_u(z) = z\Big(u - \sum_{k=1}^{\left[\frac{q-3}{2}\right]} b_k(u)|z|^{2k}\Big).$$

ii) *For $b_1(u_0) \neq 0$, the principal part of $\psi_{p/q}(z)$ is $u_0 + b_1(u_0)|z|^2$. Thus, for $\Re\big(\bar{u}_0 b_1(u_0)\big) \neq 0$, the set $\operatorname{Im}\psi_{p/q}$ of those u near u_0 for which h_u has a q-periodic orbit lies on* one *side of* $\mathbb{S}^1$.

iii) *The function $\psi_{p/q}$ is tangent to order $q-3$ at $\tilde{u}_0$ to a normal form*

$$\hat{\psi}_{p/q}(z) = u_0 + \sum_{k=1}^{\left[\frac{q-3}{2}\right]} a_k|z|^{2k} =: \chi_{p/q}(|z|), \qquad a_k \in \mathbb{C}, \quad a_1 = b_1(u_0).$$

Thus, when the first Birkhoff invariant *$b_1(u_0)$ is non-zero, restricting* $\operatorname{Dom}\psi_{p/q}$ *if necessary, the set* $\operatorname{Im}\psi_{p/q}$ *is contained near u_0 in an "Arnold tongue" $\displaystyle\bigcup_{0\le t\le\varepsilon} \big\{u \in \mathbb{C} : |u - \chi_{p/q}(t)| \le \delta_\varepsilon t^{q-3}\big\}$ along the curve $\chi_{p/q}([0, \varepsilon])$, with $\varepsilon > 0$ small and $\displaystyle\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$.*

Proof. i) follows from Lemma 2.2 b).

ii)-iii) As $\psi_{p/q}(z) = \psi_{p/q}\Big(h\big(\psi_{p/q}(z), z\big)\Big)$ by (1.1), the Taylor polynomial

$$\hat{\psi}_{p/q}(z) = u_0 + \sum_{1\le j+\ell\le q-3} c_{j\ell} z^j \bar{z}^\ell =: u_0 + \hat{c}(z)$$

satisfies $\hat{\psi}_{p/q}(z) = \hat{\psi}_{p/q}\big(P_{\hat{\psi}_{p/q}(z)}(z)\big)$ up to terms of degree greater than $q-3$. Denoting the Taylor expansion of $b_k(u_0 + v)$ at $v = 0$ by

$$\hat{b}_k(v) = \sum_{m\ge 0} b_{kmn} v^m \bar{v}^n,$$

this means that, up to terms of degree greater than $q-3$,

$$\hat{c}(z) = \hat{c}\left(z\Big(u_0 + \hat{c}(z) - \sum_{k=1}^{\left[\frac{q-3}{2}\right]} \hat{b}_k\big(\hat{c}(z)\big)|z|^{2k}\Big)\right).$$

ii) It follows that $c_{10} = u_0 c_{10} = 0$, $c_{01} = \bar{u}_0 c_{01} = 0$, $c_{20} = u_0^2 c_{20} = 0$, $c_{02} = \bar{u}_0^2 c_{02} = 0$; thus, the first $c_{j\ell}$ that can be nonzero is $c_{11} =: a_1$, and it is equal to $b_{100} = \hat{b}_1(0) = b_1(u_0)$.;

iii) Inductively, one can see that $\hat{c}(z) = \hat{c}(u_0 z)$, hence $\hat{c}(z) = \sum_{k=1}^{\left[\frac{q-3}{2}\right]} a_k |z|^{2k}$.

The reader can fill in the details as an exercise. □

Example. If $h_u(z) = z\left(u - \sum_{k=1}^{\left[\frac{q-3}{2}\right]} a_k |z|^{2k}\right)$ then $\psi_{p/q} = \hat{\psi}_{p/q}$; thus, near u_0, $\operatorname{Im}\psi_{p/q}$ is the curve $\chi_{p/q}([0,\varepsilon])$.

2.3. Opening Pandora's box wider

Question 2.5. For diophantine ω with convergents p_n/q_n, if $b_1(u_0) \neq 0$, one can wonder as in the holomorphic case whether one has the following:

- The smooth functions ψ_{p_n/q_n} tend to some $\psi_\omega : (\mathbb{C},0) \to (\mathbb{C}, u_0)$ in a uniform neighbourhood of 0; thus, the $\tilde{h}$-invariant surfaces W_{p_n/q_n} tend to the $\tilde{h}$-invariant surface $W_\omega = \{u = \psi_\omega(z)\}$.
- For small z, the periodic orbits $\left\{\left(\psi_{p_n/q_n}(z), {h_{\psi_{p_n/q_n}(z)}}^k(z)\right) : 0 \le k < q_n\right\}$ of $\tilde{h}$ tend to a closed $\tilde{h}$-invariant curve $\{\psi_\omega(z)\} \times C_{\omega z}$ such that $z \in C_{\omega z}$ and that the rotation number of $h_{\psi(z)}|_{C_{\omega z}}$ is ω.
- The standard linearisation of the $\mathbb{Z}/q_n\mathbb{Z}$-action $(m,z) \mapsto {h_{\psi_{p_n/q_n}(z)}}^m(z)$ tends to a local transformation Z_ω linearising the local diffeomorphism $z \mapsto h_{\psi_\omega(z)}(z)$. Hence, the $\mathbb{T}$-action $(\theta, z) \mapsto Z_\omega^{-1}\left(e^{2\pi i\theta} Z_\omega(z)\right)$ leaves ψ_ω invariant,[8] implying that $\operatorname{Im}\psi_\omega$ is *a curve* (with boundary), limit of the narrower and narrower Arnold tongues $\operatorname{Im}\psi_{p_n/q_n}$.

