# SOME RESULTS AND QUESTIONS ON CASTELNUOVO-MUMFORD REGULARITY

Marc Chardin

## §1. The two most classical definitions of Castelnuovo-Mumford regularity

Let $R := k[X_1, \dots, X_n]$ be a polynomial ring over a field $k$ and $M$ a finitely generated graded $R$-module.

The two most popular definitions of Castelnuovo-Mumford regularity are the one in terms of graded Betti numbers and the one using local cohomology.

— Local cohomology modules. Set $\mathfrak{m} := (X_1, \dots, X_n) = R_{>0}$, then $H^0_{\mathfrak{m}}(M) := \{x \in M \mid \exists N,\ \mathfrak{m}^N x = 0\}$ and the functors $H^i_{\mathfrak{m}}(—)$ can be defined as right derived functors of $H^0_{\mathfrak{m}}(—)$ in the category of $R$-modules. A more concrete way of considering these modules, is to see them as the cohomology modules of the Čech complex $\mathcal{C}^\bullet_{\mathfrak{m}}$ :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & \bigoplus_i M_{X_i} & \xrightarrow{\psi} & \bigoplus_{i<j} M_{X_iX_j} & \longrightarrow \cdots \longrightarrow & M_{X_1\cdots X_n} & \longrightarrow & 0\ . \\ & & \| & & \| & & \| & & \| & & \\ & & \mathcal{C}^0_{\mathfrak{m}}(M) & & \mathcal{C}^1_{\mathfrak{m}}(M) & & \mathcal{C}^2_{\mathfrak{m}}(M) & & \mathcal{C}^n_{\mathfrak{m}}(M) & & \end{array}$$

This is how we will view them in this article.

Recall that, from a more geometric point of view, one has graded isomorphismes

$$\Gamma M := \ker(\psi) \simeq \bigoplus_{\mu} H^0(\mathrm{Proj}(R), \widetilde{M}(\mu)),$$

where $\widetilde{M}$ is the sheaf of modules associated to $M$, and

$$H^i_{\mathfrak{m}}(M) \simeq \bigoplus_{\mu} H^{i-1}(\mathrm{Proj}(R), \widetilde{M}(\mu)),\ \forall i \geq 2.$$

There are two fundamental results. First, Grothendieck's theorem that asserts the vanishing of $H^i_{\mathfrak{m}}(M)$ for $i > \dim(M)$ and $i < \mathrm{depth}\,(M)$, as well as the non vanishing of these modules for $i = \dim(M)$ and $i = \mathrm{depth}\,(M)$. Second Serre vanishing theorem that implies the vanishing of graded pieces $H^i_{\mathfrak{m}}(M)_\mu$ for any $i$ and $\mu$ big enough. The Castelnuovo-Mumford regularity is a measure of this vanishing degree. Set

$$a_i(M) := \max\{\mu \mid H^i_{\mathfrak{m}}(M)_\mu \neq 0\},$$

if $H^i_{\mathfrak{m}}(M) \neq 0$ and $a_i(M) := -\infty$ else. Then,

$$\mathrm{reg}(M) = \max_i\{a_i(M) + i\}.$$

The maximum over the positive $i$'s is also an interesting invariant :

$$\mathrm{greg}(M) := \max_{i>0}\{a_i(M) + i\} = \mathrm{reg}(M/H^0_{\mathfrak{m}}(M)).$$

— Graded Betti numbers. Let $F_\bullet$ be a minimal graded free $R$-resolution of $M$,

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$$

with $F_i = \oplus_j R[-j]^{\beta_{ij}}$. Notice that $p = \mathrm{pdim}(M) = n - \mathrm{depth}\,(M)$. The maps of $F_\bullet \otimes_R k$ being zero maps, $\mathrm{Tor}_i^R(M,k) = H_i(F_\bullet \otimes k) = F_i \otimes k$ and therefore $\beta_{ij} = \dim_k \mathrm{Tor}_i^R(M,k)_j$. Set

$$b_i(M) := \max\{\mu \mid \mathrm{Tor}_i^R(M,k)_\mu \neq 0\}$$

if $\mathrm{Tor}_i^R(M,k) \neq 0$ and $b_i(M) := -\infty$ else. By definition, $b_i(M)$ is the maximal degree of a minimal generator of $F_i$, and therefore of the module of $i$-th syzygies of $M$.

The Castelnuovo-Mumford regularity is also a measure of the maximal degrees of generators of the modules $F_i$ :

$$\mathrm{reg}(M) = \max_i\{b_i(M) - i\}.$$

## §2. A lemma from homological algebra and the equivalence of the definitions

One way of proving the equality $\max_i\{a_i(M)+i\} = \max_i\{b_i(M)-i\}$ is to use a double complex relating the Tor modules and the local cohomology. We will now give a lemma which is useful in this approach, and which have further applications.

We will use the following notations :

— For $F^\bullet$ (resp. $F_\bullet$), a bounded graded complex of finitely generated free $R$-modules, we set $b(F^\bullet) := \max_q\{b_0(F^q) + q\}$ (resp. $b(F_\bullet) := \max_q\{b_0(F_q) - q\}$).

— If $H$ is a graded module such that $H_\mu = 0$ for $\mu \gg 0$, we set $\mathrm{end}(H) := \max\{\mu \mid H_\mu \neq 0\}$ if $H \neq 0$, and $\mathrm{end}(0) := -\infty$.

A complex $C$ of free $R$-modules is called minimal if the differentials of $C \otimes_R k$ are zero.

**Lemma 2.1.** *Let $F^\bullet$ be a minimal graded complex of finitely generated free $R$-modules, with $F^i = 0$ for $i < 0$, $M$ be a finitely generated graded $R$-module and $T^\bullet := \mathcal{C}^\bullet_\mathfrak{m} M \otimes_R F^\bullet$.*

*Then $H^\ell(T^\bullet)_\mu = 0$, for $\mu \gg 0$ and any $\ell$. Moreover, for any $\ell$,*

(i)

$$\mathrm{end}(H^\ell(T^\bullet)) \leq \max_{p+q=\ell}\{a_p(H^q(F^\bullet \otimes_R M))\}$$

*and equality holds if $\dim H^q(F^\bullet \otimes_R M) \leq 1$ for all $q$ or if there exists $q_0$ such that $\dim H^q(F^\bullet \otimes_R M) \leq 1$ for $q < q_0$ and $H^q(F^\bullet \otimes_R M) = 0$ for $q > q_0$.*

(ii)

$$\mathrm{end}(H^\ell(T^\bullet)) \leq \max_{p+q=\ell}\{a_p(M) + b_0(F^q)\}.$$

*If further, $b(F^\bullet) = \max\{b_0(F^0), b_0(F^1) + 1\}$, then*

$$\max_{j \leq \ell}\{\mathrm{end}(H^j(T^\bullet)) + j\} = \max_{p+q \leq \ell}\{a_p(M) + b_0(F^q) + p + q\}.$$

The proof of this lemma is given in the technical appendix. Let us prove the equivalence of the definitions, and a little more, as an application :

**Corollary 2.2.** *If $M$ is a finitely generated graded $R$-module, then for any $\ell$*

$$\max_{p \leq \ell}\{a_p(M) + p\} = \max_{q \geq n-\ell}\{b_q(M) - q\}.$$

*As a consequence,*

$$\mathrm{reg}(M) = \max_q\{b_q(M) - q\} = \max_{\mathrm{pdim} M \geq q \geq \mathrm{codim} M}\{b_q(M) - q\}.$$

*Proof.* Take $F^\bullet := K^\bullet(X_1, \dots, X_n; R)$. One has $b(F^q) = -q$ and $\dim H^q(F^\bullet \otimes_R M) \le 0$ for all $q$. Therefore, by the lemma,

$$\max_{q \le \ell}\{a_0(H^q(F^\bullet \otimes_R M)) + q\} = \max_{p \le \ell}\{a_p(M) + p\}$$

but $H^q(F^\bullet \otimes_R M) \simeq H_{n-q}(K_\bullet(X_1, \dots, X_n; M)[n]) \simeq \mathrm{Tor}^R_{n-q}(M, k)[n]$. It follows that $a_0(H^q(F^\bullet \otimes_R M)) = b_{n-q}(F^M_\bullet) - n$.

Setting $q' := n - q$ the max on the left hand side can be rewritten as $\max_{q' \ge n-\ell} b_{q'}(F^M_\bullet) - n + (n - q')$.

The second claim follows from Grothendieck's vanishing theorem. $\square$

### §3. Other definitions and further applications of the lemma.

By the definition of the regularity in terms of the Betti numbers, $\mathrm{reg}(M) = \mathrm{indeg}(M)$ if and only if $M$ is generated in a single degree and the matrices of the maps in its minimal free $R$-resolution have only linear forms as entries. Such a resolution (generators in a single degree and maps given by linear forms) is called a *linear resolution.*

A first application of the equivalence of the definitions above is the following third definition :

**Proposition 3.1.** *For a finitely generated graded R-module,*

$$\mathrm{reg}(M) = \min\{\mu \mid M_{\ge\mu} \textit{ has a linear resolution}\}.$$

*Proof.* The modules $M$ and $M' := M_{\ge\mu}$ coincides on the punctured spectrum, therefore $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}}(M')$ for $i > 1$ and $\Gamma M = \Gamma M'$. Moreover, the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M)_\nu \longrightarrow M_\nu \longrightarrow \Gamma M_\nu \longrightarrow H^1_{\mathfrak{m}}(M)_\nu \longrightarrow 0$$

compared to the corresponding one for $M'$ shows that :

— $a_0(M') = a_0(M)$ if $\mu \le a_0(M)$ and $a_0(M') = 0$ else,

— $a_1(M') = \max\{a_1(M), \mu\}$,

which implies that $\mathrm{reg}(M') = \max\{\mathrm{reg}(M), \mu\}$ and the proposition. $\square$

We will see in Proposition 9.1 (v) and Proposition 9.5 that one also have :

**Proposition 3.2.** *For a finitely generated graded R-module,*

$$\mathrm{reg}(M) = \min\{\mu \ge \min\{a_0(M), b_0(M)\} \mid H^i_{\mathfrak{m}}(M)_{\mu-i} = 0,\ \forall i\} - 1$$

*and for a graded ideal $I$ such that $\sqrt{I} \ne \mathfrak{m}$,*

$$\mathrm{reg}(I) = \min\{\mu \mid H^i_{\mathfrak{m}}(R/I)_{\mu-i} = 0,\ \forall i\}.$$

This proposition is a persistence theorem, in which the shift by $i$ in the $i$-th cohomology reveals its usefulness.

Also recall that, by local duality, $a_i(M) = -\mathrm{indeg}(\mathrm{Ext}^{n-i}_R(M, \omega_R)) = -\mathrm{indeg}(\mathrm{Ext}^{n-i}_R(M, R)) - n$ and therefore

$$\mathrm{reg}(M) = -\min_i\{\mathrm{indeg}(\mathrm{Ext}^i_R(M, R)) + i\}.$$

## §4. Regularity and Gröbner bases

**Remark 4.1.** Taking $F^\bullet = K^\bullet(f;M)$ in Lemma 2.1, or applying directly Theorem 5.1 with $N := R/(f)$, gives the well-known fact that

$$\operatorname{reg}(M) = \max\{\operatorname{reg}(0 :_M (f)), \operatorname{reg}(M/(f)M) - \deg f + 1\}$$

if $\dim(0 :_M (f)) \leq 1$. More generally, Theorem 5.1 shows that, if $f := (f_1, \dots, f_s)$ is a $s$-tuple of forms and $H_i := H_i(K_\bullet(f;M))$, one has

$$\operatorname{reg}(M) = \max_i\{\operatorname{reg}(H_i)\} - \sum_j (\deg f_j - 1),$$

if $\dim H_i \leq 1$ for $i > 0$.

For a rev-lex order one has $\operatorname{in}(I : (X_n)) = \operatorname{in}(I) : (X_n)$ and $\operatorname{in}(I + (X_n)) = \operatorname{in}(I) + (X_n)$, and this may be extended to graded modules (see for instance [E, §15.7]). As a consequence, the modules $I : (X_n)/I$ and $\operatorname{in}(I) : (X_n)/\operatorname{in}(I)$ have the same Hilbert function $H$. Therefore, if $I : (X_n)/I$ has dimension zero these two modules have the same regularity: the last degree were $H$ is not 0 (or $-\infty$ if $I : (X_n) = I$). By the remark above, if $\dim(I : (X_n)/I) \leq 1$

$$\operatorname{reg}(R/I) = \max\{\operatorname{reg}(I : (X_n)/I), \operatorname{reg}(R/I + (X_n))\}$$

and

$$\operatorname{reg}(R/\operatorname{in}(I)) = \max\{\operatorname{reg}(\operatorname{in}(I) : (X_n)/\operatorname{in}(I)), \operatorname{reg}(R/\operatorname{in}(I + (X_n))\}.$$

It follows that if $(I + (X_n, \dots, X_{i+1})) : (X_i)/(I + (X_n, \dots, X_{i+1}))$ has finite length for $i$ from $n$ to 1 then by induction (for $i = 1$, $I = \operatorname{in}(I) = \mathfrak{m}$) we have $\operatorname{reg}(R/I) = \operatorname{reg}(R/\operatorname{in}(I))$. After a general linear change of coordinates, these modules are indeed all of finite length. This proves the first part of the following theorem of Bayer and Stillman (that also extends to modules)

**Theorem 4.2.** [BS1, 2.4 & 2.9] *In general coordinates, for a rev-lex order,*

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I)),$$

*and if furthermore $k$ is of characteristic zero,* $\operatorname{reg}(\operatorname{in}(I)) = b_0(I)$.

The second part of the theorem can be deduced from the fact that, in characteristic zero, the generic initial ideal $J$ of $I$ is a strongly stable monomial ideal (*i.e.* for any monomial $M$, $X_iM \in J \Rightarrow X_jM \in J$, $\forall j \leq i$) and strongly stable monomial ideals have regularity equal to their maximal degree of generator. More precisely, Eliahou and Kervaire provided in [EK] a minimal free $R$-resolution of strongly stable monomial ideals, from which this is easy to deduce. One should also notice that, for any monomial ideal $K$, using a resolution due to Diana Taylor, one has

$$b_i(K) \leq \max_{m_0,\dots,m_i \in S} \deg(\gcd(m_0, \dots, m_i)) \leq (i+1)b_0(K),$$

were $S$ is a minimal set of monomial generators of $K$. In this sense, the discrepency between $b_0(K)$ and $\operatorname{reg}(K)$ is quite under control in the case of monomial ideals, as compared with arbitrary ideals.

