

Liaison of varieties of small dimension and deficiency modules

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1 Introduction

Liaison relates the cohomology of the ideal sheaf of a scheme to the cohomology of the canonical module of its link. We here refer to Gorenstein liaison in a projective space over a field: each ideal is the residual of the other in one Gorenstein homogeneous ideal of a polynomial ring. Assuming that the linked schemes (or equivalently one of them) are Cohen-Macaulay, Serre duality expresses the cohomology of the canonical module in terms of the cohomology of the ideal sheaf. Therefore, in the case of Cohen-Macaulay linked schemes, the cohomology of ideal sheaves can be computed one from another: up to shifts in ordinary and homological degrees, they are exchanged and dualized. In terms of free resolutions this means that, up to a degree shift, they may be obtained one from another by dualizing the corresponding complexes (for instance, the generators of one cohomology module corresponds to the last syzygies of another cohomology module of the link).

If the linked schemes are not Cohen-Macaulay, this property fails. Nevertheless, experience on a computer shows that these modules are closely related. We here investigate this relation for surfaces and three-dimensionnal schemes. To describe our results, note that the graded duals of the lo-

cal cohomology modules are Ext modules (into the polynomial ring or the Gorenstein quotient that provides the linkage), let us set $—^*$ for the graded dual and $D_i(M) := H_{\mathfrak{m}}^i(M)^*$, the i -th deficiency module when $i \neq \dim M$, while $\omega_M = D_{\dim M}(M)$.

In the case of surfaces, we have to understand the behaviour under liaison of the Hartshorne-Rao module D_1 and the module D_2 . Note that D_2 has a finite part $H_{\mathfrak{m}}^0(D_2) \simeq D_0(D_2)^*$ and a quotient $D_2/H_{\mathfrak{m}}^0(D_2) \simeq D_1(D_1(D_2))$ that is either 0 or a Cohen-Macaulay module of dimension one. For a module M of dimension $d \geq 3$, there is an isomorphism $H_{\mathfrak{m}}^2(\omega_M) \simeq H_{\mathfrak{m}}^0(D_{d-1}(M))$, which gives $H_{\mathfrak{m}}^2(\omega) \simeq H_{\mathfrak{m}}^0(D_2)$ and, together with the liaison sequence, shows how these three modules behave under linkage.

In the case of dimension three, there are seven Cohen-Macaulay modules, obtained by iterating Ext's on the deficiency modules, that encodes all the information on the deficiency modules. Five of these modules are permuted by liaison, up to duality and shift in degrees, but this is not the case for the two others that sits in an exact sequence with the corresponding two of the link. The key map in this four terms sequence of finite length modules generalizes the composition of the linkage map with Serre duality map and seems an interesting map to investigate further.

We begin with a short review on duality, and present some extensions of Serre duality that we couldn't find in the literature. For instance if X is an equidimensional scheme such that

$$(*) \quad \text{depth } \mathcal{O}_{X,x} \geq \dim \mathcal{O}_{X,x} - 1, \quad \forall x \in X,$$

then the isomorphisms given by Serre duality (in the Cohen-Macaulay case) are replaced by a long exact sequence (Corollary 2.7) in which one module over three is zero in the Cohen-Macaulay case. In the context of liaison, it turns out that condition $(*)$ is equivalent to the fact that the S_2 -ification of a link (or equivalently any link) is a Cohen-Macaulay scheme (Proposition 3.3).

2 Preliminaries on duality in a projective space

Let k be a commutative ring, n a positive integer, $A := \text{Sym}(k^n)$, $\mathfrak{m} := A_{>0}$ and $\omega_{A/k} := A[-n]$. Let $—^*$ denote the graded dual into k :

$$—^* := \text{Homgr}_A(—, k) = \bigoplus_{\mu \in \mathbf{Z}} \text{Hom}_k((—)_{\mu}, k)$$

and $—^{\vee}$ the dual into $\omega_{A/k}$: $—^{\vee} := \text{Hom}_A(—, \omega_{A/k})$.

