

Projective schemes: What is Computable in low degree?

Marc Chardin

To Professor Wolmer Vasconcelos, who inspired this work in many ways.

Introduction

Let I be a homogeneous ideal in $A := k[X_0, \dots, X_n]$ (where k is a field) given in terms of its generators

$$I = (f_1, \dots, f_s)$$

where f_i is a form of degree d_i .

This I defines a scheme

$$\mathcal{Z}_I \subset \mathbf{P}_n(k),$$

and there is a one to one correspondance between the subschemes of $\mathbf{P}_n(k)$ and the homogeneous ideals up to saturation (the saturation I^* of I consists of elements f in A such that for some m , $X_i^m f$ is in I for all i).

We will mention few ideas for algorithms to compute geometric informations on \mathcal{Z}_I from the generators of I and speak about one aspect of their complexity. Algebraic geometry told us that many geometric invariants may be computed from objects that have a more algebraic flavor: finite free resolutions, cohomology, Hilbert function, etc.

There are many ways of splitting the algorithmic problems into parts, we will choose the following one:

- 1) Provide algorithms that are easy to program,
- 2) Estimate their complexity in terms of the output and/or the input,
- 3) Bound the complexity of the output in terms of the input.

As we are not at all expert in complexity theory, we will choose a measure of complexity that we know : Castelnuovo-Mumford regularity. It bounds the degree (in A) where most algebraic questions reduces to linear algebra problems.

1. Main ingredients of two simple algorithms.

We consider $I = (f_1, \dots, f_s)$ a homogeneous ideal in $A := k[X_0, \dots, X_n]$ (k , a field) set $d_i := \deg f_i$ and assume for simplicity that $d_1 \geq \dots \geq d_s \geq 1$ and that k is infinite.

We choose two very simple algorithms as illustrations of what we look for, there are more details (and other algorithms) in [Ch1] for the first one and the second is based on a particular case of [Ch2, 5.2].

The first one relies in part on the following lemma ([Ch1, 20])

Lemma 1.1. *Let g_1, \dots, g_t be a homogeneous regular sequence in A with $g_i = f_i + \sum_{j>i} h_{i,j} f_j$ and let J be the ideal they generate. Set $\sigma := d_1 + \dots + d_r - r$. The following are equivalent,*

(1) $\text{codim}(I) > t$,

(2) *there exists $g_{t+1} = f_{t+1} + \sum_{j>t+1} h_{t+1,j} f_j$ such that g_1, \dots, g_{t+1} is an homogeneous regular sequence in A ,*

(3) *the map*

$$(A/J)_\sigma \xrightarrow{(f_{t+1}, \dots, f_s)} \bigoplus_{j=t+1}^s (A/J)_{\sigma+d_j}$$

is injective.

Steps of algorithm 1:

- Step 1: Construct a sequence g_1, \dots, g_r as in the lemma with $r = \text{codim}(I)$, using (3) to determine if $t = r$ and elementary transformations in the matrix representing this k -linear map to construct g_{t+1} as in (2) if $t < r$.

- Step 2: Choose an homogeneous element h in the kernel of

$$(A/J) \xrightarrow{(f_{r+1}, \dots, f_s)} \bigoplus_{j=r+1}^s (A/J)[d_j]$$

of degree at most $\sigma := d_1 + \dots + d_r - r$ (there exists such an element by (3) and the minimal degree of such an element provides an interesting invariant of \mathcal{Z}_I : the a -invariant of the coordinate ring of \mathcal{Z}_I).

