WORKSHOP ON RATIONAL POINTS AND BRAUER-MANIN OBSTRUCTION

OBSTRUCTION GUYS

ABSTRACT. Here we present an extended version of the notes taken by the seminars organized during the winter semester of 2015. The main goal is to provide a quick introduction to the theory of Brauer-Manin Obstructions (following the book of Skorobogatov and some more recent works). We thank Professor Harari for his support and his active participation in this exciting workshop.

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1. TALK 1: A FIRST GLANCE, PROFESSOR HARARI

Notes taken by Gregorio Baldi

The aim of the lecture is to define Hasse principle, weak approximation and give a few easy examples. Then we recall some basic stuff on the Brauer group, finally the definition on Brauer-Manin obstruction, with more examples, results and conjectures.

The prerequisites should be:

a) some basic algebraic geometry (definition of a scheme, first properties of morphisms [Har77])
b) basics in étale cohomology ([Tam06])
c) some global class field theory, say the main results in [SN86]

1.1. **Galois and Étale Cohomology.**

1.1.1. *Group Cohomology.* Let G be a finite group. Let A be a G-module (an abelian group with an action of G, given by an automorphism of G). Then one can define, as usual, the cohomology groups

$$H^i\left(G, A\right)\ , i \geq 0$$

1.1.2. *Modified/Tate Group.* One can also have

$$\hat{H}^0\left(G, A\right) = A^G / N_G A, \quad N_G(x) = \sum_{g \in G} gx$$

And can also define

$$\hat{H}^i\left(G, A\right), i \leq 0$$

using homology instead of cohomology.

1.1.3. *Restriction-inflation.* If H is a normal subgroup there is a spectral sequence (Hochschild-Serre)

$$H^p\left(G/H, H^q(H, A)\right) \Rightarrow H^{p+q}(G, A)$$

From the five degree exact sequence we have

$$0 \to H^1(G/H, A^H) \xrightarrow{inf} H^1(G, A) \xrightarrow{res} H^1(H, A)$$

Under the assumption

$$H^i(H, A) = 0 \text{ for } 1 \leq i \leq q-1$$

one has also the exactness of

$$0 \to H^q(G/H, A^H) \xrightarrow{inf} H^q(G, A) \xrightarrow{res} H^q(H, A)$$

Reference 1.1.1. [Har12]

We list here some useful results.

Proposition 1.1.2. Let G a finite group, for any G-module A the groups $H^i(G, A)$ are of $card(G)$-torsion. Moreover, if A is an abelian group of finite type, then $H^i(G, A)$ are finite.

Corollary 1.1.3. *Let G be a finite group and A a G-module which is uniquely divisible, then $H^i(G, A)$ are zero.*

Example 1.1.4. Those results, together with the long exact sequence induced by

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

give us $H^2(G, \mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z})$ and $H^1(G, \mathbb{Z}) = 0$.

1.1.4. *Profinite Groups.* Let k be a field, L a finite Galois extension and G its Galois Group acting on L^*. We have $H^0(G, L^*) = (L^*)^G = k^*$.

Recall Hilbert's 90: $H^1(G, L^*) = 0$. But in general $H^i(G, L^*)$ for $i \geq 2$ is not zero (!!Brauer Group).

A group G is called *profinite* if $G = \varprojlim G_i$ (which is contained in the product of the G_i's) i.e. the projective limit of finite group. One can give also a purely topological description: G is profinte iff it is a compact, in the french sense, completely disconnected topological group.

Example 1.1.5. Let K be a separable extension of k. The main example is $G = Gal(K, k) := \varprojlim Gal(L, k)$ where the limit runs over finite extension of k contained in K.

Let A be a discrete G module (i.e. there is an action of G on A and every stabilizer is open). If $G = \varprojlim G_i$ this means that for every $x \in A$ there exists L/k finite Galois such that the action of G on x factorizes thorough $Gal(L, k)$.

Example 1.1.6. Let n be an integer coprime with char k. Actions of $G = Gal(K, k)$ on $\overline{k}^*$, on $\mu_n = \{x \in \overline{k}^*, x^n = 1\}$ or the trivial action on $\mathbb{Z}/n$.

If a profinite group G acts on A, define

$$i \geq 0 \;\; H^i(G, A) = \varinjlim_{U \triangleleft G, open} H^i(G/U, A^U)$$

Notice that G profinite implies G/U finite (by compactness). It is possible to give a description in terms cocycles: you have to take continuous cocycle w.r.t. profinite topology.

1.1.5. *Galois Case.* Consider the absolute Galois group of k, which will be denoted G (or G_k, or even Γ_k in what follows), acting on A, then

$$i \geq 0 \;\; H^i(G, A) = \varinjlim_{k \subseteq L} H^i(Gal(L, k), A^{Gal(L,k)})$$

Example 1.1.7. $H^0(G, \overline{k}^*) = k^*$, $H^1(G, \overline{k}^*) = 0$, $H^2(G, \overline{k}^*) = \mathrm{Br}\,(k)$.

Example 1.1.8. Some examples of Brauer groups: $\mathrm{Br}\,(\mathbb{R}) = \mathbb{Z}/2$, $\mathrm{Br}\,(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$. Here one needs some local class field theory.

1.1.6. *Étale Cohomology.* It is an extension of Galois cohomology in some sense which will be explained in the next talk.

Setting: let X be a scheme, $\mathcal{F}$ an étale sheaf of abelian groups over X_{et}. Then you can define étale cohomology groups

$$H^i(X, \mathcal{F})$$

Special case: $\mathcal{F}$ represented by a smooth commutative group scheme over X, say G; then we denote it with $H^i(X, G_X) = H^i(X, G \times_k X)$. One of the main example is given by the linear groups, i.e. the subgroups of GL_n. The most important case will be the one of groups scheme coming from k.

Reference 1.1.9. [Mil13], [Tam06]

1.1.7. *Non abelian Cohomology.* If $G_k = Gal(\overline{k}, k)$ act on a group A, not necessary commutative we have

$$H^0(G_k, A) = A^{G_k}$$

and we can define $H^1(G_k, A)$ using cocyles. It will be not a group but just a pointed set. For any exact sequence of G_k modules

$$1 \to A \to B \to C \to 1$$

induces a "long but not so long" exact sequence of pointed set:

$$1 \to A^{G_k} \to B^{G_k} \to C^{G_k} \to H^1(G_k, A) \to H^1(G_k, B) \to H^1(G_k, C)$$

Achtung. *Notice that having trivial kernel does not imply the injectivity of the map. (To do this one has to twist!)*

If A is contained in the center of B then it is possible to continue the exact sequence with an arrow to $H^2(G_k, A)$. But, even if C is abelian then the map $H^1(G_k, C) \to H^2(G_k, A)$ is not a morphism of group.

Reference 1.1.10. [Ser73]

In the next talk will be given the same construction with the étale cohomology, in particular we will interpret the pointed set $H^1(X, G)$ for G a smooth group scheme. (When G is not smooth one use the fppf site). The idea is the following: Let X be a scheme, G_X a smooth group scheme over X (G defined over k), then $H^1(X, G)$ classifies the G_X-torsor over G. Intuitively a torsor is $Y \xrightarrow{f} X$ where

- f is a faithfully flat.
- and the action of G_X on Y is compatible with f.
- *Naive definition* in the fibers the action is simply transitive, i.e. if $x \in X(\overline{k})$, exists a unique $g \in G(\overline{k})$ $gy_1 = y_2$ for any $y_1, y_2 \in X_x(\overline{k})$. Notice that we do not take $x \in X(k)$, it could be empty.

Reference 1.1.11. *Skorobogatov* [Sko01], *first chapter.*

1.2. **Picard Group, Brauer Group.** Let X be a noetherian regular integral scheme (e.g. a smooth variety over a field). Under this assumptions we have

$$\mathrm{Pic}(X) = \text{group of Weil's divisors = Cartier divisors} / \mathrm{Sym}$$

And we always have the "Hilbert's 90"

$$\mathrm{Pic}(X) = H^1(X_{et}, \mathbb{G}_m) = H^1(X_{zar}, \mathcal{O}_X^*)$$

If X is a proper and smooth variety over a filed of characteristic 0 one can define a subgroup of $\mathrm{Pic}(X)$: $\mathrm{Pic}^0(X)$=divisors algebraically equivalent to 0.

Example 1.2.1. If X is a smooth and proper curve then $\mathrm{Pic}^0(X) = \{D \in \mathrm{Pic}(X), \deg(D) = 0\}$. Where

$$\deg\left(\sum_{x,\ closed} m_x x\right) = \sum_x m_x [k(x) : k]$$

Proposition 1.2.2. Let X be a proper, smooth scheme over k. There exists an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

a) $\mathrm{NS}(X)$ is finitely generated (and isomorphic to $\mathbb{Z}$ if X is a curve, here the last map is just the degree)
b) If L is a field extension of k then $\mathrm{Pic}^0(X_L) = A(L)$. ($L$ big enough, to have a rational point) Where A is the Picard variety associated to X: it is an abelian variety of dimension equal to the dimension $dim(H^1(X, \mathcal{O}_X))$

Achtung. *If $char(k) \neq 0$ this is a bit more complicated.*

Reference 1.2.3. *More on abelian varieties will be discussed in one of the following talks.*

Theorem 1.2.4. *Let X be an abelian variety, then $\mathrm{NS}(X)$ is torsion free.*

This is not true in general, e.g. for Enriques surfaces.

Definition 1.2.5. Let X be a regular and integral scheme. Its Brauer Group is $\mathrm{Br}\,(X) = H^2(X, \mathbb{G}_m)$.

Remark 1.2.6. This is the cohomological Brauer group. Another definition with "Azumaya algebras" can be given, it is smaller. A result of Gabber says that if X is noetherian, quasi projective variety, then both definition are the same. But we will work just with the cohomological one. For more details see [dJ11].

Theorem 1.2.7. *Let X be integral and regular,* $\mathrm{Br}\,(X) \subset \mathrm{Br}\,(k(X))$. *In particular it is a torsion group (Galois cohomology).*

Example 1.2.8. Let k be a field of char 0. It is not hard to prove that $\mathrm{Br}\,(\mathbb{P}_k^n) \cong \mathrm{Br}\,k$. Moreover one can also prove that $\mathrm{Br}\,(\mathbb{A}_k^n)$ is isomorphic to $\mathrm{Br}\,(k)$. For $\mathbb{P}_k^n$ see Proposition 6.9.9. of [Poo11], and Theorem 7.5 of [AG60] for $\mathbb{A}_k^n$. It is a nice exercise to deduce the result for $\mathbb{P}^n$ from the affine case.

We list some properties.

Reference 1.2.9. [Gro68]

- Let X, Y are projective and smooth varieties over a field k (of zero char). If $X \cong Y$ then $\mathrm{Br}\,(X) = \mathrm{Br}\,(Y)$.
- Exact sequence for X smooth k-variety with $n \neq 0$ mod $char(k)$:
$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \to 0$$
gives us
$$0 \to \mathrm{Pic}(X)/n \to H^2(X, \mu_n) \to_n \mathrm{Br}\,(X) \to 0$$
- H.S. Spectral sequence:
$$H^p(G_{\overline{k}}, H^q(X_{\overline{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$$

Remark 1.2.10. "Azumaya Brauer Group" corresponds to $H^1(X, PGL_n)$ moving n.

1.3. **Obstructions.** Notation: Let k be a number field. We let Ω_k denote the set of its places. The completion of k at a place v is denoted k_v.

Definition 1.3.1 (Hasse Principle)**.** Let k be a number field, the Hasse principle fails for a k-variety X if $\prod_{v\in\Omega_k} X(k_v) \neq \emptyset$ and $X(k) = \emptyset$.

Recall that given a topological field (addition, product and inverse map are continuous) k (eg. a local field) and X a k variety we can give a topology over $X(k)$, called the analytic topology in this way. Given $\mathbb{A}^n(k) = k \times \cdots \times k$ the product topology, if X is a closed subvariety of $\mathbb{A}^n$, the we can give $X(k) \subset A^n(k)$ the subspace topology. Notice that a map of k-varieties $X \to Y$ induces a continuous map $X(k) \to Y(k)$.

Definition 1.3.2 (Weak approximation)**.** Let k be a number field. Weak approximation holds for a k-variety X if the image of the diagonal map
$$X(k) \to \prod_{v\in\Omega_k} X(k_v)$$
is dense in the right hand side equipped with the product topology.

If the right hand side is not empty, this amounts to say that $X(k)$ is not empty and that for any finite set $S \subset \Omega_k$, the image
$$X(k) \to \prod_{v\in S} X(k_v)$$
is dense.

We end this section stating some theorems we will use often in the following talks.

Theorem 1.3.3. *Let k be a number field, Ω_k its set of places. There are embeddings*
$$i_v : \mathrm{Br}\,(k_v) \to \mathbb{Q}/\mathbb{Z}$$
For non-archimedean v, the map i_v is an isomorphism. For a real place v, the map inv_v *induces* $\mathrm{Br}\,(k_v) = \mathbb{Z}/2\mathbb{Z}$*. For a complex place v,* $\mathrm{Br}\,(k_v) = 0$*.*

The following is a deep theorem form Class Field Theory.

Theorem 1.3.4. *Let k be a number field, Ω_k its set of places. Then there is the following exact sequence*
$$0 \to \mathrm{Br}\,(k) \to \bigoplus_{v\in\Omega_k} \mathrm{Br}\,(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{1.1}$$

Remark 1.3.5. From the Injectiveness of the first map (and the interpretation of the two torsion of the Brauer groups in terms of quaternion algebras) it follows that the conics in two variables over a number field satisfy the Hasse principle.

Moreover we will use often also the following generalization.

Theorem 1.3.6. *Let k be a global or a local field, then $H^3(k, \overline{k}^*)$ is zero.*

All the proof can be found in [SN86].

1.4. **Miscellaneous.**

1.4.1. *Weil Restrictions.* Let k a field of characteristic 0[1], and L/k a finite extension of degree d. We want to explain how to get, from a variety Y defined over L, a variety X defined over k. This procedure is called *Weil restriction* or restriction of scalars, and we write $X = R_{L/k}(Y)$. Moreover we want to do this in a functorial way s.t. group objects go to group objects. More formally we are interested in the (representative of the) adjoint functor of the *extension of scalars*:

$$\mathrm{Hom}_L(V_L, W) \cong \mathrm{Hom}_k(V, R_{L/k}(W))$$

This means, in the affine case, that $R_{L/k}(Y)$ is uniquely described (if it exists) by the following property:

$$R_{L/k}(Y)(A) = Y(A_L)$$

for any $A \in k$-algebra.

We just give the first step of the construction:

$$R_{L/k}(\mathbb{A}_L^N)) = \mathbb{A}_k^{Nd}$$

and this is done choosing a basis of L/k and a "change of variables".

We are ready to produce many examples in a natural way. The proof is an easy exercise.

Proposition 1.4.1. The following hold

- Let X_1, X_2 biregularly isomorphic varieties, X_1 has strong approximation/WA if and only X_2 does.
- Let $X = X_1 \times X_2$, the existence of strong approximation/WA for X is equivalent to the existence of the same type of approximation n both factors.
- Let $X = R_{L/k}(Y)$, the existence of strong approximation/WA for X over k w.r.t $S \subset \Omega_k$ is equivalent to the existence of the same type of approximation in Y over L w.r.t $S' \subset \Omega_L$ given by extending the valuations of S.

1.4.2. *Induced Module are acyclic.* Recall the definition of *Induced module*. Let G a finite group, $H \leq G$. From an H-module A there is a natural way to obtain a G-module: we define $I_G^H(A) := \{\varphi : G \to A,\ \text{H-invariant}\}$ with the action of G given by $(g.\varphi)(g') := \varphi(g'.g)$. $I_G^H(A)$ are called induced module.

For $H \leq G$, $I_G(A)$ is H-induced: we can write $G/H = \{[g_i]\}$ then $I_H\left(\prod_{[g_i]} A_{g_i}\right)$ and $I_G(A) = \prod_{g\in G} A_g = \prod_{h\in H}\prod_{[g]\in G/H} A_{[g]}$

Lemma 1.4.2. *$I_G(A)$ are acyclic.*

Proof. Consider the following:

$$H^i(G, I_G^H(A)) \xrightarrow{res} H^i(H, I_G^H(A)) \xrightarrow{inf} H^i(H, A)$$

We claim that this composition gives an isomorphism. The composition at degree 0 is an isomorphism and both are derived functor, hence universal, hence they coincide. Where we the composition rule of (total) derived functors, $R^n(F \circ G) = R^n(F) \circ G$, holds since $(-)^G$ and $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], -)$ send injectives to injectives thanks to the existence of an exact left adjoint. $\square$

Notice that in the profinite case the result is still true by taking the limit.

[1] In the purely inseparable case one needs to use some "devissage".

2. Talk 2: Torsors, Emiliano Ambrosi

This talk is divided in two parts. In the first we will define the notion of *torsor* and discuss some properties. In the second we will see how this notion is related to the study of rational points.

2.1. **Torsors.** We start with an example.

Example 2.1.1. Fix $k=\mathbb{Q}$, and $G=\mathrm{Gal}(\overline{\mathbb{Q}},\mathbb{Q})$, we want to study how far is k from having the roots of its elements. This is parametrized by $H^1(k,\mu_n)$, since from the exact sequence $0\to\mu_n\to\mathbb{G}_m\to\mathbb{G}_m\to 0$ and the fact that $H^1(k^*,\mathbb{G}_m)=0$, we obtain that $H^1(k,\mu_n)=\frac{k^*}{k^{*n}}$. Torsors arise naturally as a geometric interpretation of this cohomology group. In fact, for every $a\in\frac{k^*}{k^{*n}}$ we cam construct the k-algebra $B_a:=\frac{k[x]}{(x^n-a)}$ and this algebra depends, up to iso, only on the class of a in the cohomology group. Moreover this algebra has a natural action of $\mu_n=\frac{k[x]}{x^n-1}$, given by the map $B_a\to\mu_n\otimes B_a$ that send x to $x\otimes x$. Observe that this algebra is trivial (isomorphic to μ_n with the action of multiplication over itself) if and only if a is the neutral element of $H^1(k,\mu_n)$. However the algebra is locally trivial in the ètale topology, since it is trivialized by the étale-covering $Spec(k(\zeta_n)\to k$. So the failure of k from having the roots of its elements is parametrized by some geometric objects with an action of a group, that are locally trivial in the ètale topology, but not trivial.

Definition 2.1.2 (Torsors)**.** If X is a scheme, $G\to X$ an fppf group scheme (not necessary commutative) over X, we define a X (right) torsor under G as a map $Y\to X$, with a (right) G action, such that there exists a covering $(U_i\to X)$ in the fppf topology that trivialized Y, in the sense that $Y\times U_i\simeq G\times U_i$ compatible with the action of G.
A sheaf of torsors is a sheaf of G sets, locally trivial in the fppf topology. A sheaf of torsors is a torsor if only if it is representable.