The homology of the Čech complex $\mathcal{C}_{\mathfrak{m}}^\bullet(-)$ will be denoted by $H_{\mathfrak{m}}^\bullet(-)$; notice that this homology is the local cohomology supported in \mathfrak{m} (for any commutative ring k), as \mathfrak{m} is generated by a regular sequence.

The key lemma is the following,

LEMMA 2.1 *If L_\bullet is a graded complex of finitely generated free A -modules, then*

- (i) $H_{\mathfrak{m}}^i(L_\bullet) = 0$ for $i \neq n$,
- (ii) *there is a functorial homogeneous isomorphism of degree zero:*

$$H_{\mathfrak{m}}^n(L_\bullet) \simeq (L_\bullet^\vee)^*.$$

Proof: Claim (i) is standard and (ii) directly follows from the classical description of $H_{\mathfrak{m}}^n(A)$ (see e.g. [5, §62] or [1, Ch. III, Theorem 5.1]). \square
For an A -module M , we set $D_i(M) := \text{Ext}_A^{n-i}(M, \omega_{A/k})$ and

$$\Gamma M := \ker(\mathcal{C}_{\mathfrak{m}}^1(M) \rightarrow \mathcal{C}_{\mathfrak{m}}^2(M)) = \bigoplus_{\mu \in \mathbf{Z}} H^0(\mathbf{P}_k^{n-1}, \widetilde{M}(\mu)),$$

so that we have an exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(M) \rightarrow M \rightarrow \Gamma M \rightarrow H_{\mathfrak{m}}^1(M) \rightarrow 0.$$

We will also use the notation $\omega_M := D_{\dim M}(M)$ when M is finitely generated and k is a field.

The above lemma gives,

PROPOSITION 2.2 *Assume that M has a graded free resolution $L_\bullet \rightarrow M \rightarrow 0$ such that each L_ℓ is finitely generated, then for all i there are functorial maps*

$$H_{\mathfrak{m}}^i(M) \xrightarrow{\phi^i} H_{n-i}((L_\bullet^\vee)^*) \xrightarrow{\psi_i} D_i(M)^*,$$

where ϕ^i is an isomorphism. Furthermore ψ_i is an isomorphism if k is a field.

Proof: The two spectral sequences arising from the double complex $\mathcal{C}_{\mathfrak{m}}^\bullet L_\bullet$ degenerates at the second step (due to Lemma 2.1 (i) for one, and the exactness of localisation for the other) and together with Lemma 2.1 (ii) provides the first isomorphism. If k is a field, $-^*$ is an exact functor, so that the natural map $H_{n-i}((L_\bullet^\vee)^*) \xrightarrow{\psi_i} H^{n-i}(L_\bullet^\vee)^* = D_i(M)^*$ is an isomorphism. \square

PROPOSITION 2.3 *Assume that M posses a graded free resolution $L_\bullet \rightarrow M \rightarrow 0$ such that each L_ℓ is finitely generated, then there is a spectral sequence*

$$H_m^i(D_j(M)) \Rightarrow H_{i-j}(L_\bullet^*).$$

In particular, if k is a field, then there is a spectral sequence

$$H_m^i(D_j(M)) \Rightarrow \begin{cases} M^* & \text{if } i - j = 0 \\ 0 & \text{else.} \end{cases}$$

Proof: The two spectral sequences arising from the double complex $C_m^\bullet L_\bullet^\vee$ have as second terms $'E_2^{ij} = H_m^i(\text{Ext}_A^j(M, \omega_{A/k})) = H_m^i(D_{n-j}(M))$, $''E_2^{ij} = 0$ for $i \neq n$ and $''E_2^{nj} = H^j(H_m^n(L_\bullet^\vee)) \simeq H^j((L_\bullet^{\vee\vee})^*) \simeq H^j(L_\bullet^*)$. This gives the first result. If k is a field, $H^j(L_\bullet^*) = H_j(L_\bullet)^*$ which gives the second claim. \square

