

# Implicitizing rational hypersurfaces using approximation complexes

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## Abstract

In this paper we describe an algorithm for implicitizing rational hypersurfaces in case there exists at most a finite number of base points. It is based on a technique exposed in Busé and Jouanolou [2002], where implicit equations are obtained as determinants of certain graded parts of a so-called approximation complex. We detail and improve this method by providing an in-depth study of the cohomology of such a complex. In both particular cases of interest of curve and surface implicitization we also yield explicit algorithms which only involves linear algebra routines.

## 1. Introduction

The implicitization problem asks for an implicit equation of a rational hypersurface given by a parameterization map  $\phi : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^n$ , with  $n \geq 2$ . This problem received recently a particular interest, especially in the cases  $n = 2$  and  $n = 3$ , because it is a key-point in computer aided geometric design and modeling. There are basically three kinds of methods to compute such an implicit equation of rational curves and surfaces. The first methods use Gröbner bases computations. Even if they always work, they are known to be quite slow in practice and are hence rarely used in geometric modeling (see e.g. Hoffmann [1989]). The second methods are based on resultant matrices. Such methods have the advantage to yield square matrices whose determinant is an implicit equation. This more compact formulation of an implicit equation is very useful

<sup>\*</sup>Partially supported by european projects GAIA II IST-2001-35512, and ECG IST-2000-26473.

and well behaved algorithms are known to work with. However such methods are only known if there is no base point (see Jouanolou [1996]), and in case base points are isolated and local complete intersection with some additional technical hypothesis (see Busé [2001]). The third methods are based on the syzygies of the parameterization. They were introduced in Sederberg and Chen [1995] as the method of “moving surfaces”. For curve implicitization this method, also called “moving lines”, reveals very efficient and general. This generality is no longer true for surface implicitization, but the method remains very efficient. In Cox et al. [2000] and D’Andréa [2001] its validity was proved in the absence of base point. An extension in the presence of base points is exposed in Busé et al. [2002], assuming five base points hypothesis, one being that the base points are isolated and locally complete intersection.

A similar approach involving syzygies was recently explored in Busé and Jouanolou [2002] using more systematic tools from algebraic geometry and commutative algebra, among them the so-called approximation complexes (see Vasconcelos [1994]). A new method was given to solve the hypersurface (and hence curve and surface) implicitization problem in case base points are isolated and locally complete intersection, without any additional hypothesis.

In this paper we detail and improve the results exposed in Busé and Jouanolou [2002] for implicitizing rational hypersurfaces in case the ideal of base points is finite and a projective local almost complete intersection. We also give some other properties based on the exactness on these so-called approximation complexes. Section 2 and section 3 provide a complete and detailed description of our results in the particular cases of interest of respectively curve and surface implicitizations. Algorithms are completely describe and we present some examples. Section 4 deals with the general case of hypersurface implicitization, and is much more technical that both previous one. In particular all the proofs are given in this section.

Hereafter  $\mathbb{K}$  denotes any field.

## 2. Implicitization of rational parametric curves

In this section we present a method to implicitize any parameterized plane curve. Let  $f_0, f_1, f_2$  be three homogeneous polynomials in  $\mathbb{K}[s, t]$  of the same degree  $d \geq 1$ , and consider the rational map

$$\begin{aligned} \phi : \quad \mathbb{P}_{\mathbb{K}}^1 &\rightarrow \mathbb{P}_{\mathbb{K}}^2 \\ (s : t) &\mapsto (f_0(s, t) : f_1(s, t) : f_2(s, t)). \end{aligned}$$

We denote by  $x, y, z$  the homogeneous coordinates of  $\mathbb{P}^2$ . The algebraic closure of the image of  $\phi$  is a curve in  $\mathbb{P}^2$  if and only if  $\phi$  is generically finite onto its image, that is  $\delta := \gcd(f_0, f_1, f_2)$  is not of degree  $d$ . We assume this minimal hypothesis and denote by  $\mathcal{C}$  the scheme-theoretic closed image of  $\phi$ . It is known that  $\mathcal{C}$  is an irreducible and reduced curve of degree  $(d - \deg(\delta))/\beta$ , where  $\beta$

denotes the degree of  $\phi$  onto its image (that is the number of points in a general fiber). Thus any implicit equation of  $\mathcal{C}$  is of the form  $P(x, y, z)$  where  $P$  is an irreducible homogeneous polynomial of degree  $(d - \delta)/\beta$ .

In what follows we describe an algorithm to compute explicitly an implicit equation  $P$  of  $\mathcal{C}$  (in fact we will compute  $P^\beta$ ) without any other hypothesis on the polynomials  $f_0, f_1, f_2$ . In the case  $\deg(\delta) = 0$  we recover the well-known method of moving lines (see Sederberg and Chen [1995], Cox et al. [1998]).

### 2.1. The method

Let us denote by  $A$  the polynomial ring  $\mathbb{K}[s, t]$ . We consider it with its natural graduation obtained by setting  $\deg(s) = \deg(t) = 1$ . From the polynomials  $f_0, f_1, f_2$  of the parameterization  $\phi$ , we can build the following well-known graded Koszul complex (the notation  $[-]$  stands for the degree shift in  $A$ )

$$0 \rightarrow A[-3d] \xrightarrow{d_3} A[-2d]^3 \xrightarrow{d_2} A[-d]^3 \xrightarrow{d_1} A, \quad (1)$$

where the differentials are given by

$$d_3 = \begin{pmatrix} f_2 \\ -f_1 \\ f_0 \end{pmatrix}, d_2 = \begin{pmatrix} f_1 & f_2 & 0 \\ -f_0 & 0 & f_2 \\ 0 & -f_0 & -f_1 \end{pmatrix}, d_1 = (f_0 \ f_1 \ f_2).$$

In what follows we will not consider exactly this complex, but the complex obtained by tensorizing it by  $A[x, y, z]$  over  $A$ . This complex, that we denote  $(K_\bullet(f_0, f_1, f_2), u_\bullet)$ , is of the form

$$0 \rightarrow A[x, y, z][-3d] \xrightarrow{u_3} A[x, y, z][-2d]^3 \xrightarrow{u_2} A[x, y, z][-d]^3 \xrightarrow{u_1} A[x, y, z],$$

where the matrices of the differentials  $d_i$  and  $u_i$  are the same, for all  $i = 1, 2, 3$ . Note that the ring  $A[x, y, z]$  is naturally bi-graded, having a graduation coming from  $A = \mathbb{K}[s, t]$ , and another one coming from  $\mathbb{K}[x, y, z]$  with  $\deg(x) = \deg(y) = \deg(z) = 1$ ; we hereafter adopt the notation  $(-)$  for the degree shift in  $\mathbb{K}[x, y, z]$ .

We form another bi-graded Koszul complex on  $A[x, y, z]$ , the one associated to the sequence  $(x, y, z)$ . We denote it by  $(K_\bullet(x, y, z), v_\bullet)$ , it is of the form

$$0 \rightarrow A[x, y, z](-3) \xrightarrow{v_3} A[x, y, z](-2)^3 \xrightarrow{v_2} A[x, y, z](-1)^3 \xrightarrow{v_1} A[x, y, z],$$

and the matrices of its differentials are obtained from the matrices of the differentials of (1) by replacing  $f_0$  by  $x$ ,  $f_1$  by  $y$  and  $f_2$  by  $z$ . Observe that since  $(x, y, z)$  is a regular sequence in  $A[x, y, z]$ , the previous complex  $K_\bullet(x, y, z)$  is acyclic, that is to say all its homology groups  $H_i(K_\bullet(x, y, z))$  vanish for  $i > 0$ .

We can now construct a new bi-graded complex of  $A[x, y, z]$ -modules, denoted  $\mathcal{Z}_\bullet$ , from both Koszul complexes  $(K_\bullet(f_0, f_1, f_2), u_\bullet)$  and  $(K_\bullet(x, y, z), v_\bullet)$  (observe that these complexes differ only by their differentials). Define  $Z_i := \ker(d_i)$  for all  $i = 0, \dots, 3$  (with  $d_0 : A \rightarrow 0$ ), and set  $\mathcal{Z}_i := Z_i[id] \otimes_A A[x, y, z]$  for  $i = 0, \dots, 3$ ,

which are bi-graded  $A[x, y, z]$ -modules. The map  $v_1$  induced obviously the bi-graded map, that we denote also by  $v_1$ ,

$$\begin{aligned} \mathcal{Z}_1(-1) &\xrightarrow{v_1} \mathcal{Z}_0 = A[x, y, z] \\ (g_1, g_2, g_3) &\mapsto g_1x + g_2y + g_3z. \end{aligned}$$

Using the differential  $v_2$  of  $K_\bullet(x, y, z)$  we can map  $\mathcal{Z}_2$  to  $A[x, y, z](-d)^3$ , but since  $u_1 \circ v_2 + v_1 \circ u_2 = 0$  (which follows from a straightforward computation), we have  $v_2(\mathcal{Z}_2) \subset \mathcal{Z}_1$ . And in the same way we can map  $\mathcal{Z}_3$  to  $\mathcal{Z}_2$  with the differential  $v_3$ , since  $u_2 \circ v_3 + v_2 \circ u_3 = 0$ . Thus we obtain the following bi-graded complex (it is a complex since  $(K_\bullet(x, y, z), v_\bullet)$  is) :

$$(\mathcal{Z}_\bullet, v_\bullet) : 0 \rightarrow \mathcal{Z}_3(-3) \xrightarrow{v_3} \mathcal{Z}_2(-2) \xrightarrow{v_2} \mathcal{Z}_1(-1) \xrightarrow{v_1} \mathcal{Z}_0 = A[x, y, z].$$

This complex is known as the *approximation complex of cycles* associated to the polynomials  $f_0, f_1, f_2$  in  $\mathbb{K}[s, t]$ . It was originally introduced in Simis and Vasconcelos [1981] for studying Rees algebras through symmetric algebras (see also Vasconcelos [1994]).

