

Implicitization using approximation complexes

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1. Introduction

We present in this short account a method to compute the image of a rational map from \mathbf{P}^{n-1} to \mathbf{P}^n , under suitable hypotheses on the base locus and on the image.

The formalism we use is due to Jean-Pierre Jouanolou, who gave a course on this approach in the University of Strasbourg during the academic year 2000–2001. In his joint article with Laurent Busé [BJ], this formalism is explained in details and applications to the implicitization problem are given.

The idea of using a matrix of syzygies for the implicitization problem goes back to the work of Sederberg and Chen [SC] and was at the origin of several important contributions to this approach (see for instance [Co], [CSC], [CGZ], [D'] and the articles on this subject in the volume of the 2002 conference on Algebraic Geometry and Geometric Modeling [AGGM02]).

Most of this note is dedicated to presenting the method, the geometric ideas behind it and the tools from commutative algebra that are needed. Some references to classical textbooks are given for the concepts and theorems we use for the presentation. In the last section, we give the most advanced results we know related to this approach. We illustrate this technique on an example that we carry out in details all along the article.

References are given to the publication that fits best our statements. They may not be the first place where a similar result appeared –for instance, many results were first proved for $n = 2$ or $n = 3$ –.

2. General setting

Given

$$\phi : \mathbf{P}^{n-1} \dashrightarrow \mathbf{P}^n,$$

a rational map defined by $f := (f_0, \dots, f_n)$, $f_i \in R := k[X_1, \dots, X_n]$ homogeneous of degree $d \geq 1$, such that the closure of its image is a hypersurface \mathcal{H} , the goal is to compute the equation H of this hypersurface.

We let :

- $I := (f_0, \dots, f_n) \subset R$ be the ideal generated by the f_i 's,
- $X := \text{Proj}(R/I) \subset \mathbf{P}^{n-1}$ be the subscheme defined by I .

A specific example : We will illustrate in this article the different steps and constructions on an example, taken from [BCD, Example 3.2] that we will call example E :

$$\phi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^3,$$

given by $f := (ac^2, b^2(a+c), ab(a+c), bc(a+c))$ with $R := \mathbf{Q}[a, b, c]$. The ring of the target will be $R' := \mathbf{Q}[x, y, z, t]$.

3. The algebro-geometric intuition

If $\Gamma_0 \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ is the graph of $\phi : (\mathbf{P}^{n-1} - X) \rightarrow \mathbf{P}^n$ and Γ the Zariski closure of Γ_0 , one has :

$$\mathcal{H} = \overline{\pi(\Gamma_0)} = \pi(\Gamma)$$

where $\pi : \mathbf{P}^{n-1} \times \mathbf{P}^n \rightarrow \mathbf{P}^n$ is the projection, and the bar denotes the Zariski closure (or equivalently the closure for the usual topology in the case $k = \mathbf{C}$).

The first equality directly follows from the definition of \mathcal{H} , and the second from the fact that π is a projective morphism (so that the image of a variety is a variety).

On the algebraic side [Ha, II §7, Va **], one has

$$\Gamma = \text{Proj}(\mathcal{R}_I)$$

with $\mathcal{R}_I := R \oplus I \oplus I^2 \oplus \dots$ and the embedding $\Gamma \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ corresponds to the natural graded map :

$$\begin{aligned} S := R[T_0, \dots, T_n] &\xrightarrow{s} \mathcal{R}_I \\ T_i &\longmapsto f_i \in I = (\mathcal{R}_I)_1 \end{aligned}$$

If $\mathfrak{P} := \ker(s)$, \mathfrak{P}_1 (the degree 1 part of \mathfrak{P}) is the module of syzygies of the f_i 's :

$$a_0 T_0 + \dots + a_n T_n \in \mathfrak{P}_1 \iff a_0 f_0 + \dots + a_n f_n = 0.$$

