Torsion points of abelian varieties and F-isocrystals

(joint w/ Ambrosi)

Abelian varieties

An elliptic curve over $\mathbb{C}$ is a projective curve $E \subseteq \mathbb{P}^2_{\mathbb{C}}$ defined by an equation of the form $y^2 z = x^3 + axz^2 + bz^3$ with $4a^3 + 27b^2 \neq 0$.

As a complex manifold E is a genus 1 Riemann surface.

$$E \simeq_{hol} \mathbb{C}/\Lambda \simeq_{top} (S^1)^2$$

$$\underset{\text{lattice}}{\Lambda \subseteq \mathbb{C}}$$

E admits an algebraic group law

$$m : E \times E \to E$$

compatible with the previous iso's

In particular, $E[n] \simeq (\mathbb{Z}/n)^{\oplus 2}$.

Def An abelian variety over a field K is a smooth irreducible projective alg. variety A over K endowed with an alg. commutative group law

$$m: A \times A \longrightarrow A .$$

Th (Mordell-Weil)
Let $K/\mathbb{Q}$ be a finite field extension. For every abelian variety A/K, the group $A(K)$ of K-solutions is finitely generated.

Question: What does it happen in positive characteristic?

- Over a finite field only finitely many solutions.
- Over $\overline{\mathbb{F}_p}$ the group of sol's is a torsion group.

$\ell \neq p$ prime number $\quad A(\overline{\mathbb{F}}_p)[\ell^n] \simeq (\mathbb{Z}/\ell^n)^{2g}$

$g := \dim A$

This is enough to see that $A(\overline{\mathbb{F}}_p)$ is <u>not</u> finitely generated.

What about the p-torsion?

$$A(\overline{\mathbb{F}}_p)[p^n] \simeq (\mathbb{Z}/p^n)^{a}$$

$0 \leq a \leq g$
p-rank

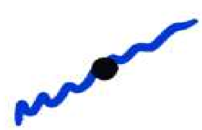
Something is missing!

One can construct a finite commutative group scheme $A[p^n]/\overline{\mathbb{F}}_p$ which is the kernel of $A \xrightarrow{\cdot p^n} A$. This scheme might be a point without being trivial!

<u>Example</u> (group scheme with only a point$/\overline{\mathbb{F}}_p$)

$$\mu_{p^n} := \ker\left(\mathbb{G}_m \longrightarrow \mathbb{G}_m\right), \quad x \longmapsto x^{p^n}$$

$$\mu_{p^n} = \operatorname{Spec}\left(\overline{\mathbb{F}}_p[x]/_{x^{p^n}-1}\right)$$



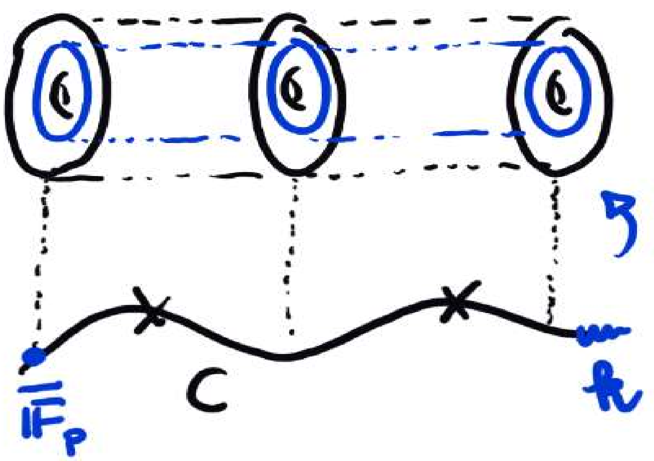
$$\mu_{p^n}(\overline{\mathbb{F}}_p) = 1 \quad , \quad \mu_{p^n}\left(\overline{\mathbb{F}}_p[\varepsilon]/_{(\varepsilon^2)}\right) = 1 + \overline{\mathbb{F}}_p\,\varepsilon$$

Let $k/\overline{\mathbb{F}}_p(t)$ be a finite field ext'n and A/k an abelian variety.
k is the field of rational functions of an algebraic curve $C/\overline{\mathbb{F}}_p$.