Example (KAM invariant curves). Assume that h possesses the following properties near some $u_0 = e^{2\pi i\omega_0}$ with $\omega_0 \in \mathbb{R}\setminus\mathbb{Q}$:

i) If $|u| = 1$, the transformation h_u preserves the area.
ii) $h_u = |u|\, h_{u/|u|}$, hence h_u multiplies the area by $|u|^2$.
iii) One has $b_1(u_0) \neq 0$.

By ii), no h_u with $|u| \neq 1$ can have a closed invariant curve near 0. Thus, if the answer to Question 2.5 is positive, then every ψ_ω has modulus one, hence $b_1(u_0) = i\lambda u_0$, $\lambda \in \mathbb{R}$—which already follows from i). Figure 2 shows what happens for $(u,z) = (e^{2\pi i\omega}, z) \in \mathbb{S}^1 \times \mathbb{C}$ close to $\tilde{u}_0$, in local coordinates (ω, z). The ω-axis is in red and the "paraboloids" are the surfaces W_ω with ω Diophantine, which do lie in $\mathbb{S}^1 \times \mathbb{C}$ as $|\psi_\omega(z)| = 1$. These surfaces intersect the slice $u = u_0$ at the h_{u_0}-invariant closed curves ("KAM circles") $C_{\omega z}$ with $\psi_\omega(z) = u_0$, which occupy most of the room near $z = 0$, with maybe complicated dynamics in between.

---

[8] Conjugating everything by Z_ω, one can assume $h_{\psi_\omega(z)}(z) = e^{2\pi i\omega} z$, hence (1.1) reads $\psi_\omega(z) = \psi_\omega(e^{2\pi i\omega} z)$, which yields $\psi_\omega(z) = \psi_\omega(e^{2\pi ik\omega} z)$ for every integer k and therefore $\psi_\omega(z) = \psi_\omega(e^{2\pi i\theta} z)$ for all $\theta \in \mathbb{T}$ by density.

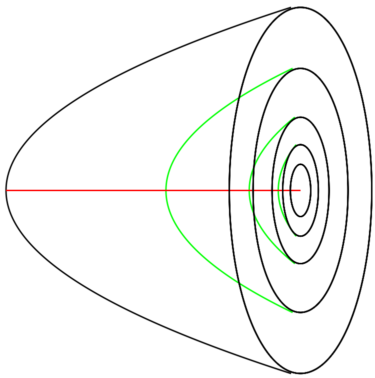

Figure 2.

Note. The limit surfaces W_ω and the linearisations Z_ω in Question 2.5 might be obtained as in [18] (where, however, the typical situation is $b_1(e^{2\pi i\omega}) \notin i\mathbb{R}$, yielding normally hyperbolic invariant circles). Figure 2, which I like a lot, most probably follows from standard KAM theory [22].

3. Higher dimensions

3.1. Statement of the hypotheses

Hypotheses. Given $u_0 \in \mathbb{C}^d$, $d > 1$, whose components are nonzero and all different, set $\tilde{u}_0 := (u_0, 0) \in \mathbb{C}^d \times \mathbb{C}^d$ and let $h : (\mathbb{C}^d \times \mathbb{C}^d, \tilde{u}_0) \to (\mathbb{C}^d, 0)$ be a smooth local map $(u, x) \mapsto h_u(x)$ such that

$$h_u(0) = 0 \qquad \text{and} \qquad Dh_u(0) = \operatorname{diag} u : z \mapsto \big(u_1 z_1, \dots, u_d z_d\big).$$

The case where h is holomorphic will be referred to as *the holomorphic case.*

Note. A general situation reduces to these hypotheses. Let $h : (u, x) \mapsto h_u(x)$ be a smooth local map $(\mathbb{R}^{2d} \times \mathbb{R}^{2d}, 0) \to (\mathbb{R}^{2d}, 0)$ such that the eigenvalues of $Dh_0(0)$ are simple and not real. Near 0, the fixed points of $\tilde{h}$ form the graph $x = \varphi(u)$ of a smooth implicit function, *which we may assume to be* 0.

There is [15, 12] a smooth local map J of $\mathbb{R}^{2d}$ into the space of $\mathbb{R}$-linear isomorphisms $\mathbb{R}^{2d} \to \mathbb{C}^d$, defined near 0, such that each $J(u)Dh_u(0)J(u)^{-1}$ is a diagonal automorphism $\operatorname{diag} \alpha(u) : z \mapsto \big(\alpha_1(u) z_1, \dots, \alpha_d(u) z_d\big)$ of $\mathbb{C}^d$ (thus, the eigenvalues $\alpha_j(u), \overline{\alpha_j(u)}$ of $Dh_u(0)$, $1 \le j \le d$, depend smoothly on u). *Via the identification* $(u, x) \mapsto \big(u, J(u)x\big)$, *we can view h as a local map* $(\mathbb{R}^{2d} \times \mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ *such that* $Dh_u(0) = \operatorname{diag} \alpha(u)$.