*Remark:* Point out that in the language of the moving lines method, the image of a triple  $(g_1, g_2, g_3) \in \mathcal{Z}_{1[\nu](0)}$  by  $v_1$  is nothing but a moving line of degree  $\nu$  following the surface parameterized by the polynomials  $f_0, f_1, f_2$  (see e.g. Cox [2001]).

Since we have supposed that  $f_0, f_1, f_2$  are all non-zero we have  $\mathcal{Z}_3 = 0$ , and consequently  $\mathcal{Z}_3 = 0$ , and the following theorem (recall  $\delta := \gcd(f_0, f_1, f_2)$ ) :

**THEOREM 2.1:** *The determinant of the graded complex of free  $\mathbb{K}[x, y, z]$ -modules*

$$0 \rightarrow \mathcal{Z}_{2[d-1]}(-2) \xrightarrow{v_2} \mathcal{Z}_{1[d-1]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[d-1]} = A[x, y, z]_{[d-1]},$$

*is  $P(x, y, z)^\beta$ , where  $P$  is an implicit equation of the curve  $\mathcal{C}$ .*

*Moreover, if  $\deg(\delta) = 0$  then  $\mathcal{Z}_{2[d-1]} = 0$ ; thus the  $d \times d$  determinant of the map  $\mathcal{Z}_{1[d-1]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[d-1]}$  equals  $P(x, y, z)^\beta$ .*

*Proof:* See section 5 in Busé and Jouanolou [2002] for a proof. We only mention how the last statement follows from the description of the approximation complex  $\mathcal{Z}_\bullet$ . If  $\deg(\delta) = 0$  then  $\text{depth}_{(s,t)}(f_0, f_1, f_2) = 2$ , which means that  $f_0, f_1, f_2$  have no base points in  $\mathbb{P}^1$ . This implies that not only the third homology group of the Koszul complex (1) vanishes, but also the second. It follows that  $\mathcal{Z}_2 \simeq A[-3d]$ , and hence  $\mathcal{Z}_2 \simeq A[x, y, z][-d]$ .  $\square$

The second statement of this theorem gives exactly the matrix constructed by the method of moving lines. The first statement show that this method can be extended even if we do not assume  $\deg(\delta) = 0$ ,  $P^\beta$  being obtained as the quotient of two determinants of respective size  $d$  and  $\delta$ . In Busé and Jouanolou [2002] it is in fact proved that for any integer  $\nu \geq d - 1$  the determinant of the complex  $(\mathcal{Z}_\bullet)_{[\nu]}$  equals  $P^\beta$ . In case  $\deg(\delta) = 0$ , this and the graded isomorphism  $\mathcal{Z}_2 \simeq A[x, y, z][-d]$  explain clearly, in our point of view, why the method of moving lines works so well with and *only* with moving lines of degree  $d - 1$ .

## 2.2. The algorithm

We here convert theorem 2.1 into an explicit algorithm where each step reduces to well-known and efficient linear algebra routines.

ALGORITHM (for implicitizing rational parametric curves):

**Input :** Three homogeneous polynomials  $f_0(s, t), f_1(s, t), f_2(s, t)$  of the same degree  $d \geq 1$  such that  $\delta = \gcd(f_0, f_1, f_2)$  is not of degree  $d$ .

**Output :** Either :

- a square matrix  $\Delta_1$  such that  $\det(\Delta_1)$  equals  $P^\beta$ , in case  $\deg(\delta) = 0$ ,
- two square matrices  $\Delta_1$  and  $\Delta_2$ , respectively of size  $d$  and  $\deg(\delta)$ , such that  $\frac{\det(\Delta_1)}{\det(\Delta_2)}$  equals  $P^\beta$ .

1. Compute the matrix  $F_1$  of the first map of (1) :  $A_{d-1}^3 \xrightarrow{d_1} A_{2d-1}$ . Its entries are either 0 or a coefficient of  $f_0, f_1$  or  $f_2$ , and it is of size  $2d \times 3d$ .
2. Compute a kernel matrix  $K_1$  of the transpose of  $F_1$ . It has  $3d$  columns and  $\text{rank}(\mathcal{Z}_{1[d-1]})$  lines.
3. Construct the matrix  $Z_1$  by  $Z_1(i, j) = xK_1(j, i) + yK_1(j, i + d) + zK_1(j, i + 2d)$ , with  $i = 1, \dots, d$  and  $j = 1, \dots, \text{rank}(\mathcal{Z}_{1[d-1]})$ . It is a matrix of the map  $\mathcal{Z}_{1[d-1]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[d-1]}$ .
4. **If**  $Z_1$  is square **then** set  $\Delta_1 := Z_1$  **else**

(a) Compute a list  $L_1$  of  $d$  integers indexing  $d$  independent columns in  $Z_1$ . Let  $\Delta_1$  be the  $d \times d$  submatrix of  $Z_1$  obtained by removing columns not in  $L_1$ .

(b) Compute the matrix  $F_2$  of the second map of (1) :  $A_{d-1}^3 \xrightarrow{d_2} A_{2d-1}^3$ , and a kernel matrix  $K_2$  of its transpose. The matrix  $K_2$  has  $3d$  columns and  $\text{rank}(\mathcal{Z}_{2[d-1]})$  lines.

(c) Construct the matrix  $Z'_2$  by, for all  $j = 1, \dots, \text{rank}(\mathcal{Z}_{2[d-1]})$ ,

$$\begin{aligned} i = 1, \dots, d & : Z'_2(i, j) = yK_2(j, i) + zK_2(j, i + d), \\ i = d + 1, \dots, 2d & : Z'_2(i, j) = -xK_2(j, i - d) + zK_2(j, i + d), \\ i = 2d + 1, \dots, 3d & : Z'_2(i, j) = -xK_2(j, i - d) - yK_2(j, i). \end{aligned}$$

(d) Construct the  $\text{rank}(\mathcal{Z}_{1[d-1]}) \times \text{rank}(\mathcal{Z}_{2[d-1]})$  matrix  $Z_2$  whose  $j^{\text{th}}$  column  $Z_2(\bullet, j)$  is the solution of the linear system  ${}^tZ_2(\bullet, j) \cdot K_1 = Z'_2(\bullet, j)$ . It is a matrix of the map  $\mathcal{Z}_{2[d-1]}(-1) \xrightarrow{v_2} \mathcal{Z}_{1[d-1]}$ .

(e) Define  $\Delta_2$  to be the square submatrix of  $Z_2$  obtained by removing the lines indexed by  $L_1$ .

**endif**

*Remark:* In order to keep this algorithm easily understandable we kept the symbolic variables  $t$  and  $z$ , but they are of course useless and should be specialized to 1.

### 2.3. An example

As we have already said, the previous algorithm is exactly the well-known method of moving lines in case  $\deg(\delta) = 0$ . The only thing new is its extension to the case where  $\deg(\delta) > 0$ . We illustrate it with the following (very simple) example.

Let  $f_0(s, t) = s^2$ ,  $f_1(s, t) = st$  and  $f_2(s, t) = t^2$ . Applying our algorithm we found that the matrix  $Z_1$  (step 3) is square :

$$Z_1 = \begin{pmatrix} -y & -z \\ x & y \end{pmatrix}.$$

We deduce that an implicit equation is given by  $xz - y^2$ . Now let us multiply artificially each  $f_i$  by  $s$ . We obtain new entries for our algorithm which are :  $f_0(s, t) = s^3$ ,  $f_1(s, t) = s^2t$  and  $f_2(s, t) = st^2$ . In this case the matrix  $Z_1$  is not square, it is the  $3 \times 4$  matrix :

$$Z_1 = \begin{pmatrix} -y & -z & -z & 0 \\ x & 0 & y & -z \\ 0 & x & 0 & y \end{pmatrix}.$$

The matrix  $\Delta_1$  can be chosen to be the submatrix of  $Z_1$  given by  $\begin{pmatrix} -y & -z & -z \\ x & 0 & y \\ 0 & x & 0 \end{pmatrix}$ ,

with determinant  $-x^2z + xy^2$ . Continuing the algorithm we obtain the matrices  $Z_2 = \begin{pmatrix} -z \\ y \\ 0 \\ -x \end{pmatrix}$ , and  $\Delta_2 = (-x)$ . It follows that an implicit equation is  $-xz + y^2$ .