The equality $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I))$ in the theorem has been refined by Bayer, Charalambous and Popescu, who proved in [BCP] that the so-called extremal Betti numbers of $I$ and $\operatorname{in}(I)$ coincide. It is also part of folklore that these coincides with what one can call extremal local cohomologies by analogy. We will represent on a typical picture what this means and some comparison between graded Betti numbers and dimensions of graded pieces of local cohomology modules. For a finitely generated graded module $M$ we set $\beta'_{i,j} := \dim_k \mathrm{Tor}_i^R(M,k)_{j+i}$ and $\alpha'_{i,j} := \dim_k H^i_{\mathfrak{m}}(M)_{j-i}$ and put in a table, indexed by $i$ and $j$ the numbers $\beta'_{i,j}$ and $\alpha'_{i,j}$. Then the tabular have the following shapes for both $M$ and its generic initial ideal :

Betti numbers :

| | | $c-1$ | $c$ | $c+1$ | $\cdots$ | $\cdots$ | $i$ | $\cdots$ | $\cdots$ | $p$ | $p+1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | $\cdots$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| reg + 1 | $\cdots$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| reg | $\cdots$ | $\boxtimes$ | $\boxtimes$ | $\blacksquare$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| reg − 1 | $\cdots$ | $*$ | $*$ | $\square$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $\cdots$ | $*$ | $*$ | $\square$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $\cdots$ | $*$ | $*$ | $\boxplus$ | $\boxtimes$ | $\boxtimes$ | $\blacksquare$ | 0 | 0 | 0 | 0 |
| | $\cdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | $\boxplus$ | $\boxtimes$ | $\boxtimes$ | $\blacksquare$ | 0 |
| | $\cdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $\square$ | 0 |
| | $\cdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $\square$ | 0 |
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

with $p := \mathrm{pdim}(M)$ and $c := \mathrm{codim}(M)$.

Local cohomology :

| | $q-1$ | $q$ | $\cdots$ | $\cdots$ | $i$ | $\cdots$ | $\cdots$ | $d-1$ | $d$ | $d+1$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |
| reg + 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |
| reg | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\blacksquare$ | $\boxtimes$ | 0 | $\cdots$ |
| reg − 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\square$ | $*$ | 0 | $\cdots$ |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\square$ | $*$ | 0 | $\cdots$ |
| | 0 | 0 | 0 | 0 | $\blacksquare$ | $\boxtimes$ | $\boxtimes$ | $\boxplus$ | $*$ | 0 | $\cdots$ |
| | 0 | $\blacksquare$ | $\boxtimes$ | $\boxtimes$ | $\boxplus$ | $*$ | $*$ | $*$ | $*$ | 0 | $\cdots$ |
| | 0 | $\square$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | 0 | $\cdots$ |
| | 0 | $\square$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | 0 | $\cdots$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

with $q := \mathrm{depth}\,(M) = n - p$ and $d := \dim(M) = n - c$.

The numbers at the spots marked by $\blacksquare$ are not zero and called the corners of the Betti diagram. Notice that $i, j$ is a corner of the Betti diagram if and only if $n - i, j$ is a corner of the local cohomology diagram.

By the result of Bayer, Charalambous and Popescu, they are unchanged when passing to the initial ideal for the rev-lex order in general coordinates.

The numbers at the spots marked by $\blacksquare$, $\boxtimes$, $\boxplus$ or $\square$ in both diagrams are related by $\tilde{\alpha}'_{i,j} \leq \beta'_{i,j} \leq \tilde{\alpha}'_{i,j} + \sum_{\ell>0} \binom{n}{\ell} \alpha'_{i-\ell,j}$, where $\tilde{\alpha}'_{i,j} := \dim_k(\mathrm{Socle}\,(H^i_{\mathfrak{m}}(M))_{j-i})$. This shows that :

— at spots marked by $\blacksquare$ : $\alpha'_{i,j} = \beta'_{i,j} \neq 0$,

— at spots marked by $\boxtimes$ : $\alpha'_{i,j} \leq \beta'_{i,j} \leq \sum_{\ell \geq 0} \binom{n}{\ell} \alpha'_{i,j-\ell}$,

— at spots marked by $\square$ : $\beta'_{i,j} = \tilde{\alpha}'_{i,j}$,

— at spots marked by $\boxplus$ : $\tilde{\alpha}'_{i,j} \leq \beta'_{i,j} \leq \tilde{\alpha}'_{i,j} + \sum_{\ell>0} \binom{n}{\ell} \alpha'_{i-\ell,j}$.

These facts comes from the study of the spectral sequence $\oplus_j \mathrm{Tor}^R_{i+j}(H^j_{\mathfrak{m}}(M), k) \Rightarrow \mathrm{Tor}^R_i(M, k)$. See the notes of Schenzel's lectures in Barcelona in [Bar] for more details.

In another direction, Bermejo and Gimenez discovered that the Castelnuovo-Mumford regularity may also be computed from the initial ideal under weaker conditions on the genericity of the coordinates (see [BG]), and that this genericity condition can be checked on the initial ideal as well. An algorithm based on this idea was implemented in Singular to compute the regularity following this approach. This idea also lead to other developments by Caviglia and Sbarra [CS].

The study of generic initial ideals for different orders is an active subject of research. A good introduction to the subject is the notes of Green from the summer school in Barcelona [Bar]. An important recent result is the determination of the regularity of the generic initial ideal, for the lexicographic order, of smooth complete intersection curves in $\mathbf{P}^3$ :

**Theorem 4.3.** [CS, 1.1] *Consider a smooth complete intersection curve in $\mathbf{P}^3$, intersection of two surfaces of degrees $a$ and $b$ with $a, b > 1$. Then the regularity of its generic initial ideal for the lexicographic order is equal to $\frac{a(a-1)b(b-1)}{2} + 1$ unless $a = b = 2$ in which case it is equal to 4.*

It is interesting to compare this value with the regularity of the lex-segment ideal associated to the complete intersection ideal, which is $\frac{a(a-1)b(b-1)}{2} + ab$. The difference is (relatively) small. Their proof relies on the use of Green's partial elimination ideals. The value of the regularity is governed by the first partial elimination ideal, which defines the singular points of a generic projection of the curve. This singular loci consists of $\frac{a(a-1)b(b-1)}{2}$ nodes.

An interesting result that they prove on the way is the following proposition for points :

**Theorem 4.4.** *Let $I$ be the defining ideal of a set of $s$ points in sufficiently general position. Then the generic initial ideal for the lexicographic order is equal to the lex-segment ideal associated to $I$.*

For a precise statement on the "general position" condition and a generalization to other orders, see [CS, 5.6].

The study of monomial ideals and their resolutions is a very active domain of research, with many links to combinatorics and many interesting recent results. We will not go further into this field in this short note.

## §5. On the regularity of Tor modules

A first important result is a theorem of Caviglia, who proved in [Cav] that $\mathrm{reg}(M \otimes_R N) \leq \mathrm{reg}(M) + \mathrm{reg}(N)$ if $\dim \mathrm{Tor}^R_1(M, N) \leq 1$. This work was a continuation of previous results of Conca and Herzog [CH] and of Sidman [Si].

The regularity of Tor modules was subsequently studied in details by Eisenbud, Huneke and Ulrich in [EHU], where they prove a result [EHU, 2.3] which is quite comparable to the following one :

**Theorem 5.1.** *Let $M, N$ be two finitely generated graded $R$-modules, set $p := \mathrm{pdim}(M)$ and $p' := \mathrm{pdim}(N)$. Assume that $\dim \mathrm{Tor}^R_1(M, N) \leq 1$. Then,*

$$\max_i \{\mathrm{reg}(\mathrm{Tor}^R_i(M, N)) - i\} \leq \mathrm{reg}(M) + \mathrm{reg}(N).$$

*Moreover, equality holds if either $\mathrm{reg}(M) = \max_{i=p,p-1}\{b_i(M) - i\}$ or $\mathrm{reg}(N) = \max_{i=p',p'-1}\{b_i(N) - i\}$. This is in particular the case if either $\mathrm{pdim} M \leq \mathrm{codim} M + 1$ or $\mathrm{pdim} N \leq \mathrm{codim} N + 1$.*

*Proof.* Let $F^N_\bullet$ be a minimal graded free $R$-resolution of $N$. We apply Lemma 2.1 to $F^\bullet := F^N_{r-\bullet}$ and $M$. It follows from Lemma 2.1 (i) with $q_0 := r$ and the first claim of Lemma 2.1 (ii) that

$$\max_{p,q}\{a_p(\mathrm{Tor}_{r-q}(M, N)) + p + q\} \leq \max_{p,q}\{a_p(M) + b_{r-q}(N) + p + q\}$$

which is equivalent to

$$\max_{p,q}\{a_p(\mathrm{Tor}_q(M,N))) + p - q\} \leq \max_{p,q}\{a_p(M) + b_q(N) + p - q\}$$

and the right hand side is equal to $\mathrm{reg}(M) + \mathrm{reg}(N)$.

Let us assume that $\mathrm{reg}(N) = \max_{i=p',p'-1}\{b_i(N)-i\}$, that we can rewrite $b(F^\bullet) = \max\{b(F^0), b(F^1)+1\}$ (by Corollary 2.2, this equality holds if $\mathrm{pdim} N \leq \mathrm{codim} N + 1$). The second statement of Lemma 2.1 (ii) implies the equality we claim.

If $\mathrm{reg}(M) = \max_{i=p,p-1}\{b_i(M) - i\}$, we reverse the roles of $M$ and $N$ in the above proof. $\square$

Caviglia was probably the first to give an example, in his thesis (see [EHU, 4.4]), where $\mathrm{reg}(M \otimes_R N) > \mathrm{reg}(M) + \mathrm{reg}(N)$ when $\dim(\mathrm{Tor}_1^R(M,N)) = 2$. We will explain this example in a more general context in section 13.

Remark 4.1 applied to $M = R$ gives bounds on the regularity of all Koszul homology modules when $\dim R/(f_1,\ldots,f_s)$ is at most 1. The same kind of arguments are used in [Ch2, 3.1] to show the following :

**Theorem 5.2.** *Let $M, M_1, \ldots, M_s$ be finitely generated graded R-modules, $T_i := \mathrm{Tor}_i^R(M, M_1, \ldots, M_s)$, $d := \dim M$, and $b_\ell := \max_{i_1+\cdots+i_s=\ell}\{b_{i_1}(M_1) + \cdots + b_{i_s}(M_s)\}$. If $\dim T_1 \leq 1$, then*

(i) $a_p(T_0) \leq \max_{0\leq i\leq d-p}\{a_{p+i}(M) + b_i\}$ *for* $p \geq 0$,

(ii) $a_0(T_q) \leq \max_{0\leq i\leq d}\{a_i(M) + b_{q+i}\}$, *for* $q \geq 0$,

(iii) $a_1(T_q) \leq \max_{0\leq i\leq d}\{a_i(M) + b_{q+i-1}\}$, *for* $q \geq 1$.

*In particular, if $\dim(M_1 \otimes \cdots \otimes M_s) \leq 1$, then*

$$\mathrm{reg}(M_1 \otimes \cdots \otimes M_s) \leq \mathrm{reg}(M_1) + \cdots + \mathrm{reg}(M_s).$$

The last inequality may be extended to all the higher multiple Tor modules, and the condition may be weakened to $\dim(\mathrm{Tor}_1^R(M_1,\ldots,M_s)) \leq 1$. The vanishing of the latter module may be controlled by the following result, which is the natural generalization for multiple Tor modules of a result of Serre (recall that the formation of Tor commutes with localization) :

**Theorem 5.3.** *Let $R$ be a regular local ring containing a field, $M_1, \ldots, M_s$ be finitely generated R-modules. The following are equivalent,*

(i) $\mathrm{Tor}_1^R(M_1,\ldots,M_s) = 0$ *and $M_1 \otimes_R \cdots \otimes_R M_s$ is Cohen-Macaulay,*

(ii) *the codimension of $M_1 \otimes_R \cdots \otimes_R M_s$ is the sum of the projective dimensions of the $M_i$'s,*

(iii) *the intersection of the $M_i$'s is proper and every $M_i$ is Cohen-Macaulay.*

The form of the bound in Theorem 5.2 is also important to notice. Let us look at the case were we have two modules $M$ and $N$. We then have, for instance,

$$a_0(M \otimes_R N) \leq \max_i\{a_i(M) + b_i(N)\}$$

and this max can be reduced to the range $\mathrm{depth}\,(M) \leq i \leq \min\{\dim(M), \mathrm{pdim}(N)\}$. Notice also that the roles of $M$ and $N$ may be reversed.

## §6. The behaviour of regularity relatively to sums, products and intersections of ideals

The behaviour relatively to these three operations are all related to the study of Tor modules. Indeed, for any pair $I, J$ of ideals, $R/(I+J) = R/I \otimes_R R/J = \mathrm{Tor}_0^R(R/I, R/J)$, and there are two exact sequences :

$$0 \longrightarrow R/(I \cap J) \longrightarrow R/I \oplus R/J \longrightarrow \mathrm{Tor}_0^R(R/I, R/J) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Tor}_1^R(R/I, R/J) \longrightarrow R/IJ \longrightarrow R/(I \cap J) \longrightarrow 0.$$

It follows that :
— $\mathrm{reg}(I+J) > \max\{\mathrm{reg}(I), \mathrm{reg}(J)\}$ if and only if $\mathrm{reg}(I \cap J) > \max\{\mathrm{reg}(I), \mathrm{reg}(J)\} + 1$, and in this case

$$\mathrm{reg}(I \cap J) = \mathrm{reg}(I+J) + 1 = \mathrm{reg}(\mathrm{Tor}_0^R(R/I, R/J)) + 2,$$

— one has

$$\begin{aligned}\mathrm{reg}(R/IJ) &\leq \max\{\mathrm{reg}(R/(I \cap J)), \mathrm{reg}(\mathrm{Tor}_1^R(R/I, R/J))\} \\ &\leq \max\{\mathrm{reg}(R/I), \mathrm{reg}(R/J), \mathrm{reg}(\mathrm{Tor}_0^R(R/I, R/J)) + 1, \mathrm{reg}(\mathrm{Tor}_1^R(R/I, R/J))\}.\end{aligned}$$

Using the results on Tor, it immediately follows that

**Theorem 6.1.** *If $(I \cap J)/IJ$ is a module of dimension at most one, then*
(i) $\mathrm{reg}(I+J) \leq \mathrm{reg}(I) + \mathrm{reg}(J) - 1$,
(ii) $\mathrm{reg}(I \cap J) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$,
(iii) $\mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$.