COROLLARY 2.4 *If k is a field, and M is a finitely generated graded A -module of dimension d , then there are functorial maps*

$$d_j^i := 'd_2^{i, n-j} : H_m^i(D_j(M)) \rightarrow H_m^{i+2}(D_{j+1}(M))$$

such that

(i) $d_{d-1}^0 : H_m^0(D_{d-1}(M)) \rightarrow H_m^2(\omega_M)$ *is an isomorphism if $d \geq 3$,*

(ii) *if $d = 3$ and M is equidimensionnal and satisfies S_1 , there is an exact sequence,*

$$0 \rightarrow H_m^1(D_2(M)) \xrightarrow{d_2^1} H_m^3(\omega_M) \rightarrow \Gamma M^* \rightarrow 0,$$

(iii) *if $d \geq 4$ there is a exact sequence,*

$$\begin{aligned} 0 \longrightarrow H_m^1(D_{d-1}(M)) \xrightarrow{d_{d-1}^1} H_m^3(\omega_M) \xrightarrow{e} H_m^0(D_{d-2}(M)) \\ \xrightarrow{d_{d-2}^0} H_m^2(D_{d-1}(M)) \xrightarrow{d_{d-1}^2} H_m^4(\omega_M) \longrightarrow C \longrightarrow 0 \end{aligned}$$

where e is the composed map

$$e : H_m^3(\omega_M) \xrightarrow{\text{can}} \text{coker}(d_{d-1}^1) \xrightarrow{('d_3^{3, n-d})^{-1}} \ker(d_{d-2}^0) \xrightarrow{\text{can}} H_m^2(D_{d-2}(M))$$

such that

(a) *if $d = 4$ and M is equidimensionnal and satisfies S_1 , C sits in an exact sequence*

$$0 \rightarrow H_m^1(D_2(M)) \xrightarrow{'d_3^{1, n-2}} C \rightarrow \Gamma M^* \rightarrow 0,$$

(b) *if $d \geq 5$, C sits in an exact sequence*

$$0 \rightarrow \ker(d_{d-2}^1) \rightarrow C \rightarrow \ker[\ker(d_{d-3}^0) \rightarrow \ker(d_{d-1}^3)/\text{im}(d_{d-2}^1)] \rightarrow 0.$$

Proof: We may assume that M does not have associated primes of dimension ≤ 1 . Then these statements are direct consequences of the spectral sequence of Proposition 2.3 applied to ΓM . Note that if M is equidimensional and satisfies S_1 , then $\dim D_i(M) < i$ for $i < d$ so that $H_{\mathfrak{m}}^i(D_i(M)) = 0$ for $i \neq d$. \square

REMARK 2.5 One may dualize all the above maps into k . By Proposition 2.2 it gives rise to natural maps $\theta_{ij} : D_{i+2}(D_{j+1}(M)) \rightarrow D_i(D_j(M))$ satisfying the “dual statements” of (i), (ii) and (iii). Note also that the map e in (iii) gives a map $\epsilon : D_0(D_{d-2}(M)) \rightarrow H_{\mathfrak{m}}^0(D_3(\omega_M)) \subseteq D_3(\omega_M)$. Also in (iii)(a) the graded dual K of C sits in an exact sequence

$$0 \rightarrow \Gamma M \xrightarrow{\tau} K \xrightarrow{\eta} D_1(D_2(M)) \rightarrow 0,$$

where τ is the graded dual over k of the transgression map in the spectral sequence and η is the graded dual over k of $'d_3^{1,n-2}$ (both composed with isomorphisms given by Proposition 2.2).