Setting $\mathcal{S}_I := \text{Sym}_R(I)$ and $V := \text{Proj}(\mathcal{S}_I)$, we have natural onto maps

$$S \rightarrow S/(\mathfrak{P}_1) \quad \text{and} \quad \mathcal{S}_I \simeq S/(\mathfrak{P}_1) \rightarrow S/\mathfrak{P} \simeq \mathcal{R}_I$$

which corresponds to the embeddings

$$\Gamma \subseteq V \subset \mathbf{P}^{n-1} \times \mathbf{P}^n.$$

As \mathcal{R}_I is the bigraded domain defining Γ , the projection $\pi(\Gamma)$ is defined by the graded domain $\mathcal{R}_I \cap k[T_0, \dots, T_n]$. We have assumed that $\pi(\Gamma)$ is the hypersurface $H = 0$, so that this may be rewritten :

$$(H) = \mathfrak{P} \cap k[T_0, \dots, T_n].$$

In example E , with $S := R[x, y, z, t] = \mathbf{Q}[a, b, c, x, y, z, t]$:

$$\mathfrak{P} = (ay - bz, at - cz, bt - cy, act - b(a + c)x) + (bx(z + t) - at^2) + (xy(z + t) - zt^2)$$

where we have separated the (minimal) generators of degrees 1, 2 and 3 for simplicity. Of course it follows that $H = xy(z + t) - zt^2$. Also, by definition,

$$\mathcal{S}_I \simeq S/(\mathfrak{P}_1) = S/(ay - bz, at - cz, bt - cy, act - b(a + c)x).$$

The fact that \mathcal{R}_I and \mathcal{S}_I , as well as the canonical map $\mathcal{S}_I \rightarrow \mathcal{R}_I$, do not depend on generators of I are useful to prove the following :

Theorem. $\Gamma = V$ if X is locally a complete intersection.

In example E , the saturation of I is the complete intersection ideal $(ac^2, b(a + c))$ –because $b(a + c).(a, b, c) \subset I$ so that $I \subseteq (ac^2, b(a + c)) \subseteq I^{sat}$, and $(ac^2, b(a + c))$ is saturated–. Therefore X is locally a complete intersection (is it even globally a complete intersection).

More refined criteria exist to insure that $\Gamma = V$, but we will stick here to this one. This is partly justified by the following result :

Proposition. If $\dim X = 0$, $\Gamma = V$ if and only if X is locally a complete intersection.

The theorem above explains the key role of syzygies in computing H : they are equations of definition of Γ when X is locally a complete intersection.

A more algebraic way to state the theorem is the following :

Theorem. The prime ideal \mathfrak{P} is the saturation of the ideal generated by its elements of degree 1 in the T_i 's (the syzygies) if X is locally a complete intersection.

In example E , $\mathfrak{P}_2 \subset (\mathfrak{P}_1) : (a, b, c)$ and $\mathfrak{P} = (\mathfrak{P}_1) : (a, b, c)^2$.

Nevertheless, as it is clear from this other formulation of the theorem, one should not forget that even if $\Gamma = V$, it need not be the case that $\mathcal{R}_I = \mathcal{S}_I$. In fact, the equality $\mathcal{R}_I = \mathcal{S}_I$ may only hold in trivial cases in our context, because H is a minimal generator of \mathfrak{P} . The difference between these algebras (which is the torsion part of \mathcal{S}_I , because \mathcal{R}_I is a domain) is a key point when one uses the syzygies to compute H . This is very much similar to the fact that a homogeneous ideal defining a variety in the projective space need not be saturated.

The way the method proceeds is somehow parallel to determinantal methods for computing resultants : it uses graded pieces of a resolution of \mathcal{S}_I to compute $\pi(V)$.

The connection between the elimination theory viewpoint, which looks at H as the generator of $\mathfrak{P} \cap k[T_0, \dots, T_n]$, and the determinantal approach that computes H from graded pieces of a resolution of \mathcal{S}_I is shown by the following :

Proposition. [BJ, 5.1] *Assume that $\Gamma = V$ and let η be such that $H_{\mathfrak{m}}^0(\mathcal{S}_I)_\mu = 0$ for all $\mu \geq \eta$. Then,*

$$\text{ann}_{k[T_0, \dots, T_n]}(\mathcal{S}_I^\eta) = \mathfrak{P} \cap k[T_0, \dots, T_n].$$

Here $\mathfrak{m} := (X_1, \dots, X_n)$ and $H_{\mathfrak{m}}^0(M) := \{m \in M \mid \exists \ell, X_i^\ell m = 0 \forall i\}$. The graded pieces of \mathcal{S}_I will be described below, and we will provide estimates for η satisfying the vanishing condition. Notice that $H_{\mathfrak{m}}^0(\mathcal{S}_I)$ is the torsion part of \mathcal{S}_I when $\Gamma = V$.