More refined results in terms of Betti numbers can be found in [EHU, 2.2].

The possibility of extending the second and third inequalities to any number of ideals is still unclear. For the first one, the previous results on mutiple Tor modules gives such an extension.

Notice that the condition $\dim(I \cap J)/IJ \leq 1$ is implied by —and equivalent to in some cases, for instance if $\dim(R/I+J) = 2$— a more geometric one : $\forall \mathfrak{p} \supseteq I+J$ such that $\dim(R/\mathfrak{p}) \geq 2$, $(R/I)_\mathfrak{p}$ and $(R/J)_\mathfrak{p}$ are Cohen-Macaulay and $\mathrm{codim}_{R_\mathfrak{p}}(I_\mathfrak{p} + J_\mathfrak{p}) = \mathrm{codim}_{R_\mathfrak{p}}(I_\mathfrak{p}) + \mathrm{codim}_{R_\mathfrak{p}}(J_\mathfrak{p})$. In other terms : locally at primes of dimension two containing $I+J$ it corresponds to a proper intersection of Cohen-Macaulay schemes.

**Theorem 6.2.** *Let $I_1, \ldots, I_s$ be graded $R$-ideals and $J$ be their sum. If $\forall \mathfrak{p} \supseteq J$ such that $\dim(R/\mathfrak{p}) \geq 2$, $(R/I_i)_\mathfrak{p}$ is Cohen-Macaulay for all $i$ and $\mathrm{codim}_{R_\mathfrak{p}}(J_\mathfrak{p}) = \sum_i \mathrm{codim}_{R_\mathfrak{p}}((I_i)_\mathfrak{p})$, then*

$$\mathrm{reg}(R/J) \leq \sum_i \mathrm{reg}(R/I_i).$$

Questions of this type were also studied in particular contexts.

— Conca and Herzog proved in [CH] that if ideals $I_1, \ldots, I_s$ are generated by linear forms, then

$$\mathrm{reg}(I_1 \cdots I_s) = \sum_i \mathrm{reg}(I_i) = s,$$

— Derksen and Sidman proved in [DS] that if ideals $I_1, \ldots, I_s$ are generated by linear forms, then

$$\mathrm{reg}(I_1 \cap \cdots \cap I_s) \leq \sum_i \mathrm{reg}(I_i) = s.$$

In their article, Conca and Herzog ask if their result may be extended to an inequality for complete intersection ideals. It is shown in [CMT, 3.3] that for monomial complete intersection ideals one has

$$\mathrm{reg}(I_1 \cap \cdots \cap I_s) \leq \sum_i \mathrm{reg}(I_i),$$

and the same holds for the product if $s = 2$ by [CMT, 1.1]. Also it follows from a lemma due to Hoa and Trung [HT, 3.1] that

$$\mathrm{reg}(I_1 \cdots I_s) \leq \sum_i \mathrm{reg}(I_i) + \sum_i \mathrm{codim}(I_i) - \mathrm{codim}(I_1 \cdots I_s) - s + 1$$

which is close to the expected bound in this monomial case.

These bounds do not hold for general complete intersection ideals, a geometric approach for constructing counter-examples is given by the following result [CMT, 1.2] :

**Theorem 6.3.** *Let $\mathcal{C}$ in $\mathbf{P}^3$ be a curve which is defined by at most 4 equations at the generic points of its irreducible components. Consider 4 elements in $I_\mathcal{C}$, $f_1, f_2, g_1, g_2$ such that $I := (f_1, f_2)$ and $J := (g_1, g_2)$ are complete intersection ideals and $I_\mathcal{C}$ is the unmixed part of $I + J$. Then, if $-\eta := \min\{\mu \mid H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(\mu)) \neq 0\} < 0$, one has*

$$\mathrm{reg}(IJ) = \mathrm{reg}(I) + \mathrm{reg}(J) + \eta - 1.$$

We will see that one can choose for $\mathcal{C}$ the locally complete intersection curve with $I_\mathcal{C} := (x^m t - y^m z) + (z,t)^n$ for $m, n > 1$, in which case $\eta = (m-1)(n-1)$, and take for instance $I := (z^n, t^n)$ and $J := (x^m t - y^m z, f)$ with $f \in I_\mathcal{C}$ not multiple of $x^m t - y^m z$ (e.g. $f = z^n$).

## §7. The regularity of the ordinary powers of an ideal

Applying Theorem 5.2 with $M := R/I^m$ and $M_1 := R/I$, so that $T_1 = \mathrm{Tor}_1^R(R/I, R/I^m) \simeq I^m/I^{m+1}$ for an homogeneous ideal $I$ with $\dim(R/I) \leq 1$, one gets the following estimate (see [Ch2, 0.4], or [EHU, 7.9] for a slightly different bound) that improves the estimate $\mathrm{reg}(I^m) \leq m\,\mathrm{reg}(I)$ proved by Chandler in [Chan] and by Geramita, Gimigliano and Pitteloud in [GGP] :

**Theorem 7.1.** *Let $I$ be an homogeneous ideal of $R$ such that $\dim(R/I) \leq 1$. Then, for any $m \geq 1$,*

$$\mathrm{reg}(I^m) \leq \max\{\mathrm{reg}(I^{sat}) + b_1(I) - 1, \mathrm{reg}(I) + b_0(I)\} + (m-2)b_0(I);$$

*in particular, unless $I$ is principal, $\mathrm{reg}(I^m) \leq \mathrm{reg}(I) + b_1(I) - 1 + (m-2)b_0(I)$.*

When $\dim(R/I) \geq 2$, the inequality $\mathrm{reg}(I^m) \leq m\,\mathrm{reg}(I)$ do not hold in general. A first counter-example was given by Terai, in characteristic different from 2 : the Stanley-Reisner ideal of the minimal triangulation of the real projective plane. This ideal is a monomial ideal with ten minimal generators of degree 3. Further investigations by Sturmfels in [St] showed that any monomial ideal $M$ with at most 7 generators that has a linear resolution is such that $M^2$ also has a linear resolution, which is equivalent to $\mathrm{reg}(M^2) = 2\,\mathrm{reg}(M)$. On the other hand, Sturmfels exhibited a monomial ideal with 8 generators for which $\mathrm{reg}(M^2) > 2\,\mathrm{reg}(M)$, in any characteristic.

Also, reasonable bounds for the regularity of the square of an ideal of dimension two may be proved for generically complete intersection ideals :

**Theorem 7.2.** [Ch2, 0.5] *Let $I$ be an homogeneous $R$-ideal such that $\dim(R/I) = 2$. Assume that $I_\mathfrak{p} \subset R_\mathfrak{p}$ is a complete intersection for every prime $\mathfrak{p} \supseteq I$ such that $\dim R/\mathfrak{p} = 2$. Then,*

$$\begin{aligned}\mathrm{reg}(I^2) &\leq \max\{\mathrm{reg}(I) + \max\{b_0(I), b_1(I) - 1, b_2(I) - 2\}, a_2(R/I) + 2b_0(I) + 1\} \\ &\leq \mathrm{reg}(I) + \max\{\mathrm{reg}(I), 2b_0(I) - 2\}.\end{aligned}$$

**Question 7.3.** *Does the inequaltity*

$$\operatorname{reg}(I^2) \leq \operatorname{reg}(I) + \max\{b_0(I), b_1(I) - 1, b_2(I) - 2\}$$

*hold under the hypotheses of the theorem ?*

By [Ch2, 4.3], this is equivalent to $a_2(\operatorname{Tor}_2(R/I, R/I)) \leq \operatorname{reg}(R/I) + \max\{b_0(I), b_1(I) - 1, b_2(I) - 2\}$. Of course one may ask if the weaker bound $\operatorname{reg}(I^2) \leq 2\operatorname{reg}(I)$ holds, which is equivalent to $a_2(\operatorname{Tor}_2(R/I, R/I)) \leq 2\operatorname{reg}(R/I) + 1$.

In arbitrary dimension, Swanson first proved in [Sw1] that for any graded ideal $I$, there exists $N$ such that $\operatorname{reg}(I^m) \leq mN$ for any $m$. Later, the asymptotic behaviour of $\operatorname{reg}(I^m)$ was proved to be a linear function of $m$ by Kodiyalam in [Ko] and by Cutkosky, Herzog and Trung in [CHT] :

**Theorem 7.4.** *Let $I$ be a graded $R$-ideal. There exists* $\operatorname{indeg}(I) \leq a \leq b_0(I)$*, $b$ and $c$ such that*

$$\operatorname{reg}(I^m) = am + b, \quad \forall m \geq c.$$

The key point in the proof is the fact that the Rees algebra $\mathcal{R}_I$ has bigraded finite free resolution, which encodes the regularity of all the powers of $I$. The same type of behaviour also holds for the integral closures of the powers of $I$ or for the symmetric powers of an ideal.

The numbers $b$ and $c$ can be estimated from the shifts in the bigraded resolution of the Rees algebra (see [CHT, 2.4]). A particular case, which is easier to state is the following

**Proposition 7.5.** [CHT, 3.7] *Let $I$ be an ideal generated by $s$ forms of degree $d$. Then for all $m \geq c := \operatorname{reg}(\mathcal{R}_I) + s + 1$ one has :*

$$\operatorname{reg}(I^m) = (m - c)d + \operatorname{reg}(I^c).$$

Here $\mathcal{R}_I$ is considered as a graded quotient of $S := R[T_1, \ldots, T_s]$ by the epimorphism sending $T_i$ to the $i$-th generator of $I$, the grading of $S$ being defined by $\deg(T_i) = 1$ for any $i$ and $\deg(f) = 0$ for $f \in R$. With this definition,

$$\begin{aligned}\operatorname{reg}(\mathcal{R}_I) &= \min\{\mu \mid H^i_{S_+}(\mathcal{R}_I)_{>\mu-i} = 0,\ \forall i\}\\ &= \min\{\mu \mid \operatorname{Tor}^S_i(\mathcal{R}_I, R)_{>\mu+i} = 0,\ \forall i\}.\end{aligned}$$

On the other hand, the regularity of saturations of the powers of an ideal may have a very irregular behaviour, as shown by examples in [CHT] based on previous constructions of Cutkosky and Srinivas [CSr], and examples by Cutkosky [Cu].

**Examples 7.6.** [CHT, 4.4, Cu Thm. 10]

— For any prime $p$ congruent to 2 mod 3, there exists a field of characteristic $p$ and an ideal $I$ in $k[x, y, z]$ such that $\operatorname{greg}(R/I^{5m+1}) = 29m + 6$ if $m$ is not a power of $p$ and $\operatorname{greg}(R/I^{5m+1}) = 29m + 7$ else.

— In arbitrary characteristic, there exists a regular curve in $\mathbf{P}^3$ such that its defining ideal $I$ satisfies $\operatorname{greg}(R/I^m) = [(9 + \sqrt{2})m] + \sigma(m)$ where $\sigma(m)$ is 1 unless $m$ belongs to a sparse subsequence $q_n$ of the integers defined recursively by $q_0 = 1$, $q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$, in which case $\sigma(m) = 0$.

Further work by Cutkosky, Ein and Lazarsfeld in [CEL] gives a geometric approach to the understanding of the asymptotic behavior of $\frac{\operatorname{greg}(R/I^m)}{m}$ and other invariants of these powers (notably their arithmetic degree).

When the symbolic blow-up is finitely generated, the asymptotic behavior is given by a finite number of linear functions, each of them corresponding to a congruence of the exponent. If the symbolic blow-up is finite over the Rees ring the regularity is eventually linear.

A recent example due to Conca also shows a very interesting phenomena :

**Example 7.7.** [Co, 3.1] Let $d > 1$ and $J_d := (xz^d, xt^d, yz^{d-1}t) + (z,t)^{d+1} \subset k[x,y,z,t]$, for any field $k$. Then $\mathrm{reg}(I^m) = m(d+1)$ (*i.e.* $I^m$ has a linear resolution) for $m < d$ and $b_1(J^d) \geq d(d+2)$. It follows that $\mathrm{reg}(I^d) \geq d(d+2) - 1 > d(d+1)$.

Therefore, even for a monomial ideal in 4 variables, an arbitrary number of powers may have a linear resolution without forcing all the powers to verify the same property. The article of Conca [Co] contains in its section 2 an interesting collection of other examples from different sources.

## §8. Geometric estimates on Castelnuovo-Mumford regularity

The first estimates for the Castelnuovo-Mumford regularity are probably :
– the bound for the regularity of smooth projective curves by Castelnuovo,
– the bound for the regularity of schemes in terms of their Hilbert polynomial by Mumford.

Both results have been extended and better understood in later works, and Mumford's technique was adaptated to prove regularity bounds in terms of degrees of defining equations.

In the direction of Castelnuovo's result, there is a famous conjecture that suggests the following bound for reduced and irreducible schemes :

**Conjecture 8.1.** [Eisenbud and Goto] *If $\mathcal{S}$ is a non degenerate reduced and irreducible projective scheme over an algebraically closed field, then*

$$\mathrm{reg}(\mathcal{S}) \leq \deg \mathcal{S} - \mathrm{codim}\, \mathcal{S}.$$

(Non degenerate means $\mathcal{S} \not\subset H$ for any hyperplane $H$.)
We recall that if $\mathcal{S} = \mathrm{Proj}(R/I)$, $\mathrm{reg}(\mathcal{S}) := \mathrm{reg}(R/I^{sat}) = \mathrm{greg}(R/I)$.

This result was known for curves when the conjecture was made. It was first established for smooth curves by Castelnuovo [Cast], and then for reduced curves with no degenerate component by Gruson, Lazarsfeld and Peskine (over a perfect field) in [GLP]. Recently, Noma improved the bound for curves of sufficiently high genus in [No1] and [No2]. There has been a lot of work on regularity of curves, in particular on monomial curves (for instance L'vovsky's bound [Lv]) and on the cases were the bound is close to be reached (see for instance [BSc]).

There is some evidence that the Eisenbud-Goto conjecture should be true at least for smooth schemes in characteristic zero: it is true for smooth surfaces (Pinkham and Lazarsfeld) and (up to adding small constants) in dimension at most six, by the work of several people including Lazarsfeld, Ran and Kwak.

