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# Applications of some properties of the canonical module in computational projective algebraic geometry

MARC CHARDIN

*Institut de Mathématiques, UMR 7586 du CNRS*

*Université Pierre et Marie Curie*

*Case 247, 4, place Jussieu, F–75252 Paris CEDEX 05*

*chardin@math.jussieu.fr*

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*Introduction.* The aim of this article is to show how one may use interesting properties of the canonical module for computational applications.

It seems that the ideas we develop here are not used in computational algebraic geometry today. We do not claim any major improvement on the theoretical complexity of the problems we address, however we have the feeling that this alternative approach for projective schemes is quite promising, for two reasons. Firsts the algorithms are extremely simple, and everything may be computed either via Gröbner basis computations or linear algebra in the polynomial ring (which may have some advantages, as pointed out by J.-C. Faugère). Secondly, the complexity of the algorithms are essentially controlled by the complexity of the output : for example the Castelnuovo-Mumford regularity of the top dimensional part and of its dualizing module, in the computation of the top dimensional component. We give here some bounds for the Castelnuovo-Mumford regularity in small dimension; much more is done in [6] giving greater evidence for "reasonable" bounds on the regularity of the top dimensional component, and thereby on the complexity of our algorithms.

In a first part we recall, sometimes with (sketch of) proofs, the classical results that we will need in the sequel.

We then give in part 2 a first example of applications by showing how a result of Hochster (and local duality) gives an easy way to compute the dimension of a projective scheme from generators of the defining ideal. This is a classical problem, see for example [11] or [15], which also treats the more delicate affine case.

In part 3 we present a method to compute the Hilbert polynomial of Cohen-Macaulay schemes for which nice vanishing theorems are known for the dualizing module. The re-

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mark is that graded pieces of the dualizing module are easily accessible by linear algebra. This is done *via* the Koszul homology of any set of polynomials that defines a scheme whose top dimensional part coincides with the scheme you investigate on.

A particular, but interesting, case is given by almost complete intersection whose top dimensional component have a cohomologically nice dualizing module (*e.g.* the ideal given by $\operatorname{codim}(\mathcal{Y})+1$ general elements in the ideal of a smooth scheme $\mathcal{Y}$), in this case we remark that the Hilbert function of the almost complete intersection has a very good regularity, so that it is computable in low degree, and one can recover the Hilbert polynomial of the top dimensional component very easily. The most surprising fact is perhaps that we do not need to know the defining equations! Not even equations defining it up to saturation, it should just coincide at the top dimensional part. Of course we do not have all the information (the Hilbert function for example), but what to do without a defining ideal, even using the nice result of Bertram-Ein-Lazarsfeld on the Castelnuovo-Mumford regularity of $\mathcal{Y}$?

In section 4, we give a new method to describe the top dimensional part of a scheme that requires only linear algebra computations in low degree in the polynomial ring. The top dimensional components are described in a new way that we will now describe. First we compute a set of forms contained in the ideal that forms a complete intersection of the same dimension. Then for each associated prime $\mathfrak{P}_i$ of maximal dimension, we give a form $f_i$ such that an element $x$ is in $\mathfrak{P}_i$ if and only if $xf_i$ is in the complete intersection. This has a very reasonable complexity because the sum of the degrees of the elements in the complete intersection (which are of degree at most the ones of the generators) bounds at the same time the degrees of the $f_i$'s and the Castelnuovo-Mumford regularity of the complete intersection (that agrees with the maximal degree of a generator of a Gröbner basis in general coordinates). Note that it is only a set theoretical description of the top dimensional component.

Also, we describe an algorithm to compute the top dimensional part of a scheme, which is controlled by intrinsic invariants of the top dimensional part itself (*see* 4.6) and a very simple algorithm that computes a subscheme of the top dimensional part of the same dimension (therefore the top part itself if it is irreducible and reduced) with a very good control on degrees (4.7). These algorithms are important simplifications of the algorithm 1.4 in [9], however they are essentially based on the same mathematical framework. We also give in 4.8 a way of determining if the top dimensional part is smooth, with a quite reasonable control on the degree of the computation.

We then give some regularity bounds in low dimension using liaison. This part is the first step of a more general study that we investigate further with Bernd Ulrich in [6], generalizing both the results of this article and the one of [2]. The basic idea is to link the scheme you investigate on with a scheme for which nice vanishing theorems are known (here reduced curves or normal surfaces). The bound for curves (Theorem 5.1) only requires quite mild hypotheses. We hope for a similar result for surfaces, Theorem 5.5 is a first step in this direction.

## 1. Some standard properties of the canonical module

In this paragraph, we gather some classical properties of the canonical module, and sometimes gives (sketch of) proofs or else refer to the literature.

Let $k$ be a field and set $A := k[X_0,\dots,X_n]$ so that $\mathbf{P}_n(k) = \operatorname{Proj}(A)$ for the standard

grading $\deg X_i = 1$. A standard algebra is, by definition, the quotient of such a ring by a homogeneous ideal. We will note $\mathfrak{m} := (X_0, \ldots, X_n)$ the homogeneous maximal ideal of $A$, and denote by the same letter its image in the homogeneous quotients of A, if no confusion is possible. The letter $\mathfrak{a}$ will denote a homogeneous ideal of $A$ and $\mathbf{a} = (a_1, \ldots, a_s)$ a $s$-tuple of forms generating $\mathfrak{a}$. Also $H^i_{\mathfrak{m}}(M)$ is the $i$-th local cohomology module of the $A$-module $M$ with support in $\mathfrak{m}$ and $H^i(\mathbf{a}; M)$ the Koszul cohomology of the sequence $\mathbf{a}$ over $M$.

If $B = A/\mathfrak{a}$ is a standard algebra, we will define:

Definition 1.1.

$$\omega_B := \mathrm{Ext}^r_A(B, \omega_A),$$

*where* $r := \dim A - \dim B$ *and* $\omega_A = A[-n-1]$ *is the canonical (or dualizing) module of* $A$.

This module only depends on $B$. Let us set $\mathcal{Z} := \mathrm{Proj}(B)$ and $\mathcal{Y} := \mathrm{Proj}(A/\mathfrak{a}^{top})$ be the top dimensional part of $\mathcal{Z}$.

Fact 1.2. *The associated primes of* $\omega_B$ *are the ones of minimal codimension of* $B$*, and* $\omega_B = \omega_{A/\mathfrak{a}} \simeq \omega_{A/\mathfrak{a}^{top}}$.

Proof. This is easily seen from the definition above, as $A$ is Cohen-Macaulay. □

Fact 1.3. $\mathrm{depth}(\omega_B) \geqslant \min\{2, \dim B\}$.

Proof. If $\dim B > 0$, notice that $\omega_B = \omega_{B/H^0_{\mathfrak{m}}(B)}$, so we may assume that $H^0_{\mathfrak{m}}(B) = 0$. Now if $x$ is non zero divisor in $B$, the long exact sequence of Ext's corresponding to $0 \to B \xrightarrow{\times x} B \to B/xB \to 0$ gives an injection

$$\omega_B/x\omega_B \hookrightarrow \omega_{B/xB}.$$

As $\omega_{B/xB}$ is unmixed, so is $\omega_B/x\omega_B$, and this proves our claim. □

Fact 1.4. *For all integer* $\nu$*,* $H^i_{\mathfrak{m}}(\omega_B)_\nu \simeq H^{i-1}(\mathcal{Y}, \omega_{\mathcal{Y}}(\nu))$ *if* $i \geqslant 2$ *and if* $\dim B \geqslant 2$*,* $(\omega_B)_\nu \simeq H^0(\mathcal{Y}, \omega_{\mathcal{Y}}(\nu))$.