COROLLARY 2.6 *If k is a field and M is a finitely generated equidimensional graded A -module of dimension d such that \widetilde{M} satisfies S_ℓ for some $\ell \geq 1$, then, for $1 < i \leq \ell$, there are functorial surjective maps*

$$f_i : H_{\mathfrak{m}}^{d+1-i}(\omega_M) \longrightarrow D_i(M) = H_{\mathfrak{m}}^0(D_i(M))$$

which are isomorphisms for $1 < i < \ell$. And there is an injection $D_1(M) \rightarrow H_{\mathfrak{m}}^d(\omega_M)$ whose cokernel is $(M/H_{\mathfrak{m}}^0(M))^*$ if $\ell \geq 2$, so that in this case $H_{\mathfrak{m}}^d(\omega_M) \simeq \Gamma M^*$.

Proof: The condition on \widetilde{M} implies that $H_{\mathfrak{m}}^i(D_j(M)) = 0$ if $i > \max\{0, j - \ell\}$ and $j \neq d$. These vanishings together with the spectral sequence of Proposition 2.3 gives the result. \square

The next result gives a slight generalization of Serre duality to schemes (or sheaves) that are close to be Cohen-Macaulay:

COROLLARY 2.7 *Let k be a field and M is a finitely generated graded A -module which is unmixed of dimension $d \geq 3$. Set $F_M := D_{d-1}(M)$, $\mathcal{M} := \widetilde{M}$ and assume that*

$$\text{depth } \mathcal{M}_x \geq \dim \mathcal{M}_x - 1, \quad \forall x \in \text{Supp}(\mathcal{M}).$$

Then there exists a functorial isomorphism $H_{\mathfrak{m}}^0(F_M) \simeq H_{\mathfrak{m}}^2(\omega_M)$ and a long exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_m^1(F_M) & \longrightarrow & H_m^3(\omega_M) & \longrightarrow & D_{d-2}(M) \\
& & & & \swarrow & & \\
& & H_m^2(F_M) & \longrightarrow & H_m^4(\omega_M) & \longrightarrow & D_{d-3}(M) \\
& & & & \swarrow & & \\
& & & & \vdots & & \\
& & & & \swarrow & & \\
& & H_m^{d-3}(F_M) & \longrightarrow & H_m^{d-1}(\omega_M) & \longrightarrow & D_2(M) \\
& & & & \swarrow & & \\
& & H_m^{d-2}(F_M) & \longrightarrow & H_m^d(\omega_M) & \longrightarrow & \Gamma M^* \longrightarrow 0.
\end{array}$$

Proof: This is immediate from the spectral sequence of Proposition 2.3. \square

REMARK 2.8 The vanishing of certain collections of modules $H_m^i(D_j(M))$ corresponds to frequently used properties that M may have. For instance, assume that M is equidimensional of dimension $d > 0$ and consider all the possibly non zero modules $H_m^i(D_j(M))$ for $j \neq d$:

$$\begin{array}{ccccccc}
& & & \Delta_d & \cdots & \Delta_2 & \Delta_1 \\
L_d \cdots & H_m^0(D_{d-1}) & \cdots & H_m^0(D_1) & & H_m^0(D_0) & \\
\vdots & \vdots & \ddots & H_m^1(D_1) & & & \\
L_2 \cdots & H_m^{d-2}(D_{d-1}) & \ddots & & & & \\
L_1 \cdots & H_m^{d-1}(D_{d-1}) & & & & & \\
& \vdots & & & & & \\
& C_d & \cdots & C_2 & & C_1 &
\end{array}$$

Then,

(i) The modules on lines L_1, \dots, L_ℓ are zero if and only if M is Cohen-Macaulay in codimension ℓ ,

(ii) The modules on columns C_1, \dots, C_ℓ are zero if and only if M has depth at least ℓ ,

(iii) The modules on diagonals $\Delta_1, \dots, \Delta_\ell$ are zero if and only if M satisfies S_ℓ .