Setting $\alpha(u) := \big(\alpha_1(u), \dots, \alpha_d(u)\big)$ and *assuming* $D\alpha(0) : \mathbb{R}^{2d} \to \mathbb{C}^d$ *invertible*, the smooth local map $\alpha : (\mathbb{R}^{2d}, 0) \to \mathbb{C}^d$ is a local diffeomorphism. *If we view it as an identification, then* $u_0 := \alpha(0)$ *satisfies our hypotheses.*

3.2. Periodic orbits

Proposition 3.1. *Assume that* $u_0 = \rho = (\rho_1, \ldots, \rho_d)$, *where* $\rho_j = e^{2\pi i p_j/q}$, $0 < p_j < q$. *Let* $\pi : (\mathbb{R}_+ \times \mathbb{S}^{2d-1}, \{0\} \times \mathbb{S}^{2d-1}) \to (\mathbb{C}^d, 0)$ *be the oriented blowup* $\pi(r, y) := ry$ ("polar coordinates"). *Then, setting* $\breve{u}_0 := (u_0, 0) \in \mathbb{C}^d \times \mathbb{R}_+$ *and denoting by* $\mathring{\mathbb{S}}^{2d-1}$ *the complement of the coordinate hyperplanes in* $\mathbb{S}^{2d-1}$*:*

i) *Near* $\tilde{u}_0$, *the map* h *lifts to a smooth local map* $\breve{h} : (u, r, y) \mapsto \breve{h}_u(r, y)$ *of* $(\mathbb{C}^d \times \mathbb{R}_+ \times \mathbb{S}^{2d-1}, \{\breve{u}_0\} \times \mathbb{S}^{2d-1})$ *into* $(\mathbb{R}_+ \times \mathbb{S}^{2d-1}, \{0\} \times \mathbb{S}^{2d-1})$ *such that* $\pi \circ \breve{h}_u = h_u \circ \pi$ *and* $\breve{h}(\breve{u}_0, y) = \big(0, (\operatorname{diag} u_0) y\big)$ *for all* $y \in \mathbb{S}^{2d-1}$.

ii) *The* q*-periodic points of the unfolding* $\tilde{\breve{h}} : (u, r, y) \mapsto \big(u, \breve{h}_u(r, y)\big)$ *contain* $\{\breve{u}_0\} \times \mathbb{S}^{2d-1}$ *and the* $\tilde{\breve{h}}$*-invariant "graph"* $\breve{W} = \{u = \breve{\psi}(r, y)\}$ *of a smooth local map* $\breve{\psi} = \breve{\psi}_{p/q} : (\mathbb{R}_+ \times \mathring{\mathbb{S}}^{2d-1}, \{0\} \times \mathring{\mathbb{S}}^{2d-1}) \to (\mathbb{C}^n, u_0)$.

iii) *Hence, the non-fixed* q*-periodic points of* $\tilde{h}$ *contain the* $\tilde{h}$*-invariant "graph"* $W = \{u = \psi(z)\}$ *of a smooth* $\psi = \psi_{p/q} : \pi(\operatorname{Dom} \breve{\psi}) \setminus \{0\} \to \mathbb{C}^n \setminus \{u_0\}$.

iv) *In the holomorphic case,* W *is holomorphic.*

Proof. i) The relation $\pi \circ \breve{h}_u(r, y) = h_u \circ \pi(r, y)$ writes $\breve{h}_u(r, y) = (R_u, Y_u)(r, y)$ with $R_u(r, y) = |h_u(ry)|$ and $Y_u(r, y) = h_u(ry)/|h_u(ry)|$ for $r > 0$; now, by Taylor's formula, $h_u(ry) = rA_u(ry)y$, where $A_u(ry) := \int_0^1 Dh_u(try)\, dt$, hence $Y_u(r, y) = A_u(ry)y/|A_u(ry)y|$ wherever $A_u(ry)y \neq 0$, including $r = 0$ near $u = u_0$ since $A_u(0) = \operatorname{diag} u$.

ii) One has that $h_u^q(ry) = ry$ if and only if $rG_u(r, y) = 0$, where $G_u(r, y) = G(u, r, y) = A_u\big(h_u^{q-1}(ry)\big) \cdots A_u(ry)y - y$, hence in particular $G(u, 0, y) = (\operatorname{diag} u)^q y - y$. Forgetting the fixed points $r = 0$, the equation $h_u^q(ry) = ry$ reads $G(u, r, y) = 0$. Now, all $y \in \mathbb{S}^{2d-1}$ verify $G(u_0, 0, y) = 0$ and $\partial_u G(u_0, 0, y) = q \operatorname{diag}(\bar{\rho}_1 y_1, \ldots, \bar{\rho}_d y_d)$, invertible if and only if $y_1 \cdots y_d \neq 0$, i.e., $y \in \mathring{\mathbb{S}}^{2d-1}$. Hence, there exist open neighbourhoods U of u_0 in $\mathbb{C}^d$ and $\breve{V}$ of $\{0\} \times \mathring{\mathbb{S}}^{2d-1}$ in $\mathbb{R}_+ \times \mathring{\mathbb{S}}^{2d-1}$ such that the zeros of $G|_{U \times \breve{V}}$ form the "graph" of a smooth implicit map $\breve{\psi} : \breve{V} \to U$; as before, this graph $\breve{W}$ becomes $\tilde{\breve{h}}$-invariant if it is replaced by $\breve{W} \cap \tilde{\breve{h}}^{-1}(\breve{W}) \cap \cdots \cap \tilde{\breve{h}}^{1-q}(\breve{W})$.

