

NORMALISATION OF THE SMOOTH FOCUS-FOCUS: A SIMPLE PROOF

MARC CHAPERON, WITH AN APPENDIX BY JIANG KAI

ABSTRACT. We prove that the smooth focus-focus can be normalized and that the smooth “node-node” can not.

Part of Eliasson’s thesis [4] on local normal forms of completely integrable hamiltonian systems was not published. In those days, due to the existence of Eliasson’s work, I did not investigate very thoroughly the symplectic case in my study [1, 2, 3] of smooth germs of $\mathbb{Z}^k \times \mathbb{R}^\ell$ actions. The aim of this note is to show that his result on the *focus-focus* singularity in the C^∞ case follows at once from [2, 3]. In Appendix B, Jiang Kai proves that there is no similar result for the “node-node”.

1. STATEMENT OF THE RESULTS

Hypotheses: first part. Let H be a smooth Hamiltonian defined near a point a in a four-dimensional manifold M equipped with a symplectic form ω . Assume that

- a is a critical point of H , implying that the Hamiltonian vector field X_H (whose interior product ωX_H with ω equals dH) vanishes at a
- this critical point is a focus-focus singularity, meaning that the eigenvalues of $dX_H(a)$ form a quadruple of complex numbers $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ with $\Re \lambda > 0$ and $\Im \lambda > 0$.

Theorem 1. *Under those hypotheses, there exists a smooth local chart $C : (M, a) \rightarrow (\mathbb{C}^2, 0)$ such that, denoting by (x, y) the points of \mathbb{C}^2 and by z the points of \mathbb{C} ,*

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- i) $C_*\omega(x, y) = dx \wedge dy + d\bar{x} \wedge d\bar{y}$
- ii) $C_*H(x, y) = h(xy)$, where $h : (\mathbb{C}, 0) \rightarrow \mathbb{R}$ is a smooth function and $dh(0) = \lambda dz + \bar{\lambda} d\bar{z}$.

Sketch of the proof This is a special case of the ‘‘symplectic Sternberg theorem’’ (see [3], 8.3, Théorème 2 bis). Assuming $H(a) = 0$, by elementary symplectic algebra¹, there is a chart $C_0 : (M, a) \rightarrow (\mathbb{C}^2, 0)$ such that $C_{0*}\omega(x, y) = dx \wedge dy + d\bar{x} \wedge d\bar{y}$ and that $C_{0*}H(x, y)$ has second order contact at 0 with $H_0(x, y) := h_0(xy)$, where $h_0(z) := \lambda z + \bar{\lambda} \bar{z}$.

Then, by the Jordan decomposition theorem², there exists a symplectic transformation $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ tangent to the identity at 0 and sending $C_{0*}H$ to a Hamiltonian whose Taylor expansion \hat{H} at 0 satisfies $\{H_0, \hat{H}\} = 0$, i. e. $\hat{H}(x, y) = \hat{h}(xy)$ for some formal power series \hat{h} in the variables z, \bar{z} . Hence, take any smooth function h whose Taylor expansion at 0 is \hat{h} and apply Theorem 6 in [2], which asserts that two *hyperbolic* germs of Hamiltonian vector fields are symplectically conjugate if and only if this is the case formally. \square

With the notation of Theorem 1, we have that

$$C_*X_H(x, y) = \partial h(xy) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) + \bar{\partial} h(xy) \left(\bar{x} \frac{\partial}{\partial \bar{x}} - \bar{y} \frac{\partial}{\partial \bar{y}} \right),$$

a vector field whose flow φ_H^t is given by

$$\varphi_H^t(x, y) = (x e^{t\partial h(xy)}, y e^{-t\partial h(xy)});$$

its unstable manifold is the x -axis, on which it has a linear focus:

$$\varphi_H^t(x, 0) = (x e^{t\lambda}, 0);$$

its stable manifold is the y -axis, on which it has a linear focus:

$$\varphi_H^t(0, y) = (0, y e^{-t\lambda}),$$

hence the name of the singularity.

Hypotheses: second part. We consider a second smooth Hamiltonian $K : (M, a) \rightarrow \mathbb{R}$ such that $dK(a) = 0$ and $\{H, K\} = 0$ (Poisson bracket); moreover, we assume that the differential $dX_K(a)$ is linearly independent from $dX_H(a)$. The aim of this note is to prove

Theorem 2. *Under those hypotheses, the chart C of Theorem 1 can be chosen so that $C_*K(x, y) = k(xy)$ for some smooth $k : (\mathbb{C}, 0) \rightarrow \mathbb{R}$.*

¹See, e.g., [3] subsection 8.1.2.