Remark. The choices on gradings are one of the delicate points in this approach. For instance, the hypothesis $H_{\mathfrak{m}}^0(\mathcal{S}_I)_\mu = 0$ is equivalent to

$$H_{\mathfrak{m}}^0(\text{Sym}_R^j(I))_{\mu+dj} = 0, \quad \forall j,$$

if we adopt the natural grading of $\text{Sym}_R^j(I)$ making the canonical map $\text{Sym}_R^j(I) \rightarrow I^j \subset R$ a homogeneous map of degree zero.

A candidate for a resolution of \mathcal{S}_I is the \mathcal{Z} -complex introduced by Simis and Vasconcelos, we will describe it in the next section.

4. The tools from commutative algebra

The saturation of an ideal. — An ideal I in a polynomial ring $R := k[X_0, \dots, X_n]$ is *saturated* (or, more precisely \mathfrak{m} -saturated) if $I : \mathfrak{m} = I$, where $\mathfrak{m} := (X_0, \dots, X_n)$. In other words, I is saturated if $X_i f \in I, \forall i \Rightarrow f \in I$.

The ideal $I^{\text{sat}} := \bigcup_j (I : \mathfrak{m}^j)$ is saturated, it is the smallest saturated ideal containing I and is called the saturation of I .

Another way of seeing the saturation of an ideal, that directly extends to modules, is given by the remark that :

$$I^{\text{sat}} = I + H_{\mathfrak{m}}^0(R/I)$$

that one can also write $R/I^{\text{sat}} = (R/I)/H_{\mathfrak{m}}^0(R/I)$.

The saturation of a module M will be $M/H_{\mathfrak{m}}^0(M)$. As usual, one should be careful about the fact that the saturation of an ideal I then corresponds to saturating the module R/I and not the ideal considered as a module over the ring.

Seeing the saturation operation in relation with the left exact functor $H_{\mathfrak{m}}^0(-)$ naturally leads to the consideration of the derived functors $H_{\mathfrak{m}}^i(-)$, and to the cohomological approach of algebraic geometry.

There is a one-to-one correspondance between the subschemes of a projective space \mathbf{P}_k^n and the *saturated* homogeneous ideals of the polynomial ring $R := k[X_0, \dots, X_n]$. To see this notice that, by definition, two subschemes of \mathbf{P}_k^n are the same if they coincide on all the affine charts $X_i = 1$. If ϕ_i is the specialization homomorphism $X_i \mapsto 1$ then the homogeneization of $\phi_i(I)$ is the ideal $I_{(i)} := \bigcup_j (I : (X_i^j))$. It follows that I and J defines

the same schemes if and only if $I_{(i)} = J_{(i)}$ for all i , which is easily seen to be equivalent to the equality of their saturation as $I^{sat} = \bigcap_i I_{(i)}$.

When considering multigraded ideals, with respect to set of variables that are generating ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ (these ideals are never maximal unless $t = 1$ and k is a field), the operations of saturation with respect to the different ideals naturally appears. The subschemes of the corresponding product of projective spaces corresponds one-to-one to ideals that are saturated with respect to all the ideals \mathfrak{m}_i , or equivalently with respect to the product of these ideals.

The ring of sections. [Ha, II §5, Ei §A4.1] — If $R := k[X_0, \dots, X_n]$ and $B := R/I$ is the quotient of R by the homogeneous ideal I , an interesting object to consider is :

$$\Gamma B := \ker \left(\bigoplus_i B_{X_i} \longrightarrow \bigoplus_{i < j} B_{X_i X_j} \right),$$

where $B_{(f)} := \{ \frac{x}{f^j} \mid x \in B, j \in \mathbf{N} \}$ and the maps are the evident ones up to a sign chosen so that $(1, \dots, 1)$ maps to 0. One has a natural isomorphism $B_{X_i} \simeq B/(X_i - 1)$ and ΓB should be interpreted as the applications that are defined on each affine chart $X_i = 1$ and matches on the intersection of two of these charts. Notice that it is clear from the definition that replacing I by its saturation do not affect ΓB .

In a sheaf theoretic language, one has

$$\Gamma B = \bigoplus_{\mu \in \mathbf{Z}} H^0(\mathbf{P}_k^n, \mathcal{O}_X(\mu)),$$

with $X := \text{Proj}(B)$, and the natural grading of ΓB coincides with the grading of the section ring on the right hand side.