### 3. Implicitization of rational parametric surfaces

We arrive to the much more intricate problem of surface implicitization. Let  $f_0, f_1, f_2, f_3$  be four homogeneous polynomials in  $\mathbb{K}[s, t, u]$  of the same degree  $d \geq 1$ , and consider the rational map

$$\begin{aligned} \phi : \quad \mathbb{P}_{\mathbb{K}}^2 &\rightarrow \mathbb{P}_{\mathbb{K}}^3 \\ (s : t : u) &\mapsto (f_0(s, t, u) : f_1(s, t, u) : f_2(s, t, u) : f_3(s, t, u)). \end{aligned}$$

We denote by  $x, y, z, w$  the homogeneous coordinates of  $\mathbb{P}^3$ . The closure of the image of  $\phi$  is a surface if and only if the map  $\phi$  is generically finite onto its image, which we assume hereafter. Thus let  $\mathcal{S}$  denote the surface in  $\mathbb{P}^3$  obtained as the closed image of  $\phi$ . Our problem is to compute an implicit equation of  $\mathcal{S}$ . It is more difficult than curve implicitization because of the presence of base points in codimension 2 arising from the parameterization, base points in codimension 1 being easily removed by substituting each  $f_i$ ,  $i = 0, \dots, 3$ , by  $f_i/\delta$ , where  $\delta := \gcd(f_0, f_1, f_2, f_3)$ , if necessary. Consequently, we assume that the ideal  $I =$

$(f_0, f_1, f_2, f_3)$  in  $\mathbb{K}[s, t, u]$  is at least of codimension 2, that is defines  $n$  isolated points  $p_1, \dots, p_n$  in  $\mathbb{P}^2$ , and we have (see section 2 and theorem 2.5 in Busé and Jouanolou [2002] for a proof and more details) :

**THEOREM 3.1:** *Let  $e_{p_i}$  denote the algebraic multiplicity of the point  $p_i$  for all  $i = 1, \dots, n$ , and  $\beta$  denote the degree of  $\phi$  onto its image. Then*

$$d^2 - \sum_{i=1}^n e_{p_i} = \begin{cases} \beta \deg(\mathcal{S}) & \text{if } \phi \text{ is generically finite,} \\ 0 & \text{if } \phi \text{ is not generically finite.} \end{cases}$$

This theorem gives us the degree of the closed image of  $\phi$  if it is a surface (that we have denoted by  $\mathcal{S}$ ), and also that this image is not a surface if and only if  $d^2 = \sum_{i=1}^n e_{p_i}$ . We will call an *implicit equation* of  $\mathcal{S}$  an equation of its associated divisor, which is an irreducible and homogeneous polynomial  $P(x, y, z, w)$  of total degree  $(d^2 - \sum_{i=1}^n e_{p_i})/\beta$ .

In the following subsections we mainly present a method based on the approximation complexes to compute explicitly an implicit equation of  $\mathcal{S}$  (in fact we will compute  $P^\beta$ ) in case the ideal  $I$  is a local complete intersection of codimension at least 2, and we also give some other related results. Then we expose an explicit description of the algorithm it yields and illustrate it with some examples.

### 3.1. The method

We denote by  $A$  the polynomial ring  $\mathbb{K}[s, t, u]$  which is naturally graded by  $\deg(s) = \deg(t) = \deg(u) = 1$ . Let us form the Koszul complex of  $f_0, f_1, f_2, f_3$  in  $A$  :

$$0 \rightarrow A[-4d] \xrightarrow{d_4} A[-3d]^4 \xrightarrow{d_3} A[-2d]^6 \xrightarrow{d_2} A[-d]^4 \xrightarrow{d_1} A, \quad (2)$$

where the differentials are given by

$$d_4 = \begin{pmatrix} -f_3 \\ f_2 \\ -f_1 \\ f_0 \end{pmatrix}, \quad d_3 = \begin{pmatrix} f_2 & f_3 & 0 & 0 \\ -f_1 & 0 & f_3 & 0 \\ f_0 & 0 & 0 & f_3 \\ 0 & -f_1 & -f_2 & 0 \\ 0 & f_0 & 0 & -f_2 \\ 0 & 0 & f_0 & f_1 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -f_1 & -f_2 & 0 & -f_3 & 0 & 0 \\ f_0 & 0 & -f_2 & 0 & -f_3 & 0 \\ 0 & f_0 & f_1 & 0 & 0 & -f_3 \\ 0 & 0 & 0 & f_0 & f_1 & f_2 \end{pmatrix}, \quad d_1 = (f_0 \ f_1 \ f_2 \ f_3).$$

As for curve implicitization, we denote by  $(K_\bullet(f_0, f_1, f_2, f_3), u_\bullet)$  this Koszul complex tensorized by  $A[\underline{x}] := A[x, y, z, w]$  over  $A$ , which is of the form :

$$0 \rightarrow A[\underline{x}][-4d] \xrightarrow{u_4} A[\underline{x}][-3d]^4 \xrightarrow{u_3} A[\underline{x}][-2d]^6 \xrightarrow{u_2} A[\underline{x}][-d]^4 \xrightarrow{u_1} A[\underline{x}],$$

where the matrices of the differentials  $d_i$  and  $u_i$  are the same,  $i = 1, 2, 3, 4$  (here again we set  $\deg(x) = \deg(y) = \deg(z) = \deg(w) = 1$ ). We also consider the bi-graded Koszul complex on  $A[\underline{x}]$  associated to the sequence  $(x, y, z, w)$ , and denote it  $(K_\bullet(x, y, z, w), v_\bullet)$  :

$$0 \rightarrow A[\underline{x}](-4) \xrightarrow{v_4} A[\underline{x}](-3)^4 \xrightarrow{v_3} A[\underline{x}](-2)^6 \xrightarrow{v_2} A[\underline{x}](-1)^4 \xrightarrow{v_1} A[\underline{x}].$$

The matrices of its differentials are obtained from the matrices of the differentials of (2) by replacing  $f_0$  by  $x$ ,  $f_1$  by  $y$ ,  $f_2$  by  $z$  and  $f_3$  by  $w$ . Note that since  $(x, y, z, w)$  is a regular sequence in  $A[\underline{x}]$ , the complex  $K_\bullet(x, y, z, w)$  is acyclic. From both Koszul complexes  $(K_\bullet(f_0, f_1, f_2, f_3), u_\bullet)$  and  $(K_\bullet(x, y, z, w), v_\bullet)$  we can build, as we did for curves, the approximation complex  $\mathcal{Z}_\bullet$ . We define  $Z_i := \ker(d_i)$  and  $\mathcal{Z}_i := Z_i[id] \otimes_A A[\underline{x}]$  for all  $i = 0, 1, 2, 3, 4$  (where  $d_0 : A \rightarrow 0$ ), they are naturally bi-graded  $A[\underline{x}]$ -modules. Since for all  $i = 1, 2, 3$  we have  $u_i \circ v_{i+1} + v_i \circ u_{i+1} = 0$ , we obtain the bi-graded complex :

$$(\mathcal{Z}_\bullet, v_\bullet) : 0 \rightarrow \mathcal{Z}_4(-4) \xrightarrow{v_4} \mathcal{Z}_3(-3) \xrightarrow{v_3} \mathcal{Z}_2(-2) \xrightarrow{v_2} \mathcal{Z}_1(-1) \xrightarrow{v_1} \mathcal{Z}_0 = A[\underline{x}],$$

where, since we have supposed  $d \geq 1$ ,  $\mathcal{Z}_4 = 0$ .

*Remark:* In the language of the moving surfaces method (see Sederberg and Chen [1995], Cox [2001]), an element  $(g_1, g_2, g_3, g_4) \in \mathcal{Z}_{1[\nu](0)}$  is nothing but a moving hyperplane of degree  $\nu$  following the surface  $\mathcal{S}$ .

To state the main result of this section, we need some notations. If  $p$  is an isolated base point defined by the ideal  $I$ , we denote by  $d_p$  its geometric multiplicity (also called its degree); note that we have already denoted by  $e_p$  its algebraic multiplicity. Recall that if  $M$  is a  $\mathbb{Z}$ -graded  $R$ -module, where  $R$  is a  $\mathbb{Z}$ -graded ring, its initial degree is defined as  $\text{indeg}(M) = \min\{\nu \in \mathbb{Z} : M_\nu \neq 0\}$ .