iii) Recall that π is a diffeomorphism off the "exceptional divisor" $\pi^{-1}(0)$.

iv) We can therefore "read" the equation $G_u(r, y) = 0$ via this diffeomorphism, that is, write it $g_u(z) := h_u^q(z) - z = 0$ for $z \neq 0$; as the unfolding $(u, r, y) \mapsto \big(u, G_u(r, y)\big)$ is a local diffeomorphism at every point of $\breve{W}$, so is $(u, r, y) \mapsto \big(u, rG_u(r, y)\big)$, hence the unfolding $\tilde{g} : (u, z) \mapsto \big(u, g_u(z)\big)$ is a diffeomorphism at every point of W; the map $\tilde{g}$ being holomorphic, its local inverses are, implying that W is holomorphic. □

Note. A nicer way to prove iv) is to use the complex blowup $\pi_{\mathbb{C}} : (D, z) \mapsto z$, $z \in D$, $D \subset \mathbb{C}^d$ complex line through 0;[9] the implicit function theorem yields a holomorphic $\breve{\psi}_{\mathbb{C}}$ "upstairs", defined on an open subset of the complement of the closure of $\{(D, z) : z \neq 0, z_1 \cdots z_d = 0\}$ and equal to u_0 on $\pi_{\mathbb{C}}^{-1}(0)$.

[9] In the standard j^{th} local chart of $\mathbb{CP}^{d-1}$, this blowup reads $\big(z_j, (w_k)_{k \neq j}\big) \mapsto z$ with $z_k = z_j w_k$ for $k \neq j$; the "forbidden" closed subset is the union of the hyperplanes $w_k = 0$.

Example. If $h_u(z) = \operatorname{diag}\big(u + \chi(z_1^{q_1}, \dots, z_d^{q_d})\big)z$, where q_j is the denominator of p_j/q in irreducible form and $\chi : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ is holomorphic, then $\psi(z) = \rho - \chi(z_1^{q_1}, \dots, z_d^{q_d})$, which has contact of order at least $\min q_k$ with the constant ρ at 0.

Proposition 3.2. *The automorphism* $\operatorname{diag}\rho$ *lifts via* π *to the diffeomorphism* $\breve{\operatorname{diag}}\,\rho : (r, y) \mapsto \big(r, (\operatorname{diag}\rho)y\big)$.

i) *Restricting* $\breve{W}$ *if required, there is a smooth diffeomorphism* $\breve{Z} = \breve{Z}_{p/q}$ *of* $\breve{W}$ *onto an open* $\breve{\operatorname{diag}}\,\rho$*-invariant subset* $\breve{\Omega} \supset \{0\} \times \mathring{\mathbb{S}}^{2d-1}$ *of* $\mathbb{R}_+ \times \mathring{\mathbb{S}}^{2d-1}$, *conjugating* $\breve{\tilde{h}}|_{\breve{W}}$ *to* $(\breve{\operatorname{diag}}\,\rho)|_{\breve{\Omega}}$, *with* $\breve{Z}(\breve{u}_0, y) = (0, y)$ *for all* $y \in \mathring{\mathbb{S}}^{2d-1}$.

ii) *The map* $\breve{Z}$ *induces a smooth diffeomorphism* $Z = Z_{p/q}$ *of* W *onto the open* $\operatorname{diag}\rho$*-invariant "trefoil"* $\Omega := \pi(\breve{\Omega}) \setminus \{0\}$, *conjugating* $\tilde{h}|_W$ *to* $\operatorname{diag}\rho|_\Omega$ *and tending to* 0 *when the variable in* W *tends to* $\tilde{u}_0$.

iii) *If* h *is holomorphic, so is* Z.

Proof. i) The conjugacy $\breve{Z}_{p/q}$ is as in Question 1.6, but in polar coordinates:

$$\breve{Z}_{p/q}\big(\breve{\psi}(r, y), (r, y)\big) = \left(r\,|C(r, y)|, \frac{C(r, y)}{|C(r, y)|}\right),$$

where
$$C(r, y) = \frac{1}{q}\sum_{k=0}^{q-1} (\operatorname{diag}\rho)^{-k} A_{\breve{\psi}(r,y)}\big(h^{k-1}_{\breve{\psi}(r,y)}(ry)\big) \cdots A_{\breve{\psi}(r,y)}(ry)y.$$

For all $y \in \mathring{\mathbb{S}}^{2d-1}$, one has that $C(0, y) = y$, hence $\breve{Z}(\breve{u}_0, y) = (0, y)$ and

$$D\breve{Z}_{p/q}(0, y) = \begin{pmatrix} 1 & 0 \\ * & \mathrm{id}_{y^\perp} \end{pmatrix} : \mathbb{R} \times y^\perp \to \mathbb{R} \times y^\perp$$

is invertible. It follows that $\breve{Z}_{p/q}$ is a smooth local diffeomorphism, whose domain can be made $\breve{\tilde{h}}$-invariant as usual. It is not difficult to check that it is a conjugacy, see equation (3.1) hereafter.