Theorem 2.1.3. *If X is a variety over a field and G is an group is affine, flat and locally of finite presentation over X then every right sheaf of torsor under G is representable.*

Observe that $Y\to X$ is a covering of X, since it is fppf. Moreover $Y\to X$ is a trivializing covering for $Y\to X$ since the natural map $Y\times G\to Y\times Y$, that send (y,g) to $(y,y\cdot g)$, is an iso, since this property can be checked locally in the fppf topology (i.e. after an fppf base change). We note also that an X-scheme with an action of G is isomorphic to G with the action of multiplication if and only if the action of G is free and transitive.

Example 2.1.4. $\frac{k[x]}{x^n-a}$ are μ_n torsors over $\mathbb{Q}$.

Example 2.1.5. Take $X=\mathbb{A}^1_k-\{0\}$ and $G=\mathbb{G}_{m,k}$. Then $(\frac{k[x,y]}{x-y^2})_x$ is a X torsor under $\mathbb{G}_m$.

Example 2.1.6. If $k\subseteq L$ is Galois, then L is a k torsor under the constant group $\mathrm{Gal}(L,k)$ since $L\otimes L\simeq L^n$.

Example 2.1.7. If $X\subseteq\mathbb{P}^n$ is a hypersurface, then $Cone(X)\subseteq\mathbb{A}^{n+1}-\{0\}$ is a $\mathbb{G}_m$-torsor.

As always we hope that something that is locally trivial can be parametrized by some cohomology group related to X and G. The problem is that G is not abelian in general, so that the usuals cohomology theories can not be used. Suppose that F is a sheaf of G torsors and choose a cover $(U_i\to X)$ that trivializes F. Given $y_i\in F(U_i)$, then, by definition, there exists a unique $g_{ij}\in G(U_{ij})$ such that $y_ig_{ij}=y_j$ in $F(U_{ij})$. They are such that $g_{ij}g_{jk}=g_{ik}$, since $y_ig_{ik}=y_k=y_jg_{kj}=y_ig_{ij}g_{jk}$. Moreover, if $y_i^{'}$ is another choice of y_i we know that there exist a $h_i\in G(U_i)$ such that $y_ih_i=y_j^{'}$, and hence $g_{ij}^{'}=h_i^{-1}g_{ij}h_j$ since $y_i^{'}g_ij^{'}=y_j^{'}=y_jh_j=y_ig_{ij}h_j=y_i^{'}h_ig_{ij}h_j$.

Definition 2.1.8 (Non abelian cohomology)**.** If X is a scheme, G is a sheaf of groups and $(U_i\to X)$ is a covering, we define $H^1((U_i\to X),G)$ as the pointed set of $\{g_{ij}\}\in G(U_{ij})$ such that $g_{ij}g_{jk}=g_{ik}$ modulo the relation $\{g_{ij}\}\simeq\{g_{ij}^{'}\}$ if there exist $h_i\in G(U_i)$ such that $g_{ij}^{'}=h_i^{-1}g_{ij}h_j$.
We define $H^1(X,G)$ as the direct limit on the possible cover of X of $H^1((U_i\to X),G)$.

Remark 2.1.9. If G is a sheaf of abelian group then $H^1((U_i \to X), G)$ is nothing else that the usual Čech cohomology.

Theorem 2.1.10. *There is a bijection between the set $H^1((U_i \to X), G)$ and the set of isomorphism classes of sheaves of torsors under G trivialized by $(U_i \to X)$. Passing to the direct limit we find a bijection of pointed set between $H^1(X, G)$ and the sheaves of torsors under G.*

Proof. We have constructed a map, so we have only construct an inverse map. Take a family of $\{g_{ij}\}$. We want to construct a sheaf F and so to every $V \to X$ we have to give a set $F(V)$. Consider the sheaves C^0 that send every V to $\prod G(U_i \times V)$ and $C^1(V) := \prod G(U_{ij} \times V)$. We have a map of sheaves $C^0 \to C^1$ that (h_i) to $(h_i^{-1}h_j)$. Then the canonical map $U_{ij} \times V \to U_{ij}$ gives us a map $f_{ij} : G(U_{ij}) \to G(U_{ij} \times V)$ and so we have a family $\{f_{ij}(g_{ij})\}$ of $C^1(V)$ for every V. Define now F as the subsheaf of C^0 made by the inverse image of this family. We have a natural well defined action of G on F made by $G(V) \times F(V) \to F(V)$, made by $(g.x) = g^{-1}x$ and so we have just to show that F is locally trivial. Of course the trivializing cover is $(U_i \to X)$. We want to construct an iso $G(U_i) \to F(U_i)$ and so we send a h to (hg_{ij}). This map is well defined by the cocycle condition since $g_{ij}^{-1}h^{-1}hg_{ik} = g_jk$ and, clearly, is an isomorphism. □

Now we can do some computations that will be useful in the future. Since $Y \to X$ is a trivial cover for itself, computing $H^1((Y \to X), G)$ will be pretty easy.

Lemma 2.1.11. *Suppose that G, G' are two algebraic group over k and $Y \to X$ is a torsor under G. Then $\check{H}^1((Y \to X), G')$ is the set of equivalence classes of morphisms $f : Y \times G \to G'$ such that $f(y,s)f(ys,s') = f(y,ss')$, where $f \simeq f'$ if and only if exists $g : Y \to G'$ such that $f'(y,s) = g(y)f(y,s)g(ys)^{-1}$. Moreover, if $G = G'$ the class of Y is given by the map $Y \times G \to G$.*

Proof. We know that $H^1(Y \to X, G')$ is the set of maps $f : Y \times Y \to G'$ such that $f\pi_{1,3} = f\pi_{1,2}f\pi_{2,3}$ where $\pi_{i,j}$ are the different projections $Y \times Y \times Y \to Y \times Y$. Recall that $Y \times Y \simeq Y \times G$, and $Y \times Y \times G \simeq Y \times Y \times Y$. A direct computation shows that under this isomorphism the projection $\pi_{1,3}$ becomes the map $(y, g, g') \mapsto (y, gg')$, $\pi_{1,2}$ becomes the map $(y, g, g') \mapsto (y, g)$, $\pi_{2,3}$ becomes the map $(y, g, g') \mapsto (yg, g')$, and so we are done.
f is equivalent to g if and only if exists a map $g : Y \to G'$ such that $g = \pi_1 f \pi_2^{-1}$ and we can conclude just noticing that the map π_1 becomes, under the iso, the map that send (x, y) to x and that the second projection becomes the map that send (x, g) to xg.
The last statement is clear by the previous reasoning. □

Suppose now that G is abelian and $G' = \mathbb{G}_m$. Then every character $\chi : G \to \mathbb{G}_m$ extend to a map $Y \times G \to \mathbb{G}_m$ that satisfies the cocycle condition and so it is an element of $H^1(Y \to X, G)$. Moreover χ induces a morphism between the cohomology groups that makes the following diagram commutative:

$$\begin{tikzcd}
H^1(Y \to X, G) \arrow[r] \arrow[d, "\chi^*"] & H^1(X, G) \arrow[d, "\chi^*"] \\
H^1(Y \to X, \mathbb{G}_m) \arrow[r] & H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)
\end{tikzcd}$$

So the class of $\chi_*(Y)$ in the Picard is equal to the class of the character χ in $H^1(Y \to X, \mathbb{G}_m)$. In particular we have a homomorphism of groups $\mathrm{Hom}(G, \mathbb{G}_m) \to \mathrm{Pic}(X)$.

Definition 2.1.12. The type of a Y-torsors is the morphism $\mathrm{Hom}(\overline{G}, \mathbb{G}_m) \to \mathrm{Pic}(\overline{X})$ constructed above. We say that a torsor is universal is this morphism is an isomorphism.

If G is abelian we have the Čech to derived spectral sequence. We quickly recall the general construction, which will be applied to $F = G$. We can factorize the global section functor

$$\Gamma : \mathrm{Sh}(X) \to Ab$$

by the composition of the forgetful $D : \mathrm{Sh}(X) \to \mathrm{PSh}(X)$ with the functor $\check{\mathrm{H}}^0 : \mathrm{PSh}(X) \to Ab$ that send F to $\mathrm{Ker}(\prod FU_i \to \prod FU_{ij})$. Now the i^{th} derived functor of D is just the functor that send F to the presheaf $\tilde{F}^i$ that send U to $H^i(U, F)$, the i^th derived functor of $\check{\mathrm{H}}$ is nothing else that the functor that send

F to the $\check{\mathrm{H}}^i((U_i \to X), F)$, where $H^i((U_i \to X), F)$ is the usual Čech cohomology. So the Grothendieck spectral sequence in this situation is $E_2^{p,q} = \check{\mathrm{H}}^p((U_i \to X), \tilde{F^q}) \Rightarrow H^{p+q}(X, F)$ and the exact sequence of low degree is

$$0 \to \check{\mathrm{H}}^1(X, H^0(X, F)) \to H^1(X, F) \to \check{\mathrm{H}}^0(X, H^1(X, F)) \to \check{\mathrm{H}}^2(X, H^0(X, F)) \to H^2(X, F)$$

If G is non abelian we can recover a similar exact sequence of pointed set

$$1 \to \check{\mathrm{H}}^1((X, H^0(X, F)) \to H^1(X, G) \to \check{\mathrm{H}}^0(X, \check{\mathrm{H}}^1(X, G))$$

by hand, and we leave the details to the reader.

2.2. **Torsors and cohomology.** To finish this short introduction to torsors, we'd like to present an application of this theory to cohomology. We will work with étale torsors (same definition, just change fppf with étale). First we recall an important fact about étale cohomology that will be useful in the sequel.

Proposition 2.2.1. There is an equivalence of categories that preserves cohomology, between the category of abelian sheaves over $Spec(k)$ and the category of $G = \mathrm{Gal}(\overline{k}, k)$ discrete moduli. In particular we have an isomorphism between $H^i(\mathrm{Spec}(k), G)$ and $H^i(k, \mathrm{colim}\ G(L))$.

Proof. We construct the two functors that realize the equivalence of categories.
An étale sheaf F is nothing else that a family $G(L)$ abelian groups for every separable extension of k such that $F(L) = F(L')^{\mathrm{Gal}(L',L)}$, for every $L \subseteq L'$ Galois. In fact the sheaf axiom on $L \subseteq L'$ is the exactness of the following sequence: $F(L) \to F(L') \to F(L' \otimes L')$. But $F(L' \otimes L') \simeq F(L'^n)$, F commutes with the direct sum, and so the claim is proved. So we can associate to F the G module $\mathrm{colim}\, F(L)$, where L runs over every separable extension of k. The natural action of G on $\mathrm{colim}\, F(L)$ it is clearly well defined and it is continuous since it factorizes, by the sheaf axioms, over a finite field extension.
For the other functors we take a discrete G-modules T and we associate to him the sheaf $F(L) = T^{\mathrm{Gal}(\overline{k},k)}$. It is a sheaf, by the previous discussion and the two functors are one the inverse of the other (observe that every discrete G, is the colimit of its fixed points).
For the statement about cohomology just observe that the composition of the functor from the ètale topos and the functor global sections is exactly the fixed point functors. □

Example 2.2.2. We have $H^1(G, \overline{k}^*) = H^1_{et}(Spec(k), \mathbb{G}_m) = H^1_{zar}(Spec(k), \mathbb{G}_m) = \mathrm{Pic}(k) = 0$.

Example 2.2.3. We have $H^1_{et}(Spec(k), \mathbb{Z}) = H^1(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z}) = 0$ since G is profinite.

In the same way it is possible to prove something more general.

Theorem 2.2.4. *Suppose that G is a topological group. Then the category $G - set$ of sets with a continuous action of G is equivalent to the category of sheaf over $\mathcal{C}$, where $\mathcal{C}$ is the subcategory of $G - set$ made by element in the form (G/U) where U is a open subgroup of G and all the maps are covering.*

Proof. [MM92] □

Suppose that X is a connected scheme. Then, applying this theorem with $G = \pi_1(X)$, the étale fundamental group, we get that the category of $\pi_1(X) - sets$ is equivalent to the category of sheaves over $\mathcal{C}$. But, by the Galois-theory of Grothendieck and using that G is profinite, $\mathcal{C}$ is equivalent to the category of finite étale covers of X. "Abelianizing" this, we get that the category of discrete G modules is equivalent to the category of étale abelian sheaf over the category of finite ètale covers of X. Unluckily, this does not mean that the étale cohomology is a group cohomology: the category of <u>finite</u> étale covers of X is too small to compute the ètale cohomology.

Example 2.2.5. If k is algebraically closed field, then the finite étale site of $\mathbb{P}^1_k$ is trivial and hence the higher cohomology, computed wrt the finite site, of any sheaves is 0!

Example 2.2.6. $H^1(G, \mathbb{Z}) = 0$, as explained above, but $H^1_{et}(X, \mathbb{Z})$ is in general different from zero. Consider, for example, X the nodal cubic in $\mathbb{A}^2_k$, we have an exact sequence of sheaves $0 \to \mathbb{Z} \to j_*\mathbb{Z} \to i_*\mathbb{Z} \to 0$ where i the inclusion of the singular point in X and j the normalization map. Taking global section we get $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to H^1(X, \mathbb{Z})$. Since the first map is an isomorphism, the map $\mathbb{Z} \to H^1(X, \mathbb{Z})$ is injective and hence $H^1(X, \mathbb{Z}) \neq 0$.

Suppose now that H is a finite ètale abelian group over X. Then $H^1(X, H)$ is just the abelian group of sheaf of torsors under H. Each of this is finite $\pi_1(X)$ module with a free and transitive action, i.e a H torsor (principal homogeneous space) in the usual group theoretic meaning. But they are parameterized by $H^1(\pi_1(X), H)$ and hence we get the following:

Theorem 2.2.7. *Suppose that X is a connected scheme and H a finite ètale abelian group. Then $H^1(X, H) = H^1(\pi_1(X), H)$.*

Example 2.2.8. If k is algebraically closed field, then the finite ètale site of $\mathbb{P}^1_k$ is trivial, so that $\pi_1(\mathbb{P}^1_k) = 0$ and hence $H^1(\mathbb{P}^1_k, H) = 0$ for every finite abelian group.

For the basic results abut the étale fundamental group cited above we to [Len08]

2.3. **Torsors and rational points.** We introduce a first obstruction to rational points. In the following $X \xrightarrow{f} Spec(k)$ is a variety over a perfect field k and G is commutative k-group scheme. Then we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, "\Gamma(X,-)"] \arrow[d, "f_*"] & Ab \\ \mathrm{Sh}(k) \arrow[ur, "\Gamma(k,-)"'] & \end{tikzcd}$$

So we get, from the Grothendieck spectral sequence, the Leray spectral sequence:

$$E_2^{p,q} = H^p(k, R^q f_* G) \Rightarrow H^p(X, G)$$

and hence the exact sequence of low degrees is

$$0 \to H^1(k, f_*G) \to H^1(X, G) \to H^0(k, R^1 f_* G) \to H^2(k, f_*G) \to H^2(X, G)$$

where $R^q f_* G$ is the sheaf associated to the presheaf that sends U to $H^1(X \times U, G)$. Now $H^0(k, R^1 f_* G) = H^0(\Gamma_k, \operatorname{colim} H^1(X_L, f_*G)) = H^0(\Gamma_k, H^1(\overline{X}, \overline{G})) = H^1(\overline{X}, \overline{G})^{\Gamma_k}$ (where the colimit is taken over the finite extension of k). An element in $H^1(\overline{X}, \overline{G})^{\Gamma_k}$ is nothing else that a Γ_k invariant torsor, i.e. a $\overline{X}$ torsor Y under $\overline{G}$ with a family $\{\psi_g\}_{g \in \Gamma_k}$ of automorphism such that makes the following diagram commutative:

$$\begin{tikzcd} \overline{Y} \arrow[r, "\psi_g"] \arrow[d] & \overline{Y} \arrow[d] \\ \overline{k} \arrow[r, "g"] & \overline{k} \end{tikzcd}$$

Assume the following

- $X(k) \neq \emptyset$,
- $\overline{G}(\overline{X}) = \overline{G}(\overline{k})$

Then $H^2(k, f_*G) = H^2(\Gamma_k, \operatorname{colim} f_*G(L)) = H^2(\Gamma_k, \overline{G}(\overline{X})) = H^2(\Gamma_k, \overline{G}(\overline{k}) = H^2(\Gamma_k, G)$ so that the map $H^2(k, G) \to H^2(X, G)$ is injective (it has a splitting given by the map induced by the rational point) and hence the map $H^1(X, G) \to H^1(\overline{X}, \overline{G})^{\Gamma_k}$ is surjective. Observe that every element in $H^1(X, G)$, i.e every X-torsor under G gives a Γ_k invariant torsor by base change, so that the surjectivity of that map is equivalent to the fact that every Γ_k invariant torsor comes from a torsor defined over X.
To conclude we have established the following:

Proposition 2.3.1. If G is commutative, $\overline{G}(\overline{X}) = \overline{G}(\overline{k})$ and there exists a Γ_k invariant torsor that does not come from a torsor over X, then X has no rational points.

Example 2.3.2. Suppose that X is a conic in $\mathbb{P}^2_k$ and $G = \mathbb{G}_m$. Then we the exact sequence becomes $0 \to H^1(k, \overline{k}^*) = 0 \to H^1(X, \mathbb{G}_m) = \operatorname{Pic}(X) \to H^1(\overline{X}, \mathbb{G}_m)^{\Gamma_k} = \operatorname{Pic}(\overline{X})^{\Gamma_k}$. Observe that $\operatorname{Pic}(\overline{X}) \simeq \mathbb{Z}$ (It has a rational point and hence it is $\mathbb{P}^1$) and the action of the Galois is trivial (every automorphism must send 1 to 1 or to -1, but it must send ample divisors to ample divisors and hence 1 goes to 1). So $\operatorname{Pic}(X) \simeq \mathbb{Z}$ and the map $\operatorname{Pic}(X) \to \mathbb{Z}$ send a divisor to its degree. So the map is surjective if and only if there exists a divisors of degree 1. But this is equivalent to have a rational point! In fact if X has a rational point then it clearly has a divisors of degree 1. Conversely, if X has a divisors D of degree one, then Riemann Roch theorem and Serre duality tell us that $h^0(D) - h^0(K - D) = \deg(D) + 1 - g$, but $g = 0$ and $K - D$ has negative degree, so that $h^0(K - D) = 0$. Hence $h^0(D) = 2$ and hence D has a linearly equivalent effective divisors E of degree 1. But this means that X has a rational point!

Now we will show how is possible to recover the rational points of X from the rational points of k-torsors under G. Let $Y \to X$ be a right k-torsor under G that is a quasi projective variety.

Suppose that $Spec(k) \to X$ is a k rational point. Then $Spec(k) \times Y$ is a k-torsor under G so that we have a map $\theta : X(k) \to H^1(k, G)$ and hence

$$X(k) = \bigcup_{a \in H^1(k,G)} \theta^{-1}(a).$$

We want to find a better description of this map. First of all we introduce the twist of a torsor.