**THEOREM 3.2:** *Suppose that the ideal  $I = (f_0, f_1, f_2, f_3) \subset A$  is of codimension at least 2 and  $\mathbb{K}$  is infinite. Let  $\mathcal{P} := \text{Proj}(A/I)$  and denote by  $I_{\mathcal{P}}$  the saturated ideal of  $I$  w.r.t. the maximal ideal  $\mathfrak{m} = (s, t, u)$  of  $A$ . Then,*

- *The complex  $\mathcal{Z}_\bullet$  is acyclic if and only if  $\mathcal{P}$  is locally generated by (at most) 3 elements.*

- *Assume that  $\mathcal{P}$  is locally generated by 3 elements, then for all integer*

$$\nu \geq \nu_0 := 2(d-1) - \text{indeg}(I_{\mathcal{P}})$$

*the determinant of the graded complex of free  $\mathbb{K}[\underline{x}]$ -modules*

$$0 \rightarrow \mathcal{Z}_{3[\nu]}(-3) \xrightarrow{v_3} \mathcal{Z}_{2[\nu]}(-2) \xrightarrow{v_2} \mathcal{Z}_{1[\nu]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[\nu]} = A_{[\nu]}[\underline{x}]$$

*is an homogeneous element of  $\mathbb{K}[\underline{x}]$  of degree  $d^2 - \sum_{p \in \mathcal{P}} d_p$ , and is a multiple of  $P^\beta$  independent of  $\nu$ , where  $P$  is an implicit equation of  $\mathcal{S}$ . It is exactly  $P^\beta$  if and only if  $I$  is locally a complete intersection. Moreover, for all  $\nu \in \mathbb{Z}$ ,  $\mathcal{Z}_{3[\nu]}$  is always a free  $\mathbb{K}[\underline{x}]$ -module of rank  $\max\binom{\nu-d+2}{2}, 0$ ; in particular  $\mathcal{Z}_{3[\nu]} = 0$  if and only if  $\nu \leq d-1$ .*

*Proof:* See theorem 4.1 for the complete proof. Note that an argument similar to the one given in the proof of theorem 2.1 shows easily that  $\mathcal{Z}_3 \simeq A[\underline{x}][-d]$ , and hence the last statement of the second point of the theorem.  $\square$

By standard properties of determinants of complexes (see e.g. appendix A in Gelfand et al. [1994]) we deduce the

**COROLLARY 3.1:** *Suppose that  $I = (f_0, f_1, f_2, f_3) \subset A$  is of codimension at least 2,  $\mathbb{K}$  is infinite, and  $\mathcal{P} = \text{Proj}(A/I)$  is locally generated by 3 elements. Then for all  $\nu \geq \nu_0 := 2(d-1) - \text{indeg}(I_{\mathcal{P}})$  any non-zero minor of (maximal) size  $(\nu+2)(\nu+1)/2$  of the surjective matrix*

$$\begin{array}{ccc} \mathcal{Z}_{1[\nu]}(-1) & \xrightarrow{\nu_1} & A_{[\nu]}[\underline{x}] \\ (g_1, g_2, g_3, g_4) & \mapsto & xg_1 + yg_2 + zg_3 + wg_4 \end{array}$$

*is a non-zero multiple of  $P^\beta$ . Moreover, if  $\mathcal{P}$  is locally a complete intersection, then the gcd of all these minors equals  $P^\beta$ .*

Before going through an algorithmic version of this theorem we make few remarks on the integer  $\nu_0$ . First note that  $\nu_0$  depends geometrically on the ideal  $I$ , since  $\text{indeg}(I_{\mathcal{P}})$  is the smallest degree of a hypersurface in  $\mathbb{P}^2$  containing the closed subscheme defined by  $I$ . Let us observe how the initial degree of  $I_{\mathcal{P}}$  behaves. If  $I$  has no base points, that is  $I_{\mathcal{P}} = A$ , then  $\text{indeg}(I_{\mathcal{P}}) = 0$ . If there exists base points then this initial degree is always greater or equal to 1 since  $I_{\mathcal{P}}$  is generated in degree at least 1, and is always bounded by  $d$  since the  $f_i$ 's are in  $I_{\mathcal{P}}$ . Also if  $I$  is saturated then its initial degree is exactly  $d$ . We deduce that

- $\nu_0 = 2d - 2$  if  $I$  has no base points,
- $d - 2 \leq \nu_0 \leq 2d - 3$  if  $I$  has base points,
- $\nu_0 = d - 2$  if  $I$  is saturated (note that in this case we know that  $\mathcal{Z}_{3[\nu_0]} = 0$ , and hence  $\det(\mathcal{Z}_{\bullet[\nu_0]})$  is always obtained as a single determinant or as a quotient of two determinants).

This shows in particular that the presence of base points simplify the complexity of the computation of the implicit surface. Finally recall that the explicit bound (and not the theoretical one) given in Busé and Jouanolou [2002] is only  $2d - 2$ , if there exists base points or not.

### 3.2. The algorithm

We now develop the algorithm suggested by theorem 3.2, and then discuss some computational aspects.

**ALGORITHM** (for implicitizing rational parametric surfaces with local almost complete intersection isolated base points, possibly empty):

**Input :** Four homogeneous polynomials  $f_0, f_1, f_2, f_3$  in  $A$  of the same degree  $d \geq 1$  such that the ideal  $(f_0, f_1, f_2, f_3) \subset A$  is locally generated by 3 elements

outside  $V(s, t, u)$  and at least of codimension 2. An integer  $\nu$  with default value  $\nu := 2d - 2$ .

**Output :** Either :

- a square matrix  $\Delta_1$  such that  $\det(\Delta_1)$  is the determinant of  $(\mathcal{Z}_\bullet)_{[\nu]}$ ,
- two square matrices  $\Delta_1$  and  $\Delta_2$  such that  $\frac{\det(\Delta_1)}{\det(\Delta_2)}$  is the determinant of  $(\mathcal{Z}_\bullet)_{[\nu]}$ ,
- three square matrices  $\Delta_1, \Delta_2$  and  $\Delta_3$  such that  $\frac{\det(\Delta_1)\det(\Delta_3)}{\det(\Delta_2)}$  is the determinant of  $(\mathcal{Z}_\bullet)_{[\nu]}$ .

1. Compute the matrix  $F_1$  of the first map of (2) :  $A_\nu^4 \xrightarrow{d_1} A_{\nu+d}$ , and a kernel  $K_1$  of its transpose which has  $\text{rank}(\mathcal{Z}_{1[\nu]})$  lines.
2. Set  $m := \frac{(\nu+2)(\nu+1)}{2}$ . Construct the matrix  $Z_1$  defined by :

$$Z_1(i, j) = xK_1(j, i) + yK_1(j, i + m) + zK_1(j, i + 2m) + wK_1(j, i + 3m),$$

with  $i = 1, \dots, m$  and  $j = 1, \dots, \text{rank}(\mathcal{Z}_{1[\nu]})$ . It is the matrix of the map  $\mathcal{Z}_{1[\nu]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[\nu]}$ .

3. **If**  $Z_1$  is square **then** set  $\Delta_1 := Z_1$  **else**

- (a) Compute a list  $L_1$  of integers indexing independent columns in  $Z_1$ .  $L_1$  consists in  $m$  integers. Let  $\Delta_1$  be the  $m \times m$  submatrix of  $Z_1$  obtained by removing columns not in  $L_1$ .
- (b) Compute the matrix  $F_2$  of the second map of (2) :  $A_\nu^6 \xrightarrow{d_2} A_{\nu+d}^3$ , and a kernel  $K_2$  of its transpose which has  $\text{rank}(\mathcal{Z}_{2[\nu]})$  lines.
- (c) Construct the matrix  $Z'_2$  defined by, for all  $j = 1, \dots, \text{rank}(\mathcal{Z}_{2[\nu]})$ ,

$$\begin{aligned} Z'_2(i, j) &= -yK_2(j, i) - zK_2(j, i + m) - wK_2(j, i + 3m), i = 1, \dots, m, \\ Z'_2(i, j) &= xK_2(j, i - m) - zK_2(j, i + m) - wK_2(j, i + 3m), i = m + 1, \dots, 2m, \\ Z'_2(i, j) &= xK_2(j, i - m) + yK_2(j, i) - wK_2(j, i + 3m), i = 2m + 1, \dots, 3m, \\ Z'_2(i, j) &= xK_2(j, i) + yK_2(j, i + m) + zK_2(j, i + 2m), i = 3m + 1, \dots, 4m. \end{aligned}$$

- (d) Construct the  $\text{rank}(\mathcal{Z}_{1[\nu]}) \times \text{rank}(\mathcal{Z}_{2[\nu]})$  matrix  $Z_2$  whose  $j^{\text{th}}$  column  $Z_2(\bullet, j)$  is the solution of the linear system  ${}^tZ_2(\bullet, j) \cdot K_1 = Z'_2(\bullet, j)$ . It is the matrix of the map  $\mathcal{Z}_{2[\nu]}(-1) \xrightarrow{v_2} \mathcal{Z}_{1[\nu]}$ .
- (e) Define  $\Delta'_2$  to be the submatrix of  $Z_2$  obtained by removing the lines indexed by  $L_1$ .