Note also that the spectral sequence shows that ω_M is Cohen-Macaulay if all non zero modules in this diagram are located on diagonals Δ_1 and Δ_2 (even if M is not equidimensionnal). This generalizes a little the case of a sequentially Cohen-Macaulay module (due to Schenzel), where all non zero modules are located on diagonal Δ_1 .

3 The liaison sequence

In this paragraph A is a polynomial ring over a field k . Recall that an homogeneous ideal \mathfrak{b} of A is Gorenstein if $B := A/\mathfrak{b}$ is Cohen-Macaulay and ω_B is a free B -module of rank 1 (*i.e.* B is Gorenstein), so that $\omega_B \simeq B[a]$ for some $a \in \mathbf{Z}$ called the a -invariant of B . The arithmetically Gorenstein subschemes of $\text{Proj}(A)$ are the subschemes of the form $\text{Proj}(A/\mathfrak{b})$ for an homogeneous Gorenstein ideal \mathfrak{b} ; if the scheme is not empty, such a \mathfrak{b} is unique.

PROPOSITION 3.1 *Let I and $J = \mathfrak{b} : I$ be two ideals of A , linked by a Gorenstein homogeneous ideal \mathfrak{b} of A (so that $I = \mathfrak{b} : J$). Set $B := A/\mathfrak{b}$, $R := A/I$, $S := A/J$, $d := \dim B$, $F_S := D_{d-1}(S)$ and $E_R := \text{End}(\omega_R)$. There are natural exact sequences or isomorphisms:*

$$(i) \quad 0 \longrightarrow \omega_R \otimes \omega_B^{-1} \xrightarrow{\iota} \omega_B \otimes_B \omega_B^{-1} = B \xrightarrow{s} S \longrightarrow 0$$

where $-^{-1} := \text{Hom}_B(-, B)$ and ι and s are the canonical maps;

$$(ii) \quad 0 \rightarrow \omega_S \xrightarrow{\text{Hom}_B(s, 1_{\omega_B})} \omega_B \xrightarrow{\text{Hom}_B(\iota, 1_{\omega_B})} E_R \otimes_B \omega_B \xrightarrow{\delta^r} F_S \rightarrow 0$$

where δ^r is the connecting map in the Ext sequence derived from (i), in particular $F_S \simeq (E_R/R) \otimes_B \omega_B$;

$$(iii) \quad \lambda_i : D_i(S) \xrightarrow{\sim} D_{i+1}(\omega_R) \otimes_B \omega_B \quad \forall i \leq d-2$$

given by the connecting map in the Ext sequence derived from (i).

Proof: (i) is standard (see e.g. [2] or [3]). Also (ii) and (iii) directly follows by the Ext sequence derived from (i). \square

REMARK 3.2 (i) For any B -module M , there are functorial isomorphisms:

$$\tau_i : D_i(M) \xrightarrow{\sim} \text{Ext}_B^{d-i}(M, \omega_B).$$

(ii) Choosing an isomorphism $\phi : B[-a] \rightarrow \omega_B^{-1}$, (1) gives an exact sequence,

$$0 \longrightarrow \omega_R[-a] \xrightarrow{\iota \circ (1 \otimes \phi)} B \xrightarrow{s} S \longrightarrow 0$$

PROPOSITION 3.3 *Let X and Y be two schemes linked by an arithmetically Gorenstein projective subscheme of $\text{Proj}(A)$ of dimension $d \geq 1$ and a -invariant a . Let $Y_2 := \text{Spec}(\mathcal{E}nd(\omega_Y))$ be the S_2 -ification of Y . The following are equivalent:*

- (i) $\text{depth } \mathcal{O}_{X,x} \geq \dim \mathcal{O}_{X,x} - 1$ for every $x \in X$,
- (ii) ω_Y is Cohen-Macaulay,
- (iii) Y_2 is Cohen-Macaulay,
- (iv) $H^i(X, \mathcal{O}_X(\mu)) \simeq H^{d-i-1}(Y, \mathcal{O}_{Y_2}(a - \mu))^*$, for every $0 < i < d - 1$ and every μ , where $-^*$ denotes the dual into k .