Proof. The Čech complex computes sheaf cohomology, and 1.3 says that $H^0_{\mathfrak{m}}(B) = H^1_{\mathfrak{m}}(B) = 0$ if $\dim B \geqslant 2$, which implies the second statement. □

Fact 1.5. (local duality.) *There are natural graded isomorphisms of degree 0,*

$$\mathrm{Ext}^{r+i}_A(B, \omega_A) \simeq \mathrm{Hom}(H^{\dim B - i}_{\mathfrak{m}}(B), k),$$

*in particular*

$$(\omega_B)_\nu \simeq \mathrm{Hom}(H^{\dim B}_{\mathfrak{m}}(B)_{-\nu}, k).$$

Proof. See *e.g.* ([3], 3.6.19). □

Let us also recall some equivalent definitions of the Castelnuovo-Mumford regularity of a finitely generated graded $A$-module $M$,

Fact 1.6. *The invariant* $\mathrm{reg}(M)$ *may be defined as,*

*(i)* $\mathrm{reg}(M) = \max\{\deg m,\ m \in \mathfrak{B}(M)\}$, *where* $\mathfrak{B}(M)$ *is a Gröbner basis of* $M$ *for rev-lex in general coordinates.*

*(ii)* $\mathrm{reg}(M) = \max_{i,j}\{\deg S_{i,j} - i\}$, *where the* $S_{i,j}$*'s forms a minimal set of generators of the* $i$*-th module of syzygies of* $M$.

*(iii)* $\mathrm{reg}(M) = \min\{\nu,\ \forall i,\ [H^i_{\mathfrak{m}}(M)]_{>\nu-i} = 0\}$.

*(iv)* $\mathrm{reg}(M) = \min\{\nu,\ \forall i,\ \ [H^i(f_1,\dots,f_t;M)]_{>\nu-i} = 0\}$, *if* $M/(f_1,\dots,f_t)M$ *is* $\mathfrak{m}$*-primary.*

*(v)* $\mathrm{reg}(M) = \min\{\nu,\ \mathrm{Ext}^i_A(M,A)_{<-\nu-i} = 0\}$.

Proof. The equivalence of (i) and (ii) is proved in [1], and may be derived from the equivalence of (ii) and (iv) taking $t := \dim B$ and general linear forms for the $f_i$'s. Choosing $t := n+1$ and $f_i := X_i$ the equivalence of (ii) and (iv) comes from the self-duality of the Koszul complex, as the dual computes the Tor modules. Local duality proves that (iii) and (v) are the same. Now to prove that (iii) and (iv) are the same, it suffices to study the two spectral sequences arising from the double complex $\mathcal{C}^\bullet H^\bullet(f_1,\dots,f_t;M)$, that allows to compare Čech and Koszul cohomologies, as it is done *e.g.* in [5] or [14]. □

Fact 1.7. *Let* $\mathcal{Y} = \mathrm{Proj}(B)$ *be a unmixed projective subscheme of* $\mathbf{P}_n(k)$ *and assume that one of the following conditions is verified,*

*(i)* $\dim \mathcal{Y} = 0$,

*(ii)* $\dim \mathcal{Y} = 1$ *and* $\mathcal{Y}$ *is reduced over* $\overline{k}$,

*(iii)* $\dim \mathcal{Y} = 2$, $\mathrm{Char}(k) = 0$ *and* $\mathcal{Y}$ *is normal,*

*(iv)* $\mathcal{Y}$ *is smooth and* $\mathrm{Char}(k) = 0$,

*then* $H^i_{\mathfrak{m}}(\omega_B)_\nu = 0$ *for every* $\nu > 0$ *and every* $i$, *so that*

$$0 \leqslant \mathrm{reg}(\omega_B) \leqslant \dim B.$$

Proof. One always have $\mathrm{reg}(\omega_B) \geqslant 0$ (*see* remark 1.11 below). We may assume that $H^0_{\mathfrak{m}}(B) = 0$. The vanishing result is trivial in cases (i) and (ii), as $\omega_B$ is Cohen-Macaulay in this case. In case (iii) the vanishing of $H^1(\mathcal{Y},\omega_{\mathcal{Y}}(\nu))$ in positive degrees is due to Mumford ([16]), and case (iv) is Kodaira vanishing theorem.

If $\dim \mathcal{Y} = 0$, $B$ is Cohen-Macaulay as well as $\omega_B$ and therefore the regularity of $\omega_B$ is the one of its Hilbert function. Now,

$$H^1_{\mathfrak{m}}(B) \simeq \mathrm{Hom}_A(\omega_B, k)$$

by local duality, so that it remains to prove that $H^1_{\mathfrak{m}}(B)_\nu$ is of dimension $\deg B$ for $\nu < 0$ and this comes from the exact sequence

$$0 \to B_\nu \to H^0(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}(\nu)) \to H^1_{\mathfrak{m}}(B)_\nu \to 0$$

as $B_\nu = 0$ for $\nu < 0$ and $\dim_k H^0(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}(\nu)) = \deg B$ for all $\nu$.

If $\dim \mathcal{Y} > 0$, as $\mathcal{Y}$ is reduced over $\overline{k}$, one has $H^0(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}(\nu)) = 0$ for $\nu < 0$ so that, by local duality (note that $\mathcal{Y}$ is Cohen-Macaulay), $H^{\dim \mathcal{Y}}(\mathcal{Y},\omega_{\mathcal{Y}}(\nu)) = 0$ for $\nu > 0$, and this proves the $(\dim B)$-regularity of $\omega_B$. □

Remark 1.8. *The previous result is also true if* $\mathcal{Y}$ *has rational singularities, as Kodaira vanishing is true in this case.*

Remark 1.9. *If $\mathcal{Y}$ is arithmetically Cohen-Macaulay, then* $\operatorname{reg}(\omega_B) = \dim B$, *as one easily sees by reduction to the zero-dimensional case. This of course includes the case* $\dim \mathcal{Y} = n-1$.

Fact 1.10. $H_{\mathfrak{m}}^{\dim B}(B)_{-\dim B} \neq 0$ *or equivalently* $(\omega_B)_{\dim B} \neq 0$.

Proof. (From [13], see also [3], 9.2.1). Let $B = A/\mathfrak{a}$, $d := \dim B$ and $r := \operatorname{codim} B$. The proof has two steps : reduction to the case where $\operatorname{Char}(k)=p>0$ and the proof in characteristic $p$.

If $\operatorname{Char}(k)=0$, choose a finite family $G$ of generators of $\mathfrak{a}$ so that $(a_1,\dots,a_r)$ for $a_i \in G$ forms a maximal regular sequence. Let $S \subset k$ be the finitely generated $\mathbf{Z}$-algebra generated by the coefficients of the elements of $G$. We may assume that $k = \operatorname{Frac}(S)$. In the algebra $S$, reduction modulo $p$ is well defined and we choose a prime $p \in \mathbf{Z}$ so that computing Gröbner bases of $\mathfrak{a}$ and $(a_1,\dots,a_r) : \mathfrak{a}$, all the computations staying in $S$, no leading term during these two computations is 0 modulo $p$. For such a $p$ reducing modulo $p$ the generators gives an ideal $\mathfrak{a}_p$ defined over $S/(p)$ that has same Hilbert function (therefore same dimension) and whose dual has the same property, due to the isomorphism $\omega_B \simeq ((a_1,\dots,a_r) : \mathfrak{a})/(a_1,\dots,a_r)[d_1+\cdots+d_r-n-1]$.

If $d := \dim B$ one has $H_{\mathfrak{m}}^d(B) \neq 0$ and if $\mathfrak{l} = (l_1,\dots,l_d)$, with $l_i$ a linear form, is such that $\dim B/\mathfrak{l} = 0$, one has a degree 0 graded isomorphism $H_{\mathfrak{m}}^d(B) \simeq H_{\mathfrak{l}}^d(B)$. Using the identification (or definition) $H_{\mathfrak{l}}^d(B) = \varinjlim_N B/(l_1^N,\dots,l_d^N)$ this means that there exists $N$ such that $1 \in B/(l_1^N,\dots,l_d^N)$ (that generates this module) is not zero in the limit $H_{\mathfrak{l}}^d(B)$, and so this remain true for every $N' \geqslant N$. Taking into account the morphisms in the inductive limit, this means that $\frac{1}{(l_1\cdots l_d)^N}$ is a non zero element in $H_{\mathfrak{l}}^d(B)$. Now this implies that $\frac{1}{l_1\cdots l_d} \neq 0$ as if this was the case, we would have a relation $(l_1\cdots l_d)^t = \sum_{i=1}^d a_i l_i^{t+1}$ and rising it to the $N' = p^e$-th power for $p^e \geqslant N$ gives

$$(l_1\cdots l_d)^{N't} = \sum_{i=1}^d a_i^{N'} l_i^{N'(t+1)}$$

that says $\frac{1}{(l_1\cdots l_d)^{N'}}$ is zero in $H_{\mathfrak{l}}^d(B)$, a contradiction. $\square$

Remark 1.11. *Fact 1.10 implies in particular that* $\operatorname{reg}(\omega_B) \geqslant 0$.