**If**  $\Delta'_2$  is square **then** set  $\Delta_2 := \Delta'_2$  **else**

- i. Compute a list  $L_2$  of integers indexing independent columns in  $\Delta'_2$ . Define  $\Delta_2$  to be the square submatrix of  $\Delta'_2$  obtained by removing columns not in  $L_2$ .
- ii. Construct the matrix  $F_3$  of the third map of (2) :  $A_\nu^4 \xrightarrow{d_3} A_{\nu+d}^6$ , and the kernel  $K_3$  of its transpose which has  $\text{rank}(\mathcal{Z}_{3[\nu]})$  lines.

iii. Construct the matrix  $Z'_3$  defined by, for all  $j = 1, \dots, \text{rank}(\mathcal{Z}_{3[\nu]})$ ,

$$\begin{aligned} Z'_3(i, j) &= zK_3(j, i) + wK_3(j, i + m), i = 1, \dots, m, \\ Z'_3(i, j) &= -yK_3(j, i - m) + wK_3(j, i + m), i = m + 1, \dots, 2m, \\ Z'_3(i, j) &= xK_3(j, i - 2m) + wK_3(j, i + m), i = 2m + 1, \dots, 3m, \\ Z'_3(i, j) &= -yK_3(j, i - 2m) - zK_3(j, i - m), i = 3m + 1, \dots, 4m, \\ Z'_3(i, j) &= xK_3(j, i - 3m) - zK_3(j, i - m), i = 4m + 1, \dots, 5m, \\ Z'_3(i, j) &= xK_3(j, i - 3m) + yK_3(j, i - 2m), i = 5m + 1, \dots, 6m. \end{aligned}$$

iv. Construct the  $\text{rank}(\mathcal{Z}_{2[\nu]}) \times \text{rank}(\mathcal{Z}_{3[\nu]})$  matrix  $Z_3$  whose  $j^{\text{th}}$  column  $Z_3(\bullet, j)$  is the solution of the linear system  ${}^tZ_3(\bullet, j) \cdot K_2 = Z'_3(\bullet, j)$ . It is the matrix of the map  $\mathcal{Z}_{3[\nu]}(-1) \xrightarrow{v_3} \mathcal{Z}_{2[\nu]}$ . Define  $\Delta_3$  to be the square submatrix of  $Z_3$  obtained by removing the lines indexed by  $L_2$ .  $\Delta_3$  is of size  $\frac{(\nu-d+2)(\nu-d+1)}{2}$ .

**endif**

**endif.**

In the language of the moving surfaces method, the matrix  $Z_1$  obtained at step 2 gather all the moving *hyperplanes* of degree  $\nu$  following the surface  $\mathcal{S}$ . Assume hereafter that  $\mathcal{P}$  is locally a complete intersection. From a computational point of view, corollary 3.1 implies that this matrix can be taken as a *representation* of the surface  $\mathcal{S}$ , replacing an expanded implicit equation (even if it is generally non-square). For instance, to test if a given point  $p = (x_0 : y_0 : z_0 : w_0) \in \mathbb{P}^3$  is in the surface  $\mathcal{S}$ , we just have to substitute  $x, y, z, w$  respectively by  $x_0, y_0, z_0, w_0$  in  $Z_1$  and check its rank;  $p$  is on  $\mathcal{S}$  if and only if the rank of  $Z_1$  does not drop. Of course some numerical aspects have to be taken into account here, but the use of numerical linear algebra seems to be very promising in this direction. Note also that such matrices, whose computation is very fast, are much more compact representations of implicit equations compared to expanded polynomials which can have a lot of monomials.

Even if the use of  $Z_1$  seems to be, in the opinion of the authors, the best way to work quickly with implicit equations, the previous algorithm also returns an explicit description of  $P^\beta$ , where  $P$  is as usual an implicit equation of  $\mathcal{S}$  (always if  $\mathcal{P}$  is locally a complete intersection). With the convention  $\det(\Delta_i) = 1$  if  $\Delta_i$  does not exist, for  $i = 2, 3$ ,  $P^\beta$  can be computed as the quotient  $\frac{\det(\Delta_1)\det(\Delta_3)}{\det(\Delta_2)}$ . Since  $\det(\Delta_1)$  is always a multiple of  $P^\beta$ , we have  $\det(\Delta_1) = P^\beta Q$ , where  $Q$  is an homogeneous polynomial in  $\mathbb{K}[\underline{x}]$ . It can be useful to notice that this extraneous factor  $Q$  divides  $\det(\Delta_2)$ ; in fact  $\det(\Delta_2) = Q \det(\Delta_3)$ .

### 3.3. Examples

We illustrate our algorithm with four particular examples. It has been implemented in the software MAGMA which offers very powerful tools to deal with linear algebra, and appears to be very efficient.

### 3.3.1. An example without base points

Consider the following example :

$$\begin{cases} f_0 = s^2t, \\ f_1 = t^2u, \\ f_2 = su^2, \\ f_3 = s^3 + t^3 + u^3. \end{cases}$$

It is a parameterization without base points of a surface of degree 9. Applying our algorithm we find, in degree  $\nu = 2 \cdot 3 - 2 = 4$  a matrix  $Z_1$  of size  $15 \times 24$  which represents our surface. Continuing the algorithm we finally obtain a matrix  $\Delta_1$  of size  $15 \times 15$ , a matrix  $\Delta_2$  of size  $9 \times 9$  and a matrix  $\Delta_3$  of size  $3 \times 3$ . Computing the quotient  $\frac{\det(\Delta_1)\det(\Delta_3)}{\det(\Delta_2)}$  we obtain an expanded implicit equation :

$$x^6z^3 + 3x^5y^2z^2 + 3x^4y^4z + 3x^4yz^4 + x^3y^6 + 6x^3y^3z^3 + 3x^2y^5z^2 + 3x^2y^2z^5 - x^2y^2z^2w^3 + 3xy^4z^4 + y^3z^6.$$

Trying our algorithm empirically with  $\nu = 3$  we obtain a matrix  $\Delta_1$  which is not square, it has only 9 columns (instead of the waited 10). The algorithm hence fails here, showing that in this case the degree bound  $2d - 2$  is the lowest possible.

### 3.3.2. An example where the ideal of base points is saturated

This example is taken from Sederberg and Chen [1995] (see also Busé et al. [2002]), it is the parameterization of a cubic surface with 6 local complete intersection base points :

$$\begin{cases} f_0 = s^2t + 2t^3 + s^2u + 4stu + 4t^2u + 3su^2 + 2tu^2 + 2u^3, \\ f_1 = -s^3 - 2st^2 - 2s^2u - stu + su^2 - 2tu^2 + 2u^3, \\ f_2 = -s^3 - 2s^2t - 3st^2 - 3s^2u - 3stu + 2t^2u - 2su^2 - 2tu^2, \\ f_3 = s^3 + s^2t + t^3 + s^2u + t^2u - su^2 - tu^2 - u^3. \end{cases}$$

The ideal  $I = (f_0, f_1, f_2, f_3)$  is here saturated, so that we have  $\nu_0 = d - 2 = 1$ , and we hence can apply the algorithm with  $\nu = 1$ . The matrix  $Z_1$  is then square and the algorithm stops in step 2. It is given by :

$$\begin{pmatrix} x & -z - w & y + w \\ y & x - 2y + z - 2w & 2y - z \\ z & -x - 2w & y + 2w \end{pmatrix}.$$

### 3.3.3. An example where the method of moving quadrics fails

We here consider the example 3.2 in Busé et al. [2002]. This example was introduced to show how the method of moving quadrics (introduced in Sederberg and Chen [1995]) generalized in this paper to the presence of base points may fail. Consider the parameterization

$$\begin{cases} f_0 = su^2, \\ f_1 = t^2(s + u), \\ f_2 = st(s + u), \\ f_3 = tu(s + u). \end{cases}$$

We can directly apply our algorithm in degree  $\nu = 2.3 - 2 = 4$ ; we obtain a matrix  $Z_1$  of size  $15 \times 30$ , matrices  $\Delta_1$  and  $\Delta_2$  of size  $15 \times 15$ , and a matrix  $\Delta_3$  of size  $3 \times 3$ . An expanded implicit equation is then  $xyz + xyw - zw^2$ .

If now we take into account that the ideal  $I = (f_0, f_1, f_2, f_3)$  has base points, we can apply the algorithm with  $\nu = 3$ . Even better, the saturated ideal of  $I$ , denoted  $I_{\mathcal{P}}$ , is generated by  $t(s + u)$  and  $su^2$ , showing that  $\text{indeg}(I_{\mathcal{P}}) = 2$  (since  $t(s + u)$  is of degree 2), and hence that we can apply the algorithm with  $\nu = 2$ . In this case  $\Delta_1$  is  $6 \times 6$ ,  $\Delta_2$  is  $3 \times 3$  and the algorithm stops in step (e).