²Or any other standard argument yielding the Birkhoff normal form: generating functions, path method, etc.

Hence, for every small enough open ball B centred at 0 in \mathbb{C}^2 , the leaves of the foliation of $U := B \setminus \{0\}$ generated by $C_*X_H|_U$ and $C_*X_K|_U$ are the level sets of the complex function $U \ni (x, y) \mapsto xy$.

2. PROOF OF THEOREM 2

Via a chart C as in Theorem 1, we may assume

$$(1) \quad \begin{aligned} (M, a) &= (\mathbb{C}^2, 0) \\ \omega(x, y) &= dx \wedge dy + d\bar{x} \wedge d\bar{y} \\ H(x, y) &= h(xy). \end{aligned}$$

In particular, (1) defines H in a neighbourhood of $\{xy = 0\}$, which we may assume of the form $\{|xy| < \delta\}$ and therefore saturated by the flow φ_H^t . We can restrict δ so that

$$\inf_{|z| < \delta} \Re \partial h(z) > 0$$

and extend K to the whole of $\{|xy| < \delta\}$ using the relation $\varphi_H^{t*}K = K$.

Lemma 1. *The commutation relation $\{H, K\} = 0$ implies that K has infinite contact along $\{xy = 0\}$ with a smooth function $k_0(xy)$. The hypothesis that $dX_K(a)$ is linearly independent from $dX_H(a)$ means that $dk_0(0)$ is linearly independent from $dh(0)$.*

Proof. The first assertion follows from two facts:

- it is true formally at 0, which can be checked directly;
- as $K(x, y)$ and $k_0(xy)$ have the same Taylor expansion at 0 and are first integrals of X_H , they have the same Taylor expansion along the union $\{xy = 0\}$ of its stable and unstable manifolds: this is a particular case of Théorème 1 in subsection 4.2.4 of [3].

The second assertion of Lemma 1 is obvious since X_K has infinite contact along $\{xy = 0\}$ with the Hamiltonian vector field associated to $k_0(xy)$, namely $\partial k_0(xy)(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}) + \bar{\partial} k_0(xy)(\bar{x} \frac{\partial}{\partial \bar{x}} - \bar{y} \frac{\partial}{\partial \bar{y}})$, whose differential $\partial k_0(0)(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}) + \bar{\partial} k_0(0)(\bar{x} \frac{\partial}{\partial \bar{x}} - \bar{y} \frac{\partial}{\partial \bar{y}})$ at 0 is independent from the differential $\partial h(0)(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}) + \bar{\partial} h(0)(\bar{x} \frac{\partial}{\partial \bar{x}} - \bar{y} \frac{\partial}{\partial \bar{y}})$ of X_H if and only if $dk_0(0)$ is independent from $dh(0)$. \square

As in [1, 2, 3], we now use the fact that the cylinder $\{|x| = 1, |y| < \delta\}$ is a quotient of $\{|xy| < \delta\} \setminus \{x = 0\}$ by the flow φ_H^t :

Lemma 2. *Setting $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, the mapping*

$$\begin{aligned} \mathbb{R} \times \mathbb{T} \times \{|z| < \delta\} &\xrightarrow{g_\delta} \{(x, y) \in \mathbb{C}^2 : |xy| < \delta\} \setminus \{x = 0\} \\ (t, \theta, z) &\longmapsto \varphi_H^t(e^{i\theta}, ze^{-i\theta}) = (e^{t\partial h(z)+i\theta}, ze^{-t\partial h(z)-i\theta}) \end{aligned}$$

is a diffeomorphism, such that

$$\begin{aligned} g_\delta^* H &= h(z) \\ g_\delta^* \omega &= dt \wedge dh(z) + d\theta \wedge d(iz - i\bar{z}). \end{aligned}$$

Proof. The equation $g_\delta(t, \theta, z) = (x, y) \in \{|xy| < \delta\} \setminus \{x = 0\}$ is equivalent to $z = xy$, $e^{t \operatorname{Re} \partial h(z)} = |x|$ and $e^{i\theta} = xe^{-t\partial h(z)}$, showing that g_δ is a diffeomorphism; it does satisfy

$$\begin{aligned} g_\delta^* H &= h(e^{t\partial h(z)+i\theta} z e^{-t\partial h(z)-i\theta}) = h(z) \\ g_\delta^* \omega &= d(e^{t\partial h(z)+i\theta}) \wedge d(ze^{-t\partial h(z)-i\theta}) + d(e^{t\bar{\partial} h(z)-i\theta}) \wedge d(\bar{z}e^{-t\bar{\partial} h(z)+i\theta}) \\ &= d(t\partial h(z) + i\theta) \wedge dz + d(t\bar{\partial} h(z) - i\theta) \wedge d\bar{z} \\ &= td(dh)(z) + dt \wedge dh(z) + d\theta \wedge d(iz - i\bar{z}). \end{aligned}$$