ii) is obvious; by definition, the conjugacy $Z_{p/q}$ is as in Question 1.6:

$$Z_{p/q}\big(\psi_{p/q}(z), z\big) = \frac{1}{q}\sum_{k=0}^{q-1} (\operatorname{diag}\rho)^{-k} {h_{\psi_{p/q}(z)}}^k(z). \tag{3.1}$$

iii) follows at once. □

Note. The diagonal action $e : (t, z) \mapsto e^{2\pi i \operatorname{diag} t} z$ of $\mathbb{T}^d$ on $\mathbb{C}^d$ preserves $\operatorname{diag}\rho$ and lifts to the action $\breve{e} : (t, r, y) \mapsto (r, e^{2\pi i \operatorname{diag} t} y) =: \breve{e}_t(r, y)$ of $\mathbb{T}^d$ on $\mathbb{R}_+ \times \mathbb{S}^{2d-1}$, which preserves $\breve{\operatorname{diag}}\,\rho$. The open subset $\breve{\Omega}$ becomes $\breve{e}$-invariant (and still $\breve{\operatorname{diag}}\,\rho$-invariant) if it is replaced by $\bigcap_{t \in \mathbb{T}^d} \breve{e}_t(\breve{\Omega})$, which contains $\{0\} \times \mathring{\mathbb{S}}^{2d-1}$ and is open because $\mathbb{T}^d$ is compact.

Hence, denoting again by $\breve{W}$ the inverse image of this new $\breve{\Omega}$ by $\breve{Z}$, the map $\breve{\tilde{h}}|_{\breve{W}}$ is invariant under the $\mathbb{T}^d$-action $\breve{Z}^*\breve{e} : \big(t, \breve{Z}^{-1}(r, y)\big) \mapsto \breve{Z}^{-1}\breve{e}(t, r, y)$; in particular, it preserves each orbit, which orbits constitute a foliation of $\breve{W}$ by d-tori $\breve{Z}^{-1}\big(\{r\} \times (x_1\mathbb{S}^1 \times \cdots \times x_d\mathbb{S}^1)\big)$ with $x_j > 0$ and $x_1^2 + \cdots + x_d^2 = 1$.

In general, these tori of course do not lie each in a slice $u =$ constant like the orbits of $\tilde{\breve{h}}|_{\breve{W}}$. The foliation, like the new $\breve{W}$, depends on the choice of $\breve{Z}$, which is far from unique since *the set of* $\operatorname{di\breve{a}g}\rho$*-invariant smooth diffeomorphism germs* $\left(\mathbb{R}_+ \times \mathbb{S}^{2d-1}, \{0\} \times \mathbb{S}^{2d-1}\right) \to \left(\mathbb{R}_+ \times \mathbb{S}^{2d-1}, \{0\} \times \mathbb{S}^{2d-1}\right)$ *is infinite dimensional.*[10]

However, when p/q tends to some diophantine $\omega \in [0,1]^d$, the orbits of $\tilde{\breve{h}}|_{\breve{W}_{p/q}}$ should "become denser and denser in such invariant tori":

3.3. Passing to the limit in the holomorphic case?

In the holomorphic case, if $u_0 = (e^{2\pi i\omega_1}, \ldots, e^{2\pi i\omega_d})$, $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{T}^d$, the following result may apply to h_{u_0}:

Theorem 3.3. *Assume* ω diophantine *in the sense that, for some large* τ*,*

$$\inf_{1\le j\le d} \inf_{|k|\ge 2} |k|^\tau \left|e^{2\pi ik\omega} - e^{2\pi i\omega_j}\right| > 0,$$

where $k \in \mathbb{N}^d$*,* $|k| = k_1 + \cdots + k_d$ *and* $k\omega = k_1\omega_1 + \cdots + k_d\omega_d$*. Then, every holomorphic germ* $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ *such that* $Df(0) = \operatorname{diag} u_0$ *is holomorphically linearisable: there exists a holomorphic local diffeomorphism* $Z_\omega : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ *such that* $Z_\omega \circ h_{u_0} = (\operatorname{diag} u_0)Z_\omega$*.*[11] □

As the rotation $z \mapsto (\operatorname{diag} u_0)z = (e^{2\pi i\omega_j} z_j)_{1\le j\le d}$ preserves each d-torus $T_r = \{|z_1| = r_1, \ldots, |z_d| = r_d\}$, every embedded torus $T_{\omega r} = Z_\omega^{-1}(T_r)$ with $r_j > 0$ small enough is h_{u_0}-invariant and, of course, $Z_\omega|_{T_{\omega r}}$ conjugates $h_{u_0}|_{T_{\omega r}}$ to the rotation $z \mapsto (\operatorname{diag} u_0)z$ restricted to T_r.