Remark that to have the best bound possible for the integer  $\nu$  we have to compute the initial degree of  $I_{\mathcal{P}}$ , which is not always obvious. However, it should be possible to determine some classes of parameterizations which satisfy certain geometric properties and for which we know in advance the initial degree of the saturation of  $I$ .

### 3.3.4. An example with a fat base point

With this example we would like to illustrate the theorem 3.2 when there is a base point minimally generated by three polynomials. Consider the parameterization given by

$$\begin{cases} f_0 &= s^3 - 6s^2t - 5st^2 - 4s^2u + 4stu - 3t^2u, \\ f_1 &= -s^3 - 2s^2t - st^2 - 5s^2u - 3stu - 6t^2u, \\ f_2 &= -4s^3 - 2s^2t + 4st^2 - 6t^3 + 6s^2u - 6stu - 2t^2u, \\ f_3 &= 2s^3 - 6s^2t + 3st^2 - 6t^3 - 3s^2u - 4stu + 2t^2u. \end{cases}$$

The ideal  $I = (f_0, f_1, f_2, f_3)$  defines exactly one *fat* base point  $p$  which is defined by  $(s, t)^2$ . Therefore any implicit equation of our parameterized surface is of degree  $d^2 - e_p = 9 - 4 = 5$ , and the degree of the determinant of the complex  $(\mathcal{Z}_{\bullet})_{[\nu_0]}$  is of degree  $d^2 - d_p = 9 - 3 = 6$ . Applying our algorithm with  $\nu_0 = 2(d - 1) - 2 = 2$  we find that the matrix  $Z_1$  is already square, of size  $6 \times 6$ . Expanding its determinant one obtains a product of a degree 5 irreducible polynomial (an implicit equation of our surface) and a linear (since  $e_p - d_p = 1$ ) irreducible polynomial.

## 4. Implicitization of rational parametric hypersurfaces

We now turn to the general problem of hypersurface implicitization, always using approximation complexes. In this section we prove new results to solve this problem under suitable conditions, and obtain the proof of theorem 3.2 as a particular case.

We are going to consider hypersurfaces obtained as the closed image of maps from  $\mathbb{P}^{n-1}$  to  $\mathbb{P}^n$ , where  $n \geq 3$ . Consequently we set  $A := \mathbb{K}[X_1, \dots, X_n]$ , which will be the ring of the polynomials of the parameterizations we will consider. We also denote  $\mathfrak{m} := (X_1, \dots, X_n)$ , and set  $-\vee := \text{Hom}_A(-, A[-n])$  and  $-^* := \text{Homgr}_A(-, A/\mathfrak{m})$ . Finally  $\omega_-$  will denote the associated canonical module (see

Bruno and Herzog [1993]). Our first result is a technical lemma on the cycles of certain Koszul complexes.

LEMMA 4.1: *Let  $f_0, \dots, f_n$  be  $n+1$  polynomials of positive degrees  $d_0, \dots, d_n$ ,  $I$  the ideal generated by them,  $\mathbf{K}_\bullet(f; A)$  be the Koszul complex of the  $f_i$ 's on  $A$  and denote by  $Z_i$  and  $H_i$  the  $i$ -th cycles and the  $i$ -th homology modules of  $\mathbf{K}_\bullet(f; A)$ , respectively. Denoting  $\sigma := d_0 + \dots + d_n$ , if  $\dim(A/I) \leq 1$  then we have*

- $H_i \neq 0$  for  $i = 0, 1$ ,  $H_i = 0$  for  $i > 2$  and  $H_2$  is zero if and only if  $\dim(A/I) = 0$ . If  $\dim(A/I) = 1$ ,  $H_2 \simeq \omega_{A/I}[n - \sigma]$ .
- If  $n \geq 3$ , then

$$H_m^i(Z_p) \simeq \begin{cases} 0 & \text{for } i = 0, 1 \\ H_m^0(H_{i-p})^*[n - \sigma] & \text{for } i = 2 \\ H_{i-p}^*[n - \sigma] & \text{for } 2 < i < n \\ Z_{n-p}^*[n - \sigma] & \text{for } i = n. \end{cases}$$

*Proof:* The first point is classical, see e.g. Bruno and Herzog [1993] 1.6.16 and 1.2.4. For the second point we consider the truncated complexes:

$$\mathbf{K}_\bullet^{>p} : \quad 0 \rightarrow K_{n+1} \rightarrow \dots \rightarrow K_{p+1} \rightarrow Z_p \rightarrow 0.$$

If  $p > 2$  this complex is exact, and gives rise to a spectral sequence  $H_m^\bullet(\mathbf{K}_\bullet^{>p}) \Rightarrow 0$ , which is at level 1 :

$$\begin{array}{ccccccc} 0 & & \dots & & 0 & & H_m^0(Z_p) \\ & & & & \vdots & & H_m^1(Z_p) \\ & & & & \vdots & & \vdots \\ & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & & H_m^{n-1}(Z_p) \end{array}$$

$$H_m^n(K_{n+1}) \xrightarrow{H_m^n(\partial_{n+1})} \dots \xrightarrow{H_m^n(\partial_{p+2})} H_m^n(K_{p+1}) \xrightarrow{H_m^n(\partial_{p+1})} H_m^n(Z_p),$$

and modulo the identifications

$$H_m^n(K_\bullet) \simeq (K_\bullet^\vee)^* \simeq (K^\bullet[-n])^* \simeq (K_{n+1-\bullet}[\sigma - n])^*,$$

the last line becomes

$$K_0^*[n - \sigma] \xrightarrow{\partial_1^*} K_1^*[n - \sigma] \xrightarrow{\partial_2^*} \dots \xrightarrow{\partial_{n-p}^*} K_{n-p}^*[n - \sigma] \rightarrow H_m^n(Z_p).$$

It follows

- $H_m^i(Z_p) = 0$  for  $i < p$  and for  $p + 2 < i < n$ ,

- $d_{n-p+1}^{>p} : H_0^*[n - \sigma] \xrightarrow{\simeq} H_m^p(Z_p)$ ,
- $d_{n-p}^{>p} : H_1^*[n - \sigma] \xrightarrow{\simeq} H_m^{p+1}(Z_p)$ ,
- $d_{n-p-1}^{>p} : H_2^*[n - \sigma] \xrightarrow{\simeq} H_m^{p+2}(Z_p)$ ,

where the upper index indicates from which spectral sequence the isomorphism comes, and the lower index indicates at which level of the spectral sequence the map is obtained. We also obtain a short exact sequence

$$K_{n-p-1}^*[n - \sigma] \xrightarrow{\partial_{n-p}^*} K_{n-p}^*[n - \sigma] \rightarrow H_m^n(Z_p) \rightarrow 0$$

that gives the isomorphism  $H_m^n(Z_p) \simeq Z_{n-p}^*[n - \sigma]$ .

For  $p = 2$ , we have a spectral sequence  $H_m^\bullet(\mathbf{K}_\bullet^{>2}) \Rightarrow H_m^1(H_2)$  which shows as in the previous cases that  $H_m^n(Z_2) \simeq Z_{n-2}^*[n - \sigma]$ , and that

- $H_m^i(Z_2) = 0$  for  $4 < i < n$ ,
- $d_{n-2}^{>2} : H_1^*[n - \sigma] \xrightarrow{\simeq} H_m^3(Z_2)$ ,
- $d_{n-3}^{>2} : H_2^*[n - \sigma] \xrightarrow{\simeq} H_m^4(Z_2)$ .

It also provides an exact sequence

$$0 \rightarrow H_m^1(Z_2) \xrightarrow{\text{can}} H_m^1(H_2) \xrightarrow{\tau} H_0^*[n - \sigma] \xrightarrow{d_{n-1}^{>2}} H_m^2(Z_2) \rightarrow 0, \quad (3)$$

where  $\tau$  is the transgression map of the spectral sequence. If  $H_2 \neq 0$ , we also look at the complex

$$\mathbf{K}_\bullet^{\leq 2} : 0 \rightarrow Z_2 \rightarrow K_2 \rightarrow K_1 \rightarrow I \rightarrow 0,$$

whose only homology is  $H_1$  and the corresponding spectral sequence  $H_m^\bullet(\mathbf{K}_\bullet^{\leq 2}) \Rightarrow H_m^\bullet(H_1)$ . Noticing that  $H_m^i(H_1) = 0$  for  $i > 1$ , we get that  $H_m^0(Z_2) = H_m^1(Z_2) = 0$  (this is also easily obtained by splitting  $\mathbf{K}_\bullet^{\leq 2}$  into two short exact sequences, and taking cohomology). Together with (3) and local duality, which gives  $H_m^1(H_2) \simeq \omega_{H_2}^* \simeq (A/I^{\text{sat}})^*[n - \sigma]$  where  $I^{\text{sat}}$  denotes the saturation of the ideal  $I$ , we get the asserted isomorphism for  $H_m^2(Z_2)$ .