These considerations extends to modules along the same lines. Also, the map we used to define ΓB fits into a complex, called the Čech complex,

$$0 \longrightarrow B \xrightarrow{\phi} \bigoplus_i B_{X_i} \xrightarrow{\psi} \bigoplus_{i < j} B_{X_i X_j} \longrightarrow \cdots \longrightarrow B_{X_0 \cdots X_n} \longrightarrow 0.$$

$$\begin{array}{ccccccc} & & \parallel & & \parallel & & \parallel \\ \mathcal{C}_m^0(B) & & \mathcal{C}_m^1(B) & & \mathcal{C}_m^2(B) & & \mathcal{C}_m^{n+1}(B) \end{array}$$

One has $H_m^0(B) = \ker(\phi)$ and $\Gamma B = \ker(\psi)$. It is a standard fact that $H_m^i(B)$ is isomorphic to the i -th cohomology module of this complex. This in particular gives an exact sequence :

$$0 \longrightarrow H_m^0(B) \longrightarrow B \longrightarrow \Gamma B \longrightarrow H_m^1(B) \longrightarrow 0,$$

which splits into two parts the difference between the homogeneous quotient B and the more geometric notion of the section ring attached to $X := \text{Proj}(R/I) \subseteq \mathbf{P}_k^n$.

Notice that, in the case k is a field and X is of dimension zero, $\Gamma B_\mu = H^0(\mathbf{P}_k^n, \mathcal{O}_X(\mu))$ is a k -vector space of dimension the degree of X for any μ . In particular, when $\dim X = 0$, ΓB is not finitely generated. In any dimension, it can be shown that $\Gamma(R/I)$ is finitely

generated if and only if I has no associated prime \mathfrak{p} such that $\text{Proj}(R/\mathfrak{p})$ is of dimension zero (i.e. $\dim(R/\mathfrak{p}) = 1$).

Castelnuovo-Mumford regularity. [Ei, §20.5] — The Castelnuovo-Mumford regularity is an invariant that measures the algebraic complexity of a graded ideal or module over a polynomial ring $R := k[X_0, \dots, X_n]$. The two most standard definitions are given either in terms of a minimal finite free R -resolution of the module (this resolution exists by Hilbert's theorem on syzygies) or in terms of the vanishing of the cohomology modules defined above (using a theorem of Serre [Ha, III 5.2] to show that this makes sense).

Theorem-Definition. Let $b_i(M)$ be the maximal degree of a minimal i -th syzygy of M and $a_i(M) := \inf\{\mu \in \mathbf{Z} \mid H_{\mathfrak{m}}^i(M)_\nu = 0, \forall \nu > \mu\}$, then

$$\text{reg}(M) = \max_i \{a_i(M) + i\} = \max_i \{b_i(M) - i\}.$$

Notice that if $M = R/I$, minimal 0^{th} syzygies of M are minimal generators of M (namely, the element 1), minimal 1^{st} syzygies of M are minimal generators of I , and 2^{nd} syzygies of M are syzygies between the chosen (minimal) generators of I . If one looks at I as a module, these modules are the same up to a shift in the labeling, except 0^{th} module for R/I , and one has $\text{reg}(I) = \text{reg}(R/I) + 1$.

The existence of different interpretations of the regularity is a key to many results on this invariant. It is for instance immediate from the cohomological definition that $\text{reg}(I^{\text{sat}}) \leq \text{reg}(I)$, but this is not easy to see on the definition in terms of syzygies. Also, when $\dim X = 0$ ($X := \text{Proj}(R/I)$, as above), it easily follows from the cohomological definition and the fact that $H_{\mathfrak{m}}^i(M) = 0$ for $i > \dim M$ (Grothendieck's vanishing theorem) that $\text{reg}(I)$ is the smallest integer μ such that :

- (1) $I_\mu = (I_X)_\mu$ (recall that I_X is the saturation of I),
- (2) $\dim(R/I_X)_{\mu-1} = \deg(X)$.

In case X is a set of simple points, condition (2) says that passing through the $\deg(X)$ different points of X impose linearly independent conditions on polynomials of degree $\mu-1$. An elementary account on regularity in this context is given in §4 of [AGGM02, D. Cox. Curves, surfaces, and syzygies, 131–150].

The fact that $\text{reg}(I)$ bounds the degrees of the syzygies of I shows the naturality of considering this invariant in the implicitization problem using the syzygy matrix.