Lemma 3. *Taking $E : z \mapsto h(z)$ and $I : z \mapsto iz - i\bar{z}$ as (real) coordinates on $\{|z| < \delta\}$, one has*

$$\begin{aligned} g_\delta^* \omega &= dt \wedge dE + d\theta \wedge dI \\ g_\delta^* H &= E \\ g_\delta^* K &= f(\theta, E, I). \end{aligned}$$

There exist an open subset $V \ni 0$ of $\{|z| < \delta\}$ and smooth coordinates $t_1(t, \theta, E, I) \in \mathbb{R}$, $\theta_1(\theta, E, I) \in \mathbb{T}$, $E_1 = E \in \mathbb{R}$, $I_1(\theta, E, I) \in \mathbb{R}$ on $\mathbb{R} \times \mathbb{T} \times V$ with the following properties:

- i) *The function $t_1 - t$ does not depend on t .*
- ii) *The functions $t_1 - t$, $\theta_1 - \theta$, $I_1 - I$ of (θ, E, I) have infinite contact with 0 along $\mathbb{T} \times \{(0, 0)\}$.*
- iii) *They satisfy*

$$\begin{aligned} g_\delta^* \omega &= dt_1 \wedge dE + d\theta_1 \wedge dI_1 \\ g_\delta^* H &= E \\ g_\delta^* K &= f_1(E, I_1). \end{aligned}$$

Proof. The formulae $g_\delta^* \omega = dt \wedge dE + d\theta \wedge dI$ and $g_\delta^* H = E$ are just a rephrasing of Lemma 2. The relation $g_\delta^* K = f(\theta, E, I)$ expresses the fact that the Lie derivative of $g_\delta^* K$ with respect to $g_\delta^* X_H = \frac{\partial}{\partial t}$ is zero.

The symplectic transformation $(t, \theta, E, I) \mapsto (t_1, \theta_1, E_1, I_1)$ is defined by a smooth generating function $R : \mathbb{T} \times W \rightarrow \mathbb{R}$ for some open subset $W \ni 0$ of \mathbb{R}^2 , meaning that the graph of the transformation is the set of those $((t, \theta, E, I), (t_1, \theta_1, E_1, I_1))$ such that $t_1 = t + \partial_{E_1} R(\theta, E_1, I_1)$, $\theta_1 = \theta + \partial_{I_1} R(\theta, E_1, I_1)$, $E = E_1$ and $I = I_1 + \partial_\theta R(\theta, E_1, I_1)$.

Condition (ii) will be satisfied if and only if R has infinite contact with 0 along $\mathbb{T} \times \{(0, 0)\}$. In order to fulfil it, we have to take care of

the only nontrivial requirement, namely $g_\delta^* K = f_1(E, I_1)$, which writes

$$f(\theta, E, I_1 + \partial_\theta R(\theta, E, I_1)) = f_1(E, I_1),$$

that is

$$(2) \quad \partial_\theta R(\theta, E, I_1) = f_{\theta, E}^{-1}(f_1(E, I_1)) - I_1,$$

where $f_{\theta, E}(I) := f(\theta, E, I)$: indeed, every $f_{\theta, E} : (\mathbb{R}, 0) \rightarrow \mathbb{R}$ with E small enough is a local diffeomorphism since, by Lemma 1, f has infinite contact along $\mathbb{T} \times \{(0, 0)\}$ with a function of (E, I) whose partial with respect to I at $(0, 0)$ is nonzero.

For the same reason, the averaged map $\frac{1}{2\pi} \int_0^{2\pi} f_{\theta, E}^{-1} d\theta$ is a local diffeomorphism $\tilde{f}_E^{-1} : (\mathbb{R}, 0) \rightarrow \mathbb{R}$ for small enough E . Now, as we need R to be 2π -periodic with respect to θ , we must have $\int_0^{2\pi} \partial_\theta R(\theta, E, I_1) d\theta = 0$; it follows that (2) will be satisfied if and only if

$$f_1(E, I_1) := \tilde{f}_E(I_1),$$

hence our lemma with $R(\theta, E, I_1) := \int_0^\theta (f_{\tau, E}^{-1}(\tilde{f}_E(I_1)) - I_1) d\tau$, which is readily seen to have infinite contact with 0 along $\mathbb{T} \times \{(0, 0)\}$. \square