- Step 3: Compute the kernel

$$(A/J) \xrightarrow{\times h} (A/J)[\deg h].$$

The output of the algorithm is the defining ideal of a scheme $\mathcal{S} \subset \mathcal{Z}_I$ which is purely of dimension $\dim \mathcal{Z}_I$ (*i.e.* unmixed of codimension r). If h is “general” the support of \mathcal{S} is the unmixed part of \mathcal{Z}_I . The complexity is bounded by the following result (in terms of Gröbner basis),

Lemma 1.2.[Ch1] *Set $\mathfrak{b} := (g_1, \dots, g_r, h - T^{\deg h}) \subset A[T]$. A Gröbner basis \mathfrak{B} of \mathfrak{b} for the deg-rev-lex order “contains” Gröbner bases for $\mathfrak{b} + (h)$ and $I_{\mathcal{S}} = \mathfrak{b} : (h)$. The maximal degree of an element in \mathfrak{B} , for general coordinates in the X_i ’s, is at most*

$$\max\{d_1 + \dots + d_r - r, \text{reg}(\mathcal{S}) + \deg h\},$$

except possibly for the element $T^{2 \deg h}$.

In characteristic zero, this bound is in fact achieved for “very general” coordinates.

Steps of algorithm 2: Assume that I is the defining ideal of normal scheme \mathcal{S} (*i.e.* $I = I^*$ and $\mathcal{S} = \text{Proj}(A/I)$).

- Step 1: Choose two elements f, g in the Jacobian ideal of I that such that $\text{codim}(I + (f, g)) = \text{codim}(I) + 2$.
- Step 2: Compute the A/I -module

$$H^1(f, g; A/I) = \{(x, y), fx + gy = 0\} / \{a(g, -f), a \in A\}.$$

The output of the algorithm is the local cohomology module $H_{\mathfrak{m}}^1(A/I) \simeq H^1(f, g; A/I)$ (called the Hartshorne-Rao module). In practice choosing f and g should be easy, verifying the codimension condition costs quite a lot as the degrees of f and g are not that small when the codimension increases.

Another strategy may be to compute first the last degree in which $H_{\mathfrak{m}}^1(A/I)$ is not zero (apply the same algorithm, replacing f and g by two linear forms satisfying the same codimension condition, the last non zero degree is the same) and then use linear algebra to finish the computation (in place of a Gröbner basis computation, that doesn't require *a priori* bounds).

The main common point of these algorithmes is that they produce (at least in some important cases) a module that is encoding geometric informations with a complexity controled mainly by the complexity of the ouput.

2. Castelnuovo-Mumford regularity

There are many ways to define this invariant attached to a finitely generated graded module over a polynomial ring. Let us recall some of them in a proposition and then connect it to degrees of element in a Gröbner basis.

Definition. Let A be a polynomial ring over a noetherian ring k , M be a finitely generated A -module that is graded (for the standard grading of A), and \mathfrak{m} the ideal generated by the variables. For an integer i we set

$$a_i(M) := \max\{\mu \mid H_{\mathfrak{m}}^i(M)_{\mu} \neq 0\}$$

and

$$b_i(M) := \max\{\mu \mid \text{Tor}_i^A(M, k)_{\mu} \neq 0\}$$

(with the convention $\max \emptyset = -\infty$).

We recall that $a_i(M)$ is indeed finite (Serre's vanishing theorem) and that the Tor module is equal to the Koszul homology module $H_i(\underline{x}; M)$, where \underline{x} denote the set of variables (because $K_{\bullet}(\underline{x}; A)$ provides a free A -resolution of $A/\mathfrak{m} = k$ as an A -module). Also, if k is a field and $F_{\bullet} \rightarrow M \rightarrow 0$ is a minimal free resolution of M , then $\text{Tor}_i^A(M, k)_{\mu} := H_i(F_{\bullet} \otimes_A A/\mathfrak{m}) = (F_i \otimes_A A/\mathfrak{m})_{\mu}$ (maps in F_{\bullet} are represented by matrices with entries in \mathfrak{m}) is the number of minimal generators of degree μ of F_i , because $A[-j] \otimes_A A/\mathfrak{m} = k[-j]$ is concentrated in degree j .