Note that we also get an isomorphism  $H_m^0(H_0)^*[n - \sigma] \simeq H_m^2(Z_2) \simeq H_m^0(H_1)$ , and an exact sequence:

$$0 \rightarrow H_m^1(H_1) \rightarrow H_m^3(Z_2) \rightarrow H_m^1(I) \rightarrow 0.$$

Finally for  $p = 1$ , the exact sequence

$$\mathbf{K}_\bullet^{\leq 1} : 0 \rightarrow Z_1 \rightarrow K_1 \rightarrow I \rightarrow 0$$

shows that  $H_m^{i-2}(H_0) \simeq H_m^{i-1}(I) \simeq H_m^i(Z_1)$  for  $i < n$ , so that  $H_m^0(Z_1) = H_m^1(Z_1) = 0$ ,  $H_m^2(Z_1) \simeq H_m^0(H_0) \simeq H_m^0(H_1)^*[n - \sigma]$ ,  $H_m^3(Z_1) \simeq H_m^1(H_0) \simeq H_2^*[n - \sigma]$  and  $H_m^i(Z_1) = 0$  for  $3 < i < n$ .

The spectral sequence  $H_m^\bullet(\mathbf{K}_\bullet^{>1}) \Rightarrow H_m^\bullet(H_\bullet)$  (because only  $H_m^1(H_2)$ ,  $H_m^0(H_1)$  and  $H_m^1(H_1)$  may not be zero) gives  $H_m^n(Z_1) \simeq Z_{n-1}^*[n - \sigma]$ , and this concludes the proof.  $\square$

*Remark:* Note that the last spectral sequence of the proof also identifies  $H_{\mathfrak{m}}^2(Z_1)$  in another way by providing the exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^1(H_1) \xrightarrow{\tau'} H_1^*[n - \sigma] \xrightarrow{d_{n-1}^{>1}} H_{\mathfrak{m}}^2(Z_1) \rightarrow 0$$

which shows that if  $\dim(A/I) = 1$

$$H_1/H_{\mathfrak{m}}^0(H_1) \simeq \omega_{H_1}[n - \sigma].$$

Thus in this case  $H_1/H_{\mathfrak{m}}^0(H_1)$  has a symmetric free resolution. Also, the spectral sequences derived from the complexes  $\mathbf{K}_{\bullet}^{\leq p}$  provide exact sequences

$$0 \rightarrow H_{\mathfrak{m}}^0(H_1) \rightarrow H_{\mathfrak{m}}^p(Z_p) \rightarrow H_{\mathfrak{m}}^1(H_2) \rightarrow 0$$

for  $p < n$ , and

$$0 \rightarrow H_{\mathfrak{m}}^1(H_1) \rightarrow H_{\mathfrak{m}}^{p+1}(Z_p) \rightarrow H_{\mathfrak{m}}^0(H_0) \rightarrow 0$$

for  $p < n - 1$ , as well as an isomorphism  $H_{\mathfrak{m}}^{p+2}(Z_p) \simeq H_{\mathfrak{m}}^1(H_0)$  for  $p < n - 2$ .

As we have already done in previous sections, we now consider approximation complexes. To do this, we introduce new variables  $T_1, \dots, T_{n+1}$  which represent the homogeneous coordinates of the target  $\mathbb{P}_{\mathbb{K}}^n$  of a given parameterization. Let  $f_0, \dots, f_n$  be  $n + 1$  homogeneous polynomials in  $A$  of the same degree  $d \geq 1$ . Denoting by  $Z_i$  the  $i^{\text{th}}$ -cycles of the Koszul complex of the  $f_i$ 's on  $A$ , we set  $\mathcal{Z}_i := Z_i[id] \otimes_A A[\underline{T}]$ , where  $[-]$  stands for the degree shift in the  $X_i$ 's and  $(-)$  for the one in the  $T_i$ 's. It appears that the differentials  $v_{\bullet}$  of the Koszul complex  $K_{\bullet}(T_1, \dots, T_{n+1}; A[\underline{T}])$  induce maps between the  $\mathcal{Z}_i$ 's, and hence we can define the approximation complex (note that  $\mathcal{Z}_{n+1} = 0$ )

$$(\mathcal{Z}_{\bullet}, v_{\bullet}) : 0 \rightarrow \mathcal{Z}_n(-n) \xrightarrow{v_n} \dots \xrightarrow{v_3} \mathcal{Z}_2(-2) \xrightarrow{v_2} \mathcal{Z}_1(-1) \xrightarrow{v_1} \mathcal{Z}_0 = A[\underline{T}].$$

It is naturally a bi-graded complex, and it is easy to check that  $H_0(\mathcal{Z}_{\bullet}) \simeq \text{Sym}_A(I)$ . We now give some acyclicity criteria in case  $I = (f_0, \dots, f_n)$  define isolated points in  $\text{Proj}(A)$ . Recall that a sequence  $x_1, \dots, x_n$  of elements in a ring  $R$  is said to be a *proper sequence* if

$$x_{i+1}H_j(x_1, \dots, x_i; A) = 0 \quad \text{for } i = 0, \dots, n - 1 \text{ and } j > 0,$$

where the  $H_j$ 's denote the homology groups of the associated Koszul complex.

**LEMMA 4.2:** *Suppose that  $I = (f_0, \dots, f_n)$  is of codimension at least  $n - 1$ ,  $\mathbb{K}$  is infinite, and let  $\mathcal{P} := \text{Proj}(A/I)$ . Then the following are equivalent :*

- (1)  $\mathcal{Z}_{\bullet}$  is acyclic,
- (2)  $\mathcal{Z}_{\bullet}$  is acyclic outside  $V(\mathfrak{m})$ ,
- (3)  $I$  is generated by a proper sequence,

(4)  $\mathcal{P}$  is locally defined by a proper sequence,

(5)  $\mathcal{P}$  is locally defined by  $n$  equations.

*Proof:* By Herzog et al. [1983], (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4). Moreover (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4). We will show that (4) $\Rightarrow$ (5) $\Rightarrow$ (3).

Assume (4). For each  $p \in \mathcal{P}$  there exists a non empty open set  $\Omega_p$  in  $\mathbb{K}^{(n+1)^2}$  such that if  $(a_{ij}) \in \Omega_p$  and  $g_i := \sum_j a_{ij} f_j$ , the  $g_i$ 's form a proper sequence locally at  $p$ . Taking  $(a_{ij}) \in \Omega := \bigcap_{p \in \mathcal{P}} \Omega_p$ , this gives  $g_1, \dots, g_{n+1}$  so that the  $g_i$ 's form a proper sequence outside  $V(\mathfrak{m})$ . We may also assume, by shrinking  $\Omega$  if necessary, that  $g_1, \dots, g_{n-1}$  is a regular sequence, so that  $g_1, \dots, g_n$  is clearly a proper sequence. Now  $g_1, \dots, g_{n+1}$  is proper if and only if  $g_{n+1}$  annihilates  $H_1(g_1, \dots, g_n; A) \simeq \omega_{A/J}$  (locally outside  $V(\mathfrak{m})$ , *a priori*), where  $J$  denotes the saturated ideal of  $(g_1, \dots, g_n)$  w.r.t.  $\mathfrak{m}$ . But  $\text{Ann}_A(\omega_{A/J}) = J$ , so that  $g_{n+1} \in J$ , which proves (5).

Now assume (5). One considers (in the same spirit as above)  $g_1, \dots, g_{n+1}$  so that  $g_1, \dots, g_{n-1}$  is a regular sequence and  $g_1, \dots, g_n$  defines  $\mathcal{P}$ . Then  $g_{n+1} \in I_{\mathcal{P}}$ , and therefore annihilates  $\omega_{A/I_{\mathcal{P}}}$ , so that the  $g_i$ 's forms a proper sequence.  $\square$

We are now able to state our main result on the hypersurface implicitization problem.