Definition 2.3.3 (Twist)**.** Suppose that $F \to X$ is a left torsor under G. Then we call the twist of F with Y, if it exists, the quotient of $Y \times F$ by the action of G, made by the map $(g, y, f) \mapsto (yg^{-1}, gf)$ and we denote it by ${}_YF$. It is a sheaf of torsor and if F is quasi-projective then it is representable (descent theory).

Theorem 2.3.4. *If X is a variety over a field k, G is a k-algebraic group and $Y \to X$ a right G k-torsor that is a quasi projective variety then the twist by any other k-torsor under G is representable.*

Remark 2.3.5. From the definitions we have

- ${}_GF \simeq G$
- If Z is a right torsor, we denote with Z' the left torsor associated, we have that ${}_{Z'}G$ is a group scheme, Z is a right torsor over ${}_{Z'}G$ and ${}_ZZ' ={}_{Z'} G$.
- We have a bijection $H^1(X, G) \to H^1(X,{}_{Z'} G)$ that send Y to ${}_ZY$.
- If G is commutative the bijection is just the map that sends Y to $Y - Z$.

Before proving the last theorem we observe that every equivariant map between torsors is an iso (this is clear when both are groups and it is true in general by descent). In particular a k-torsor Y under G has a rational point if and only if it is isomorphic to G.

Theorem 2.3.6. *If $f : Y \to X$ is a torsor under G, then we have the following equality*

$$X(k) = \bigcup_{Z \text{ torsors over } k} {}_{Z'}f({}_{Z'}Y(k))$$

where ${}_{Z'}f$ is the induced map ${}_{Z'}Y \to X$

Proof. It is enough to show that $\theta^{-1}(a) = \cup_{Z\in a}\, {}_{Z'}f({}_{Z'}Y(k))$.
${}_{Z'}Y \times_X Spec(k)$ has a rational point if and only if ${}_{Z'}Y \times_X Spec(k) \simeq_{Z'} G$ in and only if $Y \times_X Spec(k) \simeq Z$. So a rational point x of X lies over a rational point of ${}_{Z'}Y$ if and only if $\theta(x)$ is in the class of Z. □

Example 2.3.7. We verify the equality of the preceding theorem when $X = \mathbb{A}^1_k - \{0\}$ and $Y = Spec(\frac{k[x,y]}{(x-y^2)}_x)$, a $\frac{\mathbb{Z}}{2\mathbb{Z}} = \mu_2$ torsor, when $char k \neq 2$. First we have to understand what are the μ_2 torsors, but this is easy since $H^1(k, \mu_2) \simeq \frac{k^*}{k^{2*}}$ so that the torsors are in the form $Z_a = \frac{k[x]}{x^2-a}$ with $a \in k^*$ and Z_a is equivalent t Z_b if and only if $a = x^2b$ for some $x \in k^*$. Then we compute the twist ${}_YZ_a$. This is nothing else then spectrum of $B_a = (\frac{k[x,y]}{(x-y^2)}_x \otimes Z_a)^{\mu_2} \simeq (\frac{k[x,y,z]}{(x-y^2,z^2-a)})^{\mu_2}$ where μ_2 acts on B_a by the automorphism that send y to $-y$ and z to $-z$. The fixed point are generated by y^2, z^2yz, so that $B_a \simeq \frac{k[x,z]}{z^2-ax}$. So to compute the rational point of X we have just to take the union of the image of the rational points of B_a via the canonical map $Spec(B_a) \to X$, with $a \in \frac{k^*}{k^{2*}}$. But the rational point of B_a are just $(z - b, x - ab^2)$ so that for every $a \in \frac{k^*}{k^{2*}}$ we recover exactly all the point in the form ab^2, i.e the point in the same class of a! The union of this points with all possible a is exactly $X(k)$.

This easy theorem is the prototype of an important result we will discuss in the next talk, namely the main Theorem of Colliot-Thélène and Sansuc descent theory.

3.1. **Elementary obstruction and fundamental exact sequence.** We want to construct the *fundamental exact sequence* of Colliot-Thélène and Sansuc.

Given a Γ_k-module M of finite type, consider the Ext-spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}^p_{\Gamma_k}(M, (R^q p_*)\mathbb{G}_m) \Rightarrow \mathrm{Ext}^{p+q}_{X_{\text{ét}}}(p^*M, \mathbb{G}_m),$$

given by the composition of

$$\begin{tikzcd} \mathrm{Sh}(X_{\text{ét}}) \arrow[rr, "\mathrm{Hom}_{X_{\text{ét}}}(p^*M,-)"] \arrow[dr, "p_*"'] & & \mathcal{A}\mathrm{b} \\ & \mathrm{Sh}(k_{\text{ét}}). \arrow[ur, "\mathrm{Hom}_{\Gamma_k}(M,-)"'] & \end{tikzcd}$$

The functor p_* sends injectives in injectives as its left adjoint is exact. The low degrees exact sequence is

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\Gamma_k}(M, \overline{k}[X]^*) \to \mathrm{Ext}^1_{X_{\text{ét}}}(p^*M, \mathbb{G}_m) \to \mathrm{Hom}_{\Gamma_k}(M, \mathrm{Pic}(\overline{X})) \\ \xrightarrow{\partial} \mathrm{Ext}^2_{\Gamma_k}(M, \overline{k}[X]^*) \to \mathrm{Ext}^2_{X_{\text{ét}}}(p^*M, \mathbb{G}_m). \end{aligned} \tag{3.1}$$

We want to simplify it with the following Lemma.

Lemma 3.1.1. *Let S be an X-group scheme of multiplicative type. Then we have an isomorphism*

$$H^i_{fppf}(X, S) \simeq \mathrm{Ext}^i_{X_{\acute{e}t}}(p^*\widehat{S}, \mathbb{G}_m)$$

functorial in S and X.

Proof. We will prove it when S is smooth (and the general case?). Thanks to a Theorem of Grothendieck [Gro68, Théorème 11.7] $H^i_{fppf}(X,S) = H^i_{\text{ét}}(X,S)$. We use local-to-global Ext spectral sequence that you can find in SGA4 [AGV71, Exposé V, Théorème 6.1],

$$E_2^{p,q} := H^p(X_{\text{ét}}, \mathcal{E}\mathrm{xt}^q_{X_{\text{ét}}}(p^*\hat{S}, \mathbb{G}_m)) \Rightarrow \mathrm{Ext}^{p+q}_{X_{\text{ét}}}(p^*\hat{S}, \mathbb{G}_m),$$

with the following commutative diagram of functors:

$$\begin{tikzcd} \mathrm{Sh}(X_{\text{ét}}) \arrow[rr, "\mathrm{Hom}(p^*\hat{S},-)"] \arrow[dr, "\mathcal{H}\mathrm{om}(p^*\hat{S},-)"'] & & \mathcal{A}\mathrm{b} \\ & \mathrm{Sh}(X_{\text{ét}}). \arrow[ur, "\Gamma(X_{\text{ét}},-)"'] & \end{tikzcd}$$

The first step is to show that the spectral sequence completely degenerates, i.e. for $i \geq 1$, $\mathcal{E}\mathrm{xt}^i_{X_{\text{ét}}}(p^*\hat{S}, \mathbb{G}_m) = 0$. It's enough to show that the sheaf is zero on the stalks. Thus we take Y the spectrum of a strictly henselian local ring. As S is locally constant in the étale topology, it's enough to show that for $i \geq 1$,

$$\mathrm{Ext}^i_{Y_{\text{ét}}}(\mathbb{Z}, \mathbb{G}_m) = \mathrm{Ext}^i_{Y_{\text{ét}}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) = 0.$$

The first group is zero because it corresponds to $H^i_{\text{ét}}(Y, \mathbb{G}_m)$, but taking global sections is exact for a strictly henselian local ring. The second group is zero thanks to the vanishing of the first, using the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

□

In virtue of the previous Lemma and the exact sequence 3.1, we have proved:

Theorem 3.1.2 (Colliot-Thélène, Sansuc)**.** *If X is a geometrically integral smooth variety and S a group of multiplicative type, we have an exact sequence*

$$0 \to \operatorname{Ext}^1_{\Gamma_k}(\hat{S}, \overline{k}[X]^*) \to H^1_{fppf}(X,S) \xrightarrow{\text{type}} \operatorname{Hom}_{\Gamma_k}(\widehat{S}, \operatorname{Pic}(\overline{X}))$$
$$\xrightarrow{\partial} \operatorname{Ext}^2_{\Gamma_k}(\hat{S}, \overline{k}[X]^*) \to H^2_{fppf}(X,S),$$

functorial in S, called fundamental exact sequence. The map type *associates to a torsor its type.*

So for any Γ_k-invariant morphism $\lambda : \widehat{S} \to \operatorname{Pic}(\overline{X})$, the existence of a torsor of type λ is equivalent to $\partial(\lambda) = 0$. If $\operatorname{Pic}(\overline{X})$ is of finite type and $\widehat{S} = \operatorname{Pic}(\overline{X})$, we can take $\operatorname{Id} \in \operatorname{Hom}_{\Gamma_k}(\widehat{S}, \operatorname{Pic}(\overline{X}))$.

Definition 3.1.3 (Elementary obstruction)**.** We will call the *elementary obstruction* of X the class $\partial(\operatorname{Id})$ and we will denote it as $e(X)$.

Thanks to the fundamental exact sequence we have the following corollary.

Corollary 3.1.4. *If $\operatorname{Pic}(\overline{X})$ is of finite type the existence of universal torsors is equivalent to the vanishing of the elementary obstruction.*

Proof. Let's put $\widehat{S} = \operatorname{Pic}(\overline{X})$ in the fundamental exact sequence, if $\lambda \in \operatorname{Hom}_{\Gamma_k}(\widehat{S}, \operatorname{Pic}(\overline{X}))$, by functoriality we know that $\partial(\lambda) = \lambda^*(e(X))$. If λ is an isomorphism then $\lambda^*(e(X)) = 0$ if and only if the elementary obstruction is zero. □

Moreover we have an other simplification of the sequence:

Corollary 3.1.5. *If we add the hypothesis $\overline{k}[X]^* = \overline{k}^*$, then the fundamental exact sequence becomes:*

$$0 \to H^1(k,S) \xrightarrow{p^*} H^1(X,S) \xrightarrow{\text{type}} \operatorname{Hom}_{\Gamma_k}(\hat{S}, \operatorname{Pic}(\overline{X}))$$
$$\xrightarrow{\partial} H^2(k,S) \xrightarrow{p^*} H^2(X,S).$$

So in this case the space of torsors of a certain type is a principal homogeneous space under $H^1(k,S)$. The action of $H^1(k,S)$ is exactly the twist described by Emiliano in the previous talk2.3.3. In particular if k is algebraically closed the type identifies the torsor.
Always in the hypothesis $\overline{k}[X]^* = \overline{k}^*$, we can even describe the set of rational points using torsors of a certain type, just rewriting the Theorem 2.3.6 as

$$X(k) = \bigcup_{\text{type}(Y,f)=\lambda} f(Y(k)).$$

Now we want to proof two Theorems about the *elementary obstruction*.

Theorem 3.1.6. *Let X be a geometrically integral, smooth k-variety, then the class $-e(X) \in \operatorname{Ext}^2_{\Gamma_k}(\operatorname{Pic}(\overline{X}), \overline{k}^*)$ is represented by the 2-fold*

$$0 \to \overline{k}^* \to \overline{k}(X)^* \to \operatorname{Div}(\overline{X}) \to \operatorname{Pic}(\overline{X}) \to 0. \tag{3.2}$$

Proof. We need a general fact of Homological Algebra. If $p : X \to \operatorname{Spec}(k)$ is a k-scheme and

$$0 \to A \to B \to C \to 0$$

is an exact sequence of étale sheaves of abelian groups over X, if the sequence

$$0 \to p_*(A) \to p_*(B) \to p_*(C) \to R^1p_*(A) \to 0 \tag{3.3}$$

is exact then if we consider the Ext-spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\Gamma_k}(-, R^qp_*(A)) \Rightarrow \operatorname{Ext}^{p+q}_{X_{\text{ét}}}(p^*(-), A)$$

Lemma 3.1.7. *The transgression map ($E_2^{0,1} \to E_2^{2,0}$),*

$$\operatorname{Hom}_{\Gamma_k}(-, R^1p_*(A)) \to \operatorname{Ext}^2_{\Gamma_k}(-, p_*(A)),$$

is given by the Yoneda's pairing with the opposite of the class represented by the 2-fold (3.3).

The proof of this Lemma can be found in [CTS87], Lemma 1.A.4.

We also need a quite well known fact:

Proposition 3.1.8. Let X be an irreducible, noetherian, regular scheme and $j : \eta \to X$ the inclusion of the generic point. Then we have an exact sequence of étale sheaves

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to \bigoplus_{x \in X^{(1)}} (i_x)_*\mathbb{Z} \to 0.$$

This proposition is just a consequence of the same exact sequence in the Zariski site. We need to check that $R^1(p_*)(j_*\mathbb{G}_m)$ is zero, but we know that it is a subsheaf of $R^1(p \circ j)_*(\mathbb{G}_m)$, thanks to the convergence of the Grothendieck spectral sequence for the composition of p_* and j_*. But $R^1(p \circ j)_*(\mathbb{G}_m)$ is zero by Hilbert 90, so it is easy to check now that the 2-fold (3.3) becomes exactly (3.2). Thus we are done. □

The second important Theorem of the subsection is the following.

Theorem 3.1.9. *Let X be a geometrically integral, smooth k-variety such that $\overline{k}[X]^* = \overline{k}^*$, we have the following implications:*

$$X(k) \neq \emptyset \Rightarrow \left(\overline{k}^* \hookrightarrow \overline{k}(X)^* \text{ has a } \Gamma_k\text{-equivariant section.}\right) \Leftrightarrow e(X) = 0.$$

As a consequence of this Theorem, using the Corollary 3.1.4 we have:

Corollary 3.1.10. *If $\operatorname{Pic}(\overline{X})$ is of finite type, the existence of a rational point implies the existence of an universal torsor.*

This fact is important, we will use it in the proof of the main Theorem of Colliot-Thélène and Sansuc descent theory. Now we will prove the Theorem, we divide it in different parts.

Proposition 3.1.11. Let k be a perfect field, X a smooth, geometrically integral k-variety, such that $X(k) \neq \emptyset$, then the natural map

$$\overline{k}^* \hookrightarrow \overline{k}(X)^*$$

has a Γ_k-invariant retraction.

We need the following lemma:

Lemma 3.1.12. *Let G be a profinite group, H a closed subgroup, B a G-module, A an H-module. Then*

$$\operatorname{Ext}^n_G\left(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A, B\right) = \operatorname{Ext}^n_H(A, B_H)$$

Where B_H is a $\mathbb{Z}[H]$-module obtain restricting the action of B.

Sketch of proof. First of all we reduce to the case when G and H are finite groups. Then we choose a projective resolution of A. Since $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]}$ is exact and it sends projectives in projectives. Thus it's enough to check that

$$\operatorname{Hom}_G\left(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A, B\right) = \operatorname{Hom}_H(A, B_H),$$

but this can be done similarly to the commutative case. □

Now we can prove the Proposition.

Proof of Proposition 3.1.11. Let $P \in X(k)$ and consider the natural maps

$$\overline{k}^* \hookrightarrow \mathcal{O}^*_{\overline{X},P} \hookrightarrow \overline{k}(X)^*$$

where $\mathcal{O}^*_{\overline{X},P}$ is the Zariski stalk. The first map admits a section $g \mapsto g(P)$, so it's sufficient to find a section of the inclusion $\mathcal{O}^*_{\overline{X},P} \hookrightarrow \overline{k}(X)^*$. Because X is smooth, we have an exact sequence of Γ_k-modules

$$0 \to \mathcal{O}^*_{\overline{X},P} \to \overline{k}(X)^* \to \operatorname{Div}_P\left(\overline{X}\right) \to 0$$

where

$$\operatorname{Div}_P\left(\overline{X}\right) = \bigoplus_{\overline{x} \in \operatorname{Spec}(\mathcal{O}_{\overline{X},P})^{(1)}} \mathbb{Z}_{\overline{x}}$$

If this sequence splits then we get the missing section. To show this we show that $\mathrm{Ext}^1_{\Gamma_k}\left(\mathrm{Div}_P(\overline{X}),\ \mathcal{O}^*_{\overline{X},P}\right) = 0$. We notice that

$$\mathrm{Div}_P(\overline{X}) = \sum_{x \in \mathrm{Spec}\, \mathcal{O}_{\overline{X},P}{}^{(1)}} \sum_{\overline{x} \text{ over } x} \mathbb{Z}_{\overline{x}} = \sum_{x \in \mathrm{Spec}\, \mathcal{O}_{\overline{X},P}{}^{(1)}} \mathbb{Z}\left[\Gamma_k/H_x\right]$$

where H_x is the Kernel of the transitive action of Γ_k on the points over x, corresponding to a certain extension L_x/k. We have

$$\begin{aligned}\mathrm{Ext}^1_{\Gamma_k}\left(\mathrm{Div}_P(\overline{X}),\ \mathcal{O}^*_{\overline{X},P}\right) &= \mathrm{Ext}^1_{\Gamma_k}\left(\sum_x \mathbb{Z}\left[\Gamma_k/H_x\right],\ \mathcal{O}^*_{\overline{X},P}\right) = \\ &= \prod_x \mathrm{Ext}^1_{\Gamma_k}\left(\mathbb{Z}\left[\Gamma_k/H_x\right],\ \mathcal{O}^*_{\overline{X},P}\right).\end{aligned}$$

Since $\mathbb{Z}\left[\Gamma_k/H_x\right] = \mathbb{Z}[\Gamma_k] \otimes_{\mathbb{Z}[H_x]} \mathbb{Z}$, we can use Lemma 3.1.12 and we obtain that for any x,

$$\mathrm{Ext}^1_{\Gamma_k}\left(\mathbb{Z}\left[\Gamma_k/H_x\right],\ \mathcal{O}^*_{\overline{X},P}\right) = \mathrm{Ext}^1_{H_x}\left(\mathbb{Z},\ \mathcal{O}^*_{\overline{X},P}\right).$$

If $A := \mathcal{O}_{X_{L_x},P}$ and $p : \mathrm{Spec}(A) \to \mathrm{Spec}(L_x)$ is the structural map,

$$\mathrm{Ext}^1_{H_x}\left(\mathbb{Z},\ \mathcal{O}^*_{\overline{X},P}\right) = H^1_{\text{ét}}(\mathrm{Spec}\, L_x, p_*\mathbb{G}_{m,A})$$

because the functor $\mathrm{Hom}_{H_x}(\mathbb{Z}, -)$ is equal to the functor $M \to M^{H_x}$. By Leray spectral sequence,

$$H^1_{\text{ét}}(\mathrm{Spec}\, L_x, p_*\mathbb{G}_{m,A}) \hookrightarrow H^1_{\text{ét}}(\mathrm{Spec}\, A, \mathbb{G}_{m,A}).$$

The right group is zero by Hilbert 90 for local rings, so we are done.