Fact 1.12. *Let $\mathcal{Y}$ be a Cohen-Macaulay unmixed subscheme of $\mathbf{P}_n(k)$ of dimension $d$, then*

$$P_{\mathcal{Y}}(\nu) = (-1)^d P_{\omega_{\mathcal{Y}}}(-\nu).$$

Proof. This is a direct consequence of (the geometric version of) local duality,

$$P_{\mathcal{Y}}(\nu) = \sum_{i=0}^d (-1)^i h^i(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\nu)) = \sum_{i=0}^d (-1)^i h^{d-i}(\mathcal{Y}, \omega_{\mathcal{Y}}(-\nu)) = (-1)^d P_{\omega_{\mathcal{Y}}}(-\nu),$$

where $h^j(\mathcal{Y}, \_\_) := \dim_k H^j(\mathcal{Y}, \_\_)$. $\square$

Let $\mathbf{a} := (a_1,\dots,a_s)$ be an $s$-tuple of forms of degrees $d_1,\dots,d_s$ generating an homogeneous ideal $\mathfrak{a}$. The $i$-th cohomology module of the (graded) Koszul complex $K^\bullet(\mathbf{a}\,;A)$

is essentially independent of the generators (*see* e.g. [3] 1.6.8 and 1.6.21 for precise statements), we will denote it by $H^i(\mathbf{a}\,;A)$, fixing $K^0(\mathbf{a}\,;A) := \mathrm{Hom}_A(A,A) = A$ to determine the grading. Note that there is a degree 0 graded isomorphism $K^i(\mathbf{a}\,;A) \simeq K_{s-i}(\mathbf{a}\,;A)[d_1+\cdots+d_s]$, if one sets $K_0(\mathbf{a}\,;A) := A$.

The $a$-invariant of a graded standard algebra $B$ may be defined in several ways. Let us state some equivalent definitions, with notation as above,

Fact 1.13. *Let $B = A/\mathfrak{a}$ be a standard graded algebra, the $a$-invariant of $B$ may alternatively be defined as one of the following four numbers,*

*(1)* $a(B) := \max\{\nu \mid H^{\dim B}_{\mathfrak{m}}(B)_\nu \neq 0\}$,

*(2)* $a(B) := -\min\{\nu \mid (\omega_B)_\nu \neq 0\}$,

*(3) if $b_1,\dots,b_r \in \mathfrak{a}$ is a (maximal) regular sequence in $A$, $\mathfrak{b} := (b_1,\dots,b_r)$ and $e_i := \deg b_i$,*

$$a(B) := -\min\{\nu \mid (\mathfrak{b} : \mathfrak{a})_\nu \neq \mathfrak{b}_\nu\} + e_1 + \cdots + e_r - n - 1,$$

*(4)* $a(B) := -\min\{\nu \mid H^r(\mathbf{a}\,;A)_\nu \neq 0\} - n - 1$.

Proof. The equivalence of (1) and (2) is a direct consequence of the local duality theorem for graded modules (1.5). Now (2), (3) and (4) are the same due to the following graded isomorphisms of degree 0,

$$\begin{aligned} H^r(\mathbf{a}\,;A) &\simeq ((\mathfrak{b} : \mathfrak{a})/\mathfrak{b})[-e_1\cdots-e_r] \\ &\simeq \mathrm{Hom}_A(A/\mathfrak{a}, A/\mathfrak{b})[-e_1\cdots-e_r] \\ &\simeq \mathrm{Ext}^r_A(A/\mathfrak{a}, A) \\ &\simeq \mathrm{Ext}^r_A(A/\mathfrak{a}, \omega_A[n+1]) \\ &\simeq \omega_B[-n-1], \end{aligned}$$

which are classical (*see*, e.g. [3] 1.6.16 and 1.2.4). □

Fact 1.14. ([5], Cor. 2). *Let $\mathfrak{a}$ be an ideal of $A$ generated by forms of degrees $d_1 \geqslant \cdots \geqslant d_s$ and $\mathfrak{a}^{<k>}$ the intersection of isolated primary components of $\mathfrak{a}$ of codimension $k$. Then*

$$a(A/\mathfrak{a}^{<k>}) \leqslant d_1 + \cdots + d_k - n - 1.$$

## 2. Computation of the dimension of a projective scheme

The first application concerns the following question (notations as in §1),

$$(Q_c) \quad \text{Is } \mathrm{codim}(B) \geqslant c\ ?$$

and relies on the following lemma,

Lemma 2.1. *Let $k$ be a field and $a_i$ for $i = 1,\dots,s$ be homogeneous polynomials in $A := k[X_0,\dots,X_n]$. Let $\mathfrak{a} = (a_1,\dots,a_s) \subseteq A$ be the ideal they generate in $A$ and $B := A/\mathfrak{a}$. For $c \leqslant \min\{s, n+1\}$, the following are equivalent,*

*(1)* $\mathrm{codim}(B) \geqslant c$,

*(2)* $H^{c-1}(\mathbf{a}\,;A) = 0$,

*(3)* $H^{c-1}(\mathbf{a}\,;A)_{-c+1} = 0$,

*(4)* $\mathrm{rk}(d^{c-1}_{-c+1}) = \mathrm{genrk}(d^{c-1}_{-c+1})$,

*where $d^i_\nu$ is the degree $\nu$ part of the $i$-th differential of $K^\bullet(\mathbf{a}\,;A)$ and* $\mathrm{genrk}(d^i_\nu)$ *is the rank of $d^i_\nu$ for generic polynomials of the same degrees as the $a_i$'s.*

Proof. Let $r := \dim A - \dim B = \mathrm{codim}(B)$. It is clear that (1) and (2) are equivalent ($A$ is Cohen-Macaulay) and that (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Assume (4). If $r = c-1$, $H^{c-1}(\mathbf{a}\,;A)_{-c+1} \neq 0$ by fact 1.10, so that $\mathrm{rk}(d^{c-2}_{-c+1}) \neq \mathrm{genrk}(d^{c-2}_{-c+1})$, but this implies that $H^{c-2}(\mathbf{a}\,;A)_{-c+1} \neq 0$, which contradicts $r = c-1$.

Now remark that if $r < c-1$, replacing $a_1$ by a generic polynomial of the same degree, and extending the ground field, we have an ideal of codimension equal either to $r$ or $r+1$ and such that $\mathrm{rk}(d^{c-1}_{-c+1}) = \mathrm{genrk}(d^{c-1}_{-c+1})$. Iterating this process we will arrive at the case where $r = c-1$, because in the generic case $r = \min\{s, n+1\} \geqslant c$. Therefore $r < c-1$ is also impossible, so (1) holds. $\square$

Remark 2.2. *From the acyclicity properties of the generic Koszul complex (see e.g.* [14]*), one easily sees that, for $i \leqslant n+1$, the generic rank is given by the following generating function,*

$$\sum_\nu \mathrm{genrk}(d^i_\nu)t^\nu = \sum_{j=0}^{i}(-1)^{i-j}G_j(t)$$

*with*

$$\sum_{j=0}^{s} G_j(t)u^j = \frac{1}{(1-t)^{n+1}}\prod_{i=1}^{s}(1+t^{-d_i}u).$$

*The matrix of $d^i_\nu$ is of size $g_i(\nu)\times g_{i+1}(\nu)$ with $G_j(t) = \sum_\nu g_i(\nu)t^\nu$, and one have*

$$g_c(\nu) = \sum_{i_1<\cdots<i_c}\binom{\nu+n-d_{i_1}-\cdots-d_{i_c}}{n}.$$

Theorem 2.3. *Let $R$ be a domain with fraction field $k$ and $a_i$ for $i = 1,\dots,s$ be homogeneous polynomials in $R[X_0,\dots,X_n]$ and let $\mathfrak{a} = (a_1,\dots,a_s) \subseteq A$ be the ideal they generate in $A = k[X_0,\dots,X_n]$. If $B = A/\mathfrak{a}$ the question $(Q_c)$ may be answered by the computation of the rank of one matrix of Sylvester type with entries in $R$ (either 0 or a coefficient of one of the $a_i$'s) of size bounded by*

$$\sum_{i_1<\cdots<i_c}\binom{d_{i_1}+\cdots+d_{i_c}-c+n}{n}$$

*where $d_i := \deg a_i$.*

If $d_i \leqslant d$ for all $i$, this bound is of order $s^c d^n$, and may be bounded by $O(s^c(3d)^n)$. The computation of the rank is of polynomial time (computations staying in $R$), as the characteristic polynomial (I do not know if there is an algorithm of better complexity for the rank than for the characteristic polynomial).