Proof: We may assume $d \geq 3$. The equivalence of (i) and (ii) follows from Proposition 3.1(iii). Now $\mathcal{O}_{Y_2} = \omega_{\omega_Y}$ and $\omega_Y = \omega_{Y_2}$ which proves that (ii) and (iii) are equivalent.

On the other hand, if X satisfies (i), Corollary 2.7 and Proposition 3.1 (ii) and (iii) implies (iv).

If (iv) is satisfied, Proposition 3.1(iii) and the equality $\omega_Y = \omega_{Y_2}$ shows that $H^i(Y, \mathcal{O}_{Y_2}(\mu)) \simeq H^{d-i}(Y, \omega_{Y_2}(-\mu))^*$ for $i \neq d - 1$. Applying Corollary 2.7 with $M := \Gamma_* \mathcal{O}_{Y_2}$ then implies that F_M has finite length, and therefore (iii) is satisfied. \square

REMARK 3.4 If one of the equivalent conditions of Proposition 3.3 is satisfied, and R and S are the standard graded unmixed algebras defining respectively X and Y , then $H^{d-1}(X, \mathcal{O}_X(\mu)) \simeq H_{\mathfrak{m}}^d(R)_\mu \simeq D_d(R)_{-\mu}$ and $H_{\mathfrak{m}}^0(D_d(R)) \simeq H_{\mathfrak{m}}^2(\omega_R) \simeq H_{\mathfrak{m}}^1(S)[a]$.

4 Cohomology of linked surfaces and three-folds

Let k be a field, A a polynomial ring over k and $\omega_A := A[-\dim A]$.

As in the first section, $-^*$ will denote the graded dual into k , we set $D_i(M) := \text{Ext}_A^{\dim A - i}(M, \omega_A)$ and we introduce for simplicity the following abbreviated notation : $D_{ij\dots l}(M) := D_i(D_j(\dots(D_l(M))))$.

4.1 The surface case

Let R be a homogeneous quotient of A defining a surface $X \subset \text{Proj}(A)$. We will assume that $H_m^0(R) = 0$, as replacing R by $R/H_m^0(R)$ leave both X and $H_m^i(R)$ for $i > 0$ unchanged. Then the three Cohen-Macaulay modules $D_1(R)$, $D_{02}(R)$ and $D_{12}(R)$ sits in the following exact sequences that shows how they encode all the non-ACM character of X :

$$0 \longrightarrow R \longrightarrow \Gamma R \longrightarrow D_1(R)^* \longrightarrow 0$$

$$0 \longrightarrow D_{02}(R)^* \longrightarrow D_2(R) \longrightarrow \Gamma D_2(R) \longrightarrow D_{12}(R)^* \longrightarrow 0$$

where $\Gamma D_2(R) = \bigoplus_{\mu \in \mathbf{Z}} V_\mu$ and $\delta := \dim V_\mu$ is a constant which is zero if and only if X is Cohen-Macaulay. Also $D_{02}(R)$ is a finite length module and $D_{12}(R)$ is either zero or a Cohen-Macaulay module of dimension one and degree δ .

The next result shows how these Cohen-Macaulay modules that encodes the non-ACM character of X behave under linkage.

PROPOSITION 4.1 *Let $X = \text{Proj}(R)$ and $Y = \text{Proj}(S)$ be two surfaces linked by an arithmetically Gorenstein subscheme $\text{Proj}(B) \subseteq \text{Proj}(A)$ so that $\omega_B \simeq B[a]$. Then there are natural degree zero isomorphisms,*

$$D_1(S) \simeq D_{02}(R) \otimes_B \omega_B, \quad D_{12}(S) \simeq D_{112}(R) \otimes_B \omega_B^{-1}.$$

Note that the roles of R and S may be reversed, so that $D_{02}(S) \simeq D_1(R) \otimes_B \omega_B^{-1}$ and $D_{112}(S) = D_{12}(R) \otimes_B \omega_B$.