□

In the proof of the Proposition 3.1.11 we have showed an useful property of the Γ_k-module of divisors that can be easily generalized as follows.

Lemma 3.1.13. *Let X be a variety over a perfect field k, then the Γ_k-module of Weil divisors on $\overline{X}$, $\mathrm{Div}_{Weil}(\overline{X})$ is isomorphic to a certain sum*

$$\sum_{i \in I} \mathbb{Z}[\Gamma_k/H_i],$$

with H_i open normal subgroups (thus of finite index) of Γ_k and I not necessarily finite.

Definition 3.1.14. We will call *permutation module* a Γ_k-module that contains a basis invariant (not necessarily fixed) under the action of Γ_k. Thus it's a Γ_k-module of the form $\sum_{i \in I} \mathbb{Z}[\Gamma_k/H_i]$ with H_i closed subgroups of Γ_k and I not necessarily finite.

Thus in Lemma 3.1.13 we have showed that $\mathrm{Div}_{Weil}(\overline{X})$ is in particular a permutation module. Another fact that we will use many times, whose proof is the same as in the proof of Proposition 3.1.11 is the following.

Lemma 3.1.15. *Let A be a local ring that is a $\overline{k}[\Gamma_k]$-module, for any permutation module M, then* $\mathrm{Ext}^1_{\Gamma_k}(M, A^*) = 0$*. In particular when $A = \overline{k}$ we obtain*

$$\mathrm{Ext}^1_{\Gamma_k}(M, \overline{k}^*) = H^1(k, M) = 0.$$

Let's conclude now the proof of Theorem 3.1.9.

Proposition 3.1.16. Let X be a geometrically integral, smooth k-variety such that $\overline{k}[X]^* = \overline{k}^*$, then the inclusion

$$\overline{k}^* \hookrightarrow \overline{k}(X)^*$$

has a Γ_k-invariant retraction if and only if the 2-fold (3.2) is zero in $\mathrm{Ext}^2_{\Gamma_k}(\mathrm{Pic}(\overline{X}), \overline{k}^*)$, if and only if $e(X)$ is zero.

Proof. If the map $\overline{k}^* \to \overline{k}^*(X)$ has a retraction then it is a general fact that the 2-fold (3.2) is zero in $\mathrm{Ext}^2_{\Gamma_k}(\mathrm{Pic}(\overline{X}), \overline{k}^*)$.

The other implication is not true in general for 2-folds. Consider the short exact sequence

$$0 \to \overline{k}(X)^*/\overline{k}^* \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0 \tag{3.4}$$

and the long exact sequence given by the derived functor of $\mathrm{Hom}(-, \overline{k}^*)$. Thanks to the Lemmas 3.1.13 and 3.1.15, we know that $\mathrm{Ext}^1_{\Gamma_k}(\mathrm{Div}(\overline{X}), \overline{k}^*) = 0$, thus we have the injective connection map

$$\mathrm{Ext}^1_{\Gamma_k}(\overline{k}(X)^*/\overline{k}^*, \overline{k}^*) \hookrightarrow \mathrm{Ext}^2_{\Gamma_k}(\mathrm{Pic}(\overline{X}), \overline{k}^*)$$

given by the Yoneda pairing with the (3.4). The image of the short exact sequence

$$0 \to \overline{k}^* \to \overline{k}(X)^* \to \overline{k}(X)^*/\overline{k}^* \to 0 \tag{3.5}$$

is exactly the 2-fold (3.2), thus is zero. By the injectivity we obtain that also (3.5) is zero in $\mathrm{Ext}^1_{\Gamma_k}(\overline{k}(X)^*/\overline{k}^*, \overline{k}^*)$, thus the map $\overline{k}^* \to \overline{k}(X)^*$ has a retraction, as we wanted.

Thanks to the Theorem 3.1.6 we also know that the vanishing of the 2-fold (3.2) is equivalent to the vanishing of $e(X)$. □

3.2. Weak and strong approximation.

3.2.1. *Weak approximation.* Let k be a number field, for any ν, we can endow $X(k_\nu)$ with the topology as ν-adic space (add reference). We define the topological space

$$X(k_{\Omega_k}) := \prod_{\nu \in \Omega_k} X(k_\nu),$$

where the topology is the product of the topologies of $X(k_\nu)$. We will call this topology *weak topology*.

Of course there is a diagonal embedding of $X(k)$ in $X(k_{\Omega_k})$. If we take $X = \mathbb{A}^1_k$ we know that the diagonal map

$$k \hookrightarrow \prod_{\nu \in \Omega_k} k_\nu$$

is dominant. This is a classical result of Number Theory and it's called *weak approximation*. We can wonder if this is true even for other varieties. Thus we give the following definition.

Definition 3.2.1. We say that a smooth, geometrically integral k-variety X *satisfies weak approximation* if the diagonal map

$$X(k) \hookrightarrow X(k_{\Omega_k})$$

is dominant.

As $\mathbb{A}^1_k$ satisfies weak approximation, even $\mathbb{A}^n_k$ satisfy weak approximation for any n. The same remains true for any open subset of $\mathbb{A}^n_k$, because if U is a Zariski open subset of X, the set $U(k_\nu)$ is open in $X(k_\nu)$.

Let's focus in the opposite problem: if U is a dense open subset of X and we have weak approximation on U, what can we say about X? In virtue of the inverse function Theorem for complete fields with respect a non-trivial absolute value [Ser64] we have the following fact.

Proposition 3.2.2. Let X be a smooth k-variety of dimension n and let ν be a place of k. If P_ν is a local point of X, i.e. $P_\nu \in X(k_\nu)$, then there exists an open $U \subseteq X(k_\nu)$ that contains P_ν that is homeomorphic to a non-empty open subset of k_ν^n.

As X is smooth, by the previous Proposition, the set $U(k_\nu)$ is dense in $X(k_\nu)$. Indeed $X \setminus U$ is a Zariski closed subset of X of lower dimension, as X is irreducible, thus $X(k_\nu) \setminus U(k_\nu)$ has empty interior in the ν-adic topology.

So for any local point $P_\nu \in X(k_\nu)$ there exist local points $Q_\nu \in U(k_\nu)$ close as we want to P_ν. Then, as weak approximation holds in U, we can find rational points $R \in U(k)$, close as we want to Q_ν. We have proven the following important fact.

Proposition 3.2.3. Let X be a smooth, geometrically integral k-variety containing an open dense subset which verifies weak approximation. Then X satisfies weak approximation.

Corollary 3.2.4. *If X is a smooth, geometrically integral k-rational variety, then weak approximation holds.*

An other quite elementary example of varieties satisfying weak approximation can be found in the article of Colliot-Thélène, Sansuc and Swinnerton Dyer [CTSSD87]. At page 68, is stated the following Theorem.

Theorem 3.2.5. *Let k be a number field and let $V \subseteq \mathbb{P}^n_k$ with $n \geq 2$ be a pure codimension 2 intersection of two quadrics over k. Assume that V is geometrically integral and not a cone. Let X be V^{smooth} and assume that $X(k)$ is not empty, then weak approximation holds for X.*

Then in the last Talk 5, Professor Harari present the theory of weak approximation for linear groups.

3.2.2. *Adelic points and strong approximation.* If X it's a not proper variety one can even pay more attention to *integral points*.

Definition 3.2.6. Let X be a k-variety, we say that a separate scheme $\mathfrak{X}$ finite over $\mathrm{Spec}(\mathcal{O}_k)$ is a *model* of X if $X \simeq \mathfrak{X}_\eta$, with η the generic point of $\mathrm{Spec}(\mathcal{O}_k)$.

Usually we will suppose the model to be integral. One can show the following fact.

Proposition 3.2.7. Any two models of X are isomorphic out a finite number of places. Moreover if the variety is reduced (resp. irreducible, resp. proper) any model is reduced (resp. irreducible, resp proper) out a finite number of places.

Moreover we have:

Proposition 3.2.8. Let X be a k-variety, then there exists a model $\mathfrak{X}$ of X.

(add reference)
For any finite place ν, we have the inclusion

$$\mathfrak{X}(\mathcal{O}_\nu) \hookrightarrow X(k_\nu),$$

we will call any element in the image of this map, *integral points*.

Remark 3.2.9. The inclusion displayed above is a formal consequence of the *valutative criterion of separatedness* applied to the following diagram

$$\begin{tikzcd} \mathrm{Spec}(k_\nu) \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_\nu) \arrow[r] & \mathrm{Spec}(\mathcal{O}_k) \end{tikzcd}$$

If X is proper then for almost any place the map is even surjective thanks to the *valutative criterion of properness* applied to $\mathfrak{X}$ that is proper out a finite number of places.

Achtung. *The integral points may depend on the choice of a model of X.*

Then we define the set of *adelic points* of X by

$$X(\mathbb{A}_k) := \{(\rho_\nu)_\nu \in X(k_\nu) | \text{ all but finitely many } \rho_\nu \text{ are integrals}\}.$$

We notice that the definition does not depend on the choice of the model because any two of them are isomorphic away from a finite number of places. We will not consider $X(\mathbb{A}_k)$ with the topology of subspace of $X(k_\nu)$, we will be more interested to the topology defined by the basis of sets of the form

$$\prod_{\nu \in S} U_\nu \times \mathfrak{X}(\mathcal{O}_\nu),$$

with U_ν an open of $X(k_\nu)$ and S finite, such that $\Omega_\infty \subseteq S$. We will call this topology the *strong topology*.

Example 3.2.10. If $X = \mathbb{A}^1_k$, then $X(\mathbb{A}_k)$ are the adeles with the adelic topology; if $X = \mathbb{G}_m$, then $X(\mathbb{A}_k)$ are the ideles with the idelic topology.

If $\Sigma \subseteq \Omega_k$ we will call $X(\mathbb{A}^\Sigma_k)$ the set $X(\mathbb{A}_k)$, with ν-components removed for any $\nu \in \Sigma$. We will endow this set with the topology induced by the projection from $X(\mathbb{A}_k)$.

In analogy with weak approximation we give this definition.

Definition 3.2.11. Let X be a smooth, geometrically integral k-variety, if S is a finite subset of Ω_k, if

$$X(k) \to X(\mathbb{A}_k^S)$$

is dominant, we say that X satisfies *strong approximation off* S. If $S = \emptyset$ we will say that X satisfies *strong approximation*.

Thanks to the valuative criterion of properness we can deduce that if X is proper then $X(\mathbb{A}_k) = X(k_{\Omega_k})$ as topological spaces. Thus for proper schemes weak or strong approximation are the same.

We also have that $\mathbb{A}_k^n$ satisfies strong approximation off one place, i.e. off $S = \{\nu\}$, for any $\nu \in \Omega_k$. This is a classical result of Number Theory, more difficult than weak approximation.

What can we say about strong approximation on open subvarieties of $\mathbb{A}_k^n$? In general it doesn't hold off a finite set of places.

Example 3.2.12. Consider $\mathbb{G}_{m,\mathbb{Q}} \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$, if we had strong approximation away from infinity, $\mathbb{G}_m(\mathbb{Z})$ would be dense in $\prod_\ell \mathbb{G}_m(\mathbb{Z}_\ell)$, since if a point in $\prod_\ell \mathbb{G}_m(\mathbb{Z}_\ell)$ can be approximated in the strong topology by rational points it can be approximated with integral points. But $\mathbb{G}_m(\mathbb{Z}) = \{1, -1\}$ and it is not dense in any $\mathbb{G}_m(\mathbb{Z}_\ell)$. We can even show that strong approximation doesn't hold off a finite place p. If it had hold then as before the set $\mathbb{G}_m(\mathbb{Z}[1/p])$ would be dense in $\prod_{\ell \neq p} \mathbb{G}_m(\mathbb{Z}_\ell)$. But $\mathbb{G}_m(\mathbb{Z}[1/p]) = \langle -1, p \rangle$, so if we consider the extension $\mathbb{Q}(\sqrt{-1}, \sqrt{p})$ of $\mathbb{Q}$, by the Theorem of Chebotarev, there exists at least a prime ℓ that is totally split. This means that -1 and p are both squares modulo ℓ, thus the image of $\mathbb{G}_m(\mathbb{Z}[1/p])$ in $\mathbb{G}_m(\mathbb{Z}_\ell)$ is contained in the subgroup of squares of $\mathbb{G}_m(\mathbb{Z}_\ell)$. Obviously the result can be extended to any number field k and any finite set of places S. Thanks to Dirichlet unit Theorem the set $\mathbb{G}_m(\mathcal{O}_{k,S})$ is finitely generated, let's say by $t_1, \ldots, t_n$, then we can take $k(\sqrt{t_1}, \ldots, \sqrt{t_n})$ and we can apply the Theorem of Chebotarev again.

We also have many other similar obstructions on $\mathbb{G}_m$, just taking any étale cover

$$\mathbb{G}_m \xrightarrow{t \to t^n} \mathbb{G}_m.$$

This phenomenon can be generalized by the following theorem due to Minchev whose proof can be found in [Rap12], page 9.

Theorem 3.2.13 (Minchev 1989)**.** *Let X be an irreducible normal variety over a number field k such that $X(k) \neq \emptyset$. If there exists a nontrivial connected unramified covering $f : Y \to X$ defined over an algebraic closure $\overline{k}$, then X does not satisfy strong approximation off any finite set S of places of k.*

In particular if we take any polynomial in n variables f with coefficients in k and we take the open U in $\mathbb{A}_k^n$ that is defined by $f \neq 0$, you can then take as Y the closed variety in $\mathbb{A}^n \times \mathbb{G}_m$ defined by $f = x_{n+1}^m \neq 0$. The natural projection $Y \to U$ is unramified, thus on U we can not have strong approximation with respect to any finite set of places.

At the same time if we take X the complement of a closed subset in $\mathbb{A}_k^n$ of codimension at least two we still have strong approximation off one place. We will prove the Proposition in a slightly more general case.

Proposition 3.2.14. Let X be a smooth variety over the number field k satisfying weak approximation and let S be a finite set of places of k. Let's suppose that there exists a dense open U of X with the following property

- $\mathcal{P}$) For any $x \in U(k)$ there exists a dense open V_x of U such that for any $y \in V_x(k)$ there exists a variety $Z_{x,y}$ that satisfies strong approximation off S and a morphism $f_{x,y} : Z_{x,y} \to X$ such that $f_{x,y}$ restricted to a certain open $Z'_{x,y}$ is an immersion and the image of $Z'_{x,y}$ contains x and y.

Then X satisfies strong approximation off S.

In particular $\mathcal{P}$) is satisfied if the following property holds

- $\mathcal{P}'$) There exists a variety Z that satisfies strong approximation off S and a morphism $f : Z \to X$ such that f restricted to a certain open Z' of Z is an open immersion.

Proof. Let $\mathfrak{X}$ be a model of X and let T be a finite subset of $\Omega_k \setminus S$ containing $\Omega_\infty \setminus S$. For any adelic point

$$P = (P_\nu)_{\nu \in \Omega_k} \in \prod_{\nu \in T \cup S} X(k_\nu) \times \prod_{\nu \in \Omega_k \setminus (T \cup S)} \mathfrak{X}(\mathcal{O}_\nu)$$

we have to find a rational point of X that is near as we want to P when $\nu \in T$ and integral when $\nu \in \Omega_k \setminus (T \cup S)$.

In virtue of the implicit function Theorem, we can find local points of U that are near P_ν for $\nu \in T$. Thus by weak approximation on U we can choose rational points x of U that are close as we want to P_ν for $\nu \in T$.

A priori the local points x_ν could fail to be integral when $\nu \in \Omega_k \setminus (T \cup S)$, let's suppose that x_ν is not integral when $\nu \in T' \subseteq \Omega_k \setminus (T \cup S)$. We take V_x as in the property $\mathcal{P}$). Thanks to the implicit function Theorem, we can find local points of U near P_ν for $\nu \in T'$. As V_x satisfies weak approximation we can find a rational point y of V_x near the local points P_ν for $\nu \in T'$, in particular we can choose y as a $\mathcal{O}_\nu$-points for any $\nu \in T'$.

Now let's take $f_{x,y} : Z_{x,y} \to X$ as in the hypothesis ii), there exists on $Z'_{x,y}$ an adelic point $Q = (Q_\nu)_{\nu \in \Omega_k}$ such that $Q_\nu = x_\nu$ when $\nu \in \Omega_k \setminus (T' \cup S)$ and $Q_\nu = y_\nu$ when $\nu \in T'$. Thanks to strong approximation on $Z_{x,y}$ off S we can find a rational point z of $Z_{x,y}$ that is near Q in the strong topology of $Z_{x,y}(\mathbb{A}_k)$. As the morphism $f_{x,y} : Z_{x,y}(\mathbb{A}_k) \to X(\mathbb{A}_k)$ (add reference) is continuous $f_{x,y}(z)$ can be near Q in the strong topology of $X(\mathbb{A}_k)$ as we want.

Since Q is near P for the places $\nu \in T$ and it's integral outside $T \cup S$ we have the result. □

Corollary 3.2.15. *Let X be an open subvariety of $\mathbb{A}^n_k$ obtained removing a closed subset of codimension at least two, then X satisfies strong approximation off one place.*

Corollary 3.2.16. *Let X be an open subvariety of a smooth quadric Q of $\mathbb{P}^n_k$ obtained removing a closed subset of codimension at least two, then X satisfies strong approximation.*

3.3. **The adelic Brauer-Manin pairing.** In this section we define an important pairing, which will be fundamental to describe an obstruction to the existence of rational points. The main ingredient will be the main exact sequence of Class Field Theory, as in Theorem 1.1. For more details we refer to [Poo11], [Mil13] and [Sko01].

First of all we recall a result.

Theorem 3.3.1. *Let X a variety over a global field k. Let $A \in \mathrm{Br}(X)$, then for some $S \subset \Omega_k$ finite, there exists a scheme $\mathfrak{X}$ of finite type, defined over $\mathcal{O}_{k,S}$ and a class $\mathcal{A} \in \mathrm{Br}(\mathfrak{X})$ with a morphism*

$$i : X \to \mathfrak{X}$$

identifying X with the generic fiber $\mathfrak{X}_\eta$, s.t. $i^* : \mathrm{Br}(X) \to \mathrm{Br}(\mathfrak{X})$ sends A to $\mathcal{A}$.

For the proof see Corollary 6.6.11. of [Poo11].

Definition 3.3.2 (Evaluation)**.** Let X/k a variety and $A \in \mathrm{Br}(X)$. If L is a k-algebra and $x \in X(L)$ then, by functoriality of $\mathrm{Br}(-)$, it induces a homomorphism

$$\mathrm{Br}(X) \to \mathrm{Br}(L), A \mapsto A(x).$$

Let X/k be a smooth and geometrically integral variety over a number field k. We are interested in the pairing

$$\mathrm{Br}(X) \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$$

defined by the following rule:

$$(A, (P_v)) \mapsto \sum_{v \in \Omega_k} \mathrm{inv}_v(A(P_v)). \tag{3.6}$$

Where $A(P_v)$ makes sense thanks to the previous definition and inv_v are the *local invariant* maps appearing in the exact sequence of Theorem 1.1.