Proof. Let $r := \dim A - \dim B$, so that $(Q_c)$ holds if and only if $r \geqslant c$. Notice that $r \leqslant \min\{s, n+1\}$, so that we may assume that $c$ satisfies the same condition. By Lemma 2.1,

$$\mathrm{rk}(d^{c-1}_{-c+1}) = \mathrm{genrk}(d^{c-1}_{-c+1}) \Leftrightarrow (Q_c) \text{ holds true},$$

and the size of the matrix of $d^{c-1}_{-c+1}$ is given by 2.2, the theorem is now clear. $\square$

REMARK 2.4. *Using the acyclicity properties of the generic Koszul complex (see e.g.* [14]*) and the rigidity of Tor one easily sees that, for any* $\nu \in \mathbf{Z}$*,*

$$H^i(\mathbf{a}\,;A)_\nu \neq 0 \;\Rightarrow\; H^{i+1}(\mathbf{a}\,;A)_\nu \neq 0,\ \forall i < \min\{n, s-1\},$$

*which in some way explains the "propagation of non-acyclicity to the right" that is the key of Lemma 2.1.*

REMARK 2.5. *The hard point in the computation of the dimension is to prove a lower bound for the dimension. Reducing modulo p (if one works over the rationals) and adding m linear forms (even variables) to go down to a zero-dimensional standard algebra (something reasonable to test) gives immediately the bound* $\dim A/\mathfrak{a} \leqslant m$*. The point is that a proof of an inequality in the other direction is harder to obtain: one has to prove that reducing modulo p do not increase the dimension (in fact bound the number of "bad" p's) and that the* $m-1$ *"random" forms you choose for which the quotient is not of dimension zero are really random (so estimate the number of "bad" sets of forms).*

*With the method we suggest, one has to prove that the rank of a matrix (with coefficients in the base ring) is smaller than some value. This avoids taking random or generic linear forms, and therefore avoids change of coordinates, that may cost a lot in practice. However it seems to be now folklore that deterministic algorithms of essentially the same asymptotic complexity can be derived from the work of Giusti-Heintz ([11]) and Krick-Pardo ([15]).*

Note also the following consequence of Fact 1.10, that gives an easy way to produce a maximal regular sequence of elements in $A$, thereby computing the dimension. Having at hand a maximal regular sequence will be of importance in the next sections.

THEOREM 2.6. *Let* $B$ *be a standard graded Gorenstein algebra of positive dimension and* $N \geqslant \operatorname{reg}(B)$*. If* $f \in B$ *is an homogeneous element, the following are equivalent:*

*(1)* $f$ *is a non zero-divisor in* $B$*,*

*(2)* $B_N \xrightarrow{\times f} B_{N+\deg f}$ *is an injective map.*

PROOF. (1) obviously implies (2). Now if $f$ is a zero-divisor, then $\dim B/(f) = \dim B$ and the kernel of the map in (2) is isomorphic to $(\omega_{B/(f)})_{N-a(B)}$. As $a(B) = \operatorname{reg}(B) - \dim B$, this kernel is not zero by Fact 1.10. $\square$

## 3. Computation of the Hilbert polynomial of some Cohen-Macaulay schemes

We keep the setting and notations of §1. Let us first recall the following classical result,

THEOREM 3.1. *Assume that* $d_2 \geqslant \cdots \geqslant d_s \geqslant d_1$*. Then* $\mathcal{Z} = \emptyset$ *if and only if* $s \geqslant n+1$ *and* $\mathfrak{a}_\nu = A_\nu$ *for* $\nu = d_1 + \cdots + d_{n+1} - n$*.*

So testing if $\mathcal{Z} = \emptyset$ is done by linear algebra in quite low degree.

The aim of this section is to show that under geometric hypotheses on $\mathcal{Y}$, it is possible to compute the Hilbert polynomial of $\mathcal{Y}$ by linear algebra computations in low degree.

Let us recall the degree 0 graded isomorphisms,

$$H^i(\mathbf{a}\,;A) \simeq H_{s-i}(\mathbf{a}\,;A)[d_1+\cdots+d_s],$$

valid for every $i$, that comes from the corresponding isomorphisms of complexes (here we have set $K_0(\mathbf{a}\,;A)=A$ and $K^0(\mathbf{a}\,;A)=\mathrm{Hom}_A(A,A)=A$ to determine the gradings).

In the proof of fact 1.13, we have seen the following properties,

Lemma 3.2. *$H^i(\mathbf{a}\,;A)=0$ for $i<r$ and one has a graded isomorphism of degree 0,*

$$H^r(\mathbf{a}\,;A) \simeq \omega_B[n+1].$$

Proof. *Cf.* proof of fact 1.13. □

Theorem 3.3. *Assume that one of the hypotheses of 1.7 is satisfied. If $\dim \mathcal{Y}=0,1$ or $n-1$, $h^r(\nu):=\dim_k H^r(\mathbf{a}\,;A)_\nu$ is a polynomial function of $\nu$ for $\nu \geqslant -n$. In any case $h^r(\nu)$ is a polynomial function for $\nu \geqslant -r-1$.*

Proof. It suffices to notice that the regularity of the Hilbert function of a graded module $M$ is at most $\mathrm{reg}(M)-\mathrm{depth}(M)$, and apply 1.7 and 3.2. □

From the above theorems, the algorithm to compute the Hilbert polynomial is now clear,

Algorithm 3.4. *(1) Compute $h^i(\nu):=\dim_k H^i(\mathbf{a}\;;A)_\nu$ for $\nu=-i$ and increasing values of $i$, until the value is not 0, call $r$ this value and $d:=n-r$,*
*(2) If $r=0$, return "$\mathcal{Z}=\mathbf{P}_n$",*
*(3) If $r=n+1$, return "$\mathcal{Z}$ is empty",*
*(4) If $r=1$, return $(n-1,H(t))$ where*

$$H(t):=\binom{\nu+n}{n}-\binom{\nu+n-h^1(-1)}{n},$$

*(5) If $r=n$, return $(0,h^r(-n))$,*
*(6) If $1<r<n$, compute $h^r(-r-1+\nu)$ for $\nu=0,2,\cdots,d$ and the only polynomial $H(t)$ of degree $d$ such that $H(\nu+d)=h^r(-r-1+\nu)$ for $\nu=0,\ldots,n-r$, and return $(d,(-1)^dH(-t))$.*

This algorithm gives the Hilbert polynomial of $\mathcal{Y}$ if $\dim\mathcal{Y}=0$ or $d\geqslant n-1$ or $\mathcal{Y}$ verifies one of the hypotheses of fact 1.7. See 4.8 for an algorithm checking the smoothness.