Proof: By Proposition 3.1 (iii), $D_1(S) \simeq D_2(\omega_R) \otimes_B \omega_B$ and $D_2(\omega_R) \simeq D_{02}(R)$ by Corollary 2.4 (i), this proves the first claim.

As for the second claim, Corollary 2.4 (ii) gives an exact sequence

$$0 \longrightarrow \Gamma S \longrightarrow D_3(\omega_S) \longrightarrow D_{12}(S) \longrightarrow 0$$

where $D_3(\omega_S) \simeq E_S$. Now $\Gamma S/S \simeq D_1(S)^*$ so that we have an exact sequence,

$$0 \longrightarrow D_1(S)^* \longrightarrow E_S/S \longrightarrow D_{12}(S) \longrightarrow 0$$

but $E_S/S \simeq D_2(R) \otimes_B \omega_B^{-1}$ by Proposition 3.1 (ii) and we know that $D_1(S)^* \simeq D_{02}(R)^* \otimes_B \omega_B^{-1}$ so that the above sequence gives

$$0 \longrightarrow D_{02}(R)^* \otimes_B \omega_B^{-1} \longrightarrow D_2(R) \otimes_B \omega_B^{-1} \longrightarrow D_{12}(S) \longrightarrow 0$$

but $D_{02}(R)^* = H_m^0(D_2(R))$ and $D_{112}(R) = D_2(R)/H_m^0(D_2(R))$. □

4.2 The three-fold case

Let R be a homogeneous quotient of A defining a three-fold $X \subset \text{Proj}(A)$. As in the surface case will assume that $H_m^0(R) = 0$. Now the seven Cohen-Macaulay modules $D_1(R)$, $D_{02}(R)$ and $D_{12}(R)$, $D_{03}(R)$, $D_{013}(R)$, $D_{113}(R)$ and $D_{23}(R)$ also encode all the non-ACM character of X .

The following result essentially gives the behaviour of these modules under liaison.

PROPOSITION 4.2 *Let $X = \text{Proj}(R)$ and $Y = \text{Proj}(S)$ be two three-folds linked by an arithmetically Gorenstein subscheme $\text{Proj}(B) \subseteq \text{Proj}(A)$ so that $\omega_B \simeq B[a]$. Then there are natural degree zero isomorphisms,*

$$D_1(S) \simeq D_{03}(R) \otimes_B \omega_B, D_{12}(S) \simeq D_{113}(R) \otimes_B \omega_B^{-1}, D_{23}(S) \simeq D_{223}(R) \otimes_B \omega_B^{-1}.$$

And there is an exact sequence,

$$0 \longrightarrow D_{013}(R) \longrightarrow D_{02}(S) \otimes_B \omega_B \xrightarrow{\psi} D_{002}(R) \longrightarrow D_{0013}(S) \otimes_B \omega_B \longrightarrow 0.$$

where ψ is the composition of the maps from Remark 2.5 and Proposition 3.1 (iii):

$$D_{02}(S) \otimes_B \omega_B \xrightarrow{D_0(\lambda_2^{-1}) \otimes 1_{\omega_B}} D_{03}(\omega_R) \xrightarrow{\epsilon^*} D_{002}(R).$$

Note that the roles of R and S may be reversed, so that we have a complete list of relations between these fourteen modules. Also the last exact sequence may be written

$$0 \longrightarrow D_{013}(R) \longrightarrow D_{02}(S) \otimes_B \omega_B \xrightarrow{\psi} D_{02}(R)^* \longrightarrow D_{013}(S)^* \otimes_B \omega_B \longrightarrow 0.$$