(See [Ei, Ch. 17] or [BH, Ch. 1] for the definition and basic facts on the Koszul complexes $K_{\bullet}(\underline{z}; M)$, $K^{\bullet}(\underline{z}; M)$ associated to a module M and a tuple \underline{z} of elements of A . We will denote by $H_i(\underline{z}; M)$ and $H^i(\underline{z}; M)$ their homology (resp. cohomology) modules.)

For simplicity, we will assume the base ring to be a field in the following,

Proposition 2.1. *Let A be a polynomial ring over a field k and M be a finitely generated A -module that is graded (for the standard grading of A).*

The following definitions are equivalent,

- (1) $\text{reg}(M) := \max_i \{a_i(M) + i\}$,
- (2) $\text{reg}(M) := \max_i \{b_i(M) - i\}$,
- (3) $\text{reg}(M) := \min\{\mu \mid b_i(M_{\geq \mu}) \leq b_0(M_{\geq \mu}) + i \ \forall i\}$,

(4) *Let \underline{z} be a finite collection of homogeneous elements of $A_{>0}$ such that $M/(\underline{z})M$ is a finite dimensional vector space,*

$$\text{reg}(M) := \min\{\mu \mid H^i(\underline{z}; M)_{>\mu-i} = 0 \ \forall i\}.$$

The equivalence of (1) and (4) is easy using the standard tool for comparing two homological objects (spectral sequences), it implies the equivalence with (2) (taking the set of variables for \underline{z}). Using the equivalence of (1) and (2) and the fact that $a_i(M) = a_i(M_{\geq \mu})$ for any μ and $i > 0$, leads to the equivalence with (3).

The definition (1), despite its apparent inaccessibility (e.g. the local cohomology modules are not all finitely generated, unless M is of finite length) happens to be most tractable when one wants to estimate the regularity. As always, some familiarity with the object makes them very concrete and most of their apparent pathologies are not so bad (e.g. the graded duals of local cohomology modules are finitely generated, as they are isomorphic to some Ext modules).

Definition (2) is interesting for looking at families of schemes (the study of the Hilbert scheme). The condition is that the maps in the minimal free resolution of $M_{\geq \mu}$ over A have linear forms as entries. The sheaves associated to M and $M_{\geq \mu}$ are the same. The existence of *a priori* bounds on the regularity is one ingredient for proving the existence of Hilbert schemes (see [Mu], or [EH, Ch. VI] for an introduction).

Note that definition (4) implies in particular that if M is of dimension d and is a finitely generated B -module where $B := k[l_1, \dots, l_d]$ and the l_i 's are linear forms (in other words, we have a Noether normalisation) then the regularity of M as a B -module is the same as the one as an A -module –this is also quite immediate from (1).

One way to connect the regularity to degree of generators of a Gröbner basis, is to study how it behaves when passing modulo a “general” linear form (see e.g. [Ei, 20.20 and 20.21]). The key is the following lemma, where A is a polynomial ring over a field,

Lemma 2.2. *Let M be a finitely generated graded A -module and l a linear form. Set $K := 0 :_M(l) = \{m \in M \mid lm = 0\}$. Then $\text{reg}(M) \leq \max\{\text{reg}(K), \text{reg}(M/(l)M)\}$, and if the Krull dimension of $A/\text{Ann}(K)$ is at most one, then*

$$\text{reg}(M) = \max\{\text{reg}(K), \text{reg}(M/(l)M)\}.$$

The proof is a standard diagram chasing using the local cohomology definition of the regularity (which has the advantage that $H_{\mathfrak{m}}^i(N)$ for $i > \dim N$). This lemma gives in particular the following,

Proposition 2.3. *Let $\text{Proj}(A/I)$ be a projective surface with no degenerate irreducible component (i.e. I is unmixed, the Krull dimension of A/I is 3 and no prime associated to I contains a linear form). In any coordinates, for the deg-rev-lex order,*

$$\text{reg}(I) = \text{reg}(\text{in}(I)).$$

Here $\text{in}(I)$ is the ideal generated by the leading monomials of the polynomials in I for a given order on the monomials. We recall that \mathfrak{B} is a (minimal) Gröbner basis of I if $\{\text{in}(f) \mid f \in \mathfrak{B}\}$ are (minimal) generators of $\text{in}(I)$. Therefore, the maximal degree of an element in a minimal Gröbner basis of I is $b_0(\text{in}(I))$. The deg-rev-lex order on monomials is obtained by refining the degree order by the inverse of the lexicographic order. (See e.g. [Ei, Ch. 15] for an introduction on Gröbner bases.)