If one have at hand a maximal regular sequence of elements in the ideal, then the following result gives perhaps a more efficient way for the same computation,

Theorem 3.5. *If $\mathfrak{a}=(a_1,\ldots,a_s)$ defines a subscheme of codimension $r$ whose top dimensional component $\mathcal{Y}$ verifies the hypotheses of 1.7, and $a_1,\ldots,a_r\in\mathfrak{a}$ is a regular sequence, setting $\mathfrak{b}:=(a_1,\ldots,a_r)$, the rank of the kernel of the map*

$$\begin{array}{rcl}(A/\mathfrak{b})_\nu & \longrightarrow & \bigoplus_{j=r+1}^s (A/\mathfrak{b})_{\nu+d_j} \\ x & \longmapsto & (xa_{r+1},\dots,xa_s)\end{array}$$

*is a polynomial function in $\nu$ for $\nu \geqslant d_1+\cdots+d_s-s$, if $P(\nu)$ is this function, one has*

$$P_{\mathcal{Y}}(\nu) = (-1)^d P(-\nu+\delta)$$

*where $\delta := d_1+\cdots+d_r-n-1$.*

Proof. One has a graded isomorphisms of degree 0,

$$\omega_B[n+1] \simeq H^r(\mathbf{a}\,;A) \simeq H^0(\mathbf{a}\,;A/\mathfrak{b})[d_1+\cdots+d_s]$$

which identifies the kernel of the map with $\omega_B[-\delta]$, and we again apply 1.7 to conclude. □

Another possibility, maybe more useful in practice, relies on the following theorem,

Theorem 3.6. *If $\mathfrak{a}=(a_1,\dots,a_s)$ defines a subscheme of codimension $s-1$ whose top dimensional component verifies the hypotheses of 1.7, the regularity of the Hilbert polynomial of $A/\mathfrak{a}$ is at most $d_1+\cdots+d_s-s$, and*

$$P_{\mathcal{Y}}(\nu) = (-1)^d[P_{A/\mathfrak{a}}(-\nu+\delta) - P_{d_1,\dots,d_s}(-\nu+\delta)]$$

*where $\delta := d_1+\cdots+d_s-n-1$, and $P_{d_1,\dots,d_s}$ is the Hilbert polynomial of a complete intersection of forms of degrees $d_1,\dots,d_s$.*

Proof. Let $P_{d_1,\dots,d_s}(\nu) := \sum_{i=0}^s (-1)^i \dim_k H_i(\mathbf{a}\,;A)_\nu$, and remark that, for any set of polynomials of degrees $d_1,\dots,d_s$, one has

$$\chi_{d_1,\dots,d_s}(t) := \sum_{\nu\in\mathbf{Z}} P_{d_1,\dots,d_s}(\nu)t^\nu = \frac{\prod_{i=1}^s(1-t^{d_i})}{(1-t)^{n+1}},$$

so this Euler-Poincaré characteristic depends only on the degrees. Moreover this is equal to the Hilbert-Poincaré series of a complete intersection of codimension $s$ as long as $s \leqslant n+1$, which is the case here. Now, only $H^s(\mathbf{a}\,;A)$ and $H^{s-1}(\mathbf{a}\,;A)$ are not zero,

$$H^s(\mathbf{a}\,;A) \simeq H_0(\mathbf{a}\,;A)[d_1+\cdots+d_s] = A/\mathfrak{a}\,[d_1+\cdots+d_s]$$

and

$$H^{s-1}(\mathbf{a}\,;A) \simeq \omega_B[n+1].$$

Now, by 1.7, $\dim_k(\omega_B)_\nu$ is a polynomial function for $\nu \geqslant -n+s-1$, and for $\nu \geqslant d_1+\cdots+d_s-n$ this is also the case for $P_{d_1,\dots,d_s}(\nu)$.

As

$$P_{d_1,\dots,d_s}(\nu) = \dim_k(A/\mathfrak{a})_\nu - \dim_k(\omega_B)_{\nu-\delta},$$

the assertion is clear. □

## 4. Computation of the top dimensional components of a projective scheme

We will take the same notations as in §1.

Our method is based on fact 1.10, more precisely on the following corollary,

THEOREM 4.1. *Let $\mathfrak{P}$ be a prime ideal of codimension $r$ containing a regular sequence $(b_1, \dots, b_r)$ of forms of degrees $d_i := \deg b_i$. There exists a form $t$ of degree at most $d_1 + \cdots + d_r - r$ such that*

$$\mathfrak{P} = (b_1, \dots, b_r) : (t).$$

PROOF. Setting $\mathfrak{b} := (b_1, \dots, b_r)$, one has

$$\omega_{A/\mathfrak{P}}[n+1] \simeq ((\mathfrak{b} : \mathfrak{P})/\mathfrak{b})[d_1 + \cdots + d_r]$$

and by fact 1.10, $(\omega_{A/\mathfrak{P}})_{n+1-r} \neq 0$, so that there exists $t \in (\mathfrak{b} : \mathfrak{P}) - \mathfrak{b}$ of degree $\leqslant d_1 + \cdots + d_r - r$. Clearly $\mathfrak{P} \subseteq \mathfrak{b} : (t)$, on the other hand if $xt \in \mathfrak{b}$, localizing at $\mathfrak{P}$ one has $\overline{xt} \in \mathfrak{b}_{\mathfrak{P}}$ and as $\overline{t} \notin \mathfrak{b}_{\mathfrak{P}}$, $\overline{x}$ is not invertible and so $x \in \mathfrak{P}$. We have proved our claim. $\square$

COROLLARY 4.2. *Let $\mathfrak{a} = (a_1, \dots, a_s)$ be a homogeneous ideal of codimension $r$ and $\mathfrak{P}_i$ for $i = 1, \dots, m$ be the prime ideals associated to $A/\mathfrak{a}^{top}$. Assume that $a_1, \dots, a_r$ is a regular sequence in $A$. Then, for $i = 1, \dots, m$, there exists $t_i$ homogeneous of degree $\delta_i \leqslant d_1 + \cdots + d_r - r$ such that*

$$f \in \mathfrak{P}_i \quad \Leftrightarrow \quad t_i f \in (a_1, \dots, a_r).$$

*Moreover, if $\mathfrak{a}^{top}$ is reduced, setting $t := t_1 + \cdots + t_m$, the ideal $\mathfrak{a} + (t)$ is of codimension $r+1$ and coincides with $\mathfrak{a}$ outside the support of the $\mathfrak{P}_i$'s.*

PROOF. Applying 4.1 to the ideal $\mathfrak{P}_i$ gives the elements $t_i$ (here $b_i = a_i$). Now note that $t$ is not in any $\mathfrak{P}_i$ but in all the other primary components of $(a_1, \dots, a_r)$. $\square$

Let us set $\mathfrak{b} := (a_1, \dots, a_r)$, with the $a_i$'s as in the previous result, $\mathcal{Y} := \mathrm{Proj}(A/\mathfrak{a}^{top})$ and $\mathcal{Y}_i := \mathrm{Proj}(A/\mathfrak{P}_i)$. A Gröbner basis $\mathfrak{B}_i$ of $\mathfrak{b} + (T^{\delta_i} - t_i)$ in $A[T]$ (notice that this ideal is a complete intersection) for rev-lex, with $T$ dominating the $X_i$'s, gives, in particular, a Gröbner basis for $\mathfrak{P}_i$, moreover the following results insures that this computation will stop at the right degree,

THEOREM 4.3. *The maximal degree of an element in $\mathfrak{B}_i$, for general coordinates in the $X_i$'s, is equal to*

$$\max\{\sigma_r, \mathrm{reg}(A/\mathfrak{P}_i) + \delta_i\},$$

*with $\sigma_r := \sum_{i=1}^{r}(d_i - 1)$, except possibly for the element $T^{2\delta_i}$ (this element is in $\mathfrak{B}_i$ iff $\mathcal{Y}$ is not reduced at the generic point of $\mathcal{Y}_i$).*

For the proof, we will need two lemmas,

LEMMA 4.4. *$\mathfrak{B}_i$ has the following type of generators,*
- *$g_j(X, T)$, with $\overline{g_j}(X) := g_j(X, 0) \neq 0$,*
- *$Th_k(X, T)$, with $\overline{h_k}(X) := h_k(X, 0) \neq 0$,*
- *$T^2$ if $t_i^2 \in \mathfrak{b}$,*

