MINIMAL SLOPE AND PARABOLICITY CONJECTURE OF F-ISOCRYSTALS

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1. Tsuzuki's minimal slope theorem

1.1. **Introduction.** Let k be a perfect field of positive characteristic p, W its ring of Witt vectors and $K := \mathrm{Frac}(W)$. Let X over k be a smooth variety. We denote by $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ the category of (coherent) convergent F-isocrystals and by $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ the category of (coherent) overconvergent F-isocrystals. These are $\mathbb{Q}_p$-linear rigid monoidal abelian categories. Kedlaya proved that the natural functor $\alpha : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$ is fully faithful. The category $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ is equivalent to the category of coherent F-isocrystals on the crystalline site of X/W.

Definition 1.1.1. We say that $\mathcal{M} \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$ is *$\dagger$-extendable* if it is in the essential image of α and we write $\mathcal{M}^\dagger$ for the associated overconvergent F-isocrystal (which is unique up to isomorphism). If $\mathcal{N} \subseteq \mathcal{M}$ with $\mathcal{M}$ $\dagger$-extendable the *$\dagger$-hull* of $\mathcal{N}$ in $\mathcal{M}$ is the smallest[1] $\dagger$-extendable subobject of $\mathcal{M}$ containing $\mathcal{N}$ and it is denoted by $\overline{\mathcal{N}}$.

The reason for the name is that, roughly speaking, the F-isocrystals in the essential image of α are the ones that can be *extended* to a small p-adic polyannulus at infinity. In particular, when X is proper the functor is an equivalence. If $f : Y \to X$ is a smooth and proper morphism of smooth varieties over k, then $R^i f_{\mathrm{crys}*}\mathcal{O}_{Y,\mathrm{crys}}$ is a $\dagger$-extendable F-isocrystal.

An important concept in the theory of F-isocrystals is the concept of *slope*, which comes from the Dieudonné–Manin decomposition. The F-isocrystals over $\mathrm{Spec}(k)$ form a semi-simple category and the isomorphism classes of irreducible objects are in bijection with the rational number. The rational number associated to an irreducible object is called slope. If X is connected, after possibly passing to a dense open of X, every F-isocrystal $\mathcal{M}$ admits a unique *slope filtration*

$$0 = S_{-1}(\mathcal{M}) \subsetneq S_0(\mathcal{M}) \subsetneq \cdots \subsetneq S_n(\mathcal{M}) = \mathcal{M}$$

where for each $i \geq 0$, the quotient $S_i(\mathcal{M})/S_{i-1}(\mathcal{M})$ is *isoclinic* of slope $s_i \in \mathbb{Q}$ (i.e. at geometric points it is a direct sum of irreducible F-isocrystals of slope s_i) and $s_0 < \cdots < s_n$. We say that $n+1$ is the length of the filtration. If $\mathcal{M}$ admits the slope filtration every subquotient of $\mathcal{M}$ admits the slope filtration as well. Moreover, for every morphism $g : Z \to X$ with Z connected g^* preserves the slope filtration.

If $\mathcal{M}$ is $\dagger$-extendable, the pieces of the filtration are not in general $\dagger$-extendable. If we come back to the geometric example, after shrinking X enough, the F-isocrystal $S_0(R^i f_{\mathrm{crys}*}\mathcal{O}_{Y,\mathrm{crys}})$ is the F-isocrystal associated to the p-adic local system $R^i f_{\text{ét}*}\mathbb{Q}_p$. This is the classical example of an

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[1] Note that the category of F-isocrystals is artinian because if $\mathcal{M}' \subseteq \mathcal{M}$ have the same ranks, then $\mathcal{M}' = \mathcal{M}$.

F-isocrystal which is not †-extendable. This talk is about the interaction between the property of being †-extendable and the slope filtration.

In December 2018, reading the work of Vella on the characterization of parabolic subgroups, I noted that the following property was related to the *parabolicity conjecture* of F-isocrystals.