On the computational side, the degrees of generators of a Gröbner basis of the ideal for the degree-reverse-lex order, under a quite weak conditions on the coordinates, is bounded by $\text{reg}(I)$. This is another way of understanding the regularity as a measure of the complexity of the ideal.

The Koszul complex. [Ei, §17] — Let $x = (x_1, \dots, x_r)$ be a r -tuple of elements in a ring A . The (homological) Koszul complex $K_\bullet(x; A)$ is the complex with modules $K_p(x; A) := \bigwedge^p A^r \simeq A^{\binom{r}{p}}$ and maps $d_p : K_p(x; A) \rightarrow K_{p-1}(x; A)$ defined by :

$$g \cdot e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto g \cdot \sum_{j=1}^p (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \widehat{e_{i_j}} \cdots \wedge e_{i_p}.$$

We set $Z_p(x; A) := \ker(d_p)$ and $H_p(x; A) := Z_p(x; A)/\text{im}(d_{p+1})$.

The \mathcal{Z} -complex. [Va, **] — We consider $f_i \in R \subset S$ as elements of S and the two complexes $K_\bullet(f; S)$ and $K_\bullet(T; S)$ where $T := (T_0, \dots, T_n)$. These complexes have the same modules $K_p = \bigwedge^p S^{n+1} \simeq S^{\binom{p}{n+1}}$ and differentials d_\bullet^f and d_\bullet^T

• It directly follows from the definitions that $d_{p-1}^f \circ d_p^T + d_{p-1}^T \circ d_p^f = 0$, so that $d_p^T(Z_p(f; S)) \subset Z_{p-1}(f; S)$. The complex $\mathcal{Z}_\bullet := (Z_\bullet(f; S), d_\bullet^T)$ is the called \mathcal{Z} -complex associated to the f_i 's.

• Notice that $Z_p(f; S) = S \otimes_R Z_p(f; R)$ and

— $Z_0(f; R) = R$,

— $Z_1(f; R) = \text{Syz}_R(f_0, \dots, f_n)$,

— the map $d_1^T : S \otimes_R \text{Syz}_R(f_0, \dots, f_n) \rightarrow S$ is defined by :

$$(a_0, \dots, a_n) \mapsto a_0 T_0 + \dots + a_n T_n.$$

The following result shows the intrinsic nature of the homology of the \mathcal{Z} -complex, it is a key point in proving results on its acyclicity.

Theorem. $H_0(\mathcal{Z}_\bullet) \simeq \mathcal{S}_I$ and the homology modules $H_i(\mathcal{Z}_\bullet)$ are \mathcal{S}_I -modules that only depends on $I \subset R$, up to isomorphism.

• We let $R' := k[T_0, \dots, T_n]$ and look at graded pieces :

$$\mathcal{Z}_\bullet^\mu : \dots \rightarrow R' \otimes_k Z_2(f; R)_\mu \xrightarrow{d_2^T} R' \otimes_k Z_1(f; R)_\mu \xrightarrow{d_1^T} R' \otimes_k Z_0(f; R)_\mu \rightarrow 0$$

where $Z_p(f; R)_\mu$ is the part of $Z_p(f; R)$ consisting of elements of the form $\sum a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$ with the $a_{i_1 \dots i_p}$ all of the same degree μ .

Nota Bene. This is not the usual convention for the grading of these modules, however we choosed it here for simplicity. The usual grading (used for instance in [BC] or [Ch]) makes the Koszul maps homogeneous of degree 0, so they ask $a_{i_1 \dots i_p}$ to be homogeneous of degree $\mu - pd$ in place of being of degree μ .

We will denote the cokernel of the last map by \mathcal{S}_I^μ .

Determinants of complexes. [No **, GKZ **] — Let A be a commutative domain, for simplicity.

If $A^n \xrightarrow{\alpha} A^n$ is A -linear we can define $\det(\alpha) \in A$.

If we have a complex C_\bullet with three terms:

$$\begin{array}{ccccc} & & A^m & \longrightarrow & A^n \\ & \nearrow \beta & \oplus & \nearrow \alpha & \\ A^m & \longrightarrow & A^n & & \end{array}$$

such that $\det(\beta) \neq 0$, we set $\det(C_\bullet) := \frac{\det(\alpha)}{\det(\beta)}$. In fact $\det(C_\bullet)$ is independent of the decomposition of C_1 as a direct sum $A^m \oplus A^n$.