*and the $\overline{g_j}$'s and $\overline{h_k}$'s forms respectively minimal Gröbner bases of $\mathfrak{b} + (t_i)$ and $\mathfrak{P}_i = \mathfrak{b} : (t_i)$.*

PROOF. Notice that either $\mathfrak{b} : (t_i^2) = \mathfrak{b} : (t_i)$ or $t_i^2 \in \mathfrak{b}$, and apply ([8] Prop. 15.12 and Ex. 15.21). $\square$

Lemma 4.5. *One has* $\max\{\mathrm{reg}(A/\mathfrak{P}_i)+\delta_i, \mathrm{reg}(A/\mathfrak{b}+(t_i))\} \geqslant \sigma_r$ *and, if the inequality is strict,*

$$\mathrm{reg}(A/\mathfrak{b}+(t_i)) = \mathrm{reg}(A/\mathfrak{P}_i)+\delta_i - 1.$$

Proof. Let $d := \dim A/\mathfrak{b}$. From the short exact sequence,

$$0 \to A/\mathfrak{P}_i[-\delta_i] \xrightarrow{\times t_i} A/\mathfrak{b} \longrightarrow A/\mathfrak{b}+(t_i) \to 0,$$

the long exact sequence of local cohomology shows that, for all $\nu$, one has

$$H^j_{\mathfrak{m}}(A/\mathfrak{b}+(t_i))_\nu \simeq H^{j+1}_{\mathfrak{m}}(A/\mathfrak{P}_i)_{\nu-\delta_i}$$

if $j \leqslant d-2$ and an exact sequence,

$$0 \to H^{d-1}_{\mathfrak{m}}(A/\mathfrak{b}+(t_i)) \to H^d_{\mathfrak{m}}(A/\mathfrak{P}_i)[-\delta_i] \to H^d_{\mathfrak{m}}(A/\mathfrak{b}) \to H^d_{\mathfrak{m}}(A/\mathfrak{b}+(t_i)) \to 0.$$

Now $H^d_{\mathfrak{m}}(A/\mathfrak{b})_{\sigma_r-d} \simeq k$, which implies that either $H^d_{\mathfrak{m}}(A/\mathfrak{P}_i)_{\sigma_r-\delta_i-d} \neq 0$ or $H^d_{\mathfrak{m}}(A/\mathfrak{b}+(t_i))_{\sigma_r-d} \neq 0$, and this proves the first claim. Moreover, if the inequality is strict, in the concerned degrees we have

$$H^j_{\mathfrak{m}}(A/\mathfrak{b}+(t_i))_\nu \simeq H^{j+1}_{\mathfrak{m}}(A/\mathfrak{P}_i)_{\nu-\delta_i}$$

for all $j$, which proves the second claim. □

Proof. [Theorem 4.3.] Taking general coordinates in the $X_i$'s, Lemma 4.4, and Fact 1.6 (i), implies that the elements of $\mathfrak{B}_i$ have degree at most $\max\{\mathrm{reg}(A/\mathfrak{P}_i)+\delta_i, \mathrm{reg}(A/\mathfrak{b}+(t_i))\}$ and one element has degree $2\delta_i$ if $t_i^2 \in \mathfrak{P}_i$. Therefore Lemma 4.5 gives the conclusion. □

Remark 4.6. *Notice also that one may compute the top dimensional component* $\mathcal{Y}$ *in the following way:*

*(1) Determine a maximal regular sequence* $a_1,\dots,a_r$ *(see §2),*

*(2) Compute a Gröbner basis of the canonical module, seeing it as the kernel of the map,*

$$A/(a_1,\dots,a_r) \xrightarrow{(a_{r+1}\cdots a_s)} \bigoplus_{i=r+1}^{s} A/(a_1,\dots,a_r)[d_i],$$

*shifted in degree by* $d_1+\cdots+d_r-n-1$,

*(3) Determine the annihilator of the canonical module, via a Gröbner basis computation.*

*The nice point in this method is that the maximal degree involved in the computation is bounded in terms of the regularities of* $I_{\mathcal{Y}}$ *and* $\omega_{\mathcal{Y}}$, *and* $\sigma_r$. *As we have seen, or will see in §5, there are nice bounds in specific cases, but anyhow not having to worry about the regularity of* $\mathfrak{a}$ *is a nice thing.*

Remark 4.7. *A simpler possibility (that gives less) is the following,*

*(1) Determine a maximal regular sequence* $a_1,\dots,a_r$ *and by the way (see §2) a non zero homogeneous element* $b$ *in the kernel of,*

$$A/(a_1,\dots,a_r) \xrightarrow{(a_{r+1}\cdots a_s)} \bigoplus_{i=r+1}^{s} A/(a_1,\dots,a_r)[d_i],$$

*(2) compute by increasing degree*

$$K := \ker\Big(A/(a_1,\dots,a_r) \xrightarrow{\times b} A/(a_1,\dots,a_r)[\deg b]\Big)\,.$$

*By what we have seen we may choose $b$ such that $\deg b \leqslant \sigma_r$ and $K_\nu + (a_1,\dots,a_r)_\nu = J_\nu$ where $J \supseteq \mathfrak{a}$ is pure of codimension $r$. In particular, if the top dimensional part of $\mathfrak{a}$ is irreducible, $J = \mathfrak{a}^{top}$ ; also if $b$ is a general element of degree $\sigma_r$ in the kernel of the map in (1), $J$ has the same support as $\mathfrak{a}^{top}$ (even better if $b$ is not general, so that you separate top dimensional components !).*

*Also note that, adding $b$ to $\mathfrak{a}$ gives a new ideal having dimension one less or a smaller top dimensional part, but such that all the irreducible components of smaller dimension of $\mathfrak{a}$ are also irreducible compenents of $\mathfrak{a} + (b)$ (nevertheless there may be irreducible components of $\mathfrak{a}+(b)$ that are embedded in the top dimensional part of $\mathfrak{a}$). By iterating the process, one gets a decomposition into irreducible components. We have no clear idea of how this process (or a similar one based on Remark 4.6) may be compared with the known algorithms for decomposition into irreducible components or primary decomposition (see [10], [18], and [7] for a survey and other references).*

Remark 4.8. *In characteristic 0, one may also determine the smoothness of the top dimensional component, not going in too high degree. There are several ways of doing that from what we have already explained, let us explain one that seems reasonable in practice. A slight adaptation gives a way to determine if the top part has isolated singularities – which may replace (iii) and (iv) in Fact 1.7, see [6].*

*Start computing a Gröbner basis (for rev-lex) of the canonical module as in Remark 4.6. If the computation do not stop in the degree it should do for a smooth scheme and general coordinates (Fact 1.7) test if the Hilbert function is already a polynomial function (take, say, $2 \times \dim A/\mathfrak{a}$ values of the Hilbert function). If the function is not polynomial $\mathrm{Proj}(A/\mathfrak{a}^{top})$ is not smooth.*

*If the function seems polynomial (positive test) a second test is to perform a random change of coordinates and compute the Gröbner basis again. If the computation still do not end at the expected degree there is a great chance that it is not smooth.*

*In any case (i.e. with or without a second test), consider then the module defined by the (possibly truncated) computation of the canonical module; compute its annihilator and stop the computation in degree $\max\{\sigma_r + 1, D_{\mathcal{Y}}\}$, where $D_{\mathcal{Y}}$ is the leading coefficient of the Hilbert polynomial (or the candidate for it), written on the binomial basis. Notice that this leading coefficient is equal to the degree of $\mathcal{Y}$ if $\mathcal{Y}$ is smooth. (The computation may of course end before this degree.) Now check if this (possibly truncated) annihilator defines a smooth scheme, by the Jacobian criterion.*

*If yes, then the (possibly truncated) annihilator defines $\mathcal{Y}$, and $\mathcal{Y}$ is smooth. If not, $\mathcal{Y}$ is not smooth. (Remember that a smooth scheme is defined by equation of degree at most the degree of the scheme itself.)*

*Let us also recall that, up to dimension 4 it is proved that a smooth scheme have its Castelnuovo-Mumford regularity bounded by its degree. A well-known conjecture of Eisenbud-Goto suggests that this may be true for any irreducible reduced scheme.*

## 5. Control of regularity by liaison

To simplify notations, when $\mathcal{S} \subseteq \mathbf{P}_n(k)$ is a projective scheme, we will set $\operatorname{reg}(\mathcal{S}) := \operatorname{reg}(A/I_{\mathcal{S}})$ where $I_{\mathcal{S}}$ is the only saturated ideal such that $\mathcal{S} = \operatorname{Proj}(A/I_{\mathcal{S}})$, with $A := k[X_0, \dots, X_n]$.