Conjecture 1.1.2 ($\mathrm{MS}(X)$)**.** *Let X be a smooth connected variety over k and let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of F-isocrystals over X. If $\mathcal{M}$ is †-extendable and admits the slope filtration, then $S_0(\mathcal{N}) = S_0(\overline{\mathcal{N}})$.*

I will talk about the parabolicity conjecture and its proof next week, today we focus on $\mathrm{MS}(X)$. Initially, I could prove $\mathrm{MS}(X)$ for some F-isocrystals coming from p-divisible groups using a result of Tate on the local behaviour of étale p-divisible subgroups. Tsuzuki, independently, proved the full $\mathrm{MS}(X)$ when X is a smooth connected curve. This was an intermediate step in his proof of Kedlaya's conjecture over curves.

Theorem 1.1.3 ([Tsu19], Proposition 5.8)**.** *If X is a smooth connected curve over k, then $\mathrm{MS}(X)$ is true.*

Corollary 1.1.4. *Let X be a smooth connected curve over k and let $\mathcal{M}$ and $\mathcal{N}$ be as in $\mathrm{MS}(X)$. If $\mathcal{M}^\dagger$ is irreducible, then $S_0(\mathcal{N})$ is irreducible.*

Proof. Let $\mathcal{N}' \subseteq S_0(\mathcal{N})$ be a non-zero sub-F-isocrystal. Since $\mathcal{M}^\dagger$ is irreducible, $\mathcal{N}'$ and $S_0(\mathcal{N})$ have the same †-hull. On the other hand, $\mathcal{N}'$ is isoclinic, which implies by Theorem 1.1.3 that $\mathcal{N}' = S_0(\mathcal{N}') = S_0(\mathcal{N})$. □

Corollary 1.1.5 (Kedlaya's conjecture)**.** *Let X be a smooth connected curve over k and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two F-isocrystals over X which admit a slope filtration and that come from irreducible overconvergent F-isocrystals. If $S_0(\mathcal{M}_1) \simeq S_0(\mathcal{M}_2)$, then $\mathcal{M}_1 \simeq \mathcal{M}_2$.*

Proof. Write $\mathcal{N}$ for $S_0(\mathcal{M}_1)$, ι for the tautological inclusion $\mathcal{N} \subseteq \mathcal{M}_1$ and choose an isomorphism $\psi : \mathcal{N} \xrightarrow{\sim} S_0(\mathcal{M}_2)$. Consider $\mathcal{M} := \mathcal{M}_1 \oplus \mathcal{M}_2$ and the inclusion $\mathcal{N} \subseteq \mathcal{M}$ induced by (ι, ψ). By Theorem 1.1.3, we have that $S_0(\overline{\mathcal{N}}) = \mathcal{N} \neq \mathcal{N} \oplus \mathcal{N} = S_0(\mathcal{M})$, which implies that $\overline{\mathcal{N}}$ is strictly smaller than $\mathcal{M}$. Therefore, $\overline{\mathcal{N}}^\dagger$ is irreducible. On the other hand, $\overline{\mathcal{N}}$ admits non-trivial maps to $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, $\mathcal{M}_1 \simeq \overline{\mathcal{N}} \simeq \mathcal{M}_2$. □

Another consequence is the following p-adic refinement of the strong multiplicity one theorem for cuspidal automorphic representations.

Corollary 1.1.6. *Let X be a smooth connected curve over a finite field, let $\mathbb{A}$ be its adele ring and let r be a positive integer. The isomorphism class of a $\overline{\mathbb{Q}}_p$-linear cuspidal automorphic representation π of $\mathrm{GL}_r(\mathbb{A})$ is determined by the datum of the Hecke eigenvalues of minimal slope at all but finitely many closed points of X.*

Proof. By Abe, for any such π there exists an irreducible $\overline{\mathbb{Q}}_p$-linear overconvergent F-isocrystal $\mathcal{M}^\dagger$ defined over a certain dense open of X which corresponds to π in the sense of Langlands. After shrinking X we may assume that the associated F-isocrystal $\mathcal{M}$ admits the slope filtration. By Corollary 1.1.5, the F-isocrystal $S_0(\mathcal{M})$ determines the isomorphism class of $\mathcal{M}$. On the other hand, by Katz–Crew, the isoclinic F-isocrystal $S_0(\mathcal{M})$ is induced by a $\overline{\mathbb{Q}}_p$-linear continuous representation

ρ of the Weil group of X. Moreover, by Corollary 1.1.4, ρ is irreducible. By Chebotarev's density theorem for the étale fundamental group of X, since ρ is semi-simple, its isomorphism class is determined by the Frobenius eigenvalues at all but finitely many points. By construction, these eigenvalues are the same as the Frobenius eigenvalues of $\mathcal{M}^\dagger$ of minimal slope. Since π and $\mathcal{M}^\dagger$ correspond in the sense of Langlands, we conclude the proof. □

I will spend the rest of the talk explaining a proof of Theorem 1.1.3. The proof here is a shorter variant of the original proof in [Tsu19].