Two cases:

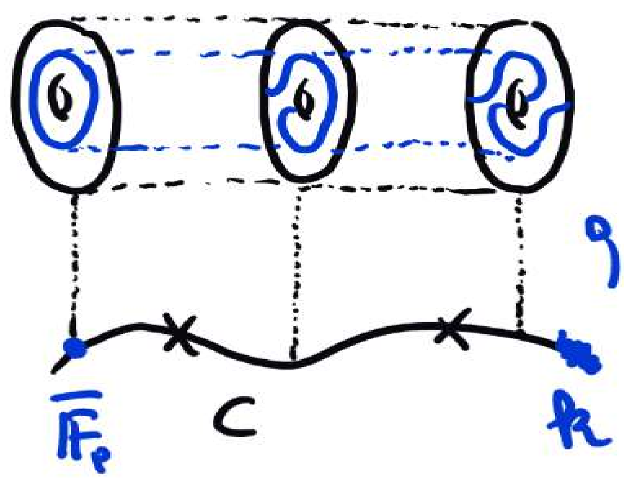
A can be defined over $\overline{\mathbb{F}}_p$

Trivial fibration



A cannot be def'd over $\overline{\mathbb{F}}_p$

Non-trivial fibration



$$A(k) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Sections of} \\ \text{the fibration} \end{array} \right\}$$

Th (Lang-Néron) For every simple abelian variety A/k which is not isogenous to an abelian variety over $\overline{\mathbb{F}_p}$, the group $A(k)$ is finitely generated.

Frobenius:

Frobenius twist of A

$$A \xrightarrow{F} A^{(p)}$$

$$[x_0 : \dots : x_n] \longmapsto [x_0^p : \dots : x_n^p]$$

Question (Esnault, '11) Consider

$$A(k)_{tors} \xhookrightarrow{F} A^{(p)}(k)_{tors} \xhookrightarrow{F} A^{(p^2)}(k)_{tors} \hookrightarrow \dots$$

By Lang-Néron this is a chain of finite groups.

Is this chain eventually stationary?

$k^{perf} := \bigcup_m k^{1/p^m}$ (Note that $A^{(p)}(k) = A(k^{1/p})$)

<u>ThA</u> (Ambrosi, D'A) Under the assumptions of the previous theorem, $A(k^{perf})_{tors}$ is a finite group.

<u>First observations</u>:

- If $\ell \neq p$, $A[\ell^n]$ is an étale group scheme, thus $A(k^{perf})[\ell^n] = A(k)[\ell^n]$ is controlled by Lang-Néron
- The group scheme $A[p^n]$ is not étale.

We have:

$$\begin{tikzcd} 0 \arrow[r] & A[p^n]^{0} \arrow[r] & A[p^n] \arrow[r] & A[p^n]^{ét} \arrow[r] & 0 \\ & & A[p^n]_{red} \arrow[u, hook] & & \end{tikzcd}$$

which does not split over k but it splits over k^{perf}.

$$\rightsquigarrow A[p^\infty](k^{perf}) = A[p^\infty]^{ét}(k^{perf}) =$$

$$= A[p^\infty]^{ét}(k) \xleftrightarrow{\text{index?}} A[p^\infty](k)$$

finite by Lang-Néron

Thm B The natural morphism

$$\mathrm{Hom}_{p\text{-div}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]) \longrightarrow \mathrm{Hom}_{p\text{-div}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]^{\text{ét}})$$

is an isomorphism.

$\mathbb{Q}_p/\mathbb{Z}_p := (\mathbb{Z}/p^n)_{n \in \mathbb{N}}$ $\quad A[p^\infty] := (A[p^n])_{n\in\mathbb{N}}$

Thm B $\Rightarrow$ Thm A

Suppose by contradiction $|A[p^\infty]^{\text{ét}}(k)| = \infty$.

