BOUNDS FOR CASTELNUOVO-MUMFORD REGULARITY IN TERMS OF DEGREES OF DEFINING EQUATIONS

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1. Foreward

This is a transcription of notes corresponding to my lecture in the NATO meeting of Sinaïa (Romania), in september 2002. We indicate for each result a reference, where a proof is given and sometimes a refined version is stated. These references are not always to the original source, but we hope that we have properly attributed the results!

The lecture was essentially a review of my (often shared) work on the subject, and does not give a general perspective on these questions. A more substential bibliography may be found in [Ch2].

2. Notations and definitions

In all these notes, k will be a field, $A := k[X_0, \dots, X_n]$ a polynomial ring over k and $\mathfrak{m} := (X_0, \dots, X_n)$ the graded maximal ideal of A.

If M is a finitely generated graded A-module, we set

$$a_i(M) := \max\{\mu \mid H^i_{\mathfrak{m}}(M)_\mu \neq 0\},$$

$$b_i(M) := \max\{\mu \mid \operatorname{Tor}_i^A(M,k)_\mu \neq 0\}.$$

With these notations, the Castelnuovo-Mumford regularity has the two following descriptions

$$\operatorname{reg}(M) = \max_i\{a_i(M) + i\} = \max_j\{b_j(M) - j\}.$$

Let us also recall that a module N has a *linear resolution* if $\operatorname{reg}(N) = \operatorname{indeg}(N)$. From the equivalence of the definitions above it folows that one have also

$$\operatorname{reg}(M) = \min\{\mu \mid M_{\geq\mu} \text{ has a linear resolution}\}.$$

The following classical lemma indicates a way of computing the regularity

Lemma. [Ch2] *Let $\ell \in A_1$ and $K := \{m \in M \mid \ell m = 0\}$, then*

$$\operatorname{reg}(M) \leq \max\{\operatorname{reg}(M/\ell M), \operatorname{reg}(K)\},$$

and equality holds if the Krull dimension of K is at most one.

As taking the initial ideal commutes with the operations $- : (X_n)$ and $- + (X_n)$, in the reverse lex order, it follows from the lemma that,

Corollary. [Ch2] *Let $I \subset A$ be an homogeneous ideal and $X := \operatorname{Proj}(A/I)$. Then*

$$\operatorname{reg}(I) = \operatorname{reg}(in_{rev-lex}(I))$$

if either:

(1) *coordinates are "general",*
(2) $\dim X \leq 0$*,*
(3) $\dim X \leq 1$ *and $X \cap \{X_n = 0\}$ is finite,*
(4) $\dim X = 2$*, X has no component of dimension one and $X \cap \{X_n = X_{n-1} = 0\}$ is finite.*

In (1), general coordinates means that this statement is true if the unit element of the linear group is in the open set of the elements ψ such that $\operatorname{reg}(in_{rev-lex}(\psi I))$ is minimal.

Result (1) is due to Dave Bayer and Mike Stillman, it was the first important result linking regularity to Gröbner bases.

These statments have computational consequences, for instance because Gröbner bases computations stops in degree $b_0(in(I)) \leq \operatorname{reg}(in(I))$. Also note that monomial ideals have explicit free resolutions (due to Diana Taylor), and even minimal ones in the "strongly stable" case (due to Eliahou and Kervaire). These resolutions show that $b_i(I) \leq (i+1)b_0(I)$ for a monomial ideal I and that $\operatorname{reg}(I) = b_0(I)$ in the "strongly stable" case.

Question: How to bound $\operatorname{reg}(I)$ in terms of $b_0(I)$?

3. General bounds on regularity

Let us first show that one can give bounds, for instance by the two methods we sketch below. The first rely on the following lemma, probably due to Mumford (Lectures on curves on an algebraic surface),

Lemma. [Ch2] *Let* $\mu \geq \max\{\mathrm{reg}(M/\ell M), b_0(M), b_1(M)-1\}$ *where* ℓ *is a general linear form. Then*

$$\mathrm{reg}(M) \leq \mu + \dim_k H^0_{\mathfrak{m}}(M)_\mu \leq \mu + \dim_k M_\mu.$$

Applied to $M := A/I$ this gives a bound $\mathrm{reg}(I) \ll b_0(I)^N$ where N is approximately $(\dim A/I)!$. (By recursion on $\dim A/I$.)

Another method uses the following results of Macaulay, Bigatti-Hulett, Pardue, and Gotzmann,

Proposition. [Macaulay, Bigatti-Hulett, Pardue] $\mathrm{reg}(I) \leq \mathrm{reg}(Lex(I))$.