Question 3.4. Applied to $f = h_{u_0}$, is this the limit of what happens near $u = (e^{2\pi ip_j/q})_{1\le j\le d}$ when $p/q \in \mathbb{Q}^d$ tends to ω?[12] Do the maps $\psi_{p/q}$ tend to $\psi_\omega = u_0$ in some uniform neighbourhood of 0? For $z \in \mathbb{C}^d$ close to 0, does the periodic orbit $\left\{\left(\psi_{p/q}(z), {h_{\psi_{p/q}(z)}}^k(z)\right) : 0 \le k < q\right\}$ of $\tilde{h}$ tend to the closed $\tilde{h}$-invariant torus $\{u_0\} \times T_{\omega r}$ such that $z \in T_{\omega r}$? More precisely, does the holomorphic linearisation (3.1) of $\tilde{h}|_{W_{p/q}}$ tend to Z_ω when $n \to \infty$?

Note. This is not as simple as Question 1.6: indeed, unless I am mistaken, the maps $\psi_{p/q}$ are not *a priori* defined in a neighbourhood of 0, so that part of the question is whether $\operatorname{Dom}\psi_{p/q}$ tends to such a neighbourhood. On the other hand, it follows from normal form theory that, as in the case $d = 1$, the map $\psi = \psi_{p/q}$ has more and more contact with u_0 at 0 when $p/q \to \omega$.[13]

---

[10] Indeed, *the set of* $\operatorname{diag}\rho$*-invariant smooth diffeomorphism germs* $(\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ *is infinite dimensional*, as *any* smooth germ $\eta : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ yields the $\operatorname{diag}\rho$-invariant germ $\frac{1}{q}\sum_1^q (\operatorname{diag}\rho)^{-k} \circ \eta \circ (\operatorname{diag}\rho)^k$.

[11] One can assume $DZ_\omega(0) = \mathrm{Id}$. Pöschel [30] attributes Theorem 3.3 to Siegel, who certainly proved its analogue for vector fields [33]. The same applies to its improvement by Brjuno. This "Siegel-Brjuno" theorem for maps and much more is proved in [31, 29, 30, 37].

[12] For example, $p_j/q = p_{jn}/q_n$ can be the n^{th} convergent of ω_j.

[13] If one prefers, $\breve{\psi}_{\mathbb{C}}$ has more and more contact with u_0 at points of $\pi_{\mathbb{C}}^{-1}(0)$.

3.4. Passing to the limit in the smooth case?

If $u_0 = (e^{2\pi i\omega_1}, \ldots, e^{2\pi i\omega_d})$, where $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{T}^d$ is *non-resonant*, meaning that $\omega_1, \ldots, \omega_d \in \mathbb{T}$ are independent over $\mathbb{Z}$, then, by normal form theory, one has the following: for each positive integer N, up to smooth local conjugacy $(u, z) \mapsto \big(u, Z_u(z)\big)$, every h_u with $u - u_0$ small enough is tangent to order $2N + 1$ at 0 to a polynomial map

$$P_u(z) = \operatorname{diag}\Big(u + \sum_{\ell=1}^{N} b_\ell(u)\big(|z_1|^2, \ldots, |z_d|^2\big)\Big) z$$

with $b_\ell(u) : \mathbb{R}^d \to \mathbb{R}^d$ homogeneous of degree ℓ, depending smoothly on u. As for $d = 1$, it follows that when p/q tends to ω the map $\psi_{p/q}$ of Proposition 3.1 is tangent to higher and higher order at 0 to a polynomial normal form[14]

$$\hat\psi_{p/q}(z) = \chi_{p/q}\big(|z_1|^2, \ldots, |z_d|^2\big), \qquad \chi_{p/q}(0) = \rho, \quad D\chi_{p/q}(0) = b_1(\rho).$$

Thus, if $b_1(u_0)$ (and therefore $b_1(\rho)$ for small $\frac{p}{q} - \omega$) is invertible then, restricting $\psi_{p/q}$, the set $\operatorname{Im}\psi_{p/q}$ lies near ρ in a thinner and thinner "Arnold tongue" along the smooth d-fold with corner $\chi_{p/q}([0, \varepsilon)^d)$ for small $\varepsilon > 0$.

Question 3.5. Assume ω *diophantine* in the sense that, for some large τ,

$$\inf_{m \neq 0} |m|^\tau \left|e^{2\pi i m\omega} - 1\right| > 0,$$

where $m \in \mathbb{Z}^d$, $|m| = m_1 + \cdots + m_d$ and $m\omega = m_1\omega_1 + \cdots + m_d\omega_d$. If $b_1(u_0)$ is invertible, one can wonder as in the holomorphic and one-dimensional cases whether one has the following when p/q tends to ω:

- The $\breve\psi_{p/q}$'s tend to a map $\breve\psi_\omega$ of $\big(\mathbb{R}_+ \times \mathring{\mathbb{S}}^{2d-1}, \{0\} \times \mathring{\mathbb{S}}^{2d-1}\big)$ into $(\mathbb{C}^n, u_0)$ in a uniform neighbourhood of $\{0\} \times \mathring{\mathbb{S}}^{2d-1}$; thus, the $\tilde{\breve h}$-invariant surfaces $\breve W_{p/q}$ tend to the $\tilde{\breve h}$-invariant $2d$-fold $\breve W_\omega = \{u = \breve\psi_\omega(r, y)\}$.
- For each (r, y), the periodic orbits $\Big\{\Big(\breve\psi_{p/q}(r, y), \breve h^k_{\breve\psi_{p/q}(r,y)}(r, y)\Big)\Big\}_{0 \le k < q}$ of $\tilde{\breve h}$ tend to a $\tilde{\breve h}$-invariant embedded d-torus $\{\breve\psi_\omega(r, y)\} \times T_{\omega r y}$ such that $(r, y) \in T_{\omega r y}$.
- The "linearisation" $\breve Z_{p/q}$ of Proposition 3.2 i) tends to a local transformation $\breve Z_\omega$ "linearising" the local diffeomorphism $(r, y) \mapsto \breve h_{\breve\psi_\omega(r,y)}(r, y)$. Hence, the $\mathbb{T}^d$-action $\big(\theta, \breve Z_\omega^{-1}(r, y)\big) \mapsto \breve Z_\omega^{-1}\big(r, e^{2\pi i \operatorname{diag}\theta} y\big)$ leaves $\breve\psi_\omega$ invariant, implying that $\operatorname{Im}\breve\psi_\omega$ is *a d-fold* (with corner), limit of the narrower and narrower subsets $\operatorname{Im}\breve\psi_{p/q}$.