Lemma 3.3.3. *The B.M. pairing is well defined, i.e. the sum of 3.6 is finite.*

Proof. Given $(P_v) \in X(\mathbb{A}_k)$ and $A \in \mathrm{Br}(X)$ we have to show that $A(P_v) = 0$ for almost all v. Thanks to Theorem 3.3.1 we can chose a finite set of places S big enough (containing all the archimedean places) such that $P_v \in \mathfrak{X}(\mathcal{O}_v)$ for all $v \notin S$ (by the definition of the adelic ring). This concludes in virtue of the following result.

Theorem 3.3.4. *Let R be the valuation ring of a non-archimedean local field k. Then* $\mathrm{Br}(R) = 0$.

$\square$

Lemma 3.3.5. *The B.M. pairing is trivial on* $\mathrm{Br}_0(X)$*, and so it can be defined also as a pairing from* $\mathrm{Br}(X)/\,\mathrm{Br}_0(X)$.

Proof. This follows immediately from the exact sequence of Theorem 1.1 and the functoriality of $\mathrm{Br}(-)$. $\square$

Definition 3.3.6. We define $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ as the right kernel of the B.M. pairing, i.e. as the subset of $X(\mathbb{A}_k)$ orthogonal to all elements of $\mathrm{Br}(X)$.

Lemma 3.3.7. *The B.M. pairing is locally constant in the adelic topology.*

For the proof see Corollary 8.2.11 of [Poo11].

Proposition 3.3.8. We have the following inclusion:

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subseteq X(\mathbb{A}_k)$$

Proof. The only non trivial inclusion is the first one. But this follows from the commutativity of the diagram

$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d] & X(\mathbb{A}_k) \arrow[d] \\ \mathrm{Br}(k) \arrow[r] & \bigoplus_{v\in\Omega_k} \mathrm{Br}(k_v) \end{tikzcd}$$

and the exact sequence of Theorem 1.1. $\square$

Remark 3.3.9. The two previous results implies that the closure of the diagonal image of $X(k)$ via the diagonal embedding in the adelic points is contained in $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$.

Remark 3.3.10 (Functoriality)**.** Let $f : X \to Y$ a k-morphism of smooth geometrically integral k-variety. Give $A \in \mathrm{Br}(X)$ and (P_v) an adelic point of Y, we have

$$\sum_{v\in\Omega_k} \mathrm{inv}_v(f^*A(P_v)) = \sum_{v\in\Omega_k} \mathrm{inv}_v(A(f(P_v)))$$

It follows

$$Y(\mathbb{A}_k)^{\mathrm{Br}(Y)} = \emptyset \Rightarrow X(\mathbb{A}_k)^{\mathrm{Br}(X)} = \emptyset.$$

Now, if the set $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ is empty of course the variety will not have rational point.

Definition 3.3.11. We will say that for a variety X *the only obstruction to Hasse principle is given by the Brauer-Manin obstruction* or some similar sentence if $X(\mathbb{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$ implies $X(k) \neq \emptyset$.

This propriety is weaker than Hasse principle (add examples).
We also introduce some other notation:

Definition 3.3.12. We will say that for a proper variety X *the only obstruction to weak approximation is given by the Brauer-Manin obstruction* if $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. If X is any variety and S is a finite subset of Ω_k we will say *the only obstruction to strong approximation off S is given by the Brauer-Manin obstruction* if $X(k)$ is dense in the image of $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ in $X(\mathbb{A}_k^S)$.

We will see in the last talk some examples of varieties satisfying these properties.

3.4. **Introduction to descent theory.**

3.4.1. *Hochschild Serre and filtration of the Brauer group.* We recall Hochschild Serre spectral sequence. If X is

$$E_2^{p,q} = H^p(k, H^q(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p,q}(X, \mathbb{G}_m).$$

The spectral sequence is a Grothendieck spectral sequence, respect to the composition of the two functors:

$$\begin{tikzcd} \mathrm{Sh}(\overline{X}_{\text{ét}}) \arrow[rr, "\Gamma(X,-)"] \arrow[dr, "\Gamma(\overline{X},-)"'] & & \mathcal{A}\mathrm{b} \\ & \mathrm{Sh}(k_{\text{ét}}). \arrow[ur, "M \to M^{\Gamma_k}"'] & \end{tikzcd}$$

The fact that the first functor sends injective sheaves to acyclic sheaves is subtle. Let $\mathcal{I}$ an injective étale sheaf on $\overline{X}$, for any finite extension L/k let's call $X_L := X \otimes L$, then we can check that

$$\check{\mathrm{H}}^i(X_L \to X, \mathcal{I}(X_L)) = H^i(\mathrm{Gal}(L/k), \mathcal{I}(X_L))$$

just verifying that the two standard complexes used to compute them are isomorphic. As $\mathcal{I}$ is injective as sheaf it is even injective in the category of presheaves, thus $\check{\mathrm{H}}^i(X_L \to X, \mathcal{I}(X_L))$ are zero when $i \geq 1$.

In virtue of the convergence of the spectral sequence, for any H^n there is a filtration

$$0 = F^{n+1}H^n \subseteq \cdots \subseteq F^0H^n = H^n$$

such that $E_\infty^{p,q} \simeq F^pH^{p+q}/F^{p+1}H^{p+q}$. We notice that $H^2 = \mathrm{Br}(X)$, we will call $\mathrm{Br}_0(X)$ the group F^2H^2 and $\mathrm{Br}_1(X)$ the group F^1H^2. We have an exact sequence

$$0 \to \mathrm{Br}_1(X) \to \mathrm{Br}(X) \to E_\infty^{0,2},$$

and

$$E_\infty^{0,2} \hookrightarrow E_2^{0,2} = H^2(\overline{X}, \mathbb{G}_m)^{\Gamma_k}.$$

This implies that $\mathrm{Br}_1(X) = \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X}))$. We also have the exact sequence

$$0 \to \mathrm{Br}_0(X) \to \mathrm{Br}_1(X) \to E_\infty^{1,1} = E_3^{1,1} = \mathrm{Ker}(E_2^{1,1} \to E_2^{3,0}).$$

At the end we obtain the exact sequence

$$0 \to \mathrm{Br}_0(X) \to \mathrm{Br}_1(X) \xrightarrow{r} H^1(k, \mathrm{Pic}(\overline{X})) \to H^3(k, \mathbb{G}_m).$$

Were r is the map defined by the spectral sequence.

Notice that when k is a number fields the last term is 0 by a not trivial result of Class Field Theory, so this last exact sequence simplifies. You can even notice that the last exact sequence can be fit in a long exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{\Gamma_k} \to \mathrm{Br}(k) \to \mathrm{Br}_1(X) \xrightarrow{r} H^1(k, \mathrm{Pic}(\overline{X})) \to H^3(k, \mathbb{G}_m)$$

For any $\lambda \in \mathrm{Hom}_{\Gamma_k}(M, \mathrm{Pic}(\overline{X}))$ we define

$$\mathrm{Br}_\lambda(X) := r^{-1}\lambda_*(H^1(k, M)).$$

As is proven in ([Ser97], I.2.2,Cor. 2) we have

$$H^1(k, \mathrm{Pic}(\overline{X})) = \bigcup_{\substack{\lambda\colon M \hookrightarrow \mathrm{Pic}(\overline{X}) \\ M \text{ of finite type}}} \lambda_*(H^1(k, M)).$$

So we have

$$\mathrm{Br}_1(X) = \bigcup_{\substack{\lambda\colon M \hookrightarrow \mathrm{Pic}(\overline{X}) \\ M \text{ of finite type}}} \mathrm{Br}_\lambda(X). \tag{3.7}$$

3.4.2. *The main theorem.* We have developed all the tools to talk about Colliot-Thélène and Sansuc descent Theory. The main goal is to show that for certain classes of k-varieties (k will always be a number field), the Brauer-Manin obstruction explains the failure of Hasse principle or weak approximation. To do this we use a description of the sets $X(\mathbb{A}_k)^{\mathrm{Br}_\lambda(X)}$ with the help of the adelic points of torsors of type λ. We will see briefly how to apply this method to tori smooth compactifications.

The following theorem was proven for torsors under tori by Colliot-Thélène and Sansuc in 1979 and generalized by Skorobogatov in 1999 for torsors under groups of multiplicative type. The proof is really long and we will see it during next talks. Let's enunciate it:

Theorem 3.4.1 (Colliot-Thélène, Sansuc, Skorobogatov)**.** *Let k be a number field, X a smooth, geometrically integral k-variety such that $\overline{k}[X]^* = \overline{k}^*$, then for any $\lambda \in \mathrm{Hom}_{\Gamma_k}(\hat{S}, \mathrm{Pic}(\overline{X}))$,*

$$X(\mathbb{A}_k)^{\mathrm{Br}_\lambda(X)} = \bigcup_{\mathrm{type}(f,Y)=\lambda} f(Y(\mathbb{A}_k)).$$

Moreover if X is proper there are only finitely many classes of isomorphism of torsors Y of a certain type such that $Y(\mathbb{A}_k) \neq \emptyset$.

The theorem is often used when $\mathrm{Pic}(\overline{X})$ is of finite type and λ is an isomorphism. In this situation we have

$$X(\mathbb{A}_k)^{\mathrm{Br}_1(X)} = \bigcup_{(f,Y)\ \text{universal}} f(Y(\mathbb{A}_k)),$$

because $\mathrm{Br}_\lambda(X) = \mathrm{Br}_1(X)$ when λ is surjective. In particular the algebraic Brauer-Manin obstruction is empty if and only if there exists an universal torsor with an adelic point.
If $\mathrm{Pic}(\overline{X})$ is not of finite type, by 3.7 we have

$$X(\mathbb{A}_k)^{\mathrm{Br}_1(X)} = \bigcap_{\substack{\lambda\colon M \hookrightarrow \mathrm{Pic}(\overline{X}) \\ M \text{ of finite type}}} \bigcup_{\mathrm{type}(f,Y)=\lambda} f(Y(\mathbb{A}_k)).$$

We can easily check the following corollary, recalling that the subsets $X(\mathbb{A}_k)^{\mathrm{Br}_\lambda(X)}$ are closed in $X(\mathbb{A}_k)$.

Corollary 3.4.2. *For X as in the theorem (not necessarily proper) and for any $\lambda \in \mathrm{Hom}_{\Gamma_k}(\hat{S}, \mathrm{Pic}(\overline{X}))$, if the X-torsors of type λ satisfy Hasse principle, then the only obstruction to Hasse principle for X is the one given by $\mathrm{Br}_\lambda(X)$, i.e. $X(\mathbb{A}_k)^{\mathrm{Br}_\lambda(X)} \neq \emptyset \Rightarrow X(k) \neq \emptyset$.*
Furthermore if X is proper, for any $\lambda \in \mathrm{Hom}_{\Gamma_k}(\hat{S}, \mathrm{Pic}(\overline{X}))$, if the X-torsors of type λ satisfy weak approximation, then the only obstruction to weak approximation for X is the one given by $\mathrm{Br}_\lambda(X)$, i.e. $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}_\lambda(X)}$.

As an application one can proof the following theorem.

Theorem 3.4.3. *Let k be a number field, X a smooth, proper k-variety that contains a k-torsor under a torus U as a dense open subset. The Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one.*

We will not give the proof because we need the local description of torsors. The idea is to show that in this situation the universal torsors are k-rational. We can do this studying their restriction to the open U, using the fact that $\mathrm{Pic}(\overline{U}) = 0$. Then, as k-rational varieties satisfy weak approximation, thanks to Corollary 3.4.2 we are done. For a proof look Skorobogatov's book [Sko01], Theorem 6.3.1.
In the case when X is not proper but we only knows that $\overline{k}[X]^* = \overline{k}^*$, we can again use descent theory to prove that strong approximation off one place with Brauer-Manin obstruction holds as it's showed in the article of Wei [Wei14].

The fact that universal torsors are k-rational gives us another interesting tool. Since the variety we are considering is proper, thanks to theorem 3.4.1 there is only a finite number of classes of isomorphism of universal torsors. Finally the set of rational points of the variety we are considering has a quite good description. It is a finite disjoint union of subsets, each one parametrized by rational points of a certain k-rational variety.

4. Talk 4: Descent theory and Poitou-Tate pairing, Gregorio Baldi

In this section k will denote a number field. If $M = G(\overline{k})$ is a Γ_k-module coming from a commutative algebraic k-group G, we will denote $H^i(\Gamma_k, G) = H^i(\Gamma_k, G(\overline{k}))$. (This will be used in the last section).

In the previous talk we discussed the main result of descent theory: it is possible to describe the adelic points orthogonal to Br_λ of a variety X, under some reasonable assumptions, in terms of the adelic points of the X-torsors of type λ. Today we will prove the existence of a torsor, given an adelic point orthogonal to (a subgroup of) Br_λ.

In the first part of the talk we will explain the notation and a generalization of the following deep result. The proof can be found in [Har12], Theorème 8.28.

Theorem 4.1 (Poitou-Tate)**.** *Let $S \subset \Omega_k$ containing the archimedean places, let k_S be the maximal unramified outside S extension of k contained in $\overline{k}$ and let G_S be the Galois group of k_S over k. For any finite Γ_k-module M (whose cardinality is invertible in $\mathcal{O}_{k,S}$) we have:*

i) For $i \geq 3$, $H^1(G_S, M) \cong \bigoplus_{v \text{ real}} H^i(k_v, M)$, in particular $Ш^i_S(k, M) = 0$.
ii) There is a nine terms exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(G_S, M) & \longrightarrow & \prod_{v\in S} H^0(k_v, M) & \longrightarrow & H^2(G_S, M^D)^D & \\
& & & & & \downarrow & \\
& H^1(G_S, M^D)^D & \longleftarrow & \mathbb{P}^1_S(k, M) & \longleftarrow & H^1(G_S, M) & \\
& \downarrow & & & & & \\
& H^2(G_S, M) & \longrightarrow & \bigoplus_{v\in S} H^2(k_v, M) & \longrightarrow & H^0(G_S, M^D)^D & \longrightarrow 0
\end{array}$$

Moreover, dualizing everything, one obtains the same sequence with M and M^D interchanged.
iii) The groups $Ш^1_S(k, M^D)$ and $Ш^2_S(k, M)$ are finite and duals.

Remark 4.2. In this talk we will deal just with the case $S = \Omega_k$ and so we will not discuss here the more general case with an arbitrary S, for this details we remand to the notes.

Even to define P.T. pairing we will need the following fact from class field theory.

Theorem 4.3. *Let k be a global or a local field, then $H^3(k, \overline{k}^*)$ is zero.*

4.1. Group Cohomology and $Ш^i(k, M)$.

4.1.1. *Cup product.* Notation: Let G be a profinite group, $C^i(G, M)$ is the group of continuous i-cochains of G with coefficients in a discrete G-module M and let $Z^i(G, M)$ and $B^i(G, M)$ defined as usual. Usually G will be the absolute Galois group of a number field k and we will denote $H^i(k, M) = Z^i(\Gamma_k, M)/B^i(\Gamma_k, M)$.

First of all we recall some generalities about the cup product in group cohomology. Let A, B be G-modules and consider the G-module $A \otimes B (:= A \otimes_{\mathbb{Z}} B)$ with the action given by $g.(a \otimes b) = g.a \otimes g.b$. This induce naturally a bilinear application at the level of continuous cochains

$$\cup : C^p(G, A) \times C^q(G, B) \to C^{p+q}(G, A \times B) \text{ with } p, q \in N$$

given by $a \cup b : (g_1, \ldots g_{p+q}) \to a(g_1 \ldots g_p) \otimes b(g_{p+1}, \ldots, g_{p+q})$. By direct computation we have

$$d(a \cup b) = da \cup b + (-1)^p (a \cup db)$$

and so it gives a bilinear application in cohomology:

$$\cup : H^p(G, A) \times H^q(G, B) \to H^{p+q}(G, A \times B) \text{ with } p, q \in \mathbb{N}$$

This is enough to define the cup product associated to any bilinear[2] application $A \times B \to C$ since it can be factored through $A \otimes B$.

[2]Not assumed to be G invariant, indeed we are considering the tensor product between A and B as abelian groups.

Achtung. *From now on we will consider the cup product associated to the valuation*
$$\mathrm{Hom}(M,\overline{k}^*)\times M\to\overline{k}^*$$
where M is a discrete Γ_k-module.

4.1.2. *Poitou-Tate Pairing. Conventions.* In what follows we have assumed the same conventions as in [Har06].

Let M be a discrete Γ_k-module, if v is a place of k we denote with Γ_v the absolute Galois group of the completion k_v. Notice that it can be identified with a subgroup (the decomposition group) of Γ_k. For every Γ_k module M, for any place v of k, one obtain an application of *restriction* $H^i(\Gamma_k,M)\to H^i(\Gamma_v,M)$. A priori this application depends on the choice of an algebraic closure of k_v (which, in general, is different from the one of k) and of the embedding $\overline{k}\hookrightarrow\overline{k}_v$, but it is always possible to make a choice such that the induced maps induced on cohomology is exactly the restriction. Notice that, even if the decomposition group is defined up to conjugation, the induced map on cohomology is uniquely determined.

Achtung (A subtle point)**.** *If G is an algebraic commutative k-group we have a restriction application $H^i(k,G)\to H^i(k_v,G)$ induced by the inclusion $G(\overline{k})\subset G(\overline{k}_v)$ and the arrow $\Gamma_{k_v}\hookrightarrow\Gamma_k$ which, as above, identifies Γ_{k_v} as a decomposition subgroup of Γ_k (for v finite). This differs from the restriction on the action of Γ_{k_v} to $G(\overline{k})$. To get rid of this problem one can work with the henselianization of k in v, which is still contained in $\overline{k}$, instead of the completion to avoid the problem of having a different algebraic closure. This will give us essentially the same duality theorems. For what follows that assumption is not mandatory and we will still consider $H^i(k_v,T)$ as the $\overline{k}$-points of T with the action of Γ_v by restriction (we will solve that problem, for example, thanks to the Rosenlicht's Lemma).*

Achtung. *In what follows $H^i(k_v,M)$ will always denote $H^i(\Gamma_v,M)$ except the case v is archimedean and $i=0$; in this case there are different conventions and we will mean the modified Tate group.*

We say that M, a discrete Γ_k-module, is not ramified if the action of the inertia group on M is trivial, i.e. the quotient $\mathrm{Gal}(k^{nr}/k)$ acts on M. We say that M is not ramified in v, a finite place of k, if it is not ramified w.r.t. the action of Γ_v on M, under this assumption we define $H^i_{nr}(k_v,M)$ as the image of $H^i(k(v),M)$ into $H^i(k_v,M)$, where $k(v)$ denotes the reside field of v. We have $H^0_{nr}(k_v,M)=H^0(k_v,M)$, $H^1_{nr}(k_v,M)=H^1(k(v),M)$ and $H^2_{nr}(k_v,M)=0$ whenever M is a torsion module[3]. (For v archimedean we define $H^i_{nr}(k_v,M)$ simply as $H^i(k_v,M)$).