1.2. **First reductions.** We reduce the problem to the case when X is a dense open of $\mathbb{P}^1_k$. The reductions are still valid for a variety of dimension greater than 1.

Lemma 1.2.1. *If* $\mathrm{MS}(X)$ *is true for* $\mathcal{N}$ *isoclinic, then it is true for every* $\mathcal{N}$*.*

Proof. The proof is by induction on the length of the slope filtration of $\mathcal{N}$. Suppose that the slope filtration of $\mathcal{N}$ has length $n+1 \geq 2$ and we already know the statement for length at most n. We want to show that $S_0(\overline{\mathcal{N}}) = S_0(\overline{S_{n-1}(\mathcal{N})})$, which implies $S_0(\overline{\mathcal{N}}) = S_0(\mathcal{N})$. We may assume $\overline{S_{n-1}(\mathcal{N})} \subsetneq \overline{\mathcal{N}}$. Thus we have to show that the minimal slope of $\overline{\mathcal{N}}/\overline{S_{n-1}(\mathcal{N})} \neq 0$ is greater than the minimal slope of $\overline{S_{n-1}(\mathcal{N})}$, which by the inductive hypothesis is s_0. If $\pi : N \to \overline{\mathcal{N}}/\overline{S_{n-1}(\mathcal{N})}$ is the natural projection, then $\pi(\mathcal{N})$ is isoclinic of slope s_n and $\overline{\pi(\mathcal{N})} = \overline{\mathcal{N}}/\overline{S_{n-1}(\mathcal{N})}$. Again, by the inductive hypothesis we have that the minimal slope of $\overline{\mathcal{N}}/\overline{S_{n-1}(\mathcal{N})}$ is $s_n > s_0$. This concludes the proof. □

Lemma 1.2.2. *If* U *is a dense open of* X*, then* $\mathrm{MS}(U) \Rightarrow \mathrm{MS}(X)$*.*

Proof. We want to prove that the operation of taking †-hull commutes with the restriction functor to U. We have by definition $\overline{(\mathcal{N}|_U)} \subseteq \overline{\mathcal{N}}|_U$. The other inclusion is a consequence of the following theorem.

Theorem 1.2.3 ([Ked07], Theorem 5.2.1 and Proposition 5.3.1)**.** *The restriction functor* $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \to \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(U)$ *is fully faithful and closed under the operation of taking subquotients.*[2]

Since $\overline{(\mathcal{N}|_U)}$ extends to some $\mathcal{M}'$ over X such that $\overline{\mathcal{N}} \subseteq \mathcal{M}' \subseteq \mathcal{M}$, we get $\overline{(\mathcal{N}|_U)} \supseteq \overline{\mathcal{N}}|_U$. Thanks to this, if $S_0(\overline{\mathcal{N}|_U}) = \mathcal{N}|_U$ then $S_0(\overline{\mathcal{N}})|_U = S_0(\overline{\mathcal{N}|_U}) = \mathcal{N}|_U$. This implies $S_0(\overline{\mathcal{N}}) = \mathcal{N}$. □

Lemma 1.2.4. *If* $f : Y \to X$ *is a finite étale Galois cover,* $\mathrm{MS}(Y) \Leftrightarrow \mathrm{MS}(X)$*.*

Proof. The result follows from the étale descent for F-isocrystals and overconvergent F-isocrystals. Write G for the Galois group of the cover. To prove $\mathrm{MS}(Y) \Rightarrow \mathrm{MS}(X)$ we prove that if $\mathcal{N}$ and $\mathcal{M}$ are over X, then $f^*\overline{\mathcal{N}} = \overline{f^*\mathcal{N}}$. The inclusion $f^*\overline{\mathcal{N}} \supseteq \overline{f^*\mathcal{N}}$ follows from the definition of †-hull. On the other hand, if $\mathcal{M}' := \overline{f^*\mathcal{N}}$, then the intersection $\bigcap_{g\in G} g^*\mathcal{M}'$ contains $f^*\mathcal{N}$ and it is †-extendable, thus it is equal to $\mathcal{M}'$. This implies that $\mathcal{M}'$ descends to X to some †-extendable F-isocrystal which contains $\mathcal{N}$. This gives the other inclusion.