Proof: $D_1(S) \simeq D_2(\omega_R) \otimes_B \omega_B \simeq D_{03}(R) \otimes_B \omega_B$ by Proposition 3.1 (iii) and Corollary 2.4 (i). Also Corollary 2.4 (iii) provides exact sequences:

$$(1) \quad 0 \longrightarrow K \xrightarrow{\text{can}} E_R \xrightarrow{\theta_{23}} D_{23}(R) \xrightarrow{\theta_{02}} D_{02}(R) \xrightarrow{e} D_3(\omega_R) \xrightarrow{\theta_{13}} D_{13}(R) \longrightarrow 0$$

and

$$(2) \quad 0 \longrightarrow \Gamma R \xrightarrow{\tau} K \xrightarrow{\eta} D_{12}(R) \longrightarrow 0.$$

From (2) we get isomorphisms $D_i(\tau) : D_i(K) \xrightarrow{\sim} D_i(R)$ for $i \geq 3$ and another exact sequence

$$(3) \quad 0 \longrightarrow D_2(K) \xrightarrow{D_2(\tau)} D_2(R) \xrightarrow{\delta} D_{112}(R) \xrightarrow{D_1(\eta)} D_1(K) \longrightarrow 0,$$

from which it follows that $D_1(K) = 0$ (note that this vanishing also follows upon taking Ext's on sequence (1)). As $D_{02}(R)$ is of dimension zero, the right end of the first sequence then shows that $D_1(\theta_{13}) : D_{113}(R) \xrightarrow{\sim} D_{13}(\omega_R)$, but we have an isomorphism $\lambda_2^{-1} \otimes 1_{\omega_B^{-1}} : D_3(\omega_R) \xrightarrow{\sim} D_2(S) \otimes_B \omega_B^{-1}$ by Proposition 3.1 (iii) and this provides the second isomorphism. Now taking Ext's on (1) also provides two exact sequences

$$(4) \quad 0 \longrightarrow D_{013}(R) \xrightarrow{D_0(\theta_{13})} D_{03}(\omega_R) \xrightarrow{\epsilon^*} D_{002}(R) \xrightarrow{can} L \longrightarrow 0$$

$$(5) \quad 0 \longrightarrow D_3(E_R) \longrightarrow D_3(R) \longrightarrow D_{223}(R) \longrightarrow D_2(E_R) \longrightarrow D_2(K) \xrightarrow{can} L' \longrightarrow 0.$$

and an isomorphism $\mu : L \rightarrow L'$. Now (5) together with (3) gives us a complex

$$(6) \quad 0 \rightarrow D_3(E_R) \rightarrow D_3(R) \rightarrow D_{223}(R) \rightarrow D_2(E_R) \rightarrow D_2(R) \rightarrow D_{112}(R) \rightarrow 0$$

whose only homology is the subquotient L of $D_2(R)$.

Taking Ext's on the exact sequence $0 \rightarrow R \rightarrow E_R \rightarrow D_3(S) \otimes_B \omega_B^{-1} \rightarrow 0$ given by Proposition 3.1 (ii), we get an exact sequence

$$(7) \quad 0 \rightarrow D_3(E_R) \rightarrow D_3(R) \rightarrow D_{23}(S) \otimes_B \omega_B \rightarrow D_2(E_R) \\ \rightarrow D_2(R) \rightarrow D_{13}(S) \otimes_B \omega_B \rightarrow 0.$$

Comparing (6) and (7) proves the third isomorphism and provides the exact sequence

$$(8) \quad 0 \rightarrow L \rightarrow D_{13}(S) \otimes_B \omega_B \rightarrow D_{112}(R) \rightarrow 0$$

showing that $L \simeq D_{0013}(S) \otimes_B \omega_B$ because $D_{112}(R) \simeq D_{1113}(S) \otimes_B \omega_B$. This last identification together with (4) provides the exact sequence of the Proposition and concludes the proof. \square

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