End of the proof. Let $\Psi_\delta : \mathbb{R} \times \mathbb{T} \times \{|z| < \delta\} \rightarrow \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2$ be the embedding $(t, \theta, z) \mapsto (t, \theta, h(z), iz - i\bar{z})$ and let Φ denote the symplectic transformation whose components are $t_1(t, \theta, E, I)$, $\theta_1(\theta, E, I)$, E and $I_1(\theta, E, I)$. Then $\mathring{C} := g_\delta \circ \Psi_\delta^{-1} \circ \Phi \circ \Psi_\delta \circ g_\delta^{-1}$ has the following properties:

- a) it is an embedding of $\{(x, y) \in \mathbb{C}^2 : xy \in U, x \neq 0\}$ into $\{(x, y) \in \mathbb{C}^2 : x \neq 0\}$ for some open subset $U \ni 0$ of \mathbb{C} ;
- b) it preserves $\omega = dx \wedge dy + d\bar{x} \wedge d\bar{y}$ and H ;
- c) it satisfies $\mathring{C}_* K(x, y) = k(xy)$, where $k(z) := f_1(h(z), iz - i\bar{z})$;
- d) it has infinite contact with the identity along the unstable manifold $\{y = 0\}$ of X_H .

The following lemma completes the proof:

Lemma 4. *The map \mathring{C} extends by continuity to a smooth embedding C of the whole of $\{(x, y) \in \mathbb{C}^2 : xy \in U\}$ into \mathbb{C}^2 having infinite contact with the identity along $\{xy = 0\}$; by (b), C preserves ω and H ; by (c), it satisfies $C_* K(x, y) = k(xy)$.*

This follows from Théorème 5 in [2], a particular case of a result playing a central role in [3], namely Théorème 2 in subsection 4.2.3. At the referee's suggestion, a direct proof is sketched in Appendix A.

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APPENDIX A. PROOF OF LEMMA 4

By (d), for all $(X, Y) \in Q := \{(x, y) \in \mathbb{C}^2 : xy = U, |x| = 1\}$ and every integer $k > 1$, Taylor's formula yields

$$\begin{aligned} \mathring{C}(X, Y) - (X, Y) &= \mathring{C}(X, Y) - \mathring{C}(X, 0) - \partial_2 \mathring{C}(X, 0)Y \\ &= \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \partial_2^k \mathring{C}(X, tY) Y^k dt; \end{aligned}$$

hence, there exist $\delta_k > 0$ and $c_k > 0$ such that

$$(3) \quad \forall (X, Y) \in Q \quad |Y| < \delta_k \Rightarrow |\mathring{C}(X, Y) - (X, Y)| \leq c_k |Y|^k.$$

Now, the cylinder Q is a quotient of $\tilde{U} := \{(x, y) \in \mathbb{C}^2 : xy \in U, x \neq 0\}$ by the flow: indeed, the relation $\varphi_H^t(x, y) \in Q$ writes $|xe^{t\partial h(xy)}| = 1$, i.e. $t = \tau(x, y)$, where

$$\tau(x, y) := \frac{-\ln|x|}{\Re \partial h(xy)}$$

is well-defined since we have $|xy| < \delta$ and therefore $\Re \partial h(xy) > 0$.

Hence, the commutation relation $\varphi_H^t \circ \mathring{C} = \mathring{C} \circ \varphi_H^t$ equivalent to (b) implies that \mathring{C} is obtained from its restriction to Q by the formula

$$(4) \quad \mathring{C}(x, y) = \varphi_H^{-\tau(x, y)} \left(\mathring{C}(\varphi_H^{\tau(x, y)}(x, y)) \right).$$

We now prove that \mathring{C} extends continuously by the identity map on $\{x = 0\}$: given a sequence (x_n, y_n) in \tilde{U} converging to $(0, y_\infty)$, if we set

$$(5) \quad (x'_n, y'_n) = \varphi_H^{\tau(x_n, y_n)}(x, y) = (x_n e^{\tau(x_n, y_n) \partial h(x_n y_n)}, y_n e^{-\tau(x_n, y_n) \partial h(x_n y_n)})$$

then, by definition,

$$|x'_n| = 1, \quad |y'_n| = |x_n y_n| \quad \text{and} \quad x'_n y'_n = x_n y_n;$$

hence, by (3), for every integer $k > 1$ there exists $n_k \in \mathbb{N}$ such that $(x''_n, y''_n) := \mathring{C}(x'_n, y'_n)$ satisfies

$$(6) \quad n \geq n_k \Rightarrow |(x''_n, y''_n) - (x'_n, y'_n)| \leq c_k |x_n y_n|^k.$$