With no geometric hypotheses on $\text{Proj}(A/I)$ one of the main early discoveries is the following result of Bayer and Stillman,

Theorem 2.4.[BS] *For any order and any coordinates, $\text{reg}(I) \leq \text{reg}(\text{in}(I))$. For the deg-rev-lex order, and in general coordinates, $\text{reg}(I) = \text{reg}(\text{in}(I))$.*

The expression “general coordinates” means that there exists a Zariski open subset of the linear group so that any matrix of this open subset gives rise to coordinates that satisfies the given property (in particular it may be that over finite fields an extension of the base field is needed to find good coordinates).

Also, Diana Taylor find an explicit resolution of monomial ideals, that in particular proves the following,

Proposition 2.5. *If J is a monomial ideal in A ,*

$$b_i(J) \leq (i + 1)b_0(J),$$

so that $\text{reg}(J) \leq (n + 1)(b_0(J) - 1) + 1$.

Note that, in this form, the result is optimal: consider the case where J is generated by the b_0 -th powers of the variables.

Mayr and Mayer, and other since then, provided examples of binomial ideals J where $b_i(J)$ is much bigger than $b_0(J)$ (say $\text{reg}(J) \gg b_0(J)^c$ where n is the number of variables and $c > 1$ is close to $\sqrt{2}$).

Bounds on the regularity follows from an inductive argument on the number of variables from the following result (see [BM, the proof of 3.8]). The bounds are more or less of the same type as the lower bounds coming from Mayr-Meyer type examples.

Proposition 2.6. *If M is a finitely generated graded A -module, for a general linear form l and $\mu \geq \max\{\text{reg}(M/(l)M) + 1, b_0(M), b_1(M) - 1\}$,*

$$\text{reg}(M) \leq \mu + \dim_k H_{\mathfrak{m}}^0(M)_{\mu} \leq \mu + \dim_k M_{\mu}$$

Idea of the proof. Notice that, for a general l the kernel $K := \ker(M \xrightarrow{\times l} M)$ is of finite length. Some diagram chasing gives $b_0(K) \leq \max\{b_0(M), b_1(M) - 1, b_2(M/(l)M) - 1\}$ so that K has no generator of degree bigger than $N := \max\{\text{reg}(M/(l)M) + 1, b_0(M), b_1(M) - 1\}$ (in particular $K_\mu = 0$ implies $K_{\mu+1} = 0$ for $\mu \geq N$).

Now the local cohomology definition shows that $\text{reg}(M/H_{\mathfrak{m}}^0(M)) \leq \text{reg}(M/(l)M)$ and gives an exact sequence $0 \rightarrow K_\mu \rightarrow H_{\mathfrak{m}}^0(M)_\mu \xrightarrow{\times l} H_{\mathfrak{m}}^0(M)_{\mu+1} \rightarrow 0$ for $\mu > \text{reg}(M/(l)M)$, so that for $\mu \geq N$, $\dim_k H_{\mathfrak{m}}^0(M)_\mu$ is a strictly decreasing function of μ until it reaches 0.

It was also remarked (and proved) by André Galligo that initial ideals have an interesting property in general coordinates and characteristic zero: they are stable, which means that if a monomial $x_i m$ is in the initial then it is also the case of $x_j m$ if $j < i$. Afterwards, Eliahou and Kervaire find a minimal free resolution for stable monomial ideals; it follows from the resolution that all graded Betti numbers of these ideals may be easily read from the minimal generators, in particular $\text{reg}(J) = b_0(J)$ for a stable monomial ideal J .