Example (KAM invariant tori). Assume that h possesses the following properties near some $u_0 = e^{2\pi i\omega_0}$ with ω_0 non-resonant:

i) If $|u_1| = \cdots = |u_d| = 1$, the transformation h_u preserves the standard symplectic form $\sigma = \frac{1}{2i}(d\bar z_1 \wedge dz_1 + \cdots + d\bar z_d \wedge dz_d)$.

[14]If one prefers, $\breve\psi$ has more and more contact with $\hat\psi_{p/q} \circ \pi$ at points of $\pi^{-1}(0)$.

ii) One has that $h_u = h_{(u_1/|u_1|,\dots,u_d/|u_d|)} \circ \mathrm{diag}(|u_1|,\dots,|u_d|)$ and therefore $h_u^*\sigma = \frac{1}{2i}(|u_1|^2 d\bar{z}_1 \wedge dz_1 + \cdots + |u_d|^2 d\bar{z}_d \wedge dz_d)$.

iii) The linear map $b_1(u_0)$ is an isomorphism.

Then, if the answer to question 3.5 is positive, it follows from ii) that every ψ_ω takes its values in $\{|u_1| = \cdots = |u_d| = 1\}$, yielding a $3d$-dimensional analogue of Figure 2, see [22, 27].[15]

4. Comments and references

My interest in this part of the program sketched in [12] awoke when I heard Abed Bounemoura talk about [4].

The dimension of both parameter and phase space, minimal here, can be much higher[16]. Proposition 1.3 has been known (at least) to me for thirty years, as well as the "blown-up" version of Proposition 2.1.[17] I have no reference for the higher dimensional results in subsection 3.4. The excision of the coordinate hyperplanes in Propositions 3.1-3.2 corresponds to the closure of manifolds of periodic orbits of lower period, which might tend to (manifolds of) lower dimensional KAM tori à la Eliasson [21, 22].

It is known that "good" periodic orbits accumulate on KAM tori. My naive hope is to do it the other way round and get the mysterious objects as limits of obvious ones, which would clarify a very intricate situation.

One of the sources of this article is an awfully biased reading of the two papers [18, 19] by Alain Chenciner, to whom my debt cannot be overestimated, though he certainly does not share my viewpoint that conservative systems are essentially meant to deny the existence of death (and birth…).

Last but not least, I thank Jacques Féjoz and Laurent Stolovitch for very useful discussions and comments.

References

[1] Arnold, V.I.: Sur une propriété topologique des applications globalement canoniques de la mécanique classique. C. R. Acad. Sc. Paris, **261**, Groupe 1, 3719–3722 (1965)

[2] ——————: Méthodes mathématiques de la mécanique classique. Mir, Moscou (1976), translated from the Russian original edition (1974)
English version: Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics, **60**, Springer, New York (1989)

[3] —, Avez, A.: Problèmes ergodiques de la mécanique classique. Gauthier-Villars, Paris (1967)

[4] Bounemoura, A., Fischler, S.: A Diophantine duality applied to the KAM and Nekhoroshev theorems. Math. Z., **275** (3), 1135–1167 (2013)

[15] Note that the slices with $\omega \in \mathbb{Q}^d$ close to ω_0 contain invariant tori "far" away from 0.

[16] In [12] one considers the tautological families $(f, x) \mapsto f(x)$, the parameter space being that of all maps f.

[17] As far as I know, the general version is due to Lino Samaniego [16].

[5] Brjuno, A.D.: Analytical form of differential equations. Transactions Moscow Math. Soc. **25** (1971), 131–288; **26**, 199–239 (1972)

[6] Chaperon, M.: Quelques questions de géométrie symplectique [d'après, entre autres, Poincaré, Arnold, Conley et Zehnder]. Séminaire Bourbaki, Astérisque, **105–106**, 231–249 (1983)

[7] ——————: Une idée du type "géodésiques brisées" pour les systèmes hamiltoniens. C. R. Acad. Sc. Paris, **298**, série A, 293–296 (1984)

[8] ——————: An elementary proof of the Conley-Zehnder theorem in symplectic geometry. In: Braaksma, B. L. J., Broer, H., Takens, F. (eds) Dynamical systems and bifurcations, 1–8. Lecture Notes in Mathematics, **1125**. Springer, Berlin Heidelberg New York (1985)

[9] ——————: Géométrie différentielle et singularités de systèmes dynamiques. Astérisque **138–139**, Société Mathématique de France, Paris, 434 pp. (1986)