More generally a bounded complex C_\bullet of free A -modules such that $\text{Frac}A \otimes_A C_\bullet$ is exact may always be decomposed in the following way

$$\begin{array}{ccccccc} \dots & & A^q & \longrightarrow & A^p & \longrightarrow & A^m & \longrightarrow & A^n \\ & \nearrow \delta & \oplus & \nearrow \gamma & \oplus & \nearrow \beta & \oplus & \nearrow \alpha & \\ \dots & & A^q & \longrightarrow & A^p & \longrightarrow & A^m & \longrightarrow & A^n \end{array}$$

with α, β, \dots having non-zero determinants. Then $\det(C_\bullet) := \frac{\det(\alpha) \cdot \det(\gamma) \cdots}{\det(\beta) \cdot \det(\delta) \cdots}$ is independent of the decomposition.

Performing the decomposition of a given complex is easy : decompose first C_1 into $A^n \oplus A^m$ so that $\det(\alpha) \neq 0$ (this amounts to choose a non zero maximal minor of the map $C_1 \rightarrow C_0 = A^n$), and apply the procedure recursively to the complex :

$$\dots \longrightarrow C_3 \longrightarrow C_2 \longrightarrow A^m \longrightarrow 0$$

Fitting ideals. [No **, Ei, §20.2] — If A is a ring and M is a module represented as the cokernel of a map $\psi : A^m \rightarrow A^n$, the ideal generated by minors of size $n - i$ of ψ only depends on M and i , this ideal is called the i -th Fitting ideal of the A -module M . One of the most important of these ideals associated to the A -module M is the 0-th Fitting ideal (*i.e.* the one generated by the maximal minors of ψ), denoted by $\text{Fitt}_A^0(M)$.

5. The method and main results

We assume hereafter that $\pi(\Gamma)$ is of codimension 1 in \mathbf{P}_k^n defined by the equation $H = 0$ and denote by δ the degree of the map π from Γ onto its image.

If J is a R' -ideal, we will denote by $[J]$ the gcd of the elements in J . It represents the component of codimension one of the scheme defined by J (its divisorial component) because R' is factorial.

With these notations, one has :

Proposition 1. [BJ, 5.2] *If $X = \emptyset$, \mathcal{Z}_\bullet is acyclic and*

$$[\text{Fitt}_{R'}^0(\mathcal{S}_I^\mu)] = \det(\mathcal{Z}_\bullet^\mu) = H^\delta,$$

for every $\mu \geq (n - 1)(d - 1)$.

The identities above are identities of principal ideals in R' , therefore it corresponds to an equality of elements of R' up to units. Recall that $[\text{Fitt}_{R'}^0(\mathcal{S}_I^\mu)]$ is the gcd of maximal minors of the map $d_1^T : (a_0, \dots, a_n) \mapsto a_0 T_0 + \dots + a_n T_n$ from the syzygies of degree μ

(each a_i is of degree μ) seen as a vector space over k to $R' \otimes_k R_\mu$. The entries of this matrix are therefore linear forms in the T_i 's with coefficients in k .

This proposition shows that the determinant of this graded part of \mathcal{Z}_\bullet actually computes the divisor $\pi_*(\Gamma) = \delta \cdot \pi(\Gamma)$ obtained as direct image of the cycle Γ (see [Fu, **] for the definition of the direct image $\pi_*(\Gamma)$ of the cycle Γ).

In the case X is of dimension zero, the situation is slightly more complicated :

Proposition 2. [BJ, 5.7, 5.10 & BC 4.1] *If $\dim X = 0$,*

- (i) *The following are equivalent :*
 - (a) *X is locally defined by at most n equations,*
 - (b) *\mathcal{Z}_\bullet is acyclic,*
 - (c) *\mathcal{Z}_\bullet^μ is acyclic for $\mu \gg 0$.*

- (ii) *If \mathcal{Z}_\bullet is acyclic, then*

$$[\text{Fitt}_{R'}^0(\mathcal{S}_I^\mu)] = \det(\mathcal{Z}_\bullet^\mu) = H^\delta G,$$

for every $\mu \geq (n-1)(d-1) - \varepsilon_X$, where $1 \leq \varepsilon_X \leq d$ is the minimal degree of a hypersurface containing X and $G \neq 0$ is a homogeneous polynomial which is a unit if and only if X is locally a complete intersection.