**Theorem 8.2.** *Let $k$ be a field of characteristic zero and $\mathcal{S}$ a non degenerate smooth irreducible projective scheme over $k$ of dimension $D \leq 6$, then*

$$\mathrm{reg}(\mathcal{S}) \leq \deg \mathcal{S} - \mathrm{codim}\, \mathcal{S} + \epsilon_D,$$

*with $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = 1$, $\epsilon_4 = 4$, $\epsilon_5 = 10$ and $\epsilon_6 = 20$.*

In any dimension, it was proved by Mumford ([BM]) —and it also follows easily from a theorem of Bertram, Ein and Lazarsfeld (Theorem 10.1 below)— that in characteristic zero a smooth scheme $\mathcal{S}$ satisfies,

$$\mathrm{reg}(\mathcal{S}) \leq (\dim \mathcal{S} + 1)(\deg \mathcal{S} - 1).$$

In positive characteristic, one has by [Ch1, 4.5] $\operatorname{reg}(\mathcal{S}) \leq \dim \mathcal{S}(\dim \mathcal{S}+1)(\deg \mathcal{S}-1)$ (Theorem 10.3 below gives a slightly weaker result). There are also quite reasonable results for schemes with isolated singularities (see [Ch1, §4]).

The conjecture is known for some classes of toric varieties. In codimension two, the result was proved by Peeva and Sturmfels in [PS] and there is several results in this direction for classes of toric varieties (see for instance [HS] for the case of simplicial toric rings).

## §9. General bounds on the regularity in terms of degrees of defining equations

A general bound in terms of degrees of defining equations and in terms of the Hilbert polynomial (for saturated ideals) may be derived from the following proposition, that studies the behavior when adding a sufficiently general linear form. This result goes back to Mumford ([Mu, Lect. 14]) for the essential key points, and have been used in several forms or variants since then. What Mumford proved is very close to points (iv) and (v) below. Set $h^i_{\mathfrak{m}}(M)_\mu := \dim_k H^i_{\mathfrak{m}}(M)_\mu$.

**Proposition 9.1.** *Let $M$ be a finitely generated graded $R$-module. For a linear form $l$ such that $K := 0 :_M (l)$ has finite length, set $\overline{M} := M/(l)M$. Then,*

(i) $\operatorname{greg}(M) \leq \operatorname{reg}(\overline{M}) \leq \operatorname{reg}(M)$,

(ii) *for $\mu \geq \operatorname{reg}(\overline{M})$, $h^0_{\mathfrak{m}}(M)_{\mu+1} \leq h^0_{\mathfrak{m}}(M)_\mu$, and the inequality is strict if $h^0_{\mathfrak{m}}(M)_\mu \neq 0$ and $\mu \geq \max\{\operatorname{reg}(\overline{M})+1, b_0(M), b_1(M)-1\}$,*

(iii) *for $\mu \geq \operatorname{greg}(M)+2$, $(M \otimes_R k)_\mu \simeq (H^0_{\mathfrak{m}}(M) \otimes_R k)_\mu$, in particular $b_0(H^0_{\mathfrak{m}}(M)) \leq \max\{\operatorname{greg}(M)+1, b_0(M)\}$,*

(iv) *for $\mu \geq \max\{\operatorname{greg}(\overline{M}), a_0(M)\}$, $h^1_{\mathfrak{m}}(M)_{\mu+1} \leq h^1_{\mathfrak{m}}(M)_\mu$ and the inequality is strict if $h^1_{\mathfrak{m}}(M)_\mu \neq 0$ and $\mu \geq \max\{\operatorname{greg}(\overline{M})+1, a_0(M), b_0(\overline{M})\}$,*

(v) *for $\mu' \geq \mu \geq \min\{a_0(M), b_0(M)\}$, $\{H^i_{\mathfrak{m}}(M)_{\mu-i} = 0,\ \forall i\} \Rightarrow \{H^i_{\mathfrak{m}}(M)_{\mu'-i} = 0,\ \forall i\}$.*

The proof is given in the technical appendix.

**Corollary 9.2.** *Let $I$ be a homogeneous ideal and $l$ be a linear form such that $K := (I : (l))/I$ has finite length, then for $\mu \geq \max\{b_0(I)-1, \operatorname{reg}(I+(l))\}$,*

$$\operatorname{reg}(I) \leq \mu + \lambda(H^0_{\mathfrak{m}}(R/I)_\mu) = \mu + \lambda(K_{\geq \mu}) \leq \mu + \lambda(K).$$

*Proof.* Let $\mu \geq \max\{b_0(I)-1, \operatorname{reg}(I+(l))\}$. By (i), $\operatorname{greg}(R/I) \leq \mu - 1$, and by (ii), $h^0_{\mathfrak{m}}(R/I)_{\mu+1} < h^0_{\mathfrak{m}}(R/I)_\mu$ if $h^0_{\mathfrak{m}}(R/I)_\mu \neq 0$. It follows that $h^0_{\mathfrak{m}}(R/I)_{\mu+i} \leq \max\{0, h^0_{\mathfrak{m}}(R/I)_\mu - i\}$, hence $a_0(R/I) \leq \mu + h^0_{\mathfrak{m}}(R/I)_\mu$. Finally notice that $\lambda(K_{\geq \mu}) = \sum_{\mu' \geq \mu}(h^0_{\mathfrak{m}}(R/I)_{\mu'} - h^0_{\mathfrak{m}}(R/I)_{\mu'+1}) = h^0_{\mathfrak{m}}(R/I)_\mu$. □

This corollary may be used directly to prove bounds on the regularity in terms of defining equations, by recursion on the dimension. The point is then to bound $\lambda(H^0_{\mathfrak{m}}(R/I)_\mu)$, for which one may use that $H^0_{\mathfrak{m}}(R/I)_\mu \subseteq (R/I)_\mu$.

A more refined way for bounding $\lambda(H^0_{\mathfrak{m}}(R/I)_\mu)$ was find by Caviglia and Sbarra [CS]. The following lemma is a key ingredient of their proof :

**Lemma 9.3.** [CS, 2.2] *If $(I : (l))/I$ has finite length, for any $j > 0$,*

$$\lambda\left(\frac{I : (l)^j}{I : (l)^{j-1}}\right) - \lambda\left(\frac{I : (l)^{j+1}}{I : (l)^j}\right) = \lambda\left(\frac{I : (l)^j + (l)}{I : (l)^{j-1} + (l)}\right)$$

Let us now sketch a variant of their proof, and of their result.

They first remark that $I^{sat} = I : (l)^N$ for $N \geq a_0(R/I)$ and that by the lemma above, $K$ has the same length as $(I^{sat} + (l))/(I + (l))$ (sum up the equalities in the lemma for $j$ from 1 to $N$).

The lemma also gives $\lambda\left(\frac{I:(l)^{j+1}}{I:(l)^j}\right) \leq \lambda\left(\frac{I:(l)}{I}\right)$ for any $j > 0$, and therefore

$$\lambda\left(I^{sat}/I\right) = \sum_{j=1}^{a_0(R/I)} \lambda\left(\frac{I:(l)^j}{I:(l)^{j-1}}\right) \leq a_0(R/I)\lambda\left(\frac{I:(l)}{I}\right).$$

Also, for a linear form $l'$ such that $(I + (l)) : (l')/(I + (l))$ has finite length,

$$\lambda\left(\frac{I:(l)}{I}\right) = \lambda\left(\frac{I^{sat}+(l)}{I+(l)}\right) \leq \lambda\left(\frac{(I+(l))^{sat}}{I+(l)}\right) \leq a_0(R/I+(l))\lambda\left(\frac{(I+(l)):(l')}{(I+(l))}\right).$$

Therefore, setting $I_i := I + (l_1, \ldots, l_i)$ and $K_i := I_i : (l_{i+1})/I_i$ for a sequence of linear forms $l_i$ such that $\lambda(K_i) < \infty$ for every $i$, one has $\dim(R/I_i) = \max\{0, \dim(R/I) - i\}$ and :

— $\mathrm{reg}(R/I_i) \leq \max\{\mathrm{reg}(R/I_{i+1}), d-2\} + \lambda(K_i)$,

— $\lambda(K_i) \leq \mathrm{reg}(R/I_{i+1})\lambda(K_{i+1})$,

which gives a way to bound $\mathrm{reg}(R/I_i)$ by recursion on $i$ from $\delta := \dim(R/I)$ to 0. Indeed for $I$ generated in degree at most $d$ :

— for $i = \delta$, $\dim(R/I_\delta) = 0$ and therefore $\mathrm{reg}(R/I_\delta) \leq (n-\delta)(d-1)$ by Theorem 12.4 (this is well-known and goes back to Macaulay, at least) and $\lambda(K_\delta) = \lambda(R/I_{\delta+1}) \leq d^{n-\delta-1}$,

— if $\delta \geq 1$, for $i = \delta - 1$, $\dim(R/I_{\delta-1}) = 1$, $\mathrm{reg}(R/I_{\delta-1}) \leq (n - \delta + 1)(d-1)$ by Theorem 12.4 and $\lambda(K_{\delta-1}) = \lambda(R/I_\delta) - \deg(R/I_{\delta-1}) \leq d^{n-\delta} - 1$.

And then by recursion it follows :

**Theorem 9.4.** *If $I$ is a graded $R$-ideal generated in degree at most $d$,*

(i) $\mathrm{reg}(R/I) \leq n(d-1)$ *if* $\dim(R/I) \leq 1$,

(ii) *if* $\delta := \dim(R/I) \geq 2$,

$$\mathrm{reg}(R/I) \leq ((n-\delta+1)(d-1)d^{(n-\delta)})^{2^{\delta-2}} < (3^{\frac{1}{3}}\, d)^{(n-\delta+1)2^{\delta-2}}.$$

Point (iv) of the proposition can be used to bound $\mathrm{greg}(R/I)$ in terms of the Hilbert polynomial $P$ of $R/I$ (this was the motivation of Mumford), as for $\mu \geq \mathrm{greg}(\overline{R/I})$ one has $P(\mu) = h^1_{\mathfrak{m}}(R/I)_\mu + \dim_k(R/I)_\mu$. This last formula follows from the equality (see for instance [BH, 4.3.5]):

$$\dim_k M_\mu = P_M(\mu) + \sum_{i\geq 0}(-1)^i h^i_{\mathfrak{m}}(M)_\mu$$

which is valid for any graded $R$-module $M$ of finite type and any $\mu$.

Another way of proving bounds in terms of the Hilbert polynomial is to remark that the regularity of the Hilbert function of $R/I^{sat}$ is strictly smaller that the one of the lex-segment ideal associated to $I^{sat}$; as a corollary, the regularity of this lex-segment ideal only depends on the Hilbert polynomial. The corresponding bound can be computed from the standard writing of the Hilbert polynomial by formulas first proved by Blancafort in her thesis (see [Bl]).

A bound on the regularity of lex-segment ideals associated to complete intersection ideals was proved in [CM], and improved and extended by Hoa and Hyry in [HoHy2], as an ingredient for proving bounds for the degrees of generators of Gröbner bases of an ideal in terms of the degrees of its generators, for any admissible order (see [CM, 3.6]).

Point (v) may be used to prove that :

**Proposition 9.5.** *If $I$ is a graded ideal which is not $\mathfrak{m}$-primary, then*

$$\mathrm{reg}(I) = \min\{\mu \mid H^i_{\mathfrak{m}}(R/I)_{\mu-i} = 0,\ \forall i\}.$$

*Proof.* Recall that if $d := \dim(R/I) > 0$, one has $H^d_{\mathfrak{m}}(R/I)_i \neq 0$ for any $i \leq d$, (see [Ho] or [BH, 9.2.4 (b)]) hence the minimum on the right is positive, and the claim follows from (v). $\square$

One of the motivations for looking at regularity of ideals was the theorem of Bayer and Stillman which asserts that, for the rev-lex order and in general coordinates $\mathrm{reg}(I) = \mathrm{reg}(\mathrm{in}(I))$, were $\mathrm{in}(I)$ is the initial ideal of $I$ (generated by the leading monomials of the elements in the Gröbner basis).

Bounds in terms of the maximal degree of the generators were expected to be of much smaller order then the ones known at that time, which were of the same magnitude as the bound we proved above (at least in characteristic zero).

A big surprise came with the example of Mayr and Meyer, who provided an ideal in a polynomial ring in $10n+2$ variables generated by polynomials of degrees at most $d+2$ with a minimal first syzygy of degree at least $d^{2^n}$ (see [BS2]).

A very interesting study of the ideals of Mayr and Meyer, and of some closely related ideals, was done by Irena Swanson in [Sw2] and [Sw3]. It shows that these ideals have minimal primes and embedded primes in different codimensions, and points to some embedded ideals that might be at the origin of their high regularity.

It is interesting to notice that these type of binomial ideals are still the unique source of exemples of ideals with very high regularity. We will see in section 13 examples with a much more geometric construction, but not with such a high regularity. It would be very interesting to provide a geometric construction of ideals with huge multiplicity as compared to degrees of generators.

## §10. Bounds in terms of degrees of defining equations in a geometric context

We now turn to bounds depending on the degrees of generators. As we mentioned in the preceeding paragraph, the regularity may be very high without imposing geometric conditions. We will see now that the situation is quite different if the ideal defines a scheme with strong geometric properties.

Let $I = (f_1, \ldots, f_s)$ be a homogeneous ideal in $R$, where $f_i$ is a form of degree $d_i$. We will assume that $d_1 \geq d_2 \geq \cdots \geq d_s \geq 1$. Let $\mathcal{Z}_I \subset \mathbf{P}^{n-1}$ be the scheme defined by $I$ and $r$ be the codimension of $I$ in $R$, which is also the one of $\mathcal{Z}_I$ as a subscheme of $\mathbf{P}^{n-1}$. Let $\mathcal{S}$ be the top dimensional part of $\mathcal{Z}_I$.

The first striking result on regularity in these terms is due to Bertram, Ein and Lazarsfeld :

**Theorem 10.1.**[BEL] *If $\mathcal{Z}_I = \mathcal{S}$ is smooth of characteristic zero, and $j > 0$,*

$$\mathrm{greg}(R/I^j) \leq jd_1 + d_2 + \cdots + d_r - r,$$

*with equality if and only if $\mathcal{S}$ is a complete intersection of degrees $d_1, \ldots, d_r$.*

This bound, in the case $j = 1$, was generalized in [CU],

**Theorem 10.2.** *Assume that $\mathcal{Z}_I$ have at most a one dimensional singular locus an is locally a complete intersection outside finitely many points. If $k$ is of characteristic zero, then*

$$\mathrm{reg}(\mathcal{S}) \leq d_1 + \cdots + d_r - r,$$

*with equality if and only if* $\mathcal{Z}_I = \mathcal{S}$ *is a complete intersection of degrees* $d_1, \ldots, d_r$.