[10] ——————: C^k–conjugacy of holomorphic flows near a singularity. Publ. Math. I.H.E.S. **64**, 143–183 (1986)

[11] ——————: On generating families. In: Hofer, H., Taubes, C. H., Weinstein, A., Zehnder, E. (eds) The Floer Memorial Volume, 283–296. Progress in Mathematics, **133**, Birkhäuser, Basel Boston (1995)

[12] ——————: Singularities in dynamics: a catastrophic viewpoint. Proceedings of the Steklov Institute **267**, 8–39 (2009)

[13] ——————: Generalized Hopf bifurcations: a birth lemma. Moscow Mathematical Journal **11** (3), 413–438 (2011)

[14] ——————: A forgotten theorem on $\mathbb{Z}^k \times \mathbb{R}^m$-action germs and related questions. Regular and Chaotic Dynamics, **18** (6), 742–775 (2013)

[15] —, López de Medrano S.: Regularities and singularities appearing in the study of polynomials and linear operators. Astérisque **323**, 123–160 (2009)

[16] —, —, Samaniego Mendoza J.L.: On sub-harmonic bifurcations. C. R. Acad. Sci., Paris, Sér. I, **340**, 827–832 (2005)

[17] —, Zehnder, E.: Quelques résultats globaux en géométrie symplectique. In: Dazord, P., Desolneux-Moulis N. (eds.) Géométrie symplectique et de contact: autour du théorème de Poincaré-Birkhoff, 71–121. Travaux en cours, Hermann, Paris (1984)

[18] Chenciner, A.: Bifurcations de points fixes elliptiques, I. Courbes invariantes. Publ. Math. I.H.E.S., **61**, 67–127 (1985)

[19] ——————: Bifurcations de points fixes elliptiques, II. Orbites périodiques et ensembles de Cantor invariants. Invent. Math., **80**, 81–106 (1985)

[20] Conley, C.C., Zehnder, E.: The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnol'd. Invent. Math., **73**, 33–49 (1983)

[21] Eliasson, L. H.: Perturbations of stable invariant tori for Hamiltonian systems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15**, no. 1, 115–147 (1988)

[22] Féjoz, J.: Introduction to KAM theory with a view to celestial mechanics. In: Bergounioux, M., Peyré, G., Schnörr, C., Caillau, J.-B., Haberkorn, T. (eds) Variational Methods. W. de Gruyter, Berlin/Munich/Boston, 387–433 (2016)

[23] A. Floer: Morse theory for Lagrangian intersections. J. Diff. Geom. **28**, 513–547 (1988)

[24] Gromov, M.: Pseudo holomorphic curves in symplectic manifolds. Invent. Math. **82**, 307–347 (1985)

[25] Hofer, H.: Lagrangian embeddings and critical point theory. Ann. Inst. Henri Poincaré, Analyse non linéaire **2**, 407–462 (1985)

[26] Laudenbach, F., Sikorav, J.-C.: Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibré cotangent. Invent. math., **82**, 349–357 (1985)

[27] Massetti, J. E.: A normal form à la Moser for diffeomorphisms and a generalization of Rüssmann's translated curve theorem to higher dimensions. Anal. PDE **11** (1), 149–170 (2018)

[28] Pérez-Marco, R.: Fixed points and circle maps. Acta Math. **179**, 243–294 (1997)

[29] Pöschel, J.: On invariant manifolds of complex analytic mappings near fixed points. Expo. Math. **4**, 97–109 (1986)

[30] ———: On the Siegel-Sternberg linearization theorem. arXiv:1702.03691v2 (2017)

[31] Rüssmann, H.: On the convergence of power series transformations of analytic mappings near a fixed point into a normal form. Preprint, I.H.E.S., 1977.

[32] Siegel, C.L.: Iteration of analytic functions. Ann. Math. **43**, 607–612 (1942)

[33] ———: Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. (German) Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1952, 21–30 (1952)

[34] Sikorav, J.-C.: Points fixes d'une application symplectique homologue l'identité. J. Differential Geometry, **22**, 49–79 (1985)

[35] ———: Sur les immersions lagrangiennes dans un fibré cotangent .... C.R. Acad. Sci. Paris, t.302, série I, n^o 3, 119–122 (1986)

[36] ———: Problèmes d'intersections et de points fixes en géométrie hamiltonienne. Comment. Math. Helv., **62**, 62–73 (1987)

[37] Stolovitch, L.: Family of intersecting totally real manifolds of $(\mathbb{C}^n, 0)$ and germs of holomorphic diffeomorphisms. Bull. Soc. math. France **143** (2), 247–263 (2015)

[38] Viterbo, C.: A proof of Weinstein's conjecture in $\mathbb{R}^{2n}$. Ann. Inst. Henri Poincaré, Analyse non linéaire, **4**, 407–462 (1987)

[39] ———: Symplectic topology as the geometry of generating functions. Math. Ann., **292**, 685–710 (1992)

[40] Yoccoz, J.-C.: Petits diviseurs en dimension 1. Astérisque, **231**, Société mathématique de France (1995)

Marc Chaperon
Université de Paris & Sorbonne Université, CNRS, IMJ-PRG, F-75006 Paris, France
e-mail: `marc.chaperon@imj-prg.fr`