Remark 3. In fact $[\text{Fitt}_{R'}^0(\mathcal{S}_I^\mu)] = \det(\mathcal{Z}_\bullet^\mu) = \pi_* V$ for $\mu \geq (n-1)(d-1) - \varepsilon_X$, and the degree of G is the sum of numbers measuring how far X is from a complete intersection at each point of X .

Remark 4. It is very fast to compute the ideal I_X with a dedicated computer algebra system (like Macaulay 2, Singular or Cocoa), and *a fortiori* to compute ε_X which is the smallest degree of an element in I_X . Moreover the following result actually implies a good bound on the complexity of this task.

Proposition 5. [Ch, 3.3] *If $J \subset R$ is a homogeneous ideal generated in degree at most d with $\dim(R/J) = 1$ and J' its saturation (in other words, the defining ideal of the zero-dimensionnal scheme $X := \text{Proj}(R/J)$), then*

$$\text{reg}(J) \leq n(d-1) + 1 \quad \text{and} \quad \text{reg}(J') \leq (n-1)(d-1) + 1.$$

In example E , $\text{reg}(I) = \text{reg}(I_X) = 4$, while the general bound above gives $\text{reg}(I) \leq 7$ and $\text{reg}(I^{\text{sat}}) \leq 5$. A minimal free R -resolution of I gives a resolution of Z_1 :

$$0 \longrightarrow R[-2] \xrightarrow{\begin{bmatrix} c \\ -b \\ -a \\ 0 \end{bmatrix}} R[-1]^3 \oplus R[-2] \xrightarrow{\begin{bmatrix} 0 & 0 & 0 & -b(a+c) \\ a & 0 & c & 0 \\ -b & -c & 0 & 0 \\ 0 & a & -b & ac \end{bmatrix}} Z_1 \subset R^4 = S_1$$

and we have seen that $I_X = (ac^2, b(a+c))$, so that $\text{indeg}(I_X) = 2$ and therefore $(n-1)(d-1) - \varepsilon_X = 2 \times (3-1) - 2 = 2$. The syzygies of degree 2 are of the form :

$$\ell_1(ay - bz) + \ell_2(at - cz) + \ell_3(cy - bt) + \lambda_4(act - b(a+c)x)$$

with $\ell_i \in R_1$ and $\lambda_4 \in R_0 = k$. Notice that they are not linearly independent, and that the relation (unique in this degree) is given by the second syzygy :

$$c(ay - bz) - b(at - cz) - a(cy - bt) = 0.$$

We may for instance choose as generators of syzygies of degree 2 the 9 syzygies, s_1 to s_9 :

$$a(ay-bz), b(ay-bz), a(at-cz), b(at-cz), c(at-cz), a(cy-bt), b(cy-bt), c(cy-bt), act-b(a+c)x$$

which gives the 6×9 matrix of linear forms (elements of R'_1) for the matrix of d_1^T in degree 2 (recall that $T = (x, y, z, t)$ with the notations of the example) :

$$\begin{array}{c} s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8 \quad s_9 \\ \begin{array}{l} a^2 \\ ab \\ ac \\ b^2 \\ bc \\ c^2 \end{array} \end{array} \begin{bmatrix} y & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ -z & y & 0 & t & 0 & -t & 0 & 0 & -x \\ 0 & 0 & -z & 0 & t & y & 0 & 0 & t \\ 0 & -z & 0 & 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & -z & 0 & 0 & y & -t & -x \\ 0 & 0 & 0 & 0 & -z & 0 & 0 & y & 0 \end{bmatrix}.$$

Now, Z_2 has a free R -resolution of the form:

$$0 \longrightarrow R[-4] \longrightarrow R[-1] \oplus R[-3]^3 \longrightarrow Z_2 \subset \bigwedge^2 S_1 = R^6.$$

As $d_2^f(\Sigma) = b(a+c)[c(ay-bz) - b(at-cz) - a(cy-bt)] = 0$, the generator of degree 1 is

$$\Sigma := c y \wedge z - b z \wedge t - a y \wedge t \in \bigwedge^2 S_1.$$

We have $d_2^T(\Sigma) = c(z \otimes y - y \otimes z) - b(z \otimes t - t \otimes z) - a(t \otimes y - y \otimes t) \in S \otimes_R \bigwedge^1 S_1$, that we may rewrite

$$d_2^T(\Sigma) = -t \otimes (ay - bz) + y \otimes (at - cz) + z \otimes (cy - bt).$$