The conditions of genericity needed for having a stable monomial ideal are not so often realized without performing a change of coordinates, they are more difficult to achieve than the ones for having $\text{reg}(I) = \text{reg}(\text{in}(I))$ (for deg-rev-lex order).

It is also important to notice that doing a generic change of coordinates have quit a big influence on the size of the computation, for several reasons: the coefficients get bigger, the polynomials became dense and the number of generators of the initial ideal increases in general. On the other hand it should be noted that the degrees of generators of the initial ideal in special coordinates may be much bigger than in general coordinates. Applying the following lemma to the Mayr-Mayer ideal provides such an example,

Lemma 2.7. *Let f_1, \dots, f_s be forms of degrees d_1, \dots, d_s in A and set $I := (f_1, \dots, f_s)$.*

(1) *If $\text{codim}(I) = s$, $\text{reg}(I) = d_1 + \dots + d_s - s + 1$.*

(2) *If $\text{codim}(I) = r < s$, there exists a graded complete intersection of degrees d_1, \dots, d_s in $A[Y_1, \dots, Y_{s-r}]$ such that the regularity of its initial ideal for the deg-rev-lex order bounds the regularity of the initial ideal of I for the deg-rev-lex order.*

The regularity of a graded ideal is always bounded in terms of the Hilbert function of the ideal, as all the graded Betti numbers are bounded above by the ones of the lex-segment ideal that only depends on the Hilbert function (see [Bi] and [Hu] for characteristic zero case, [Pa] for the general case, and [CGP] for a short argument in characteristic zero). If I is saturated, the regularity of the lex-segment ideal only depends on the Hilbert polynomial. The regularity of the lex-segment ideal may be computed ([CM, 1.3 and 2.3]) and leads for example to the following bound that is at least as bad as expected...

Corollary 2.8.[CM] *Let $I \subset A$ be an ideal generated by polynomials of degrees d_1, \dots, d_s and let r be the codimension of I . For any admissible monomial order and any coordinates,*

$$\text{reg}(\text{in}(I)) \leq 1 + [d_1 \cdots d_s]^{2^{n-r}}$$

if $r \leq n$ and $\text{reg}(\text{in}(I)) \leq d_1 + \dots + d_{n+1} - n$ if $r = n + 1$.

3. Bounds on Castelnuovo-Mumford regularity

There is a famous conjecture that suggests the following bound for reduced and irreducible schemes:

Conjecture[Eisenbud and Goto]. *If $\mathcal{S} \subset \mathbf{P}_n$ is a non degenerate reduced and irreducible scheme,*

$$\operatorname{reg}(\mathcal{S}) \leq \deg \mathcal{S} - \operatorname{codim} \mathcal{S}.$$

(Non degenerate means $\mathcal{S} \not\subset H$ for any hyperplane H .)

We recall that if $\mathcal{S} := \operatorname{Proj}(A/I)$, $\operatorname{reg}(\mathcal{S}) := \operatorname{reg}(A/I^*) = \operatorname{reg}(I^*) - 1$.

This result was known for curves when the conjecture was made. It was first established for smooth curves by Castelnuovo [Ca], and the for reduced curves with no degenerate component by Gruson, Lazarsfeld and Peskine (over a perfect field) in [GLP]. There is some evidence that this may be true at least for smooth schemes in characteristic zero: it is true for smooth surfaces (Pinkham and Lazarsfeld) and (up to adding small constants) in dimension at most six, by the work of several people including Lazarsfeld, Ran and Kwack.