The proof of both results relies on refined versions of Kodaira vanishing theorem, and uses the blow-up of $\mathcal{Z}_I$ in $\mathbf{P}^{n-1}$. In addition the proof of Theorem 10.2 uses liaison theory, as explained in section 11.

A slightly different type of result, valid in any characteristic, is the following :

**Theorem 10.3.** [Ch1, 4.4] *Assume that* $\mathcal{Z}$ *is an isolated component of* $\mathcal{Z}_I$ *of codimension* $s$ *that doesn't meet the other components of* $\mathcal{Z}_I$*, and that* $\mathcal{Z}_I$ *is smooth at all but a finite number of points of* $\mathcal{Z}$*. Then,*

$$\mathrm{reg}(\mathcal{Z}) \leq (\dim \mathcal{Z} + 1)(d_1 + \cdots + d_s - s - 1) + 1.$$

This generalizes the result of [CP] that treats the case where $\dim \mathcal{Z} = 0$. The proof relies on [CP] and a result of Hochster and Huneke, which implies that the phantom homology (which is, roughly speaking, the one that vanishes in the Cohen-Macaulay case) is uniformly killed by the Jacobian ideal. The result then follows by cutting the scheme $\mathcal{Z}$ by a sequence of parameters in the Jacobian ideal and using some homological algebra to exploite this uniform vanishing. The connection between annihilators and vanishing was already remarked and used to study the Castelnuovo-Mumford regularity of the $\ell$-Buchsbaum schemes by Miyazaki, Nagel, Schenzel and Vogel (see [Mi], [NS1], [NS2] or [MV]).

In characteristic zero, the theory of multiplier ideals shows the following

**Theorem 10.4.** *Under the hypotheses of Theorem 10.3, if further* $k$ *is of characteristic zero,*

$$\mathrm{greg}(R/I^j_{\mathcal{Z}}) \leq (s + j - 1)(d_1 - 1), \quad \forall j > 0.$$

Two other results sheds a somehow different light on these kind of result :

**Theorem 10.5.** [Ch2, 7.13] *Assume that* $\dim \mathcal{Z}_I = 2$ *and set* $\ell := \min\{s, n-1\}$*. If the component of dimension two of* $\mathcal{Z}_I$ *is a reduced surface* $\mathcal{S}$ *and, for* $x \in \mathcal{S}$*, except at most at finitely many points,* $\mathcal{Z}_I$ *is locally defined at* $x$ *by at most* $\mathrm{codim}\, \mathcal{S} + 1$ *equations, then for any* $j > 0$

$$\mathrm{greg}(R/I^j) \leq jd_1 + d_2 + \cdots + d_\ell - \ell.$$

We conjecture that the hypothesis on the local number of generators may be dropped

**Conjecture 10.6.** *Assume that* $\dim \mathcal{Z}_I = 2$ *and set* $\ell := \min\{s, n-1\}$*. If the component of dimension two of* $\mathcal{Z}_I$ *is a reduced surface, then for any* $j > 0$

$$\mathrm{greg}(R/I^j) \leq jd_1 + d_2 + \cdots + d_\ell - \ell.$$

In any case, a slightly weaker bound holds in the context of Conjecture 10.6 (see [Ch2, 5.16]). On the other hand it is necessary to assume that $\mathcal{S}$ is reduced (not having section in negative degrees may suffice).

The rather surprising point here is that the main hypothesis is on $\mathcal{S}$, while the result is on $\mathcal{Z}_I$. In particular, it need not be the case that $\mathcal{Z}_I$ is equidimensionnal, or reduced, or without embedded primes. Moreover it is unclear to us that the same upper bound holds for $\mathcal{S}$. Recall the following example [CD, 1.3 & 3.5] :

**Example 10.7.** There exists an irreducible and reduced surface $\mathcal{S} \subset \mathbf{P}^5$ and another scheme $\mathcal{S}'$ supported on $\mathcal{S}$ which coincides with $\mathcal{S}$ outside one line and such that:

(i) $\operatorname{reg}(\mathcal{S}') < \operatorname{reg}(\mathcal{S})$,

(ii) $\operatorname{depth}(R/I_{\mathcal{S}'}) > \operatorname{depth}(R/I_{\mathcal{S}})$.

The next result shows that any homogeneous ideal may be replaced by other ones –we just present one of them here, but we used another one in Theorem 10.4 above–, very much related to it on the smooth locus, that has a well-controlled regularity :

**Theorem 10.8.** [L] *Let $I$ be an homogeneous ideal of codimension $r$ in $R$, with $k$ a field of characteristic zero. We denote by $I^{top}$ the intersection of primary components of $I$ of minimal codimension. Assume that $I$ is generated in degree at most $d$. Then there exists an ideal $J$ of codimension $r$, such that*

(i) *$J \subseteq I^{top}$ and $I^r \subseteq J$,*

(ii) *for any prime $\mathfrak{p}$ containing $I^{top}$ such that $(R/I)_{\mathfrak{p}}$ is regular, $I_{\mathfrak{p}} = J_{\mathfrak{p}}$,*

(iii) *$\operatorname{reg}(R/J) \leq r(d-1)$.*

*Proof.* See [CD, 1.1]. $\square$

The ideal $J$ is the multiplier ideal $J := \oplus_\mu H^0(\mathbf{P}_k^{n-1}, \mathcal{J}(\mathcal{I}^r)(\mu))$, where $\mathcal{I}$ is the sheaf of ideals associated to $I$. Other multiplier ideals $\oplus_\mu H^0(\mathbf{P}_k^{n-1}, \mathcal{J}(\mathcal{I}^{r'})(\mu))$, with $r' \geq r$ give rise to other interesting ideals that coincides with some powers of $I$, locally on the smooth locus, in codimension $\leq r'$. We used the one with $r' := s + j - 1$ in Theorem 10.4. See [L] for an introduction to the fascinating theory of multiplier ideals, and much more.

As an application one has :

**Corollary 10.9.** *Let $\mathcal{S}$ be a reduced equidimensionnal projective scheme over a field of characteristic zero. There exists a scheme $\mathcal{S}'$, containing $\mathcal{S}$ and supported on $\mathcal{S}$, which coincides with $\mathcal{S}$ on the smooth locus of $\mathcal{S}$, and such that $\operatorname{reg}(\mathcal{S}') \leq \operatorname{codim} \mathcal{S}(\deg \mathcal{S} - 1)$.*

*Proof.* The ideal $I$ generated by forms of degrees at most $\deg \mathcal{S}$ contains all the equations of the projections of $\mathcal{S}$ on the linear subspaces of dimension $\dim \mathcal{S} + 1$. The latter ideal defines $\mathcal{S}$ on the smooth locus and defines a scheme supported on $\mathcal{S}$. Therefore, this is also the case for $I$. Applying Theorem 10.8 to $I$ provides an ideal $J$ such that the corollary holds by choosing $\mathcal{S}' := \operatorname{Proj}(R/J)$. $\square$

The following result, for schemes of dimension one,

**Theorem 10.10.** *Let $k$ be a field, $\mathcal{Z}_1, \ldots, \mathcal{Z}_s$ be closed subschemes of a projective $k$-scheme $\mathcal{Z}$ of dimension $d$. Assume that $\operatorname{reg}(\mathcal{Z}_1) \geq \cdots \geq \operatorname{reg}(\mathcal{Z}_s)$. If the intersection of the $\mathcal{Z}_i$'s is of dimension at most 1, then*

$$\operatorname{reg}(\mathcal{Z}_1 \cap \cdots \cap \mathcal{Z}_s) \leq \operatorname{reg}(\mathcal{Z}) + \sum_{i=1}^{\min\{d,s\}} \operatorname{reg}(\mathcal{Z}_i).$$

Naturally leads to the following problem :

**Question 10.11.** Let $k$ be a field, $\mathcal{Z}_1, \ldots, \mathcal{Z}_s$ be closed subschemes of a smooth projective $k$-scheme $\mathcal{Z}$ of dimension $d$. Assume that $\operatorname{reg}(\mathcal{Z}_1) \geq \cdots \geq \operatorname{reg}(\mathcal{Z}_s)$. If the intersection of the $\mathcal{Z}_i$'s is smooth outside a scheme of dimension at most one, does one have

$$\operatorname{reg}(\mathcal{Z}_1 \cap \cdots \cap \mathcal{Z}_s) \leq \operatorname{reg}(\mathcal{Z}) + \sum_{i=1}^{\min\{d,s\}} \operatorname{reg}(\mathcal{Z}_i) \ ?$$

The geometric hypotheses we choose are of course quite arbitrary, one may for instance allow isolated singularities for $\mathcal{Z}$ or on the opposite side impose more smoothness to the intersection. If $\mathcal{Z}$ is a projective space, the $\mathcal{Z}_i$'s are Cohen-Macaulay along the intersection and it is a proper intersection, the result follows from Theorem 5.2 and Theorem 5.3 (see also [Ch2, 3.2]).

### §11. Regularity and liaison

We will briefly sketched how liaison can be used to obtain results on regularity.

The essential point in liaison is the so-called liaison sequence,

$$0 \longrightarrow \omega_{A'} \otimes \omega_B^{-1} \longrightarrow B \longrightarrow A \longrightarrow 0$$

where $B := R/\mathfrak{b}$ is a homogeneous Gorenstein quotient of $R$ and $A = R/I$ and $A' = R/I'$ are homogeneous quotients such that $I' = \mathfrak{b} : I$ and $I = \mathfrak{b} : I'$ (as $\mathfrak{b}$ is Gorenstein, these equalities are equivalent if $I$ and $I'$ are unmixed of codimension $\operatorname{codim}\mathfrak{b}$).

As $B$ is Gorenstein there is a graded isomorphism $\omega_B \simeq B[a]$, and therefore $\omega_B^{-1} \simeq B[-a]$, where $a = a_{\dim B}(B)$ is called the $a$-invariant of $B$. Notice that $\operatorname{reg}(B) = a + \dim(B)$.

In the applications, $I$ and $I'$ often have no associated primes in common, and therefore $\mathfrak{b} = I \cap I'$ : this is called geometric liaison.

The key fact to derive the liaison sequence is the isomorphism $I'/\mathfrak{b} = (\mathfrak{b} : I)/\mathfrak{b} \simeq \operatorname{Hom}_R(R/I, R/\mathfrak{b})$ sending the class of $x$ to the map $1 \mapsto x$.

Recall that $\omega_R = R[-n]$ and for a finitely generated graded $R$-module $M$ whose support is of codimension $r$, one has $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for $i < r$ and $\omega_M := \operatorname{Ext}^r_R(M, \omega_R)$ is a module of finite type, which has the same dimension as $M$ and satisfies $S_2$. This module is called the canonical module of $M$. It is a dualizing module if $M = R/I$ is a Cohen-Macaulay quotient of $R$. See for instance [BH, section 3] for a general introduction to canonical modules.

Recall also the local duality isomorphismes :

$$\operatorname{Ext}^i_R(M, \omega_R) \simeq H^{n-i}_{\mathfrak{m}}(M)^*,$$

where $—^* = \operatorname{Homgr}_R(—, k)$ stands for the graded $k$-dual.

As $B$ is Cohen-Macaulay, $H^i_{\mathfrak{m}}(B) = 0$ except for $i = D := \dim B$. It follows from the liaison sequence that

$$H^i_{\mathfrak{m}}(A) \simeq H^{i+1}_{\mathfrak{m}}(\omega_{A'})[-a], \quad \forall i < D-1,$$

and local duality and the liaison sequence shows that $H^D_{\mathfrak{m}}(A) \simeq (I'/\mathfrak{b})^*[-a]$ and

$$H^{D-1}_{\mathfrak{m}}(A) \simeq \operatorname{coker}(A' \longrightarrow \operatorname{End}(\omega_{A'}))^*[-a].$$

Also recall that $\operatorname{End}(\omega_{A'})$ is the $S_2$-ification of $A'$, and therefore if the scheme $\operatorname{Proj}(A')$ satisfies $S_2$, then $H^{D-1}_{\mathfrak{m}}(A) \simeq H^1_{\mathfrak{m}}(A')^*[-a]$.

If furthermore $\operatorname{Proj}(A')$ is a Cohen-Macaulay scheme, then by Serre duality we get :

$$H^i_{\mathfrak{m}}(A) \simeq H^{D-i}_{\mathfrak{m}}(A')^*[-a], \quad \forall i < D.$$

It follows that $\operatorname{Proj}(A)$ is also Cohen-Macaulay in this case and

$$a_i(A) = a - \operatorname{indeg}(H^{D-i}_{\mathfrak{m}}(A')), \quad \forall i < D.$$

These facts are for instance explained in [Mi] or [CU, 4.1]. The liaison sequence, and the remarks above, proves the following,

**Theorem 11.1.** [CU, 4.3] *Let $I$ and $I'$ be homogeneous ideals linked by a homogeneous Gorenstein ideal $\mathfrak{b}$. The following are equivalent*

(i) $\mathrm{reg}(R/I) < \mathrm{reg}(R/\mathfrak{b})$,

(ii) $\mathrm{reg}(\omega_{R/I'}) = \mathrm{reg}(\omega_{R/\mathfrak{b}}) = \dim R/\mathfrak{b}$.

*Moreover, if one of these conditions fail, then* $\mathrm{reg}(R/I) = \mathrm{reg}(R/\mathfrak{b}) + \mathrm{reg}(\omega_{R/I'}) - \dim R/\mathfrak{b} - 1$.

A fundamental result on the canonical module is Kodaira vanishing theorem, which applied in this context gives,

**Theorem 11.2.** *Let $Y := \mathrm{Proj}(R/I')$ be a smooth scheme. If $k$ is of characteristic zero, then, for all $i$,*

$$H^i_{\mathfrak{m}}(\omega_{R/I'})_\mu = 0, \quad \forall \mu > 0.$$

Therefore, if $Y$ is smooth, $\mathrm{reg}(\omega_{R/I'}) = D := \dim(R/I')$ $(H^D_{\mathfrak{m}}(\omega_{R/I'})_0 \simeq H^0(Y, \mathcal{O}_Y) \neq 0$ is a $k$-vector space of dimension the number of connected components of $Y$ (over $\overline{k}$), and therefore $a_D(\omega_{R/I'}) = 0)$.

This theorem extends to the case where $Y$ has rational singularities outside a finite number of points, in particular to schemes with isolated singularities.