Let us first recall a well-known upper bound for the regularity of reduced curves, first proved by Castelnuovo in the smooth case ([4]), and generalized by Gruson, Lazarsfeld and Peskine in [12],

THEOREM. *Let $\mathcal{C} \subset \mathbf{P}_n(k)$ be a scheme purely of dimension 1, reduced over $\overline{k}$, then*

$$\operatorname{reg}(\mathcal{C}) \leqslant \deg \mathcal{C} - 1.$$

The result is slightly more precise if $\mathcal{C}$ is irreducible and not contained in any hyperplane. In an other direction, closer to the type of results of Bertram, Ein and Lazarsfeld in [2], which uses the degree of defining equations to bound the regularity, we have the following result,

THEOREM 5.1. *Let $\mathfrak{a} = (a_1, \dots, a_s) \subset A$ be a homogeneous ideal with $\mathcal{Z} := \operatorname{Proj}(A/\mathfrak{a})$ of dimension 1. Set $d_i := \deg a_i$, assume that $d_1 \geqslant \cdots \geqslant d_s$ and let $\mathcal{C}$ be the component of dimension 1 of $\mathcal{Z}$. Assume that $\mathcal{C}$ is locally a complete intersection at the generic points of its irreducible components. Then for every $\mathcal{C}' \subseteq \mathcal{C}$ pure of dimension 1 such that $\operatorname{Proj}(A/(I_{\mathcal{C}} : I_{\mathcal{C}'}))$ is reduced (over $\overline{k}$), one has*

$$\operatorname{reg}(\mathcal{C}') \leqslant \sum_{i=1}^{n-1} (d_i - 1).$$

*Moreover, if $\deg \mathcal{C}' \neq d_1 \cdots d_{n-1}$, the inequality is strict.*

*In particular, if $\mathcal{C}$ is reduced in $\mathbf{P}_n(\overline{k})$, for every $\mathcal{C}' \subseteq \mathcal{C}$ pure of dimension 1, the above estimate is verified.*

The proof of 5.1 will be derived from the following result,

LEMMA 5.2. *Let $\mathfrak{b}$ be Gorenstein ideal of codimension $r$ in $A$ and $\mathfrak{c}$ be an unmixed ideal of codimension $r$ strictly containing $\mathfrak{b}$. Set $\mathfrak{c}' := \mathfrak{b} : \mathfrak{c}$, $a := a(A/\mathfrak{b})$ and assume that $d := \dim A/\mathfrak{b} \geqslant 2$, then*

$$\operatorname{reg}(A/\mathfrak{c}') \leqslant \max\{a+d-1, \operatorname{reg}(\omega_{A/\mathfrak{c}})+a-1\} \quad \text{and} \quad \operatorname{reg}(\omega_{A/\mathfrak{c}}) \leqslant \max\{\operatorname{reg}(A/\mathfrak{c}')-a+1, d\}.$$

*Moreover, $\operatorname{reg}(A/\mathfrak{c}') > a + d - 1$ if and only if $\operatorname{reg}(\omega_{A/\mathfrak{c}}) > d$, and in this case,*

$$\operatorname{reg}(A/\mathfrak{c}') = \operatorname{reg}(\omega_{A/\mathfrak{c}}) + a - 1.$$

*If $\operatorname{Proj}(A/\mathfrak{c})$ is reduced over $\overline{k}$, connected, non degenerate, and $H^i_{\mathfrak{m}}(\omega_{A/\mathfrak{c}})_{>d-i-1} = 0$ for $2 \leqslant i \leqslant d-1$, then*

$$\operatorname{reg}(A/\mathfrak{c}') \leqslant a + d - 2.$$

PROOF. Set $d := \dim A/\mathfrak{b}$ and $a := a(A/\mathfrak{b})$. If $M$ is a graded $A$-module, we will set $a_i(M) := \max\{\nu, H^i_{\mathfrak{m}}(M)_\nu \neq 0\}$ so that $\operatorname{reg}(M) = \max_i\{a_i(M) + i\}$. We have an exact sequence,

$$0 \to \mathfrak{b} \to \mathfrak{c}' \to \mathfrak{c}'/\mathfrak{b} \simeq \operatorname{Hom}_A(A/\mathfrak{c}, A/\mathfrak{b}) \to 0$$

and $\mathrm{Hom}_A(A/\mathfrak{c}, A/\mathfrak{b}) \simeq \mathrm{Hom}_A(A/\mathfrak{c}, \omega_{A/\mathfrak{b}}[-a]) \simeq \omega_{A/\mathfrak{c}}[-a]$. This exact sequence leads to a long exact sequence in local cohomology, which gives

$$H^i_{\mathfrak{m}}(A/\mathfrak{c}') \simeq H^{i+1}_{\mathfrak{m}}(\omega_{A/\mathfrak{c}})[-a]$$

for $i \leqslant d-2$ so that $a_i(A/\mathfrak{c}') = a + a_{i-1}(\omega_{A/\mathfrak{c}})$ in this range, and an exact sequence

$$0 \to H^{d-1}_{\mathfrak{m}}(A/\mathfrak{c}') \to H^d_{\mathfrak{m}}(\omega_{A/\mathfrak{c}})[-a] \to H^d_{\mathfrak{m}}(A/\mathfrak{b}) \to H^d_{\mathfrak{m}}(A/\mathfrak{c}') \to 0.$$

Notice also that $H^d_{\mathfrak{m}}(A/\mathfrak{b}) \simeq H^d_{\mathfrak{m}}(\omega_{A/\mathfrak{b}})[-a]$.

By local duality, setting $\mathcal{S} := \mathrm{Proj}(A/\mathfrak{b})$ and $\mathcal{Y}' := \mathrm{Proj}(A/\mathfrak{c})$, one has $H^d_{\mathfrak{m}}(\omega_{A/\mathfrak{c}})_\mu \simeq H^0(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}(-\mu))$ and $H^d_{\mathfrak{m}}(\omega_{A/\mathfrak{b}})_\mu \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-\mu))$, and, with these identifications, the middle arrow in degree $\nu$ is dual to the natural map

$$H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-\nu + a)) \xrightarrow{\text{can}} H^0(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}(-\nu + a)).$$

This map is injective for $\nu = a$ as $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) = (A/\mathfrak{b})_0 = k$, and also for $\nu = a-1$ if $\mathcal{Y}'$ is non degenerate.

Therefore $H^d_{\mathfrak{m}}(A/\mathfrak{c}')_{>a} = 0$ and $H^d_{\mathfrak{m}}(A/\mathfrak{c}')_a = 0$ if $\mathcal{Y}'$ is non degenerate. In degree $a$, $H^d_{\mathfrak{m}}(A/\mathfrak{c}')_a = 0$ and $\dim_k H^{d-1}_{\mathfrak{m}}(A/\mathfrak{c}')_a = \dim_k H^d_{\mathfrak{m}}(\omega_{A/\mathfrak{c}})_0 - 1$. Therefore, if $a_{d-1}(A/\mathfrak{c}') = a-1$ then $a_d(\omega_{A/\mathfrak{c}}) = 0$, and if $a_{d-1}(A/\mathfrak{c}') \geqslant a$ one has $a_{d-1}(A/\mathfrak{c}') = a_d(\omega_{A/\mathfrak{c}}) + a$.

Notice also that if $a_d(\omega_{A/\mathfrak{c}}) = 0$ then either $a_{d-1}(A/\mathfrak{c}') = a-1$ (and this is always the case when $\mathcal{Y}'$ is reduced and connected) or $a_{d-1}(A/\mathfrak{c}') = a$.