In any dimension, it was prove by Mumford ([BM]) that in characteristic zero a smooth scheme \mathcal{S} satisfies,

$$\operatorname{reg}(\mathcal{S}) \leq (\dim \mathcal{S} + 1)(\deg \mathcal{S} - 1).$$

In positive characteristic, it follows from theorems that we will mention below that one has $\operatorname{reg}(\mathcal{S}) \leq (\dim \mathcal{S} + 1)^2(\deg \mathcal{S} - 1)$, and there are also quite reasonable results for schemes with isolated singularities.

We now turn to bounds depending on the degrees of generators. As we mentioned in the preceding paragraph, there is no reasonable bound on the regularity without imposing geometric conditions.

Let $I = (f_1, \dots, f_s)$ be a homogeneous ideal in A , where f_i is a form of degree d_i . We will assume that $d_1 \geq d_2 \geq \dots \geq d_s \geq 1$. Let $\mathcal{Z}_I \subset \mathbf{P}_n$ be the scheme defined by I and r be the codimension of I in A , which is also the one of \mathcal{Z}_I as a subscheme of \mathbf{P}_n . Let \mathcal{S} be the top dimensional part of \mathcal{Z}_I and Y the residual of \mathcal{S} in \mathcal{Z}_I . In algebraic terms, $I_{\mathcal{S}}$ is the intersection of the primary components of I of codimension r , and $I_Y := (I : I_{\mathcal{S}}) = \{f \in A \mid fI_{\mathcal{S}} \subset I_Y\}$.

The first striking result on regularity in these terms is due to Bertram, Ein and Lazarsfeld:

Theorem 3.1.[BEL] *If $\mathcal{Z}_I = \mathcal{S}$ is smooth of characteristic zero,*

$$\operatorname{reg}(\mathcal{S}) \leq d_1 + \dots + d_r - r,$$

with equality if and only if \mathcal{S} is a complete intersection of degrees d_1, \dots, d_r .

This linear bound was generalized in [CU],

Theorem 3.2. *Assume that \mathcal{S} have at most a one dimensional singular locus an is locally a complete intersection outside finitely many points. If the residual Y have at most isolated singularities and k is of characteristic zero,*

$$\operatorname{reg}(\mathcal{S}) \leq d_1 + \dots + d_r - r,$$

with equality if and only if $\mathcal{Z}_I = \mathcal{S}$ is a complete intersection of degrees d_1, \dots, d_r .

Note that the defining ideal of \mathcal{S} may be computed in low degree by Algorithm 1, even if the regularity of I is much bigger. They rely on liaison theory and use either Kodaira's vanishing theorem or a result of Karen Smith that enables an induction on the dimension.

More recently, we showed several other bounds. They essentially improve the ones of [CU] in positive characteristic, and provide the following result:

Theorem 3.3.[Ch2, 4.4] *Assume that \mathcal{Z} is an isolated component of \mathcal{Z}_I that doesn't meet the other components, and that \mathcal{Z}_I is smooth at all but a finite number of points of \mathcal{Z} . Then,*

$$\text{reg}(\mathcal{Z}) \leq (\dim \mathcal{Z} + 1)(d_1 + \dots + d_r - r - 1) + 1.$$

This generalizes the result of [CP] that treats the case where $\dim \mathcal{Z} = 0$. The proof relies on [CP] and a result of Hochster and Huneke, which implies that the phantom homology (which is, roughly speaking, the one that vanishes in the Cohen-Macaulay case) is uniformly killed by the Jacobian ideal. The result then follows by cutting the scheme \mathcal{Z} by a sequence of parameters in the Jacobian ideal and using some homological algebra to exploit this uniform vanishing. The connection between annihilators and vanishing was already remarked and used to study the so-called ℓ -Buchsbaum schemes by Miyazaki, Nagel, Schenzel and Vogel ([Mi], [NS1], [NS2] and [MV]).

Let us also point out the following remark that formalizes the fact that bounding the regularity in a geometric context is as difficult as bounding the degree where the Hilbert function becomes a polynomial, or bounding the degree where every global section is the restriction of a polynomial.