It was proved by Hironaka, and in this context by Peskine and Szpiro, that if $\mathrm{Proj}(R/I)$ is smooth and $\mathfrak{b}$ is a complete intersection ideal of the same codimension given by sufficiently general elements in $I$, then $\mathrm{Proj}(R/I')$ is smooth in codimension three. Therefore, Theorem 10.1 follows from Theorem 11.1 and Theorem 11.2, at least in characteristic zero, if $\mathcal{Z}_I$ is of dimension at most 4.

It is proved in [CU] that, in any dimension, the singularities of a general link of a smooth scheme are rational. In fact a more refined study of the singularities of a general link in the graded case [CU, 4.4] leads to the proof of Theorem 10.2 (see [CU, 4.7]).

## §12. Actual resolutions are not necessary to estimate the regularity

This important remark is one of the ingredients of the proof of the theorem of Gruson, Lazarsfeld and Peskine on the regularity of reduced curve (see [GLP, Lemma 1.6], and [EL] or [Ch2, §1] for refined statements of the same type). The simplest, but already very useful, result of this kind is :

**Theorem 12.1.** [Ch2, 1.1] *Let $C$ be a graded complex of finitely generated $R$-modules with $C_i = 0$ for $i < 0$. If $\dim H_i(C) \leq i$ for $i > 0$, then $a_p(H_0(C)) \leq \max_{i \geq 0}\{a_{p+i}(C_i)\}$ for any $p \geq 0$. Therefore,*

$$\mathrm{reg}(H_0(C)) \leq \max_{i=0,\ldots,n} \{\mathrm{reg}(C_i) - i\}.$$

The proof is very similar to the one of Lemma 2.1. The results on regularity of tensor product of modules follows from this result. The initial proof of Sidman (for ideals) and Caviglia (for modules) are based on this result in the case $C$ is a complex of free $R$-modules.

This result may be used directly to prove a particular case of the following conjecture, due to Katzman [Ka], on the regularity of Frobenius powers of ideals :

**Conjecture 12.2.** *If $S$ is a standard graded algebra over a field of characteristic $p > 0$ and $I$ is an homogeneous ideal of $S$, then there exists $N$ such that*

$$\mathrm{reg}(I^{[p^e]}) \leq p^e N, \quad \forall e.$$

Denoting by $\mathcal{F}$ the Frobenius functor, one has

**Theorem 12.3.** *Let $S$ be a standard graded ring over a field of characteristic $p>0$ and $M$ a finitely generated graded $S$-module. Assume that* $\dim(\mathrm{Sing}(S)\cap\mathrm{Supp}(M))\leq 1$ *and set* $b_i^S(M):=\max\{j \mid \mathrm{Tor}_i^S(M,k)_j\neq 0\}$ *if* $\mathrm{Tor}_i^S(M,k)\neq 0$*, and* $b_i^S(M):=-\infty$ *else. Then,*

$$\mathrm{reg}(\mathcal{F}^e M)\leq \max_{0\leq i\leq j\leq \dim S}\{p^e b_i^S(M)+a_j(S)+j-i\}\leq \mathrm{reg}(S)+\max_{0\leq i\leq \dim S}\{p^e b_i^S(M)-i\}.$$

*Proof.* Let $F$ be a minimal graded free $S$-resolution of $M$, therefore $F_i=\oplus_j S[-j]^{\beta_{ij}}$, where $\beta_{ij}=\dim_k \mathrm{Tor}_i^S(M,k)_j$. The complex $\mathcal{F}^e F$ is a complex of graded free $S$-modules and $\mathcal{F}^e F_i=\oplus_j S[-jp^e]^{\beta_{ij}}$. $\mathcal{F}$ is exact over regular rings and $H_i(\mathcal{F}^e F)=\mathrm{Tor}_i^S(M,S_{\mathcal{F}^e})$, where $S_{\mathcal{F}^e}$ is $S$ with the $S$-module structure given by $\mathcal{F}^e: S\longrightarrow S$. As the formation of Tor commutes with localization, the hypothesis implies that $\dim H_i(\mathcal{F}^e F_\bullet)\leq 1$ for $i>0$. Therefore, by Theorem 12.1

$$a_\ell(\mathcal{F}^e M)\leq \max_{0\leq i\leq \dim S-\ell}\{a_{i+\ell}(S)+p^e b_i^S(M)\}, \forall \ell\geq 0,$$

which proves our claim. $\square$

Applying Theorem 12.1 to the Koszul complex on a set of homogeneous elements, one gets

**Theorem 12.4.** *Let $S$ be a standard graded ring over a field and $I$ be a $S$-ideal generated by forms of degrees $d_1\geq d_2\geq\cdots\geq d_s$ and set $\ell:=\min\{s,\dim S\}$. If $\dim(S/I)\leq 1$, then*

$$\begin{aligned}\mathrm{reg}(S/I)&\leq \max_{i=0,\ldots,\ell}\{a_i(S)+d_1+\cdots+d_i\}\\ &\leq \mathrm{reg}(S)+d_1+\cdots+d_\ell-\ell.\end{aligned}$$

This is essentially a particular case of Theorem 10.10.

Another application is to the regularity of powers and symmetric powers of an $R$-ideal $I=(f_1,\ldots,f_s)$. In this case, one considers the graded piece $\mathcal{Z}_\bullet^j$ of an approximation complex $\mathcal{Z}_\bullet$, whose zeroth homology module $H_0(\mathcal{Z}_\bullet^j)$ is $\mathrm{Sym}_R^j(I)$ :

$$\cdots\longrightarrow Z_2\otimes_R \mathrm{Sym}_R^{j-2}(F)\longrightarrow Z_1\otimes_R \mathrm{Sym}_R^{j-1}(F)\longrightarrow Z_0\otimes_R \mathrm{Sym}_R^{j}(F)\longrightarrow 0,$$

where $Z_i$ is the $i$-th module of cycles of the Koszul complex $K(f_1,\ldots,f_s;R)$, and $F=K_1(f_1,\ldots,f_s;R)$ is the free $R$-module of rank $s$ providing the natural graded epimorphism $F\longrightarrow I$. Notice that this complex is of length $\min\{j,s\}$

There are quite sharp criterions for the acyclicity of this complex, mainly due to Herzog, Simis and Vasconcelos, who introduced and studied in detail these complexes in [HSV1], [HSV2] and [HSV3].

A crucial point is that, even though these complexes depend on the generators, their homology don't. In connection with Theorem 12.1, this shows that when these criterions are satisfied locally in dimension two, the cohomolgy of the symmetric powers of the ideal is controlled in terms of the cohomology of the cycles of the Koszul complex. The cohomology of the cycles can be in turn controlled from the one of the Koszul homology modules.

This is the way Busé and Jouanolou were controlling the torsion of the symmetric powers of an ideal of dimension at most one, in their study of the implicitization problem in [BJ].

This method produces pretty sharp estimates on the regularity of symmetric powers, and of powers, in small dimensions. More refined results on the torsion in the symmetric powers in terms of graded Betti numbers have been obtained by Eisenbud, Huneke and Ulrich in dimension zero in [CHU, 9.1] :

**Theorem 12.5.** *Let $I$ be a homogeneous $\mathfrak{m}$-primary ideal. For any $j$,*

— $a_0(\mathrm{Sym}^j_R(I)) \leq \mathrm{reg}(I) + b_1(I) - 1 + (j-2)b_0(I)$,

— *if the resolution of $I$ is linear for $[\frac{n}{2}]$ steps (this in particular implies that $I$ is generated in a single degree), then $H^0_{\mathfrak{m}}(\mathrm{Sym}^j_R(I))$ is concentrated in degree $jb_0(I)$ for every $j$.*

The following result gives a flavor of the results proved in [Ch2, §7] for the powers :

**Proposition 12.6.** *Let $I \subset R$ be an ideal generated in degrees $d_1 \geq \cdots \geq d_s$, with $\dim(R/I) \leq 3$ and $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \supseteq I$ such that $\dim R/\mathfrak{p} \geq 2$. If $\dim(R/I) = 3$, assume further that the unmixed part of $I$ is a radical ideal. Then for $j \geq 2$*

$$\mathrm{reg}(I^j) \leq (j-1)d_1 + \max\{\mathrm{reg}(I), d_2 + \cdots + d_\ell - n + 1\},$$

*where $\ell := \min\{s, n\}$.*

Combined with the bounds on $\mathrm{reg}(I)$, this gives relatively sharp bounds for the regularity of powers in terms of defining equations. Nevertheless, these bounds are only proved in small dimension. For smooth schemes, or schemes with isolated singularities, we have already stated much better results obtained by Bertram, Ein and Lazarsfeld, or as corollaries of vanishing theorems applied to multiplier ideals.

## §13. Some aspects of an example

This example derives from example [CD, 2.3]. The ideals $J_{n-1,n}$ were also considered by Caviglia in his thesis (see [EHU, 4.4]).

We will use the following lemma,

**Lemma 13.1.** *Let $S = R/\mathfrak{b}$ be a graded Cohen-Macaulay ring of dimension $D$ and $I$ be a $S$-ideal of codimension $r > 0$, minimally generated by $r+1$ elements of degrees $d_1 \geq \cdots \geq d_{r+1}$. Set $F := \bigoplus_{i=1}^{r+1} S[-d_i]$. Then for $j > 0$,*

$$H^i_{\mathfrak{m}}(\mathrm{Sym}^j_S(I)) \simeq H^i_{\mathfrak{m}}(I) \otimes \mathrm{Sym}^{j-1}_S(F), \quad \forall i \leq D - \max\{r, 2\}.$$

*In particular, $a_i(\mathrm{Sym}^j_S(I)) = a_i(I) + (j-1)d_1$ for $i \leq D - \max\{r, 2\}$.*

*Proof.* The approximation complex $\mathcal{Z}_\bullet(f; S)$ is exact, as $I = (f_1, \ldots f_{r+1})$ is an almost complete intersection. Also, for $q \geq 2$, $0 \to K_{r+1} \to K_r \to \cdots \to K_{q+1} \to Z_q \to 0$ is a minimal free $S$-resolution of $Z_q$. It follows that $\mathrm{depth}\,(Z_q) = D - r + q$ for $q \geq 2$.

As $\mathcal{Z}_\bullet$ is exact, this shows that $H^i_{\mathfrak{m}}(\mathcal{Z}^j_1) \simeq H^{i-1}_{\mathfrak{m}}(\mathrm{Sym}^j_S(I))$ for $i \leq \min\{D-r+1, D-1\}$. In particular $H^i_{\mathfrak{m}}(Z_1) \simeq H^{i-1}_{\mathfrak{m}}(I)$ in this range.

Now $\mathcal{Z}^j_1 = Z_1 \otimes_S \mathrm{Sym}^{j-1}_S(F)$ and we get

$$H^{i-1}_{\mathfrak{m}}(\mathrm{Sym}^j_S(I)) \simeq H^i_{\mathfrak{m}}(Z_1) \otimes_S \mathrm{Sym}^{j-1}_S(F) \simeq H^{i-1}_{\mathfrak{m}}(I) \otimes \mathrm{Sym}^{j-1}_S(F),$$

for $i \leq \min\{D-r+1, D-1\}$ which proves our claim. $\square$

In the sequel we set $R := k[x, y, z, t]$ where $k$ is a field.

Consider the surface $\mathcal{S}_m$ in $\mathbf{P}^3$ of equation $x^m t - y^m z$. This surface contains the lines defined by the ideals $J := (z, t)$ and $J' := (x, z)$.

**Lemma 13.2.** *The ideal $I_{m,n} = (x^m t - y^m z) + J^n$ is unmixed and is the saturation of $J_{m,n} := (x^m t - y^m z) + (z^n, t^n)$.*

*Proof.* First notice that the ideals $I_{m,n}$ and $J_{m,n}$ are both trivial on the affine charts $z = 1$ and $t = 1$, are equal to $(t - y^m z, z^m)$ if $x = 1$ and to $(x^m t - z, t^m)$ if $y = 1$.

In particular these ideals are locally complete intersection on the punctured spectrum, where they coincide.

Also, setting $S := R/(x^m t - y^m z)$, $JS$ is locally a complete intersection in $S$ and therefore

$$H^1_{\mathfrak{m}}(\mathrm{Sym}^n_S(J)) \simeq H^1_{\mathfrak{m}}(J^n S) \simeq H^0_{\mathfrak{m}}(R/I_{m,n}),$$

for any $n$. As $H^0_{\mathfrak{m}}(R/I_{m,1}) = H^0_{\mathfrak{m}}(k[x,y]) = 0$, Lemma 13.1 implies that $H^0_{\mathfrak{m}}(R/I_{m,n}) = 0$ for any $n$. This proves that $I_{m,n}$ is saturated. As it is a complete intersection on the punctured spectrum and equidimensionnal, the claim follows. □

We assume in the sequel that $m$ and $n$ are at least 2.

We will use liaison to study the ideals $I_{m,n}$ and $J_{m,n}$.

The complete intersection ideal $K_{m,n} := (x^m t - y^m z, z^{n+1} - x t^n)$ was introduced in [CD] to provide a counter-example to the conjecture that $\mathrm{reg}(\sqrt{I}) \leq \mathrm{reg}(I)$ :

**Example 13.3.** [CD, 2.3] For $m, n \geq 2$, $\mathrm{reg}(K_{m,n}) = m + n + 1$ and $\mathrm{reg}(\sqrt{K_{m,n}}) = m(n-1) + 2$.

One has

$$K_{m,n} = \mathfrak{b}_{m,n} \cap I_{m,n} \cap J'$$

where $\mathfrak{b}_{m,n}$ is the ideal of the monomial curve $\mathcal{C}_{m,n}$ parametrized on the affine chart $x = 1$ by $(1 : u : u^{mn} : u^{m(n+1)})$. Indeed, the containment $K_{m,n} \subseteq \mathfrak{b}_{m,n} \cap I_{m,n} \cap J'$, the equality

$$\deg(K_{m,n}) = (m+1)(n+1) = m(n+1) + n + 1 = \deg(\mathfrak{b}_{m,n}) + n + \deg(J'),$$

and the fact that $\deg(I_{m,n}) \geq n$ (notice that $\deg(I_{m,1}) = 1$ and $\deg(I_{m,\ell}) > \deg(I_{m,\ell-1})$ for $\ell > 0$) implies the equality.

Notice $\mathrm{reg}(\mathfrak{b}_{m,n}) \geq mn$ due to the two (related) following facts :

— $y^{mn} - x^{mn-1} z$ is a minimal generator of $\mathfrak{b}_{m,n}$,

— the line defined by $J$ is a secant of order $mn$ to the curve $\mathcal{C}_{m,n}$.