By (4), we have

$$\mathring{C}(x_n, y_n) = \left(x''_n e^{-\tau(x_n, y_n) \partial h(x''_n, y''_n)}, y''_n e^{\tau(x_n, y_n) \partial h(x''_n, y''_n)} \right);$$

since $\partial h(x''_n, y''_n) = \partial h(x'_n, y'_n) + O(|x_n y_n|^k) = \partial h(x_n, y_n) + O(|x_n y_n|^k)$ by (6), it does follow that, by (6)-(5),

$$\mathring{C}(x_n, y_n) \sim \left(x'_n e^{-\tau(x_n, y_n) \partial h(x_n, y_n)}, y'_n e^{\tau(x_n, y_n) \partial h(x_n, y_n)} \right) = (x_n, y_n)$$

when $n \rightarrow \infty$.

The somewhat tedious explicit proof that $\partial_x^\alpha \partial_{\bar{x}}^\beta \partial_y^\gamma \partial_{\bar{y}}^\delta \mathring{C}$ extends continuously by $\partial_x^\alpha \partial_{\bar{x}}^\beta \partial_y^\gamma \partial_{\bar{y}}^\delta \text{Id}$ on $\{x = 0\}$ for every $(\alpha, \beta, \gamma, \delta) \in \mathbb{N}^4$ is entirely similar and left to the brave reader.

APPENDIX B. A NON-NORMALIZABLE SMOOTH “NODE-NODE” BY JIANG KAI

On $\mathbb{R}^4 = O_{x_1 y_1 x_2 y_2}$ equipped with the standard symplectic form $dx \wedge dy := dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ we consider the three Hamiltonians

$$\begin{aligned} Q_1(x, y) &= x_1 y_1 + x_2 y_2 \\ Q_2(x, y) &= x_1 y_1 - x_2 y_2 \\ H(x, y) &= Q_1(x, y) + \begin{cases} \varphi(x_1 y_1) & \text{for } x_1 \geq 0 \\ -\varphi(x_1 y_1) & \text{for } x_1 \leq 0, \end{cases} \end{aligned}$$

where

$$\varphi(t) := \begin{cases} e^{-\frac{1}{t^2}} & \text{for } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

A *normal form of node-node type* is a completely integrable Hamiltonian system generated by Hamiltonians of the form

$$\begin{aligned} N_1(x, y) &= Q_1(x, y) + R_1(x_1 y_1, x_2 y_2) \\ N_2(x, y) &= Q_2(x, y) + R_2(x_1 y_1, x_2 y_2) \end{aligned}$$

where the C^∞ functions R_1, R_2 vanish to order 1 at $0 \in \mathbb{R}^2$.

In contrast with the “focus-focus”, one has the following

Proposition. *The completely integrable Hamiltonian system generated by H and Q_2 can not be symplectically C^1 -conjugated to a normal form in any neighbourhood of 0 in \mathbb{R}^4 .*

Proof. Near the origin, the two maps (N_1, N_2) and (H, Q_2) from \mathbb{R}^4 to \mathbb{R}^2 have the same singular locus $\{x_2 = y_2 = 0\} \cup \{x_1 = y_1 = 0\}$; on the component $\{x_2 = y_2 = 0\}$, the difference $H - Q_2$ is positive for $x_1 > 0$ and negative for $x_1 < 0$, whereas the two subsets $N_1 - N_2 > 0$ and $N_1 - N_2 < 0$ are invariant by symmetry with respect to the origin. \square

Note. The idea of this counterexample is due to Eliasson, but it seems that many of his readers did not extract it from [4]. The difference with the focus-focus is that, in the node-node case, the germ at $0 \in \mathbb{R}^4$ of the singular locus is not the union of the “strongly invariant manifolds” [3] of the \mathbb{R}^2 -action germ ρ^t generated by the two Hamiltonians (a s.i.m. is a manifold germ which is the unstable manifold of the diffeomorphism germ ρ^t for some $t \in \mathbb{R}^2$).

UNIVERSITÉ PARIS DIDEROT - PARIS 7, UFR DE MATHÉMATIQUES, CASE 7012,
BÂTIMENT CHEVALERET, F-75205 PARIS CEDEX 13.

E-mail address: chaperon@math.jussieu.fr, jiangk@math.jussieu.fr