We pile-up the points forming

$$0 \neq (P_i)_{i\in\mathbb{N}} \in \varprojlim_i A[p^i]^{\text{ét}}(k)$$

We define $\varphi : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]^{\text{ét}}$.

$$1/p^i \longmapsto P_i$$

By Theorem A φ lifts to

$$\tilde{\varphi} : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty].$$

This contradicts Lang-Néron

Intermediate step: spreading out.

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\substack{\text{abelian}\\ \text{scheme}}"', "\substack{\text{smooth}\\ \text{proper}}"] & & A \arrow[ll, hook'] \arrow[d] \\ X & & \operatorname{Spec} k \arrow[ll, hook'] \end{tikzcd}$$

curve

<u>Thm</u> (Berthelot-Breen-Messing)

There exists a contravariant functor

$$\mathbb{D}_{\mathbb{Q}} : \{\text{p-div gps}/X\}_{\mathbb{Q}} \longrightarrow F\text{-Isoc}(X)$$

$$\shortparallel$$

$$\{F\text{-isocrystals}/X\}$$

"p-adic local systems"

which is exact and fully faithful.

What is, for example, $F\text{-}\mathrm{Isoc}(\operatorname{Spec}\overline{\mathbb{F}}_p)$?

$W := W(\overline{\mathbb{F}}_p) \twoheadrightarrow \overline{\mathbb{F}}_p$ $\qquad$ $W \xrightarrow{\sigma} W$

Witt vectors $\qquad$ Frobenius lift

$K := \operatorname{Frac}(W)$

<u>Def</u> An F-isocrystal $/\overline{\mathbb{F}}_p$ is a finite dimensional K vectorspace V endowed with a σ-linear automorphism

$$\Phi : V \to V .$$

Coming back to Thm B.

$$\mathcal{H} := \mathbb{D}_{\mathbb{Q}}(\mathcal{A}[p^\infty]) \qquad \mathcal{H}^{\text{ét}} := \mathbb{D}_{\mathbb{Q}}(\mathcal{A}[p^\infty]^{\text{ét}})$$

$$\mathcal{H}^{\circ} := \mathbb{D}_{\mathbb{Q}}(\mathcal{A}[p^\infty]^{\circ})$$

$$\rightsquigarrow \qquad \begin{tikzcd} 0 \arrow[r] & \mathcal{H}^{\text{ét}} \arrow[r] \arrow[dr] & \mathcal{H} \arrow[r] \arrow[d, dotted, "?"] & \mathcal{H}^{\circ} \arrow[r] & 0 \\ & & \mathbb{1} & & \end{tikzcd}$$

Faithful translation of the original problem

$$\mathrm{F\text{-}Isoc}^{\dagger}(X) \subseteq \mathrm{F\text{-}Isoc}(X)$$

$$\begin{array}{c} \| \\ \left\{ \begin{array}{c} \text{overconvergent} \\ \text{F-isocrystals} \end{array} \right\} \\ \Psi \\ \mathcal{H} \end{array}$$

__Thm__ (Kedlaya) $\langle \mathcal{H} \rangle^{\otimes}_{\mathrm{F\text{-}Isoc}^{\dagger}(X)}$ is a semi-simple category. (Related to Riemann hypothesis for $\mathbb{F}_p$)

⚠ $\mathcal{H}^{\acute{e}t}$ and $\mathcal{H}^{\circ}$ are not in $\mathrm{F\text{-}Isoc}^{\dagger}(X)$.

To summarize:

$$\langle \mathcal{H} \rangle^{\otimes}_{\mathrm{F\text{-}Isoc}^{\dagger}(X)} \subseteq \langle \mathcal{H} \rangle^{\otimes}_{\mathrm{F\text{-}Isoc}(X)} \ni \mathcal{H}^{\acute{e}t}, \mathcal{H}^{\circ}$$

$$\begin{array}{ccc} \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathrm{Rep}_K(G^{\dagger}) & \subseteq & \mathrm{Rep}_K(G) \end{array}$$

$$\underset{\text{Reductive}}{G^{\dagger}} \supseteq \underset{???}{G} \quad \text{alg. gps.}/K$$

<u>Thm</u> C G contains a maximal torus of G^+.

THANK YOU
FOR THE
ATTENTION!