Equivalently one can say $H^i_{nr}(k,M)=H^i(\mathrm{Gal}(k^{nr}/k),M)$, see the end of chapter 7 of Harari's Notes. Recall also the following result:

Theorem 4.1.1. *k^{nr} is a C_1 field, in particular its Brauer Group is zero.*

Definition 4.1.2. Let M be a discrete Γ_k-module of finite type (as abelian group). Then M is not ramified (up to a finite number of places) and so we can define $\mathbb{P}^i(k,M)$ as the *restricted* product of the $H^i(k_v,M)$ w.r.t. $H^i_{nr}(k_v,M)$, i.e. the elements are $(x_v)_{v\in\Omega_k}$ with $x_v\in H^i(k_v,M)$ for any v and $x_v\in H^i_{nr}(k_v,M)$ for almost every v.

We have

- $\mathbb{P}^0(k,M)=\prod_v H^0(k_v,M)$. If M is finite then it is compact (since $H^0(k_v,M)$ is finite) .
- $\mathbb{P}^1(k,M)$ inherits the topology of the restricted product: $H^1(k_v,M)$ are discrete and a base of open neighbourhood of 0 is given by $\prod H^1_{nr}(k_v,M)$ for almost every v. Since M is of finite type then $\mathbb{P}^1(k,M)$ is locally compact (since $H^1(k_v,M)$ is torsion and finitely generated)

Notation. *For a finitely generated module M we write M^d for the sub-module* $\mathrm{Hom}(M,k^{nr*})$ *of* $M^D=\mathrm{Hom}(M,k^{sep*})$*. If M is finite then M^D is finite and M^{DD} is canonically isomorphic to M.*

Lemma 4.1.3. *Let M be a Γ_k-module of finite type, then the image of*
$$H^i(k,M)\to\prod_{v\in\Omega_k}H^i(k_v,M)$$
is contained in $\mathbb{P}^i(k,M)$. The result is true also with M^d, see Proposition (8.6.1) of [SN86].

[3] Because $cd_p(\hat{\mathbb{Z}})=1$ for any p and the theorem 3.14 of Harari's Notes.

Definition 4.1.4. Let M be a discrete Γ_k-module we define

$$Ш^i(k,M) = \mathrm{Ker}\left(H^i(k,M) \to \prod_{v\in\Omega_k} H^i(k_v,M)\right)$$

The definition, in general, makes sense also with M^d.

Thanks to the Lemma, if M is finitely generated then we have

$$Ш^i(k,M) = \mathrm{Ker}(H^i(k,M) \to \mathbb{P}^i(k,M))$$

and also, thanks to the second part of the Lemma, we have

$$Ш^i(k,M^d) = \mathrm{Ker}(H^i(k,M^d) \to \mathbb{P}^i(k,M^d))$$

We state the Theorem 4.20 of [Mil06].

Theorem 4.1.5 (Poitou-Tate)**.** *Let M be a Γ_k module of finite type, then*

- *The group $Ш^2(k,M^d)$ is finite and is dual to $Ш^1(k,M)$.*
- *There exists a six terms exact sequence of continuous homomorphisms, analogous to the one of Theorem 4.1.*
- *For $i \geq 3$ there is an isomorphism*

$$H^i(k,M^d) \to \prod_{v\ \ real} H^i(k_v,M)$$

Achtung. *Pay attention to the proof in Milne's Book!*

Remark 4.1.6. The proof of this theorem, which is highly non trivial, does not use the explicit description of the pairing. But it can be shown that it coincide with the pairing we are going to describe in terms of cocyles.

We end this section with the explicit construction of the global Poitou-Tate pairing. For any Γ_k-module M of finite type we define

$$\langle\cdot,\cdot\rangle : Ш^2(k,\mathrm{Hom}(M,\overline{k}^*)) \times Ш(k,M)^1 \to \mathbb{Q}/\mathbb{Z}$$

- Let $\beta = [b]$ with $b \in Z^2(\Gamma_k,M^d)$ and $\alpha = [a]$ with $a \in Z^1(\Gamma_k,M)$, then $b \cup a$ is an element of $Z^3(\Gamma_k,\overline{k}^*)$ which is equal to $B^3(\Gamma_k,\overline{k}^*)$ because k is a number filed. And so we can write $b \cup a = dh$ for some $h \in C^2(\Gamma_k,\overline{k}^*)$
- By hypothesis the restriction of b to Γ_v, for any place v, is trivial; hence it has the form $d\xi_v$ for some $\xi_v \in C^1(\Gamma_v,M^d)$
- By definition of h we have $\xi_v \cup a - h \in Z^2(\Gamma_v,\overline{k}^*)$ (dh is just the restriction of b). Let $\varepsilon_v \in \mathrm{Br}\,(k_v)$ be its class, we define

$$\langle\beta,\alpha\rangle := \sum_{v\in\Omega_k} \mathrm{inv}_v(\varepsilon_v) \in \mathbb{Q}/\mathbb{Z}$$

  where the sum is finite because thanks to the previous Lemma and the fact that the Brauer Group of a C_1 field is zero.

It is easy to see that the element we defined is independent of the choices made. Moreover the pairing can be defined even for arbitrary Γ_k-modules but we do not have such an explicit description (the sum is not finite) and the non degeneracy of the pairing.

More over we will need also another version of the global Poitou-Tate paring. For this we follow [SN86]. Namely the theorem (8.6.8) of page 421.

Theorem 4.1.7. *Let M be a finitely generated Γ_k-module, then there exists a perfect pairing*

$$Ш^1(\Gamma_k,M^d) \times Ш^2(\Gamma_k,M) \to \mathbb{Q}/\mathbb{Z}$$

of finite groups, which is induced by the cup product.

Also this pairing can be defined explicitly for any module M as we did before.

4.2. **Descent theory.** We make the following assumptions:

- k a number field,
- X is a smooth, geometrically integral k-variety,
- $\mathrm{Pic}(\overline{X})$ is of finite type,
- $X(\mathbb{A}_k)$ is not empty,
- $\overline{k}^* = \overline{k}(X)^*$ (this assumption will be dropped in the section about tori)
- M is a Γ_k-module of finite type, and $\lambda : M \to \mathrm{Pic}(\overline{X})$ it's type,
- S is the k-group (of multiplicative type) such that $M = \hat{S}$.

Achtung. *The assumption about the Picard group is not made in the book of Skorobogatov but is necessary for the lemma we are going to prove. It is not clear how to fix our proof in order to drop this assumption...*

4.2.1. *Statement of the Main Lemma.* Recall the theorem of Colloit-Thélène and Sansuc.

Theorem 4.2.1. *The class of $-e(x) \in \mathrm{Ext}^2_k(\mathrm{Pic}(\overline{X}, \overline{k}^*))$ coincides with the class of the following 2-fold of Γ_k-modules*

$$0 \to \overline{k}^* \to \overline{k}(X)^* \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0$$

We have the following implication: $X(k) \neq \emptyset \Rightarrow e(x) = 0 \Leftrightarrow \overline{k}^* \to \overline{k}(X)^*$ has a Γ_k-equivariant section.

We state now the main Lemma we are going to prove.

Lemma 4.2.2. $e(x)$ *can be interpreted as an element $b_X \in Ш^2(k, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}^*))$.*
For any adelic point (P_v) and $\alpha \in Ш^1(k, M)$ we have the following equality between the BM pairing and the PT pairing

$$\langle b_X, \lambda_*(\alpha)\rangle = \sum_{v \in \Omega_k} \mathrm{inv}_v(A(P_v))$$

where $A \in Б(X)$ is such that $r(A) = \lambda_*(\alpha)$

Recall that from the low degree terms of the Hochschild-Serre we have the exactness of the following sequence:

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{\Gamma_k} \to H^2(k, \overline{k}^*) \to$$
$$\to \mathrm{Ker}(\mathrm{Br}\,(X) \to \mathrm{Br}\,(\overline{X})) \xrightarrow{r} H^1(k, \mathrm{Pic}(\overline{X})) \to H^3(k, \overline{k}^*) = 0$$

And we defined

$$\mathrm{Br}_0(X) = \mathrm{Im}(\mathrm{Br}\,(k) \to \mathrm{Br}\,(X))$$
$$\mathrm{Br}_1(X) = \mathrm{Ker}(\mathrm{Br}\,(X) \to \mathrm{Br}\,(\overline{X}))$$
$$Б(X) = \{[A] \in \mathrm{Br}_1(X) \text{ s.t. when seen in } \mathrm{Br}_1(X_v) \text{ are in } \mathrm{Br}_0(k_v)\ \forall v\}$$

Thanks to the map $\lambda : M \to \mathrm{Pic}(\overline{X})$ we obtain $\lambda_* : H^1(k, M) \to H^1(k, \mathrm{Pic}(\overline{X}))$, we can define

$$\mathrm{Br}_\lambda(X) := r^{-1}(\lambda_* H^1(k, \mathrm{Pic}(\overline{X})))$$

Notice that it make sense to ask for the existence of an element $A \in \mathrm{Br}_1(X)$ such that

$$r(A) = \lambda_*(\alpha)$$

as in the statement of the lemma, because here r is surjective and α is locally trivial, by definition, and so A lives in $Б(X)$.

4.2.2. *The existence of a torsor of type λ.* Thanks to the comparison between the two paring we are now ready to prove the hardest part of the Main Result on descent theory.

Theorem 4.2.3. *If there exists an adelic point (P_v) orthogonal (w.r.t. the BM paring) to $r^{-1}(\lambda_*(Ш^1(k, M))) \subset \mathrm{Br}_\lambda(X)$ then there exists a torsor $f : Y \to X$ of type λ.*

Proof. By the fundamental exact sequence of C.T.-S. we know that there exist an X-torsor of type λ under S if and only if the image of λ, $\partial(\lambda) = \lambda^*(e(X)) \in H^2(k, S) = \mathrm{Ext}^2_k(M, \overline{k}^*)$ is zero.

The functoriality of the P.T. pairing (which makes sense even if $\mathrm{Pic}(\overline{X})$ is not of finite type) associated to λ can be displayed as the commutativity of

$$\begin{array}{ccccc}
Ш^2(k,\mathrm{Hom}(M,\overline{k}^*)) & \times & Ш^1(k,M) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\uparrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda_*} & & \\
Ш^2(k,\mathrm{Hom}(M,\overline{k}^*)) & \times & Ш^1(k,\mathrm{Pic}(\overline{X})) & \longrightarrow & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Hence, for any adelic point (P_v) as in the lemma and any $\alpha \in Ш^1(k, M)$ we have

$$0 = \sum_v \mathrm{inv}_v(A(P_v)) = \langle b_X, \lambda_*(\alpha)\rangle =$$

(by the Main lemma)

$$= \langle \lambda^* b_X, \alpha\rangle =$$

(by functoriality)

$$= \langle \lambda^*(e(X)), \alpha\rangle = \langle \partial(\lambda), \alpha\rangle = 0$$

and this implies $\partial(\lambda) = 0$ by the non degeneracy of the P.T. pairing applied to M, which is of finite type. □

4.3. **Proof of the Lemma.**

Reference 4.3.1. *Here we present a (quite long) proof using explicit cocycle computation. A modern approach can be found in* [HS10].

Step 0: understanding the map r

We want to represent the map $r : \mathrm{Br}_1(X) \to H^1(k, \mathrm{Pic}(\overline{X}))$ and to show that if $A \in Б(X)$, then $r(A)$ lives in $Ш^1(k, \mathrm{Pic}(\overline{X}))$. To do this we assume the existence of the following commutative exact diagram[4]

$$\begin{array}{ccccccc}
 & & \mathrm{Br}\,(k) & \longrightarrow & \mathrm{Br}_1(X) & \xrightarrow{r} & H^1(k,\mathrm{Pic}(\overline{X})) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H^2(k,\overline{k}(X)^*) & \xrightarrow{=} & H^2(k,\overline{k}(X)^*) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{div}_*} & & \\
H^1(k,\mathrm{Pic}(\overline{k})) & \longrightarrow & H^2(k,\overline{k}(X)^*/\overline{k}) & \longrightarrow & H^2(k,\mathrm{Div}(\overline{X})) & &
\end{array}$$

Where the firs line is from H.S., the first column is induced by the sequence $1 \to \overline{k}^* \to \overline{k}(X)^* \to \overline{k}(X)^*/\overline{k}^* \to 0$ and the bottom line comes from $\to \overline{k}(X)^*/\overline{k}^* \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0$. In particular the map from $H^1(k, \mathrm{Pic}(\overline{X}))$ to $H^2(k, \overline{k}(X)^*/\overline{k}^*)$ is injective because $H^1(k, \mathrm{Div}(\overline{X})) = 0$ (as always: it is a permutation module and so we can apply Shapiro and Hilbert's 90).

Taking the class of an element A in $\mathrm{Br}_1(X)$ we have

$$\begin{array}{ccccc}
 & & & & A \\
 & & & & \downarrow \\
 & & (f_{st}) \in Z^2(\Gamma_k,\overline{k}(X)^*) & \xrightarrow{=} & (f_{st}) \in Z^2(\Gamma_k,\overline{k}(X)^*) \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{div}_*} \\
class(D) & \longrightarrow & \pi(f_{s,t}) & \longrightarrow & dD = \mathrm{div}(f) = 0
\end{array}$$

[4]For the proof see chapter 4 of [Sko01].

where $class(D)$ is a well defined (thanks to the injectiveness discussed above) element of $Z^1(\Gamma_k, \mathrm{Pic}(\overline{X}))$. Thanks to "the shape" of the diagram one can prove the following equality:

$$class(D) = -r(A) \in H^1(k, \mathrm{Pic}(\overline{X}))$$

This is a purely homological algebra diagram chasing, for the proof see Lemma 4.3.2. of Skorobogatov's Book.

Step 1: a very useful remark

Consider the following commutative (by the definition of the action on the hom-sets) diagram:

$$\begin{tikzcd} \Gamma_k - mod \arrow[rr, "\mathrm{Hom}_k(-,\overline{k}^*)"] \arrow[dr, "\mathrm{Hom}(-,\overline{k}^*)"'] & & \mathcal{A}\mathrm{b} \\ & \Gamma_k - mod. \arrow[ur, "M \mapsto M^{\Gamma_k}"'] & \end{tikzcd}$$

Where $\mathrm{Hom}(-, \overline{k}^*)$ is the Γ_k module of group map to $\overline{k}^*$. Since $\overline{k}^*$ is injective as $\mathbb{Z}$-module we have the following relation between total derived functors:

$$(R(-)^{\Gamma_k}) \mathrm{Hom}(-, \overline{k}^*) \cong R(\mathrm{Hom}_k(-, \overline{k}^*))$$

In particular, taking the cohomology, we obtain the following isomorphism (give by the snake lemma)

$$H^i(k, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}^*)) \cong \mathrm{Ext}^i_k(\mathrm{Pic}(\overline{X}), \overline{k}^*)$$

Moreover, for $i = 2$, this isomorphism maps b_X to $e(X)$.

Achtung. *Notice that we proved something similar when we discussed the fundamental exact sequence of C.T.-S. (in the previous talk). But this result applies without any assumptions on the Γ_k-module, in particular even if $\mathrm{Pic}(\overline{X})$ is not of multiplicative type. Moreover here we obtain an explicit description of the isomorphism, and this will be essential to carry on our proof with cocyles.*

Step 2: Representation of b_X

We argue as in the theorem of C.T.-S. about the representation of the elementary obstruction, because we want to understand the cohomology class of b_X in terms of a 2-cocyle.

The proof relies on the following remark: The Yoneda pairing

$$\mathrm{Ext}^1_k(\mathrm{Div}_0(\overline{X}), \overline{k}^*) \xrightarrow{\partial} \mathrm{Ext}^2_k(\mathrm{Pic}(\overline{X}), \overline{k}^*)$$

sends the inverse of class of the extension $1 \to \overline{k}^* \to \overline{k}(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}_0(\overline{X}) \to 0$ to the class of $1 \to \overline{k}^* \to \overline{k}(X)^* \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0$. Hence it is enough to give a description in terms of cocyles of the first extension, as element of $H^1(k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}^*)$, and to follow the construction of the connecting homomorphism

$$H^1(k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}^*) \xrightarrow{\partial} H^2(k, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}^*))$$

Choose $\overline{P} \in X(\overline{k})$ and let $\mathcal{O}_{\overline{X},\overline{P}}$ be the local ring of $\overline{X}$ at $\overline{P}$. We have a section of the inclusion $\overline{k}^* \hookrightarrow \mathcal{O}_{\overline{X},\overline{P}}$ given by the association $g \mapsto g(\overline{P})$. Moreover the sequence of abelian groups

$$1 \to \mathcal{O}_{\overline{X},\overline{P}} \to \overline{k}(X)^* \to \mathrm{Div}_{\overline{P}}(\overline{X}) \to 0$$

is split, since $\mathrm{Div}_{\overline{P}}(\overline{X})$ is projective. This gives a section (of groups) from $\overline{k}(X)^*$ to $\mathcal{O}_{\overline{X},\overline{P}}$, by composition we obtain a section

$$e_{\overline{P}} : \overline{k}(X)^* \to \overline{k}^*$$

which, if g is invertible at $\overline{P}$ can be expressed by the association $g \mapsto g(\overline{P})$. Notice that if $\overline{P}$ is a k-point, i.e. $\overline{P} = P \in X(k)$, then the section $e_{\overline{P}}$ can be made Γ_k-equivariant.

Consider the extension

$$1 \to \overline{k}^* \to \overline{k}(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}_0(\overline{X}) \to 0$$

and the map

$$\sigma_{\overline{P}} : \mathrm{Div}_0(\overline{X}) \to \overline{k}(X)^*, \ \mathrm{div}(g) \mapsto \frac{g}{e_{\overline{P}}(g)}$$

This map of groups is a well defined section of div since g is uniquely determined up to scalar multiplication and this problem is eliminated by taking the ratio. From $\sigma_{\overline{P}} \in C^0(\Gamma_k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}(X)^*))$ we obtain $d\sigma_{\overline{P}} \in B^1(\Gamma_k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}(X)^*))$ given by

$$d\sigma_{\overline{P}} : a \mapsto \frac{a \cdot \sigma_{\overline{P}}}{\sigma_{\overline{P}}} = \frac{\sigma_{a \cdot \overline{P}}}{\sigma_{\overline{P}}}$$

hence $d\sigma_{\overline{P}}$ is an element of $Z^1(\Gamma_k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}^*))$. Notice that we are just following the proof of the snake lemma, and so was not necessary to prove the last assertion.