Proposition. [Gotzmann, CM] *If* $\mathrm{reg}(I) \neq \mathrm{reg}(Lex(I))$ *or* $\mathrm{reg}(A/I) \neq a_0(A/I)$ *then*

$$\mathrm{reg}(Lex(I)) = B(P),$$

where P *is the Hilbert polynomial of* A/I*, and* $B(P)$ *a function of* P *related to its Macaulay representation.*

With Guillermo Moreno-Socías, we explicitely computed $B(P)$ in terms of the coefficients of P, and the following bound follows

Proposition. [CM] *Assume* I *is generated in degrees* $d_1, \dots, d_s$*. Then, for any term order,*

$$b_0(in(I)) \leq (d_1 \cdots d_s)^{2^{\dim X}}$$

unless $X := \mathrm{Proj}(A/I) = \emptyset$. (There is a good well-known bound if $X = \emptyset$.)

Examples. Let us set $n = 2m$ for simplicity.

• Some Mayr-Meyer type examples are such that $\mathrm{reg}(I)$ is essentially equivalent to $b_0(I)^{2^m}$.

• If I is the lex-segment ideal of a complete intersection of m forms of degree d, then the regularity of the Hilbert function of A/I is essentially md while the regularity of I is essentially d^{m2^m}.

Despite these examples, there is a hope that more reasonable bounds hold if I defines a scheme which has nice geometric properties.

4. Bounds under geometric hypotheses

If $X \subseteq \mathbf{P}^n_k$ is a scheme, we will set $\mathrm{reg}(X) := \mathrm{reg}(A/I_X)$, where I_X is the unique saturated ideal such that $X = \mathrm{Proj}(A/I_X)$.

In all this section we will assume that I is an ideal of codimension $r > 0$, generated by polynomials of degrees $d_1 \geq d_2 \geq \cdots \geq d_s \geq 1$.

The first important result is the following one, due to Bertram, Ein and Lazarsfeld:

Theorem. [BEL] *If k is of characteristic zero and $X := \mathrm{Proj}(A/I)$ is smooth and purely of codimension r,*

$$\mathrm{reg}(X) \leq d_1 + \cdots + d_r - r,$$

and equality holds if and only if X is a complete intersection of degree $d_1 \cdots d_r$.

Some words about liaison.

Let $\mathfrak{b} = (f_1, \ldots, f_r)$ be a complete intersection homogeneous ideal in A. Assume that $\mathrm{Proj}(A/\mathfrak{b}) =: S = Y \cup Z$, in other words $\mathfrak{b} = J \cap K$ with J and K purely of codimension r and without common associated prime. Then Y and Z are linked by S and one has an exact sequence:

$$0 \to \omega_{A/K}[-a] \to A/\mathfrak{b} \to A/J \to 0,$$

where $\omega_{A/K} = \mathrm{Ext}^r_A(A/K, \omega_A)$, $\omega_A = A[-n-1]$ and $a = d_1 + \cdots + d_r - n - 1$.

We will say that $Z = \mathrm{Proj}(A/K)$ satisfies *weak Kodaira vanishing* (abreviated WKV) if $\mathrm{reg}(\omega_{A/K}) = \dim A/K$. This property is implied by Kodaira vanishing theorem if Z is smooth (and lifts mod p^2 in characteristic p), or has isolated irrational singularities (in characteristic zero).

Proposition. [CU] *The following are equivalent:*
(1) $\mathrm{reg}(Y) < \mathrm{reg}(S)$,
(2) *Z satisfies WKV.*

Note that $\mathrm{reg}(S) = d_1 + \cdots + d_r - r$.

Hironaka, and Peskine and Szpiro in this specific context, proved that for general polynomials of degrees $d_1, \ldots, d_r$ in I in Theorem [BEL], X is linked to a scheme Y that is regular in codimension 3. As a corollary, this shows the theorem in dimension at most four. In fact Ulrich and I have shown that Y have at most rational singularities.

Theorem. [CU] *If $X = \mathrm{Proj}(A/I)$ is locally a complete intersection and have at most isolated irrational singularities, then a generic link Y of X have rational singularities.*

(In positive characteristic there are several possible extensions of the notion of rational singularity: one is the notion of F-rational singularity introduced by Hochster and Huneke, another one is by requiring the existence of a resolution of singularities for which the usual requirements are satisfied. For both notions the Theorem holds.)

If $X = \mathrm{Proj}(A/I)$ is not unmixed we need to define for the next theorem: X^{top}: the unmixed part of X, and X^{low}: the closure of $X - X^{top}$.