Gathering the information above, we have,

- $a_i(A/\mathfrak{c}') = a + a_{i+1}(\omega_{A/\mathfrak{c}})$ for $i \leqslant d-2$,
- $a_d(A/\mathfrak{c}') \leqslant a-1$,
- $a_d(A/\mathfrak{c}') \leqslant a-2$ if $\mathcal{Y}'$ is non degenerate,
- $a_{d-1}(A/\mathfrak{c}') = a-1 \;\Rightarrow a_d(\omega_{A/\mathfrak{c}}) = 0$,
- $a_d(\omega_{A/\mathfrak{c}}) = 0 \;\Rightarrow\; \{a_{d-1}(A/\mathfrak{c}') = a-1 \text{ or } a_{d-1}(A/\mathfrak{c}') = a\}$,
- $\{a_{d-1}(A/\mathfrak{c}') \geqslant a \text{ or } a_d(\omega_{A/\mathfrak{c}}) > 0\} \;\Rightarrow\; a_{d-1}(A/\mathfrak{c}') = a_d(\omega_{A/\mathfrak{c}}) + a$,

from these facts, the assertion is clear. □

Let us state a definition,

Definition 5.3. *Let $\mathfrak{c}$ be an ideal that is pure of codimension $r$, $d := (d_1, \ldots, d_r)$ an $r$-tuple of positive integers. Then $\mathfrak{c}$ is $d$-residually reduced (resp. normal) if there exists $r$ forms $a_1, \ldots, a_r$ of respective degrees $d_1, \ldots, d_r$ forming a regular sequence in $A$, such that* $\mathrm{Proj}(A/(a_1, \ldots, a_r) : \mathfrak{c})$ *is reduced in $\mathbf{P}_n(\overline{k})$ (resp. normal). An ideal of the form $(a_1, \ldots, a_r) : \mathfrak{c}$ will be called a $d$-link of $\mathfrak{c}$.*

Now 5.1 is a consequence of the following lemma,

Lemma 5.4. *Let $\mathfrak{a}$ be an ideal of codimension $r$ generated by forms of degrees $d_1 \geqslant \cdots \geqslant d_s$, and let $\mathfrak{c}$ be the unmixed part of codimension $r$ of $\mathfrak{a}$. If $\mathfrak{c}$ is a complete intersection locally in codimension $r$, there exists $a_1, \ldots, a_r$ in $\mathfrak{a}$ of respective degrees $d_1, \ldots, d_r$ such that $(a_1, \ldots, a_r) = \mathfrak{c} \cap \mathfrak{c}'$ where $\mathfrak{c}'$ is reduced. In particular $\mathfrak{c}$ is $(d_1, \ldots, d_r)$-residually reduced if the ground field is perfect.*

Proof. By Bertini theorem, there exists an open subset $\Omega$ of the affine space $\mathbf{A}^N :=$

$\mathfrak{a}_{d_1} \times \cdots \times \mathfrak{a}_{d_r}$ such that for every corresponding $r$-tuple of polynomials $(a_1, \ldots, a_r)$ one has $(a_1, \ldots, a_r) = \mathfrak{c}^* \cap \mathfrak{c}$ with $\mathfrak{c}$ reduced and $\mathfrak{c}^*$ having the same support as $\mathfrak{c}$. Now locally at each prime $\mathfrak{P}$ of codimension $r$ the classes of the generators $g_i$ of $\mathfrak{a}$ forms a set of generators of the complete intersection $\mathfrak{c}_{\mathfrak{P}}$. As any minimal set of generators have the same number of elements, there exists $i_1^{\mathfrak{P}}, \ldots, i_r^{\mathfrak{P}} \in [s]$ such that $\mathfrak{c}_{\mathfrak{P}} = (g_{i_1^{\mathfrak{P}}}, \ldots, g_{i_r^{\mathfrak{P}}})$. This implies that for each such prime there exists an open set $\Omega_{\mathfrak{P}} \subseteq \mathfrak{a}_{d_1} \times \cdots \times \mathfrak{a}_{d_r}$ such that for every corresponding $r$-tuple of polynomials $(a_1, \ldots, a_r)$ one has $(a_1, \ldots, a_r)_{\mathfrak{P}} = \mathfrak{c}_{\mathfrak{P}}$. Now taking an $r$-tuple in the intersection $\Omega \cap \Omega_{\mathfrak{P}_1} \cap \cdots \cap \Omega_{\mathfrak{P}_t}$ where the $\mathfrak{P}_i$'s are the primes associated to $\mathfrak{c}$, we get the desired complete intersection. $\square$

Proof. [Theorem 5.1] By lemma 5.4, $\mathcal{C}'$ is $(d_1, \ldots, d_{n-1})$-residually reduced. Applying Lemma 5.2 and Fact 1.7 (ii) gives the result, as $a(A/\mathfrak{b}) = (\sum_{i=1}^{n-1} d_i) - n - 1$ in this case. $\square$

If $\mathcal{Z}$ is a 1-dimensional scheme, defined by equations of degrees $d_1 \geqslant \cdots \geqslant d_s$ in $\mathbf{P}_n$, we have:

(1) If the unmixed part $\mathcal{C}$ of $\mathcal{Z}$ is reduced, every irreducible component of $\mathcal{C}$ has regularity at most $\sum_{i=1}^{n-1}(d_i - 1)$,

(2) In any case, if $a_1, \ldots, a_{n-1}$ are general forms of degrees $d_1, \ldots, d_{n-1}$ in $I_{\mathcal{Z}}$ and $\mathcal{C}'$ the component of the scheme they define supported by $\mathcal{C}$ one has

$$\operatorname{reg}(\mathcal{C}') \leqslant \sum_{i=1}^{n-1}(d_i - 1),$$

in particular, there exists an ideal contained in the ideal of $\mathcal{C}$ and having the same radical with "small" regularity.

Let us also recall from 1.14,

(3) If $\mathcal{X}$ is the zero dimensional part of $\mathcal{Z}$,

$$\operatorname{reg}(\mathcal{X}) \leqslant \sum_{i=1}^{n}(d_i - 1).$$

In dimension 2, we have the following result,

Theorem 5.5. *Let $\mathfrak{a} = (a_1, \ldots, a_s) \subset A$ be a homogeneous ideal. Assume that $\mathcal{Z} := \operatorname{Proj}(A/\mathfrak{a})$ is of dimension 2 with no component of dimension 1 (embedded or not), and that the top dimensional part $\mathcal{S}$ of $\mathcal{Z}$ is reduced, Cohen-Macaulay and locally a complete intersection in codimension $n-1$ (e.g. $\mathcal{S}$ is a normal surface). Set $d_i := \deg a_i$, assume that $\operatorname{Char}(k) = 0$ and $d_1 \geqslant \cdots \geqslant d_s$, then*

$$\operatorname{reg}(\mathcal{S}) \leqslant \sum_{i=1}^{n-2}(d_i - 1),$$

*and the inequality is strict, unless $\mathcal{S}$ is of degree $d_1 \cdots d_{n-2}$.*

Lemma 5.6. *With the hypotheses of 5.5, $\mathcal{S}$ is $(d_1, \ldots, d_{n-2})$-residually normal.*

Proof. First, by Bertini theorem, it is clear that a general $(d_1, \ldots, d_{n-2})$-link of $\mathcal{S}$ is irreducible. Such a link is Cohen-Macaulay as $\mathcal{S}$ is. The fact that a general link in these degrees is regular in codimension 1 is proved in ([17], 4.1, and the remark before). $\square$

Proof. [Theorem 5.5] By lemma 5.6, $\mathcal{S}$ is $(d_1, \ldots, d_{n-2})$-linked to a normal surface $\mathcal{S}'$. By Fact 1.7 (iii), $\operatorname{reg}(\omega_{\mathcal{S}'}) \leqslant 3$, and the conclusion follows from 5.2, as $a = (\sum_{i=1}^{n-2} d_i) - n - 1$ in this case. $\square$

**Note.** The results of this paragraph are extended in several directions in our forthcoming joint work with Bernd Ulrich [6].

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