By construction the class of $d\sigma_{\overline{P}} \in H^1(k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}^*))$ corresponds to the inverse of the extension

$$1 \to \overline{k}^* \to \overline{k}(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}_0(\overline{X}) \to 0$$

where we changed sign because of the relation

$$\sigma_{\overline{P}}(\mathrm{div}(-)) + e_{\overline{P}}(-) = Id_{\overline{k}(X)^*}(-)$$

(The assertion "by construction" should be clear: $e_{\overline{P}}$ is a section of the map $\overline{k}^* \to \overline{k}(X)^*$ which lives in the preimage of the identity when you apply the functor $\mathrm{Hom}(,\overline{k}^*)$.) We are almost done, following the strategy explained at the beginning: With the following exact sequences in mind

$$0 \to \mathrm{Div}_0(\overline{X}) \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0$$

$$0 \to \mathrm{Hom}(\mathrm{Pic}(\overline{X})) \to \mathrm{Hom}(\mathrm{Div}(\overline{X}), \overline{k}^*) \to \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}^*) \to 0$$

we do a bit of diagram chasing: by injectivity of $\overline{k}^*$ we extend $d\sigma_{\overline{P}}$ to an element $\psi_{\overline{P}} \in C^1(\Gamma_k, \mathrm{Hom}(\mathrm{Div}(\overline{X}), \overline{k}^*))$, and we obtain $d\psi_{\overline{P}} \in B^2(\Gamma_k, \mathrm{Hom}(\mathrm{Div}(\overline{X}), \overline{k}^*))$ whose restriction to $\mathrm{Div}_0(\overline{X})$ is $d^2\sigma_{\overline{P}} = 0$. And so we have $d\psi_{\overline{P}} \in Z^2(\Gamma_k, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}^*))$, which is the element that represents the class of $-b_X$.

Step 3: P.T. pairing, global factor
Since $X(\mathbb{A}_k) \neq \emptyset$ we have that $e(X)$ goes to zero under the restrictions from k to k_v for all places v (we have a Γ_v-equivariant section), and so b_X belongs to $Ш^2(k, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}^*))$.

To finish the proof we have to compute the P.T pairing

$$Ш^2(k, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}^*)) \times Ш^1(k, \mathrm{Pic}(\overline{X})) \to \mathbb{Q}/\mathbb{Z}$$

of $\langle b_X, r(A)\rangle$ but, thanks to the previous steps, we have $b_X = -d\psi_{\overline{P}}$ and $r(A) = -class(D)$, so it is enough to compute the pairing $\langle d\psi_{\overline{P}}, class(D)\rangle$. With the notation of *Step 0* we define

$$h := \psi_{\overline{P}} \cup D - e_{\overline{P}} \cup f \in C^2(\Gamma_k, \overline{k}^*)$$

since, by direct computation, we have $dh = d\psi_{\overline{P}} \cup D$, as wanted.

Step 4: P.T. pairing, local factor
We notice that the support of $\mathrm{div}(f_{s,t})$ is "small": $f_{s,t}$ are continuous map from $\Gamma_k \times \Gamma_k$ (with the profinite topology) to $\overline{k}(X)^*$ (with the discrete one), since the image of a compact in a discrete space is finite we have that $\mathrm{div}(f_{s,t})$ is supported by finitely many principal divisors. By the implicit function theorem (X is smooth) we can choose $P_v \in X(k_v)$away from $\mathrm{div}(f_{s,t})$ for any place v.

As in *Step 2* we consider the Γ_v-equivariant map σ_{P_v}, and we define

$$\theta_v : \mathrm{Div}_0(\overline{X}) \to \overline{k}_v^*$$

as the composition of the inclusion of $\mathrm{Div}_0(\overline{X})$ in $\mathrm{Div}_0(\overline{X}_v)$ and the map $\frac{\sigma_{\overline{P}}}{\sigma_{P_v}} : \mathrm{Div}_0(\overline{X}_v) \to \overline{k}_v^*$. Whenever $g \in \overline{k}(X)^*$ is invertible at $\overline{P}$ and P_v we can write

$$\theta_v(\mathrm{div}(g)) = \frac{g}{g(\overline{P})}\frac{g(P_v)}{g} = \frac{g(P_v)}{g(\overline{P})}$$

(if not this still make sense but we do not have an explicit description).

Since σ_{P_v} is equivariant we have $d\sigma_{P_v} = 0$ and so, by the derivation rule, $d(\sigma_v)$ is the "restriction" of $d\sigma_{\overline{P}} \in Z^1(\Gamma_k, \mathrm{Hom}(\mathrm{Div}_0(\overline{X}), \overline{k}^*))$ to k_v. As above $\overline{k}_v^*$ is an injective abelian group and so $\theta_v : \mathrm{Div}_0(\overline{X}) \to \overline{k}_v^*$ extends to a map

$$\mu_v : \mathrm{Div}(\overline{X}) \to \overline{k}^*$$

Our candidate for the local factor is

$$\xi_v := \psi_{\overline{P}} - d\mu_v \in C^1(\Gamma_v, \mathrm{Hom}(\mathrm{Div}(\overline{X}), \overline{k}_v^*))$$

As in *Step 0* ξ_v is trivial on $\mathrm{Div}_0(\overline{X})$:

- the restriction of $\psi_{\overline{P}}$ to $\mathrm{Div}_0(\overline{X})$ is $d\sigma_{\overline{P}}$
- the restriction of $d\mu_v$ to $\mathrm{Div}_0(\overline{X}_v)$ is $d\sigma_v$ which is the restriction of $d\sigma_{P_v}$.

Hence ξ_v belongs to $C^1(\Gamma_v, \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \overline{k}_v^*))$. Moreover $d\xi_v$ is just the restriction of $d\psi_{\overline{P}}(\leftrightarrow -b_x)$ to Γ_v (since $d\xi_v = d\psi_{\overline{P}} - dd\mu_v$). As claimed ξ_v is our local factor.

Step 5: last computation

We define $\epsilon_v \in \mathrm{Br}\,(k_v)$ as the class of the cocyle

$$\xi_v \cup class(D) - h = \psi_{\overline{P}} \cup D - d\mu_v \cup D - \psi_{\overline{P}} \cup D + e_{\overline{P}} \cup f =$$
$$= e_{\overline{P}} \cup f - d\mu_v \cup D = -d(\mu_v \cup D) + \mu_v \cup dD + e_{\overline{P}} \cup f$$

Hence ϵ_v can be also represented by $\theta_v \cup \mathrm{div}(f) + e_{\overline{P}} \cup f$. But now we know how to compute them:

$$\sigma_v \cup \mathrm{div}(f) = \frac{e_{P_v}(f)}{e_{\overline{P}}(f)}, \text{ and } e_{\overline{P}} \cup f = e_{\overline{P}}(f)$$

Thanks to our choice of the P_vs we can write that ϵ_v is the class of $f(P_v) = f_{s,t}(P_v)$. We proved

$$\mathrm{inv}_v(\xi_v \cup D - h) = \mathrm{inv}_v(A(P_v)), \text{ with } A = (f_{s,t}).$$

4.4. **Torsor under tori.**

4.4.1. *Groups of Multiplicative type.* Recall that a group of multiplicative type S over k is a commutative linear k-group which is an extension of a finite group by a torus. The module of characters of S is the abelian group $\hat{S} = \mathrm{Hom}(\overline{S}, \mathbb{G}_m)$, equipped with the action of the Galois group Γ_k.

We state here some results we used many times during this Workshop.

Theorem 4.4.1 (Rosenlicht's Theorem)**.** *Let X, Y be geometrically irreducible k-varieties, then $\overline{k}[X]^*/\overline{k}^*$ is a free abelian group of finite rank and there is an isomorphism of groups*

$$\overline{k}[X \times_k Y]^*/\overline{k}^* \cong \overline{k}[X]^*/\overline{k}^* \times \overline{k}[Y]^*/\overline{k}^*$$

Theorem 4.4.2. *Let G be a k-group. The association $G \rightsquigarrow \hat{G}$ gives an equivalence of category between the category of k groups of multiplicative type and the category of discrete Γ_k-modules of finite type. Moreover a sequence of groups of multiplicative type is exact iff the dual of Γ_k-modules of characters is exact.*

4.4.2. *The only obstruction to the Hasse principle.* Let G be an algebraic group of the following list

(1) a torus,
(2) a semisimple group,
(3) an abelian variety.

Then for G-torsors under X the only obstruction to the Hasse principle is the one attached to Б. In this situation, following the proof of the Main Lemma, one obtain a simplified relation between the adelic Manin pairing and the P.T. pairing using, in the corresponding situations, the following facts:

(1) "Γ_k-equivariant" Rosenlicht's Theorem, i.e. the exactness of

$$1 \to \overline{k}^* \to \overline{k}[X]^* \to \hat{G} \to 0.$$

(2) The exact sequence

$$1 \to F \to G^{sc} \to G^{ss} \to 1$$

and the bijection $H^1(k, G^{sc}) \cong \prod_{v\ real} H^1(k_v, G^{sc})$ of the Kneser-Harder-Chernousnov's Theorem.

(3) The Cassel-Tate pairing:

$$Ш(S) \times Ш(S^t) \to \mathbb{Q}/\mathbb{Z}.$$

Today we will discuss just the first case, the theory of semisimple groups will be discussed in the two talks of Professor Harari (see the next section), with many generalisations and the case of abelian varieties is described in the last section of the notes.

As stated above our main goal is to prove the following.

Theorem 4.4.3. *Let X be a torsor under a torus T, then the Manin obstruction related to $Б(X)$ is the only obstruction to the Hasse principle.*

Achtung. *We drop the assumption $\overline{k}^* = \overline{k}[X]^*$.*

Thanks to the result of Rosenlicht we have

$$\hat{T} = \overline{k}[T]^*/\overline{k}^* \cong \overline{k}[X]^*/\overline{k}^*$$

where the last isomorphism is given because, over $\overline{k}$, X and T are isomorphic (the category of torsors is a groupoid!). It is quite easy to show that it does not depend on the choice of a $\overline{k}$-point and that it is Γ_k-equivariant: we can represent X as a cocyle $c \in Z^1(\Gamma_k, T(\overline{k}))$ such that the action of $g \in \Gamma_k$ on $s \in T(\overline{k}) = X(\overline{k})$ is given by

$$g(s) = c(g) \cdot^g s$$

Moreover the action of $s \in T(\overline{k})$ on $\overline{k}[T]^*/\overline{k}^* = \hat{T}$ is just the character multiplied by its value in s, hence it is trivial.

This gives the exactness of

$$1 \to \overline{k}^* \to \overline{k}[X]^* \xrightarrow{\gamma} \hat{T} \to 0$$

as Γ_k-modules.

Achtung. *Thanks to Rosenlicht we also have that $\hat{T} = \overline{k}[X]^*/\overline{k}^* = \overline{k}_v[X]^*/\overline{k}_v^*$.*

Lemma 2.4.3 Skorobgatov ensures that the class of such extension in $\mathrm{Ext}^1_k(\hat{T}, \overline{k}^*)$ corresponds to the class of $-[X]$ in $H^1(k, T)$.

Consider the low degree exact sequence from the H.S.

$$\mathrm{Pic}(\overline{X})^{\Gamma_k} \to H^2(k, \overline{k}[X]^*) \to \mathrm{Br}_1(X) \to H^1(k, \mathrm{Pic}(\overline{X})) \to H^3(k, \overline{k}[X]^*)$$

Since $\mathrm{Pic}(\overline{T}) = \mathrm{Pic}(\overline{X}) = 0$ we get $\mathrm{Br}_1(X) \cong H^2(k, \overline{k}[X]^*)$. Moreover the long exact sequence induced by $1 \to \overline{k}^* \to \overline{k}[X]^* \xrightarrow{\gamma} \hat{T} \to 0$ gives

$$\mathrm{Br}\,(k) \to H^2(k, \overline{k}[X^*]) = \mathrm{Br}_1(X) \to H^2(k, \hat{T}) \to H^3(k, \overline{k}^*) = 0$$

Thanks to the surjectivity of the last map we obtain an isomorphism

$$i : Б(X)/\mathrm{Br}_0(X) \cong Ш^2(k, \hat{T})$$

and this isomorphism allows as to state the "comparison lemma" between the two pairing.

Lemma 4.4.4. *Let k be a number field and X a k-torsor under the action of a torus T such that $X(\mathbb{A}_k) \neq \emptyset$. Then the class of X in $H^1(k, T)$ belongs to $Ш^1(k, T)$. Let $A \in Б(X)$, for any adelic point (P_v) we have the following equality:*

$$\sum_{v \in \Omega_k} \mathrm{inv}_v(A(P_v)) = -\langle [X], i(A) \rangle.$$

Achtung. *Notice that there the pairing of the LHS is not exactly the P.T. pairing of the previous section, but it is the one Theorem 4.1.7. This will not be a big problem since it has a very similar description in terms of cocyles (as proved in Neukirch's Book). Both of them are not degenerate!*

Proof of the Theorem. From the equation between the pairings we get that the Manin obstruction attached to $\text{Б}(X)$ is the only one:

$$X(\mathbb{A}_k)^{\text{Б}(X)} \neq \emptyset \Rightarrow X(k) \neq \emptyset$$

Indeed, given (P_v) such that $\sum_{v\in\Omega_k} \operatorname{inv}_v(A(P_v)) = 0$ for any $A \in \text{Б}(X)$ we have

$$-\langle [X], i(A)\rangle = 0, \quad \forall A \in \text{Б}(X)$$

and the non degeneracy of (the second) P.T. pairing this implies $[X] = 0$ and so there exists a rational point (if we see X in $\operatorname{Ext}^1_k(\hat{T}, \overline{k}^*)$ it corresponds to a Γ_k-equivariant section). □

Proof of the Lemma. The Proof is essentially the same of the other "comparison lemma" but more easy: one considers the two group sections

$$e_{\overline{P}} : \overline{k}[X]^* \to \overline{k}^*$$
$$\sigma_{\overline{P}} : \hat{T} \to \overline{k}[X]^*$$

We have $\sigma_{\overline{P}}(\gamma(-)) + e_{\overline{P}}(-) = 1$ and so $d\sigma_{\overline{P}} = -de_{\overline{P}}$ and $d\sigma_{\overline{P}}$ represents, in $\operatorname{Ext}^1_k(\hat{T}, \overline{k}^*)$ the class of $1 \to \overline{k}^* \to \overline{k}[X]^* \xrightarrow{\gamma} \hat{T} \to 0$ (following the proof of the snake lemma), which corresponds $-[X]$.

To describe explicitly the pairing $\langle [X], i(A)\rangle$ with $A = (f_{s,t}) \in Z^2(\Gamma_k, \overline{k}[X]^*)$ we define

$$h = -e_{\overline{P}} \cup f = f(\overline{P})^{-1}$$

because $dh = -de_{\overline{P}} \cup f = d\sigma_{\overline{P}} \cup f$. The local factor is just

$$\sigma_v : \hat{T} \to \overline{k}_v^*, \;\; \gamma(g) \mapsto \frac{g(P_v)}{g(\overline{P})}$$

σ_{P_v} is Γ_v-equivariant, hence $d\sigma_{P_v} = 0$. Thanks to this we have that $d\sigma_v \in Z^1(\Gamma_v, \operatorname{Hom}(\hat{T}, \overline{k}_v^*))$ is the restriction to k_v of the cocyle $d\sigma_{\overline{P}}$, whose class in $H^1(k, \operatorname{Hom}(\hat{T}, \overline{k}^*))$ is $-[X]$. We are done since ϵ_v corresponds to the class of

$$\sigma_v \cup \gamma(f) - h = \sigma_v \cup \gamma(f) + e_{\overline{P}} \cup f = \frac{f(P_v)}{f(\overline{P})} + f(\overline{P}) = f(P_v).$$

□

5. Talk 5: Weak Approximation on linear groups, Professor Harari

Notes taken by Gregorio Baldi

5.1. **Weak approximation on tori.**

5.1.1. *Statement of the main theorem.* Let k be a number field, T a torus and $\hat{T}$ the Galois module of its characters. $\overline{T(k)}$ is defined as the closure of $T(k)$ into $\prod_{v\in\Omega_k} T(k_v)$ with the direct product topology. And $\overline{T(k)}^S$ the closure inside $\prod_{v\in S} T(k_v)$ (S finite set of places).

Recall the Tate's local pairing (given by cup product):

$$H^0(k_v,T)\times H^2(k_v,\hat{T})\to \mathrm{Br}\,(k_v)$$

where $H^0(k_v,T)=T(k_v)$ and $\hat{H}^0(k_v,T)$ if $v|\infty$. $\hat{T}$ is not finite and so it induces a duality (perfect pairing) on the profinite completions of $H^0(k_v,T)$ and $H^2(k_v,\hat{T})$.

This pairing induces a map $\prod_{v\in\Omega_k} T(k_v)\xrightarrow{\theta} Ш^2_\omega(\hat{T})^D$ where

$$Ш^2_\omega(\hat{T})=\{\alpha\in H^2(k,\hat{T}),\alpha_v=0\in H^2(k_v,\hat{T})\text{ for almost all } v\}$$

It contains the usual $Ш^2(\hat{T})$. Moreover we will denote

$$Ш^2_S(\hat{T})=\{\alpha\in H^2(k,\hat{T}),\alpha_v=0\text{ for } v\notin S\}\subset Ш^2_\omega(\hat{T})$$

And the map is given by

$$\theta:(t_v)\mapsto(\alpha\mapsto\sum_v(t_v,\alpha_v))\text{ (Tate Pairing)}$$

it is a sum of finitely many terms by the definition of $Ш^2_\omega$.

Theorem 5.1.1 (Voskresenskii, Sansuc)**.** *There is an exact sequence*

$$0\to\overline{T(k)}\xrightarrow{i}\prod_{v\in\Omega_k}T(k_v)\xrightarrow{\theta}Ш^2_\omega(\hat{T})^D\to Ш^1(T)\to 0$$

for S finite the sequence becomes

$$0\to\overline{T(k)}^S\xrightarrow{i}\prod_{v\in S}T(k_v)\xrightarrow{\theta}Ш^2_S(\hat{T})^D\to Ш^1(T)\to 0$$

The important part of the sequence are the first three terms.

5.1.2. *Consequences.*

Theorem 5.1.2.

a) The "defect of W.A." (=the cokernel of i) is finite. "It almost satisfies WA".
b) T satisfies "weak weak approximation"= there exists $S_0\subset\Omega_f$ such that $T(k)$ is dense in $\prod_{v\in S}T(k_v)$ for every finite S s.t. $S\cap S_0=\emptyset$.
c) Let $T^c\supset T$ a smooth compactification of T, BM obstruction to WA on T^c is the only one.

Remark 5.1.3. Part $a)$ is far from being true for abelian varieties: the group that takes in to account the defect of WA is huge: H^1 of the dual abelian varieties. Part $c)$ is hard to translate on a function field: we do not have resolution of singularities.

Exercise 5.1.4. Projective, not simply connected then the condition $b)$ can not hold.