The fact that a scheme with a secant of high order have a minimal generator of high degree is the main geometric source of varieties with high regularity. Gruson, Lazarsfeld and Peskine showed in [GLP] that they are the only curves attaining the bound they prove on the regularity of curves. It is conjectured that this also holds for smooth surfaces (at least in characteristic zero). Marie-Amélie Bertin proved in [Be] that smooth varieties with an extremal secant satify Eisenbud-Goto conjecture.

Let us sketch the simple argument showing this well-known fact :

**Proposition 13.4.** *If $X$ is a projective scheme and $L$ a line with $L \cap X$ of dimension zero and degree $d$, then the defining ideal of $X$ contains a minimal generator of degree at least $d$.*

*Proof.* As $L$ is not contained in $X$, there exists a minimal generator $F$ of $I_X$ not contained in $I_L$. Let $H := \mathrm{Proj}(R/(F))$. One has $X \cap L \subseteq H \cap L$. Therefore $d \leq \deg(H \cap L) = \deg F$, which proves the claim. □

It follows from a theorem of Bresinsky, Curtis, Fiorentini and Hoa [BCFH], who determines the regularity of all curves with a parametrization $(1 : u : u^a : u^b)$, that $\mathrm{reg}(\mathfrak{b}_{m,n}) = mn$.

This may be easily proved in this special case using the fact that the regularity can be computed from the one of the Hilbert function (due to the fact that $\mathrm{reg}(\mathfrak{b}_{m,n}) \geq mn$ and $a_2(R/\mathfrak{b}_{m,n}) \leq m+n-3$).

[The Hilbert function of the monomial curve $R/\mathfrak{b}_{m,n}$ in degree $\mu$ is the number of non negative integers of the form $a + bmn + cm(n+1)$ with $a$, $b$ and $c$ non negative integers and $a+b+c \leq \mu$. This is the same as counting the integers of the form $a + (b'n + c')m$ with $a + b' \leq \mu$ and $c' \leq b'$. It is not hard to see that this function increases exactly by $mn$ from $\mu - 1$ to $\mu$ if and only if $\mu \geq mn$.]

**Lemma 13.5.** *For $m$ and $n$ at least 2 and any $j > 0$,*

$$\mathrm{reg}(R/J^j_{m,n}) = (m+1)n - 2 + (j-1)\max\{m+1, n\}.$$

*Proof.* As $xy^{mn} - x^{mn}z$ is a minimal generator of $\mathfrak{b}_{m,n} \cap J'$, we have $\mathrm{reg}(\mathfrak{b}_{m,n} \cap J') \geq mn + 1$, and the equality holds due Theorem 6.1 (ii). By Theorem 11.1 we have

$$\mathrm{reg}(\omega_{R/I_{m,n}}) = mn - m - n + 3.$$

As $\omega_{R/I_{m,n}}$ is Cohen-Macaulay of dimension two and $H^2_{\mathfrak{m}}(\omega_{R/I_{m,n}})$ is the graded $k$-dual $\Gamma(R/I_{m,n})$, it follows that

$$\mathrm{indeg}(H^1_{\mathfrak{m}}(R/I_{m,n})) = -(m-1)(n-1).$$

It is an easy particular case of a duality result on Koszul homology (see [Ch2, 5.9]) that

$$H^0_{\mathfrak{m}}(R/J_{m,n})_\mu \simeq H^1_{\mathfrak{m}}(R/I_{m,n})_{-\mu+2n+m-3}$$

and therefore $a_0(R/J_{m,n}) = (m+1)n - 2$. As $\mathfrak{b}_{m,n} \cap J'$ is radical, it follows from Theorem 11.1 that $\mathrm{greg}(R/J_{m,n}) = \mathrm{reg}(R/I_{m,n}) < \mathrm{reg}(R/K_{m,n}) = m + n$. Therefore $\mathrm{reg}(R/J_{m,n}) = (m+1)n - 2$.

As $J_{m,n}$ is a complete intersection ideal on the punctured spectrum, $a_i(\mathrm{Sym}^j_R(I)) = a_i(I^j)$ for $i > 0$. Therefore, by Lemma 13.1, $a_i(R/I^j) = a_i(R/I) + (j-1)\max\{m+1, n\}$ for $i = 0, 1$. Also $a_2(R/I^j) \leq a_2(R/I) + (j-1)\max\{m+1, n\}$ (for instance by Theorem 5.2 with $M := R/I^j$ and $M_1 := R/I$, by recursion on $j$). The claim follows. $\square$

**Example 13.6.** Let $I_n := (z^n, t^n)$, $J_m := (x^{m-1}t - y^{m-1}z)$ and $K_m := (x^{m-1}t - y^{m-1}z, z^m)$. Then, for $m, n \geq 3$,

(i) $\mathrm{reg}(I_n + J_m) = mn - 1 > m + 2n - 2 = \mathrm{reg}(I_n) + \mathrm{reg}(J_m) - 1$,

(ii) $\mathrm{reg}(I_n \cap J_m) = mn > m + 2n - 1 = \mathrm{reg}(I_n) + \mathrm{reg}(J_m)$,

(iii) if $m \geq n$, $\mathrm{reg}(I_n K_m) = mn + m - 1 > 2(m + n - 1) = \mathrm{reg}(I_n) + \mathrm{reg}(K_m)$.

*Proof.* Property (i) follows from Lemma 13.5, and (ii) is given by the exact sequence

$$0 \longrightarrow I_n \cap J_m \longrightarrow I_n \oplus J_m \longrightarrow I_n + J_m \longrightarrow 0\ .$$

Finally (iii) follows from Theorem 6.3 with $\eta := -\mathrm{indeg}(H^1_{\mathfrak{m}}(S/I_{m-1,n})) = (m-2)(n-1) > 0$ by the proof of Lemma 13.5. $\square$

## §14. Technical appendix

We first give a proof of Lemma 2.1, based on a classical spectral sequence argument. See for instance Schenzel's lectures in [Bar] for a similar type of argument.

*Proof of Lemma 2.1.* The homology of $T^\bullet$ is the aboutment of a two spectral sequences that have as first terms :

$$ {}^{v}_{1}E^{p,q} = H^p_{\mathfrak{m}}(M) \otimes_R F^q, \qquad {}^{v}_{2}E^{p,q} = H^q(H^p_{\mathfrak{m}}(M) \otimes_R F^\bullet), $$

$$ {}^{h}_{1}E^{p,q} = \mathcal{C}^p_{\mathfrak{m}}(H^q(M \otimes_R F^\bullet)), \qquad {}^{h}_{2}E^{p,q} = H^p_{\mathfrak{m}}(H^q(M \otimes_R F^\bullet)). $$

For (i), we consider the spectral sequence obtained with the horizontal filtration. The inequality immediatly follows because $({}^{h}_{\infty}E^{p,q})_\mu$ is a subquotient of $({}^{h}_{2}E^{p,q})_\mu$ for any $p$, $q$ and $\mu$. Also, if one of the two other conditions holds, then ${}^{h}_{\infty}E^{p,q} \simeq {}^{h}_{2}E^{p,q}$ for every $p$ and $q$, which shows the equality.

For (ii) we consider the spectral sequence obtained with the vertical filtration. The inequality immediatly follows, because $\mathrm{end}({}^{v}_{1}E^{p,q}) = a_p(M) + b_0(F^q)$.

Set $e_\ell := \max_{p+q\leq \ell}\{a_p(M) + b(F^q) + p + q\}$.

We choose $q$ minimal such that $p+q \leq \ell$ and $a_p(M)+b_0(F^q)+p+q = e_\ell$ and let $\mu := a_p(M)+b_0(F^q)$. As $b(F^\bullet) = \max\{b_0(F^0), b_0(F^1)+1\}$, we have $q \leq 1$. It suffices to show that $({}^{v}_{1}E^{p,q})_\mu \simeq ({}^{v}_{\infty}E^{p,q})_\mu$. For any $r \geq 0$, ${}^{v}_{r+1}E^{p,q}$ is isomorphic to the homology of the sequence

$$ {}^{v}_{r}E^{p+r-1,q-r} \xrightarrow{{}^{v}_{r}d^{p+r-1,q-r}} {}^{v}_{r}E^{p,q} \xrightarrow{{}^{v}_{r}d^{p,q}} {}^{v}_{r}E^{p-r+1,q+r} $$

Now, $({}^{v}_{r}E^{p-r+1,q+r})_\mu$ is a subquotient of $({}^{v}_{1}E^{p-r+1,q+r})_\mu \simeq (H^{p-r+1}_{\mathfrak{m}}(M) \otimes_R F^{q+r})_\mu$ which is zero for $r \geq 2$ because

$$ \begin{aligned} a_{p-r+1}(M) + b^{q+r}(F^\bullet) &\leq a_{p-r+1}(M) + b(F^\bullet) - q - r \\ &\leq a_{p-r+1}(M) + \max\{b_0(F^0) - q - r, b_0(F^1) - q - r + 1\} \\ &\leq e_\ell - p - q - 1 \\ &\leq \mu - 1 \end{aligned} $$

and ${}^{v}_{r}E^{p+r-1,q-r} = 0$ for $r \geq 2$ because $q \leq 1$. It follows that $({}^{v}_{2}E^{p,q})_\mu \simeq ({}^{v}_{\infty}E^{p,q})_\mu$.

If $r = 1$ and $q = 0$, ${}^{v}_{1}E^{p,-1} = 0$ and $a_p(M) + b_1(F^\bullet) \leq \mu - 1$ which implies that $({}^{v}_{1}E^{p,1})_\mu = 0$ and shows that $({}^{v}_{2}E^{p,0})_\mu \simeq ({}^{v}_{1}E^{p,0})_\mu$, as claimed.

Finally, if $r = 1$ and $q = 1$, then $b_0(F^2) < b_0(F^1)$ so that $({}^{v}_{1}E^{p,2})_\mu = 0$. Also $b_0(F^0) \leq b_0(F^1)$ as we choosed $q$ minimal. If $b_0(F^0) < b_0(F^1)$, $({}^{v}_{1}E^{p,0})_\mu = 0$ which implies that $({}^{v}_{2}E^{p,1})_\mu \simeq ({}^{v}_{1}E^{p,1})_\mu$ and proves our claim. Else, let $w := b_0(F^0) = b_0(F^1)$ and write $F^0 = R[-w]^\alpha \oplus G^0$ and $F^1 = R[-w]^\beta \oplus G^1$ with maximal shifts of $G^0$ and $G^1$ bigger than $-w$. Now $({}^{v}_{2}E^{p,1})_\mu$ is the homology of the sequence

$$ (H^p_{\mathfrak{m}}(M) \otimes_R R[-w]^\alpha)_\mu \longrightarrow (H^p_{\mathfrak{m}}(M) \otimes_R R[-w]^\beta)_\mu \longrightarrow 0 $$

but, as $F^\bullet$ is minimal, the map $R[-w]^\alpha \longrightarrow R[-w]^\beta$ induced by $f^0 : F^0 \longrightarrow F^1$ is the zero map. Therefore, in this case also $({}^{v}_{2}E^{p,1})_\mu \simeq ({}^{v}_{1}E^{p,1})_\mu$. $\square$

We now sketch the proof of Proposition 9.1. All these facts are known, but the proofs are a bit scattered in the literature, and quite often restricted to the case were the module $M$ is cyclic.

*Sketch of the proof of Proposition 9.1.* Some diagram chasing using the exact sequence

$$ 0 \longrightarrow K \longrightarrow M \xrightarrow{\times l} M[1] \longrightarrow \overline{M}[1] \longrightarrow 0 $$

shows that $b_0(K) \leq \max\{b_0(M), b_1(M)-1, b_2(\overline{M})-1\}$. This 4 terms exact sequence also gives rise to a long exact sequence in cohomology

$$(\#) \qquad \begin{tikzcd} 0 \arrow[r] & K \arrow[r] & H^0_{\mathfrak{m}}(M) \arrow[r] & H^0_{\mathfrak{m}}(M)[1] \arrow[r] & H^0_{\mathfrak{m}}(\overline{M})[1] \arrow[dll] \\ & & H^1_{\mathfrak{m}}(M) \arrow[r] & H^1_{\mathfrak{m}}(M)[1] \arrow[r] & H^1_{\mathfrak{m}}(\overline{M})[1] \arrow[dll] \\ & & H^2_{\mathfrak{m}}(M) \arrow[r] & H^2_{\mathfrak{m}}(M)[1] \arrow[r] & \cdots \end{tikzcd}$$

Sequence (#) shows (i) (recall that $H^i_{\mathfrak{m}}(M)_\mu = 0$ for $\mu \gg 0$ and any $i$) and gives an exact sequence $0 \to K_\mu \to H^0_{\mathfrak{m}}(M)_\mu \stackrel{\times l}{\to} H^0_{\mathfrak{m}}(M)_{\mu+1} \to 0$ for $\mu \geq \operatorname{reg}(\overline{M})$, which implies directly the first part of (ii) and using that $b_0(K) \leq \max\{b_0(M), b_1(M)-1, b_2(\overline{M})-1\}$ we get the second claim of (ii).

For (iii), we use the spectral sequence ${}^1E^i_j = H^i_{\mathfrak{m}}(M) \otimes K_j \Rightarrow \operatorname{Tor}_{j-i}(M,k)$, where $K_\bullet$ is the Koszul complex on the variables of $R$ —which is a minimal graded free $R$-resolution of $k$— and the fact that $(H^i_{\mathfrak{m}}(M) \otimes K_j)_\mu = 0$ for $\mu > a_i(M)+j$. It follows that $({}^\infty E^0_0)_\mu = ({}^2E^0_0)_\mu = (H^0_{\mathfrak{m}}(M) \otimes k)_\mu$ if $\mu \geq \operatorname{greg}(M)+2$ and $({}^\infty E^i_i)_\mu = ({}^1E^i_i)_\mu = 0$ for $i > 0$ if $\mu \geq \operatorname{greg}(M)+1$. This proves (iii).

We now turn to (iv) and notice that for any $\mu \geq \max\{a_0(M), \operatorname{greg}(\overline{M})\}$ we have an exact sequence $0 \to H^0_{\mathfrak{m}}(\overline{M}) \to H^1_{\mathfrak{m}}(M)_\mu \stackrel{\times l}{\to} H^1_{\mathfrak{m}}(M)_{\mu+1} \to 0$ and, by (iii), $b_0(H^0_{\mathfrak{m}}(\overline{M})) \leq \max\{\operatorname{greg}(\overline{M})+1, b_0(\overline{M})\}$.