**THEOREM 4.1:** *Let  $n \geq 3$ . Let  $I$  be the ideal  $(f_0, \dots, f_n)$ ,  $\mathcal{P} := \text{Proj}(A/I)$  and  $I_{\mathcal{P}}$  the saturation of  $I$  w.r.t.  $\mathfrak{m}$ . Assume that  $\mathbb{K}$  is infinite and that  $\dim \mathcal{P} \leq 0$ , that is  $\mathcal{P}$  define a finite number of base points in  $\text{Proj}(A)$ , possibly empty. Then we have*

- $\mathcal{Z}_{\bullet}$  is acyclic if and only if  $\mathcal{P}$  is locally defined by  $n$  equations.
- Let  $\nu \geq \nu_0 := (n-1)(d-1) - \text{indeg}(I_{\mathcal{P}})$ . Then  $H_{\mathfrak{m}}^0(\text{Sym}_A(I))_{[\nu]} = 0$ . Moreover  $(\mathcal{Z}_{\bullet})_{[\nu]} \otimes_{\mathbb{K}[\underline{T}]} \mathbb{K}(\underline{T})$  is acyclic if and only if  $\mathcal{Z}_{\bullet}$  is acyclic.
- Let  $\nu \geq \nu_0$  and assume that  $\mathcal{P}$  is locally defined by  $n$  equations. Then  $D := \det((\mathcal{Z}_{\bullet})_{[\nu]})$  is a non zero homogeneous element of  $\mathbb{K}[\underline{T}]$ , independent of  $\nu$  (modulo  $\mathbb{K}^{\times}$ ), of degree  $d^{n-1} - \sum_{p \in \mathcal{P}} d_p$ . Denoting by  $\mathcal{H}$  the closed image of the rational map  $\phi : \mathbf{P}^{n-1} \rightarrow \mathbf{P}^n$ ,  $D = H^{\text{deg}(\phi)} G$  where  $H$  is an implicit equation of  $\mathcal{H}$ . Moreover  $G \in \mathbb{K}^{\times}$  if and only if  $\mathcal{P}$  is locally of linear type, and if and only if  $\mathcal{P}$  is locally a complete intersection.

Before giving the proof of the theorem we recall that, by definition,  $\mathcal{P}$  is said to be locally of linear type if  $\text{Proj}(\text{Sym}_A(I)) = \text{Proj}(\text{Rees}_A(I))$ . By Busé and Jouanolou [2002] theorem 2.5,  $\text{deg}(H^{\text{deg}(\phi)}) = d^{n-1} - \sum_{p \in \mathcal{P}} e_p$ , and consequently we always have  $\text{deg}(G) = \sum_{p \in \mathcal{P}} (e_p - d_p)$ . Also  $e_p \geq d_p$  with equality if and only if  $\mathcal{P}$  is locally a complete intersection at  $p \in \mathcal{P}$ .

*Proof:* The first point follows from lemma 4.2. For the second point we consider the two spectral sequences associated to the double complex  $H_{\mathfrak{m}}^{\bullet}(\mathcal{Z}_{\bullet})$ , both abouting to the hypercohomology of  $\mathcal{Z}_{\bullet}$ . One of them abouts at level two with:

$${}_2'E_q^p = {}_\infty'E_q^p = \begin{cases} H_m^p(H_q(\mathcal{Z}_\bullet)) & \text{for } p = 0, 1 \text{ and } q > 0 \\ H_m^p(\text{Sym}_A(I)) & \text{for } q = 0 \\ 0 & \text{else.} \end{cases}$$

The other one gives at level one:

$${}_1''E_q^p = H_m^p(Z_q)[qd] \otimes_A A[\underline{T}](-q).$$

By lemma 4.1,  $H_m^p(Z_q) = 0$  for  $p < q$  and for  $p < 2$ . This shows that  $H_m^p(H_q(\mathcal{Z}_\bullet)) = 0$  for  $q > 0$  except possibly for  $p = q = 1$ , by comparing the two spectral sequences. Therefore,  $H_q(\mathcal{Z}_\bullet) = 0$  for  $q > 1$  and  $H_m^0(H_1(\mathcal{Z}_\bullet)) = 0$ .

If  $p > 2$

$${}_1''E_p^p \simeq H_0^*[n - (n + 1 - p)d] \otimes_A A[\underline{T}](-p)$$

so that  $({}_1''E_p^p)_\nu = 0$  if  $\nu > (n - 2)d - n$ . Also,

$$({}_1''E_2^2) \simeq (I_{\mathcal{P}}/I)^*[n - (n - 1)d] \otimes_A A[\underline{T}](-2),$$

so that  $({}_1''E_2^2)_\nu = 0$  for  $\nu > (n - 1)d - n - \text{indeg}(I_{\mathcal{P}}/I)$ .

Therefore, if  $\nu \geq \nu_0$ ,  $({}_1''E_q^p)_\nu = 0$  for  $p \leq q$ . Note also that we have the equalities

$$\min\{d, \text{indeg}(I_{\mathcal{P}}/I)\} = \min\{d, \text{indeg}(I_{\mathcal{P}})\} = \text{indeg}(I_{\mathcal{P}}).$$

By comparing with the other spectral sequence, we have  $H_m^1(H_1(\mathcal{Z}_\bullet))_\nu = 0$  and  $H_m^0(\text{Sym}_A(I))_\nu = 0$ .

As  $H_m^0(\text{Sym}_A(I))_\nu = 0$  for  $\nu \geq \nu_0$  we have

$$\text{Ann}_{\mathbb{K}[\underline{T}]}(\text{Sym}_A(I)_\nu) = \text{Ann}_{\mathbb{K}[\underline{T}]}(\text{Sym}_A(I)_{\nu_0})$$

for any  $\nu \geq \nu_0$  (see for instance Busé and Jouanolou [2002], the proof of 5.1), so that this module is torsion if and only if  $\mathcal{P}$  is locally defined by at most  $n$  equations (because  $\text{Ann}_{\mathbb{K}[\underline{T}]}(\text{Sym}_A(I)_\nu)$  is torsion for  $\nu \gg 0$  if and only if  $I$  is defined by  $< n + 1$  equations outside  $V(\mathfrak{m})$ ; one may also use the study of the minimal primes of  $\text{Sym}_A(I)$  in Huneke and Rossi [1986]). Also in the case where  $\mathcal{P}$  is locally defined by at most  $n$  equations, the divisor associated to this module is independant of  $\nu \geq \nu_0$ . This finishes the proof of point two.

We come to the third point. Note that we have just shown that  $D$  is independant of  $\nu$  (up to an element of  $\mathbb{K}^\times$ ). To compute the degree of  $D$ , we may then take  $\nu \gg 0$ .

The matrices of the maps of  $\mathcal{Z}_\bullet$  have entries which are linear forms in the  $T_i$ 's, so that the determinant of  $(\mathcal{Z}_\bullet)_{[\nu]}$  is a form in the  $T_i$ 's of degree

$$\delta := \sum_{i=1}^n (-1)^{i+1} i \dim_{\mathbb{K}}(Z_{i[\nu+id]}).$$

We have canonical exact sequences,  $i = 0, \dots, n$ ,

$$0 \rightarrow Z_{i+1} \rightarrow K_{i+1} \rightarrow B_i \rightarrow 0$$

and

$$0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0$$

which shows that  $\delta$  can be expressed in terms of the Hilbert polynomials of the  $K_i$ 's and the  $H_i$ 's. The contribution of the  $K_i$ 's only depends on  $n$  and  $d$ , and the one of the  $H_i$ 's only comes from  $Z_1$  and  $Z_2$  and is :

$$(H_0)_{\nu+d} - 2((H_1)_{\nu+2d} - (H_0)_{\nu+2d})$$

as  $\deg H_1 = 2 \deg \mathcal{P}$  (one may use for example that  $(H_0)_\nu - (H_1)_\nu + (H_2)_\nu = 0$  for  $\nu \gg 0$  and that  $H_2 \simeq \omega_{R/I_{\mathcal{P}}}$  up to a degree shift), this contribution is equal to  $-\deg \mathcal{P}$  for  $\nu \gg 0$ . The contribution of the  $K_i$ 's is  $d^{n-1}$  (for  $\nu \gg 0$ ) in the case where the  $H_i$ 's are 0 for  $\nu \gg 0$ , and we get  $\delta = d^{n-1} - \deg \mathcal{P}$ .

Let  $X := \text{Proj}(\text{Rees}_I(A)) \subseteq Y := \text{Proj}(\text{Sym}_A(I)) \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ . The scheme  $X$  is the closure of the graph of  $\phi$  in  $\mathbf{P}^{n-1} \times \mathbf{P}^n$ . Therefore, the closed image of  $\phi$  is  $p_2(X)$ , and  $p_2(X) \subseteq p_2(Y)$ . As  $D \neq 0$ ,  $p_2(Y) \neq \mathbf{P}^n$ . If  $\mathcal{P}$  is locally of linear type, then  $X = Y$ , so that  $G \in \mathbb{K}^\times$  which implies that  $d_p = e_p$  for any  $p \in \mathcal{P}$  and therefore  $\mathcal{P}$  is locally a complete intersection. If  $\mathcal{P}$  is not locally a complete intersection at  $p \in \mathcal{P}$ ,  $d_p \neq e_p$  so that  $\deg G > 0$  and  $X \neq Y$ . Note that these last facts also follows from the study of the minimal primes associated to  $\text{Sym}_A(I)$  in Huneke and Rossi [1986].  $\square$

This theorem yields easily an algorithm to implicitize parameterized hypersurfaces under suitable assumptions, using only linear algebra routines. The case  $n = 3$  have been completely explicitated in section 3.

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