Proof. a). $Ш^2_\omega(\hat{T})$ finite? Suppose T is split, i.e. $\hat{T}=\mathbb{Z}^n$. Then

$$Ш^2_\omega(\mathbb{Z})=Ш^1_\omega(\mathbb{Q}/\mathbb{Z})$$

because $H^2(k,\mathbb{Z})\cong H^1(k,\mathbb{Q}/\mathbb{Z})$ (from the sequence $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$). Moreover

$$Ш^1_\omega(\mathbb{Q}/\mathbb{Z})=0$$

By Chebotarev (Proposition 9.2 of Harari's notes) and writing $\mathbb{Q}/\mathbb{Z}$ as limit of finite groups. Take L/k galois s.t. T splits over L:

$$Ш^2_\omega(\hat{T}) \subset \operatorname{Ker}(H^2(k,\hat{T}) \to H^2(L,\hat{T})) = H^2(\operatorname{Gal}(L/k),\hat{T})$$

Where the first inclusion is because, by the split case, we have $Ш^2_\omega(L,\hat{T}) = 0$; and the last equality follows from the inflation-restriction exact sequence (thanks to the assumption $H^1(L,\hat{T}) = H^1(L,\mathbb{Z}^n) = 0$, as in Corollaire 1.24 of Harari's Notes). But $\hat{T}$ is of finite type and so the galois cohomology group is finite. $\square$

b). $Ш^2_\omega(\hat{T})$ finite and so there exists $S_0 \subset \Omega_k$ s.t. for any $\alpha \in Ш^2_{\Omega_k}(\hat{T})$, $\alpha_v = 0$ for every $v \notin S_0$. We conclude since if $S \cap S_0 = \emptyset$, then $Ш^2_S(\hat{T}) = Ш^2(\hat{T})$ and so there is no obstruction to W.W.A. outside S_0 (by the second exact of the main theorem sequence with such an S). $\square$

c). $T \hookrightarrow T^c$, as in the last section of the previous talk we prove

$$Ш^2_\omega(\hat{T}) \cong \operatorname{Br}_1(T^c)/\operatorname{Br}(k)$$ [5]

We have $\operatorname{Pic}(\overline{T}) = 0$ since $\overline{T} \cong \mathbb{G}_m^n$. We have the following exact sequence (Rosenlicht's Lemma)

$$0 \to \hat{T} = \overline{k}[T]^*/\overline{k}^* \xrightarrow{\operatorname{div}} P \to \operatorname{Pic}(\overline{T}^c) \to 0$$

Where P is the permutation module of divisors at the infinity, and so $\operatorname{Br}_1(T^c)/\operatorname{Br}(k) \cong H^1(k,\operatorname{Pic}(\overline{T}^c))$. $H^1(k,P) = 0$, $Ш^2_\omega(P) = 0$ ($Ш^2_\omega(\mathbb{Z}) = 0$)

$$0 \to \overline{T(k)} \to \prod T(k_v) \xrightarrow{\theta} (\operatorname{Br}_1(T^c)/\operatorname{Br}(k))^D$$

Now we are left to check the compatibility of the BM pairing with θ. See Sansuc. $\square$

$\square$

5.1.3. *Proof of the main theorem.* a) $T = \mathbb{G}_m$ ok, $Ш^2_\omega(\hat{T}) = Ш^2_\omega(\mathbb{Z}) = 0$ and $\mathbb{G}_m$ satisfies WA (it is an open subset of the affine space). And $Ш^1(\mathbb{G}_m) = 0$ by Hilbert's 90. Moreover it is the same for quasi-trivial tori, by Shapiro.
b) Ono's Lemma: T a torus, there exist two quasi-trivial tori R, R' (i.e. $\hat{R}$ is permutation module and $R \cong \prod R_{k_i|k}(\mathbb{G}_m) \hookrightarrow \mathbb{A}^m_k$) such that the following is exact

$$0 \to F \to R \to T^m \times R' \to 0$$

with F finite. So we can assume the existence of a sequence

$$0 \to F \to R \to T \to 0$$

with R quasi-trivial and F finite. Since if the theorems holds for $T^m \times R'$ then holds also for T, because taking the powers is not a problem and R' is quasi trivial and so does not count by the previous step, as explained by Proposition 1.4.1.
c) S finite contained in Ω_k, we have the following exact sequence

$$H^1(k,F) \to \prod_{v\in S} H^1(k_v,F) \to Ш^1_S(\hat{F})^D$$

thanks to the proposition at the end of the discussion [this part holds also for F of finite type].
d) From the sequence $0 \to F \to R \to T \to 0$ we get a big diagram with exact rows

$$\begin{tikzcd}
R(k) \arrow[r] \arrow[d] & T(k) \arrow[r] \arrow[d] & H^1(k,F) \arrow[r] \arrow[d] & H^1(k,R)=0 \\
\prod_{v\in S} R(k_v) \arrow[r] & \prod_{v\in S} T(k_v) \arrow[r] \arrow[d] & \prod_{v\in S} H^1(k_v,F) \arrow[d] & \\
 & Ш^2_S(\hat{T})^D \arrow[r] & Ш^1_S(\hat{F})^D &
\end{tikzcd}$$

[5] One could be even more precise: it is equal to $\operatorname{Br}(T^c)/\operatorname{Br}(k)$, because there are not trascendental Brauer classes.

And the last map between the Sha groups is an isomorphism because $H^1(k,\hat{R}) = Ш^2_S(k,\hat{R}) = 0$, by step a) and the fact that the map is the one induced by $0 \to \hat{T} \to \hat{R} \to \hat{F} \to 0$.

Remark 5.1.5. The idea is to reduce the statement about torus to the case of a finite group, where we can apply Poitou-Tate!

By easy diagram chasing and the fact that $\overline{R(k)}^S = \prod_{v\in S} R(k_v)$ we get the exactness of

$$0 \to \overline{T(k)} \to \prod_{v\in S} T(k_v) \to Ш^2_S(\hat{T})^D$$

Moreover, by definition of $Ш^2_S$ we have the exactness of

$$0 \to Ш^2(\hat{T}) \to Ш^2_S(\hat{T}) \to \bigoplus_v H^2(k_v,\hat{T})$$

Dualizing, using the iso $(Ш^2(\hat{T}))^D = Ш^1(T)$, given by Poitou-Tate applied to a torus and the isomorphism given by the Local Tate pairing (?? paying attention to the archimedean places) we get the right cokernel.
e) Recall that $Ш_\omega$ is defined as the limit of $Ш_S$ with S finite. So, taking the limit over S, gives the exactness of

$$0 \to \overline{T(k)} \to \prod_{v\in\Omega_k} T(k_v) \to Ш^2_\omega(\hat{T})^D$$

Proposition 5.1.6. Given F a finite Γ_k-module then the following is exact:

$$H^1(k,F) \to \prod_{v\in S} H^1(k_v,F) \to Ш^1_S(\hat{F})^D$$

Proof. By the exact mid three terms of the nine terms exact sequence given by the global Poitou-Tate duality (as stated at the beginning of the previous section) we have the exactness of

$$H^1(k,M) \to \mathbb{P}^1_S(k,M) \to H^1(k,\hat{M})^D$$

This implies the exactness of

$$Ш^1_S(M) \to \prod_{v\in S} H^1(k_v,M) \to H^1(k,\hat{M})^D$$

Which, applied to $M = \hat{F}$ and dualizing we obtain

$$H^1(k,F) \to \prod_{v\in S} H^1(k_v,\hat{F})^D \to Ш^1_S(\hat{F})^D$$

The local Tate duality induced by cup product (for finite Γ_{k_v}-modules) gives us the isomorphism

$$H^1(k_v,\hat{F})^D \cong H^1(k_v,F)$$

Hence the result is proved. $\square$

5.2. **Arithmetic of linear algebraic groups.**

Reference 5.2.1. *The book* [VA94] *and the article of Sansuc* [San81].

5.2.1. *A few Remainders. All you have to know about the structure of linear algebraic groups.*

k field (of $char(k) = 0$) [in positive char it is more easy to assume also smoothness].

Definition 5.2.2. A *linear* algebraic group G over k which is a Zariski closed of GL_n. Equivalently G affine group scheme over k.

Example 5.2.3. GL_n, SL_n, PGL_n a torus.

Assume G connected.

- G^u = unipotent radical of G. Unipotent group means that one can embed it into the group of unipotent matrices. Not very interesting for us: U unipotent implies $H^1(k,U) = 0$ as a variety $U \cong \mathbb{A}^m_k$.

- G/G^u is a reductive group

Assume G reductive, G is an extension

$$1 \to G^{ss} \to G \to T \to 1$$

with G^{ss} semi simple. Notice that a semisimple group is equal to its derived subgroup $G' = [G, G]$, and this implies $\hat{G^{ss}} = 1$ character free ($\mathbb{G}_m$ is abelian, $\mathrm{Hom}(G, \mathbb{G}_m) = \mathrm{Hom}(G/[G, G], \mathbb{G}_m) = 0$) and so $\overline{k}[G^{ss}]^* = \overline{k}^*$ (by Rosenlicht).

Reference 5.2.4. *See* [ABD+66] *(very general) or Milne's notes* [Mil11]*. Also Chapter 5 of* [Poo11] *contains a short overview.*

For a semisimple group there exists a universal covering G^{sc} which is semi-simple, simply connected (there are not non trivial geometric étale covering, i.e. over the algebraic closure) and with $\mathrm{Pic}(\overline{G^{sc}}) = 0$ that fits in the following exact sequence:

$$1 \to F \to G^{sc} \to G^{ss} \to 1$$

F finite, central. This exact sequence induces (Thanks to Corollary 6.11 of [San81]) the following:

$$0 \to \hat{G^{ss}}(\overline{k}) \to \hat{G^{sc}}(\overline{k}) \to \hat{F}(\overline{k}) \to \mathrm{Pic}(\overline{G^{ss}}) \to \mathrm{Pic}(\overline{G^{sc}}) \to \mathrm{Pic}(\overline{F}) = 0$$

Since G^{sc} and G^{ss} are semi-simple, the firs two terms are zero and so one get also $\mathrm{Pic}(\overline{G^{ss}}) = \hat{F}$.

Reference 5.2.5. *For the existence of the universal covering see Sansuc's Crelle or "*On Picard groups of algebraic fibre spaces*" of R. Fossum and B. Iversen.*

Example 5.2.6. SL_n is semisimple, GL_n is reductive but not semisimple.

5.2.2. *Arithmetic of G^{sc}.* Let G^{sc} be a semi-simple and simply connected group over a filed k.

Theorem 5.2.7 (Harder)**.** *Assume k a p-adic field (i.e. finite extension of $\mathbb{Q}_p$). Then $H^1(k, G^{sc}) = 0$*

Every principal homogeneous space over the p-adic field under a semi simple and simply connected group is trivial, i.e. has a rational points. Not easy to prove: you have to know the classification of those groups.

Example 5.2.8. $G = SL_n$ and its twisted forms SL_D.

Assume k a number field.

Theorem 5.2.9 (Platonov)**.** *G^{sc} satisfies WA.*

And finally the harder theorem.

Theorem 5.2.10 (Kneser-Harder-Chernousnov)**.** *Every principal homogeneous space of G^{sc} satisfies Hasse principle:*

$$H^1(k, G^{sc}) \to \prod_{v\ real} H^1(k_v, G^{sc})$$

is a bijection.

5.2.3. *Main Theorem.*

Theorem 5.2.11 (Sansuc, 1981)**.** *Let G be a connected linear group over a number field k. Then*

(1) Let X be a principal homogeneous space of G, then the BM obstruction to the Hasse Principle associated to $\text{Б}(X)$ is the only one.
(2) Let G^c be a smooth compactification of G, then the BM obstruction to WA is the only one for G^c.

Reference 5.2.12. *The first part of* [San81] *(It is a fundamental paper!).*

5.2.4. *W.A. by Galois Cohomology.* k a number field, G connected (reductive) linear group.

Remark 5.2.13. At least in *char* 0 reductive is not a restrictive assumption: the unipotent part plays no role.

We try to mimic the property of Ono's Lemma as follows.

Definition 5.2.14. A special covering of G is an exact sequence of k-groups[6]

$$1 \to B \to G' \to G \to 1 \tag{5.1}$$

with B finite (as k-group), $G' \cong R \times G_0$ where R is a quasi trivial torus (i.e. isomorphic to the product of Weil restrictions: $R \cong \prod R_{k_i/k}\mathbb{G}_m$) and G_0 is semi simple, simply connected. Then we have:

- B is central in G',
- $H^1(k, G') \cong \bigoplus_{v\ real} H^1(k_v, G') \cong \bigoplus_{v\ real} H^1(k_v, G_0)$ since $H^1(k,R) = H^1(k_v,R) = 0$ by Shapiro's Lemma and Hilbert's 90, and Kneser-Harder-Chernusov.

The proof of the following Lemma is just a generalization of Ono's Lemma.

Lemma 5.2.15. *There exists $m > 0$ and a quasi trivial torus R_0 s.t. $G^m \times R_0$ has a special covering.*

Thanks to this, from now on, we can assume (5.1) does exists.

Theorem 5.2.16 (Sansuc)**.** *Let $S \supset \Omega_\infty$ be a finite set of places of k. Then*

$$\mathrm{Coker}\left[\overline{G(k)} \to \prod_{v\in S} G(k_v)\right] \cong \mathrm{Coker}\left[H^1(k,B) \to \bigoplus_{v\in S} H^1(k_v, B)\right]$$

So the defect of weak approximation at S is given by

$$Ш^1_S(\hat{B})^D = \mathrm{Ker}[H^1(k,\hat{B}) \to \prod_{v\in S} H^1(k_v,\hat{B})]^D$$

where $\hat{B} = \mathrm{Hom}(B, \mathbb{G}_m)$.

Taking the limit over S, we obtain the following Corollary (exactly as in the case of a torus).

Corollary 5.2.17. *There is an exact sequence*

$$0 \to \overline{G(k)} \xrightarrow{i} \prod_{all\ v} G(k_v) \xrightarrow{\theta} Ш^1_\omega(\hat{B})^D \to Ш^2(B) \to 0$$

(the last arrow is from global P.T.). Then Coker i *is finite and G satisfies WWA.*

The idea is to abelianize the cohomology (the first two terms are not abelian groups, but the coker does!).

Proof. Thanks to (5.1) we obtain an exact sequence of pointed set:

$$\begin{array}{ccccccc} G'(k) & \longrightarrow & G(k) & \longrightarrow & H^1(k,B) & \longrightarrow & H^1(k,G') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod_{v\in S} G'(k_v) & \longrightarrow & \prod_{v\in S} G(k_v) & \longrightarrow & \prod_{v\in S} H^1(k_v,B) & \longrightarrow & \prod_{v\in S} H^1(k_v,G') \end{array}$$

The last vertical arrow is an isomorphism by K.H.C. together with the fact that G' satisfies WA, since $G' \cong R \times G_0$ which is ss, sc by Platonov.
Now diagram chasing as in the previous section and Proposition 5.2.19. conclude.
"Non commutative diagram chasing, but you don't need a torsion argument because it is central!" □

Achtung. *To see that the map $\prod_{v\in S} G'(k_v) \to \prod_{v\in S} G(k_v)$ is continuous one has to check something.*

Definition 5.2.18. G is split if a maximal torus T of G is split ($\cong \mathbb{G}_m^n$).

In particular G split implies that $B \cong \prod_i \mu_{n_i}$, and so $\hat{B}$ has trivial Galois action.

[6] Isogenies.

Proposition 5.2.19. Let B a finite Galois module. Take K/k finite Galois extension s.t. the action of $\mathrm{Gal}(\overline{k}, k)$ factorize through $\mathrm{Gal}(K, k)$. Then

$$Ш^1_S(\hat{B}) = Ш^1_{S\ non\ syc}(\hat{B})$$

with $S_{non\ syc} \subset S$ consists of places v s.t. the decomposition subgroup of v for K/k is not cyclic.

Proof. See Lemme 1.1 and 1.2 of [San81] (page 18). It is just a consequence of Chebotarev: every cyclic subgroup of the Galois is conjugated to a decomposition group! □

Let G be connected, linear. Take K/k Galois finite s.t. G splits over K.

Example 5.2.20.

a) Assume that for $v \in S$, every decomposition subgroup of K/k at v is cyclic. Then G satisfies WA at S because $Ш^1_S(\hat{B}) = Ш^1(\hat{B})$.
b) In particular if $v|\infty$, G satisfies WA at v.
c) For $R'_{K/k}\mathbb{G}_m$ with K/k bycilic (i.e. product of two cyclic group), then there are counterexamples.

5.2.5. *Obstruction to the Hasse Principle.* We consider (5.1):

$$1 \to B \to G' \to G \to 1$$

and the map

$$\partial : H^1(k, G) \to H^2(k, B)$$

because B is central in G'. So we have a map on the completion, and it make sense (= it respects "being locally trivial") to consider

$$\delta : Ш^1(G) \to Ш^2(B)$$

Theorem 5.2.21 (Sansuc)**.** *$\delta : Ш^1(G) \to Ш^2(B)$ is injective (actually a bijection).*

BM obstruction is an abelian obstruction, so to relate obstruction to BM you have to abelianize the phenomena as we are doing.

Proof. As usual $G' \cong R \times G_0$, and we have:

$$\begin{tikzcd}
H^1(k,B) \arrow[r] \arrow[d] & H^1(k,G') \arrow[r] \arrow[d] & H^1(k,G) \arrow[r] \arrow[d] & H^2(k,B) \arrow[d] \\
\prod_{v\ real} H^1(k_v,B) \arrow[r] & \prod_{v\ real} H^1(k_v,G') \arrow[r] & \prod_{all\ v} H^1(k_v,G) \arrow[r] & \prod_{all\ v} H^2(k_v,B)
\end{tikzcd}$$

Notice that it is important to restrict ourself to finitely many places (i.e. $v\ real$ in the left). Diagram chasing: consider $g \in H^1(k, G)$ s.t.

$$\begin{tikzcd}
H^1(k,B) \arrow[r] \arrow[d] & g' \arrow[r] \arrow[d] & g \arrow[r] \arrow[d] & 0 \\
b_v \arrow[r] & (g'_v) \arrow[r] & 0 &
\end{tikzcd}$$

Notice that the map

$$H^1(k, B) \to \prod_{v\ real} H^1(k_v, B)$$

is surjective since its cokernel is $Ш^1_s(\hat{B})$, which is zero by the previous lemma, since S is the set of real places (all the decomposition groups are cyclic!) and the fact that $Ш^1_\omega(\hat{B})^D = 0$.

Considering the part (g'_v) makes sense even if it is not zero because can be modified using the surjectivity discussed above, and then an usual torsion argument applies. □

Corollary 5.2.22. *Let X be a principal homogeneous space of G. Assume X has a point in extensions k_i/k with coprime $[k_i : k]$. Then X has a k-point.*

Proof. We prove it just in the case $\Omega_{\mathbb{R}} = \emptyset$. We call α the class of X in $H^1(k, G)$. Consider

$$0 \to H^1(k_v, G) \xrightarrow{\partial} H^2(k_v, B)$$

it is injective because $H^1(k_v, G') = 0$. We have that $\partial(\alpha_v) = 0$ by Restriction-Corestriction. So

$$Ш^1(G) \hookrightarrow Ш^2(B)$$

So $\partial(\alpha) = 0$ by Res-Cores. $\square$

Remark 5.2.23. If G is not connected this is an open question (G not abelian, if not Res-Cor works).

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