Theorem. [CU] *Assume that k is of characteristic zero and*
(1) *Outside finitely many points, X is locally a complete intersection and X^{low} have rational singularities.*
(2) *The locus of irrational singularities of X is of dimension at most one.*
Then,

$$\mathrm{reg}(X^{top}) \leq d_1 + \cdots + d_r - r - 1,$$

unless X is a complete intersection of degree $d_1 \cdots d_r$.
Moreover we show that equality holds if and only if the general residual Y as above is degenerate or disconnected.

From a computational point of view, the complexity of the determination of X^{top} is essentially controlled by its regularity.

We also proved the following result, valid in any characteristic,

Theorem. [CU] *Assume* $\dim X \geq 1$. *If X is locally a complete intersection and have rational singularities (char. 0) or F-rational singularities (positive char.), then*

$$\mathrm{reg}(A/I) \leq \frac{(\dim X + 2)!}{2}(d_1 + \cdots + d_r - r).$$

Using an extension of the notion of regularity, Philippon and myself proved the following exotic result,

Theorem. [CP] *Let Z be the zero dimensionnal component of X, then*

$$\mathrm{reg}(Z) \leq d_1 + \cdots + d_r - r.$$

It was remarked by several authors (Miyazaki, Nagel, Schenzel and Vogel for instance) that knowledge on cohomological annihilators (k-Buchsbaumness in their work) gives bounds on the Castelnuovo-Mumford regularity. Following this idea and using a result of Hochster and Huneke that implies that elements of the Jacobian ideal provides strong cohomological annihilators, we derived several bounds, one of which is given below.

One easy remark that shows why annihilators are useful for bounding the regularity is the following:

Remark. Let M be a finitely generated graded A-module and x an homogeneous element. Assume that $[xH^i_{\mathfrak{m}}(M)]_\mu = 0$ and that the Krull dimension of $0 :_M x$ is at most i, then

$$H^i_{\mathfrak{m}}(M)_\mu \subseteq H^i_{\mathfrak{m}}(M/xM)_\mu.$$

Theorem. [Ch1] *Let J be an intersection of primary components of I, of the same codimension t, and set $S := \mathrm{Proj}(R/J) \subset X$. Assume that*
(1) *X have at most isolated singularities locally on S,*
(2) *S do not meet the closure of $X - S$.*
Then,

$$\mathrm{reg}(S) \leq (\dim S + 1)(d_1 + \cdots + d_t - t - 1) + 1.$$

Remark. Condition (2) is necessary. An example where the result fails with (1) only is $I = (x^m t - y^m z, z^{m+1} - xt^m)$ and J the ideal of the monomial curve contained in the complete intersection defined by I, namely $(u^{m(m+1)} : u^{m(m+1)-1}v : u^m v^{m^2} : v^{m(m+1)})$. Then $\mathrm{reg}(J) = m^2$, which is bigger then $2(2m-1)+2$ if $m > 4$.

The theory of multiplier ideals provides interesting results on regularity, the following one that is derived from results in the notes of Lazarsfeld is an example:

Theorem. [L, CD] *If k is of characteristic zero, there exists an ideal I' of same codimension r so that*
(1) *I' is contained in the unmixed part of I,*
(2) *$I^r \subseteq I'$,*
(3) *for all primes $\mathfrak{p}$ containing the unmixed part of I such that $(A/I)_{\mathfrak{p}}$ is regular, $I_{\mathfrak{p}} = I'_{\mathfrak{p}}$,*
(4) $\mathrm{reg}(I') \leq r(b_0(I) - 1) + 1$.

The same theory shows that there exists an ideal I'' such that $\sqrt{I''} = \sqrt{I}$ and $\mathrm{reg}(I'') \leq n(b_0(I) - 1) + 1$.

This naturaly leads to the following question: Is it possible that embedded primes improves the regularity?

Together with Clare D'Cruz, we discovered simple examples where not only the regularity decreases while embedding a prime, but also the depth is increasing!

Example. [CD] Let $S \subseteq \mathbf{P}^5$ be the monomial surface parametrized by

$$(a^{13} : ab^{12} : a^5c^8 : a^5bc^7 : a^7b^5c : b^9c^4)$$

let I be the corresponding defining ideal (which is prime), and J the subideal generated by the polynomials of degrees at most 21 belonging to I. Then, with the help of the computer algebra software Macaulay 2, in characteristic zero (and some finite ones), one finds that:

(1) $\mathrm{reg}(I) = 32$,

(2) $\mathrm{reg}(J) = 24$, and $J = I \cap K$, where K is (X_1, X_2, X_3, X_5)-primary,

(3) $\mathrm{depth}(R/I) = 1$ and $\mathrm{depth}(R/J) = 2$.

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