For (v) we induct on $\dim(M)$. The case of dimension zero is clear as then $M = H^0_{\mathfrak{m}}(M)$ and $H^i_{\mathfrak{m}}(M) = 0$ for $i > 0$. Notice that the hypothesis and conclusion are unaffected by extending the field $k$ if it is finite, so that one may always assume that there exists an element $l$ such that $K := 0 :_M (l)$ has finite length. In this case $\dim(\overline{M}) < \dim(M)$ unless $\dim(M) = 0$.

For the inductive step we may therefore assume that $\dim(M) > 0$, $\{H^i_{\mathfrak{m}}(M)_{\mu-i} = 0,\ \forall i\}$ and choose such an $l$. By the exact sequence (#), we have $\{H^i_{\mathfrak{m}}(\overline{M})_{\mu-i} = 0,\ \forall i\}$.

If $\mu \geq b_0(M)$, the induction hypothesis and the fact that $b_0(\overline{M}) \leq b_0(M)$ gives $\{H^i_{\mathfrak{m}}(\overline{M})_{\mu'-i} = 0,\ \forall i\}$ for $\mu' \geq \mu$ which implies using (#) that $H^i_{\mathfrak{m}}(M)_{\mu'-i+1} \simeq H^i_{\mathfrak{m}}(M)_{\mu'-i}$ for all $i$ and $\mu' \geq \mu$, and completes this first case.

If $\mu \geq a_0(M)$, the induction hypothesis applied to $\overline{M}/H^0_{\mathfrak{m}}(\overline{M})$ shows that $\{H^i_{\mathfrak{m}}(\overline{M})_{\mu'-i} = 0,\ \forall i > 0\}$ for $\mu' \geq \mu$. Hence, using (#), it follows that for $\mu' \geq \mu$ : $H^i_{\mathfrak{m}}(M)_{\mu'-i+1} \simeq H^i_{\mathfrak{m}}(M)_{\mu'-i}$ for all $i \geq 2$ and $H^1_{\mathfrak{m}}(\overline{M})_{\mu'-1} = 0 \Rightarrow H^1_{\mathfrak{m}}(\overline{M})_{\mu'} = H^0_{\mathfrak{m}}(\overline{M})_{\mu'} = 0$. This shows that $a_0(\overline{M}) \leq \mu$ and $a_1(M) \leq \mu - 1$, and completes the proof. $\square$

**References**


[BS1] Bayer, David; Stillman, Michael. A criterion for detecting $m$-regularity. Invent. Math. 87 (1987), no. 1, 1–11.

[BS2] Bayer, David; Stillman, Michael. On the complexity of computing syzygies. Computational aspects of commutative algebra. J. Symbolic Comput. 6 (1988), no. 2-3, 135–147.

[BCP] Bayer, Dave; Charalambous, Hara; Popescu, Sorin. Extremal Betti numbers and applications to monomial ideals. J. Algebra 221 (1999), no. 2, 497–512.

[BM] Bayer, Dave; Mumford, David. What can be computed in algebraic geometry? Computational algebraic geometry and commutative algebra (Cortona, 1991), 1–48, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.

[BG] Bermejo, Isabel; Gimenez, Philippe. Computing the Castelnuovo-Mumford regularity of some subschemes of $\mathbf{P}^n_K$ using quotients of monomial ideals. Effective methods in algebraic geometry (Bath, 2000). J. Pure Appl. Algebra 164 (2001), no. 1-2, 23–33.

[Be] Bertin, Marie-Amélie. On the regularity of varieties having an extremal secant line. J. Reine Angew. Math. 545 (2002), 167–181.

[BEL] Bertram, Aaron; Ein, Lawrence; Lazarsfeld, Robert. Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. J. Amer. Math. Soc. 4 (1991), no. 3, 587–602.

[Bl] Blancafort, Cristina. Hilbert functions of graded algebras over Artinian rings. J. Pure Appl. Algebra 125 (1998), no. 1-3, 55–78.

[BCFH] Bresinsky, H.; Curtis, F.; Fiorentini, M.; Hoa, L. T. On the structure of local cohomology modules for monomial curves in $P^3_K$. Nagoya Math. J. 136 (1994), 81–114.

[BSc] Brodmann, Markus; Schenzel, Peter. On projective curves of maximal regularity. Math. Z. 244 (2003), no. 2, 271–289.

[BSh] Brodmann, Markus; Sharp, Rodney. Local cohomology: an algebraic introduction with geometric applications. Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge, 1998.

[BH] Bruns, Winfried; Herzog, Jürgen. Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[BJ] Busé, Laurent; Jouanolou, Jean-Pierre. On the closed image of a rational map and the implicitization problem. J. Algebra 265 (2003), no. 1, 312–357.

[Cast] Castelnuovo, Guido. Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica. Rend. Circ. Mat. Palermo 7 (1893), 89–110.

[Cav] Caviglia, Giulio. Bounds on the Castelnuovo-Mumford regularity of Tensor Products. Preprint 2003. *To appear in* Proc. Amer. Math. Soc.

[CS] Caviglia, Giulio; Sbarra, Enrico. Characteristic-free bounds for the Castelnuovo-Mumford regularity. math.AC/0310122

[Chan] Chandler, Karen A. Regularity of the powers of an ideal. Comm. Algebra 25 (1997), no. 12, 3773–3776.

[CP] Chardin, Marc; Philippon, Patrice. Régularité et interpolation. J. Algebraic Geom. 8 (1999), no. 3, 471–481.

[CU] Chardin, Marc; Ulrich, Bernd. Liaison and Castelnuovo-Mumford regularity. Amer. J. Math. 124 (2002), no. 6, 1103–1124.

[CM] Chardin, Marc; Moreno-Socí as, Guillermo. Regularity of lex-segment ideals: some closed formulas and applications. Proc. Amer. Math. Soc. 131 (2003), no. 4, 1093–1102

[CD] Chardin, Marc; D'Cruz, Clare. Castelnuovo-Mumford regularity: examples of curves and surfaces. J. Algebra 270 (2003), no. 1, 347–360.

[Ch1] Chardin, Marc. Cohomology of projective schemes: from annihilators to vanishing. J. Algebra 274 (2004), no. 1, 68–79.

[Ch2] Chardin, Marc. Regularity of ideals and their powers. Prépublication 364. Institut de mathématiques de Jussieu, Mars 2004.

[CMT] Chardin, Marc; Minh, Nguyen Cong; Trung, Ngô Viêt. On the regularity of products and intersections of complete intersections. math.AC/0503157

[CSi] Conca, Aldo; Sidman, Jessica. Generic initial ideals of points and curves. math.AC/0402418

[CH] Conca, Aldo; Herzog, Jürgen. Castelnuovo-Mumford regularity of products of ideals. Collect. Math. 54 (2003), no. 2, 137–152.

[Co] Conca, Aldo. Regularity jumps for powers of ideals. Commutative Algebra with a focus on Geometric and Homological Aspects, Lecture Notes in Pure and Appl. Math., Marcel Dekker *(to appear).*

[CSr] Cutkosky, S. Dale; Srinivas, Vasudevan. On a problem of Zariski on dimensions of linear systems. Ann. of Math. (2) 137 (1993), no. 3, 531–559.

[Cu] Cutkosky, S. Dale. Irrational asymptotic behaviour of Castelnuovo-Mumford regularity. J. Reine Angew. Math. 522 (2000), 93–103.

[CHT] Cutkosky, S. Dale; Herzog, Jürgen; Trung, Ngô Viêt. Asymptotic behaviour of the Castelnuovo-Mumford regularity. Compositio Math. 118 (1999), no. 3, 243–261.

[CEL] Cutkosky, Steven Dale; Ein, Lawrence; Lazarsfeld, Robert. Positivity and complexity of ideal sheaves. Math. Ann. 321 (2001), no. 2, 213–234.

[DS] Derksen, Harm; Sidman, Jessica. A sharp bound for the Castelnuovo-Mumford regularity of subspace arrangements. Adv. Math. 172 (2002), no. 2, 151–157.

[EL] Ein, Lawrence; Lazarsfeld, Robert. Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension. Invent. Math. 111 (1993), no. 1, 51–67.

[E] Eisenbud, David. Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[EG] Eisenbud, David; Goto, Shiro. Linear free resolutions and minimal multiplicity. J. Algebra 88 (1984), no. 1, 89–133.

[EHU] Eisenbud, David; Huneke, Craig; Ulrich, Bernd. The Regularity of Tor and Graded Betti Numbers. math.AC/0405373

[EK] Eliahou, Shalom; Kervaire, Michel. Minimal resolutions of some monomial ideals. J. Algebra 129 (1990), no. 1, 1–25.

[GGP] Geramita, Anthony V.; Gimigliano, Alessandro; Pitteloud, Yves. Graded Betti numbers of some embedded rational $n$-folds. Math. Ann. 301 (1995), no. 2, 363–380.

[GLP] Gruson, Laurent; Lazarsfeld, Robert; Peskine, Christian. On a theorem of Castelnuovo, and the equations defining space curves. Invent. Math. 72 (1983), no. 3, 491–506.

[HHT] Herzog, Jürgen; Hoa , Lê Tuân; Ngô Viêt Trung. Asymptotic linear bounds for the Castelnuovo-Mumford regularity. Trans. Amer. Math. Soc. 354 (2002), no. 5, 1793–1809

[HSV1] Herzog, Jürgen; Simis, Aron; Vasconcelos, Wolmer V. Approximation complexes of blowing-up rings. J. Algebra 74 (1982), no. 2, 466–493.

[HSV2] Herzog, Jürgen; Simis, Aron; Vasconcelos, Wolmer V. Approximation complexes of blowing-up rings. II. J. Algebra 82 (1983), no. 1, 53–83.

[HSV3] Herzog, Jürgen; Simis, Aron; Vasconcelos, Wolmer V. Koszul homology and blowing-up rings. Commutative algebra (Trento, 1981), pp. 79–169, Lecture Notes in Pure and Appl. Math., 84, Dekker, New York, 1983.

[HTM] Hoa, Lê Tuân; Miyazaki, Chikashi. Bounds on Castelnuovo-Mumford regularity for generalized Cohen-Macaulay graded rings. Math. Ann. 301 (1995), no. 3, 587–598.

[HS] Hoa, Lê Tuân; Stückrad, Jürgen. Castelnuovo-Mumford regularity of simplicial toric rings. J. Algebra 259 (2003), no. 1, 127–146.

[HT] Hoa, Lê Tuân; Trung, Ngô Viêt. On the Castelnuovo-Mumford regularity and the arithmetic degree of monomial ideals. Math. Z. 229 (1998), no. 3, 519–537.

[HoHy1] Hoa, Lê Tuân; Hyry, Eero. On local cohomology and Hilbert function of powers of ideals. Manuscripta Math. 112 (2003), no. 1, 77–92.

[HoHy2] Hoa, Lê Tuân; Hyry, Eero. Castelnuovo-Mumford regularity of initial ideals (private communication).

[Ho] Hochster, Melvin. Contracted ideals from integral extensions of regular rings. Nagoya Math. J. 51 (1973), 25–43.

[Ka] Katzman, Mordechai. The complexity of Frobenius powers of ideals. J. Algebra 203 (1998), no. 1, 211–225.

[Ko] Kodiyalam, Vijay. Asymptotic behaviour of Castelnuovo-Mumford regularity. Proc. Amer. Math. Soc. 128 (2000), no. 2, 407–411.

[L] Lazarsfeld, Robert. Positivity in algebraic geometry. I & II. Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. 48 & 49. Springer-Verlag, Berlin, 2004.

[Lv] L'vovsky, S. On inflection points, monomial curves, and hypersurfaces containing projective curves. Math. Ann. 306 (1996), no. 4, 719–735.

[Mi] Migliore, Juan C. Introduction to liaison theory and deficiency modules. Progress in Mathematics, 165. Birkhäuser Boston, Inc., Boston, MA, 1998.

[Mi] Miyazaki, Chikashi. Graded Buchsbaum algebras and Segre products. Tokyo J. Math. 12 (1989), no. 1, 1–20.

[MV]ÊMiyazaki, Chikashi; Vogel, Wolfgang. Bounds on cohomology and Castelnuovo-Mumford regularity. J. Algebra 185 (1996), no. 3, 626–642.

[NS1] Nagel, Uwe; Schenzel, Peter. Degree bounds for generators of cohomology modules and Castelnuovo-Mumford regularity. Nagoya Math. J. 152 (1998), 153–174.

[NS2] Nagel, Uwe; Schenzel, Peter. Cohomological annihilators and Castelnuovo-Mumford regularity. Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), 307–328, Contemp. Math., 159, Amer. Math. Soc., Providence, RI, 1994.

[No1] Noma, Atsushi. A bound on the Castelnuovo-Mumford regularity for curves. Math. Ann. 322 (2002), no. 1, 69–74.

[No2] Noma, Atsushi. Castelnuovo-Mumford regularity for nonhyperelliptic curves. Arch. Math. (Basel) 83 (2004), no. 1, 23–26.

[PS] Peeva, Irena; Sturmfels, Bernd. Syzygies of codimension 2 lattice ideals. Math. Z. 229 (1998), no. 1, 163–194.

[Si] Sidman, Jessica. On the Castelnuovo-Mumford regularity of products of ideal sheaves. Adv. Geom. 2 (2002), no. 3, 219–229.

[SSw] Singh, Anurag K.; Swanson, Irena. Associated primes of local cohomology modules and of Frobenius powers. Int. Math. Res. Not. 2004, no. 33, 1703–1733.

[Bar] Six lectures on commutative algebra. Papers from the Summer School on Commutative Algebra held in Bellaterra, July 16–26, 1996. Edited by J. Elí as, J. M. Giral, R. M. Miró-Roig and S. Zarzuela. Progress in Mathematics, 166. Birkhäuser Verlag, Basel, 1998.

[St] Sturmfels, Bernd. Four counterexamples in combinatorial algebraic geometry. J. Algebra 230 (2000), no. 1, 282–294.

[Sw1] Swanson, Irena. Powers of ideals. Primary decompositions, Artin-Rees lemma and regularity. Math. Ann. 307 (1997), no. 2, 299–313

[Sw2] Swanson, Irena. The minimal components of the Mayr-Meyer ideals. J. Algebra 267 (2003), no. 1, 137–155.

[Sw3] Swanson, Irena. On the embedded primes of the Mayr-Meyer ideals. J. Algebra 275 (2004), no. 1, 143–190.

Marc Chardin

Institut de Mathématiques de Jussieu,

CNRS & Université Paris VI,

4, place Jussieu, 75005 Paris, France

*e-mail :* chardin@math.jussieu.fr