Remark 3.4.[CM 2.5] Let \mathcal{P} be a property of embedded projective schemes and $N(X)$ a numerical invariant attached to such a scheme X . Assume that if $X \subseteq \mathbf{P}_n$ satisfies \mathcal{P} and H is a general hyperplane, then $X \cap H \subseteq H \simeq \mathbf{P}_{n-1}$ satisfies \mathcal{P} and $N(X \cap H) \leq N(X)$. We denote by $I_X \subseteq R$ the defining ideal of X and by H_X the Hilbert function of R/I_X . Then the following are equivalent,

- (i) If X satisfies \mathcal{P} , $\text{reg}(R/I_X) \leq N(X)$.
- (ii) If X satisfies \mathcal{P} , $\text{reg}(H_X) \leq N(X) - 1$.
- (iii) If X satisfies \mathcal{P} , $(R/I_X)_\mu = H^0(X, \mathcal{O}_X(\mu))$ for $\mu \geq N(X)$,

where $\text{reg}(H_X)$ is the last degree where H_X differs from the Hilbert polynomial P_X .

Examples for property \mathcal{P} are: X satisfies S_k , X is smooth in codimension ℓ , X is irreducible, X is equidimensional, or any conjunction of some of these properties. For $N(X)$ one may choose the degree of X , or the degree of X minus the embedding codimension of X if X is irreducible and reduced, or the minimum over the sets of equations defining X of the maximal degree of these equations.

Another important point is to notice that even if we are not able to bound the regularity in many cases, a big part of the information is sometimes available in an indirect way. For example, if the top dimensional component \mathcal{S} of \mathcal{Z}_I have at most isolated singularities the canonical module $\omega_{\mathcal{S}}$ of \mathcal{S} have a small regularity (at least in characteristic

zero, thanks to Kodaira’s vanishing theorem) and is easily computable (as the kernel of the map in Step 2 of Algorithm 1). From $\omega_{\mathcal{S}}$ we may compute the Hilbert polynomial of \mathcal{S} or its cohomology modules using Serre duality (at least if \mathcal{S} is Cohen-Macaulay), or test if an element is in $I_{\mathcal{S}}$. This in turn gives a way to check if $\text{reg}(\mathcal{S}) \leq N$ by linear algebra computation in degree at most N plus a linear function of the degrees of generators (in general coordinates, the criterion of [BS] gives such a test).

Acknowledgements. A major part of my understanding of Castelnuovo-Mumford grew out of the many ideas learned from or shared with several people. It is partially reflected in the references below: Patrice Philippon and Bernd Ulrich for the “geometric” results and Vesselin Gasharov, Irena Peeva and Guillermo Moreno for our work on lex-segment ideals. Many others should be mentioned: Dale Cutkosky, David Eisenbud, Philippe Gimenez, Robert Lazarsfeld, Mike Stillman, Wolmer Vasconcelos, ... A special thanks also to Martin Sombra for reading and commenting a first draft of these notes.

References.