In degree 2, the matrix of $d_2^T : R' \otimes_k \Sigma R_1 \longrightarrow R' \otimes_k Z_1(f; R)_2$ on the bases $(a\Sigma, b\Sigma, c\Sigma)$ for the source and (s_1, \dots, s_9) for the target is therefore the transpose of

$$\begin{array}{c} s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8 \quad s_9 \\ \begin{array}{l} a\Sigma \\ b\Sigma \\ c\Sigma \end{array} \end{array} \begin{bmatrix} -t & 0 & y & 0 & 0 & z & 0 & 0 & 0 \\ 0 & -t & 0 & y & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & -t & y & t & 0 & z & 0 \end{bmatrix}.$$

We now choose a maximal non zero minor of this matrix, for instance the minor Δ_2 given by lines 3, 4 and 5 of the matrix of d_2^T , and the minor Δ_1 of the matrix of d_1^T obtained by erasing columns 3,4 and 5. We get the formula :

$$H = \frac{\Delta_1}{\Delta_2} = \frac{\begin{vmatrix} y & 0 & 0 & 0 & 0 & 0 \\ -z & y & t & 0 & 0 & -x \\ 0 & 0 & y & 0 & 0 & t \\ 0 & -z & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & y & -t & -x \\ 0 & 0 & 0 & 0 & y & 0 \end{vmatrix}}{\begin{vmatrix} y & 0 & 0 \\ 0 & y & -t \\ 0 & 0 & y \end{vmatrix}} = \frac{-y^3(xyz + xyt - t^2z)}{y^3}.$$

Computations of the free R -resolutions of Z_1 and Z_2 were done using the dedicated software Macaulay 2 by Dan Grayson and Mike Stillman [M2]. In the case $n = 3$, this computation goes very fast, even for pretty high degree d , and Macaulay 2 performs degree truncations to speed up the computation, if needed. The graded pieces that we need to know can also easily be computed using linear algebra routines, as detailed in [BC] and implemented in [Bu].

When the dimension of the base locus X of the map ϕ increases, the situation becomes harder to analyze. In dimension 1, the situation is pretty well understood :

Proposition 6. [Ch, 8.2, 8.3] *Assume that $\dim X = 1$ and let \mathcal{C} be the union of components of dimension 1 of X (its “unmixed part”). Then,*

(i) *The following are equivalent :*

(a) *X is locally defined by at most n equations and \mathcal{C} is defined on a dense open subset by at most $n - 1$ equations,*

(b) *\mathcal{Z}_\bullet^μ is acyclic for $\mu \gg 0$.*

(ii) *If \mathcal{Z}_\bullet^μ is acyclic for $\mu \gg 0$, the following are equivalent :*

(a) *\mathcal{Z}_\bullet is acyclic,*

(b) *\mathcal{C} is arithmetically Cohen-Macaulay,*

(b') *every section $f \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(\mu))$ is the restriction to \mathcal{C} of a polynomial function of degree μ , for every $\mu \in \mathbf{Z}$.*

(iii) *If \mathcal{Z}_\bullet^μ is acyclic for $\mu \gg 0$ and $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(\mu)) = 0$ for all $\mu < -d$ —for instance if \mathcal{C} is reduced— then \mathcal{Z}_\bullet^μ is acyclic for $\mu \geq (n - 1)(d - 1)$. If further X is defined by at most $n - 1$ equations locally on the support of \mathcal{C} , then*

$$[\text{Fitt}_{R'}^0(\mathcal{S}_I^\mu)] = \det(\mathcal{Z}_\bullet^\mu) = H^\delta G,$$

for every $\mu \geq (n - 1)(d - 1)$, where G is a homogeneous polynomial such that the support of $\pi(V)$ is the zero set of GH .

Here also, more precisely, $\det(\mathcal{Z}_\bullet^\mu)$ represents the divisor π_*V .

Remark 7. It is perhaps true that $\det(\mathcal{Z}_\bullet^\mu)$ represents the divisor π_*V for $\mu \geq (n-1)(d-1)$ when \mathcal{Z}_\bullet^μ is acyclic for $\mu \gg 0$ and $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(\mu)) = 0$ for all $\mu < -d$, but we needed the slightly stronger hypothesis above to prove it in [Ch].

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