- [BM] D. Bayer, D. Mumford, *What can be computed in algebraic geometry?* in *Computational algebraic geometry and commutative algebra (Cortona, 1991)*, 1–48. Sympos. Math. XXXIV, Cambridge Univ. Press, Cambridge, 1993.
- [BS] D. Bayer, M. Stillman, *A criterion for detecting m -regularity.* Invent. Math. **87** (1987), 1–11.
- [BEL] A. Bertram, L. Ein, R. Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties.* J. Amer. Math. Soc. **4** (1991), 587–602.
- [BH] W. Bruns, J. Herzog. *Cohen-Macaulay Rings.* Cambridge stud. in adv. math **39**. Cambridge Univ. Press, 1993.
- [Bi] A. Bigatti, *Upper bounds for the Betti numbers of a given Hilbert function.* Comm. in Algebra **21** (1993), 2317–2334.
- [Ca] G. Castelnuovo, *Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica.* Rend. Circ. Mat. Palermo **7** (1893), 89–110.
- [Ch1] M. Chardin, *Applications of some properties of the canonical module in computational projective algebraic geometry.* J. Symbolic Comput. **29** (2000), 527–544.
- [Ch2] M. Chardin, *Cohomology of projective schemes: from annihilators to vanishing.* (preprint 290, U. Paris 6)
- [CGP] M. Chardin, V. Gasharov, I. Peeva, *Maximal Betti numbers.* Proc. Amer. Math. Soc. (to appear).
- [CM] M. Chardin, G. Moreno, *Regularity of lex-segment ideal: some closed formulas and applications,* (preprint 292, U. Paris 6).
- [CP] M. Chardin, P. Philippon, *Régularité et interpolation.* J. Algebraic Geom. **8** (1999), 471–481.
- See also the erratum at the address <http://www.math.jussieu.fr/~chardin/textes.html>.
- [CU] M. Chardin, B. Ulrich, *Liaison and Castelnuovo-Mumford regularity.* Amer. J. Math. (to appear).

- [DGP] W. Decker, G.-M. Greuel, G. Pfister, *Primary decomposition: algorithms and comparisons*, in *Algorithmic Algebra and Number Theory*, 187-220, Springer Verlag, Heidelberg, 1998.
- [Ei] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 1995.
- [EH] D. Eisenbud, J. Harris. *The geometry of schemes*, Springer-Verlag, 1999.
- [EHV] D. Eisenbud, C. Huneke, W. Vasconcelos, *Direct methods for primary decomposition*. *Invent. Math.* **110** (1992), 207–235.
- [Gr] M. Green, *Generic initial ideals*, in *Six lectures on commutative algebra*, Birkhäuser, Progress in Mathematics **166**, (1998), 119–185.
- [GLP] L. Gruson, R. Lazarsfeld, C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*. *Invent. Math.* **72** (1983), 491–506.
- [Hu] H. Hulett, *Maximum Betti numbers of homogeneous ideals with a given Hilbert function*. *Comm. in Algebra* **21** (1993), 2335–2350.
- [KP] T. Krick, L. M. Pardo, *A computational method for Diophantine approximation*. Algorithms in algebraic geometry and applications (Santander, 1994), 193–253, *Progr. Math.*, **143**, Birkhäuser, Basel, 1996.
- [L] R. Lazarsfeld, *Multiplier ideals for algebraic geometers*. Notes for the ICTP School on Vanishing Theorems, August 2000.
- [MV] C. Miyazaki, W. Vogel, *Bounds on cohomology and Castelnuovo-Mumford regularity*. *J. Algebra* **185** (1996), 626–642.
- [Mu] D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. Stud. **59**, Princeton University Press (1966).
- [NS1] U. Nagel, P. Schenzel, *Cohomological annihilators and Castelnuovo-Mumford regularity*. *Contemp. Math.* **159** (1994), 307–328.
- [NS2] U. Nagel, P. Schenzel, *Degree bounds for generators of cohomology modules and Castelnuovo-Mumford regularity*. *Nagoya Math. J.* **152** (1998), 153–174.
- [Pa] K. Pardue, *Deformation classes of graded modules and maximal Betti numbers*. *Illinois J. Math.* **40** (1996), 564–585.
- [V1] W. Vasconcelos, *Constructions in commutative algebra*, in *Computational algebraic geometry and commutative algebra (Cortona, 1991)*, 151–197, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.
- [V2] W. Vasconcelos, *Arithmetic of blowup algebras*. London Mathematical Society Lecture Note Series, 195. Cambridge University Press, Cambridge, 1994.
- [V2] W. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry*. Algorithms and Computation in Mathematics, 2. Springer-Verlag, Berlin, 1998.

Marc Chardin, Institut de Mathématiques, CNRS & Université Paris 6,
 4, place Jussieu, F-75252 Paris CEDEX 05, France
 chardin@math.jussieu.fr