[2] Note that the analogous result is true for complex local systems or lisse $\mathbb{Q}_\ell$-sheaves, since for normal connected varieties the fundamental group of dense opens surjects to the fundamental group of the entire variety. In general, F-isocrystals instead might acquire new subquotients when you shrink X as they may acquire the slope filtration only after shrinking X.

We prove now $\mathrm{MS}(X) \Rightarrow \mathrm{MS}(Y)$. If $\mathcal{N}$ and $\mathcal{M}$ are over Y, then by the assumption $S_0(\overline{f_*\mathcal{N}}) = f_*\mathcal{N}$. By the previous argument this implies $S_0(\overline{f^*f_*\mathcal{N}}) = f^*f_*\mathcal{N}$. Since $f^*f_*\mathcal{N} = \bigoplus_{g\in G} g^*\mathcal{N}$ and all the $g^*\mathcal{N}$ have equal minimal slopes we conclude that $S_0(\overline{\mathcal{N}}) = \mathcal{N}$. □

Putting together the two observations it is enough to prove Theorem 1.1.3 when $C \subseteq \mathbb{P}^1_k$ is a dense affine open obtained by removing a set of rational points $Z = \{a_1, \dots, a_s\}$[3]. Let $\mathrm{Spf}(\hat{A}_C) = \mathcal{C} \subseteq \widehat{\mathbb{P}}^1_W$ be a smooth formal lift of C.

1.3. **A local version.** We prove now a local version of Theorem 1.1.3. The main ingredient in the proof is the *reverse slope filtration*. Consider the Cohen ring

$$\mathcal{O}_{\mathcal{E}} := W((t))^\wedge = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \,\middle|\, a_i \in W,\ \lim_{i\to-\infty} v_p(a_i) = \infty \right\}.$$

It is a complete discrete valuation ring unramified over W with residue field $k((t))$. Let $\mathcal{O}_{\mathcal{E}^\dagger} \subseteq \mathcal{O}_{\mathcal{E}}$ be the subring of series which converge in some annulus $* \leq |t| < 1$. These two rings are both endowed with a Frobenius lift $\varphi(t) = t^p$ and a derivation ∂_t. Write $\mathcal{E}$ and $\mathcal{E}^\dagger$ for the fraction fields.

Definition 1.3.1 ((φ, ∇)-modules)**.** If R is either $\mathcal{E}$ or $\mathcal{E}^\dagger$, we say that a finite dimensional vector space M over R is a (φ, ∇)-*module* if it is endowed with a φ-linear isomorphism $F : M \xrightarrow{\sim} M$ and an additive morphism $\nabla_{\partial_t} : M \to M$ which satisfies the Leibniz rule and such that $\nabla_{\partial_t} \circ F = pt^{p-1} F \circ \nabla_{\partial_t}$.

The category $\mathbf{F}\text{-}\mathbf{Isoc}(k((t)))$ is the category of (φ, ∇)-modules over $\mathcal{E}$ and $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(k((t)))$ is the category of (φ, ∇)-modules over $\mathcal{E}^\dagger$. Kedlaya's full faithfulness says that the extension-of-scalars functor $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(k((t))) \to \mathbf{F}\text{-}\mathbf{Isoc}(k((t)))$ is fully faithful.

Proposition 1.3.2 (Kedlaya, [Tsu19, Theorem 2.14])**.** *If $N \subseteq M$ $\mathbf{F}\text{-}\mathbf{Isoc}(k((t)))$ and M is $\dagger$-extendable, then $S_0(N) = S_0(\overline{N})$.*

Construction 1.3.3. Let $Q^\dagger$ be the image of

$$(M^\dagger)^\vee := \mathrm{Hom}_{\mathcal{E}^\dagger}(M^\dagger, \mathcal{E}^\dagger) \to \mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \to \mathrm{Hom}_{\mathcal{E}}(N, \mathcal{E}) := N^\vee.$$

Note that we have natural maps

$$M^\vee = (M^\dagger)^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E} \to Q^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \to N^\vee.$$

The first arrow is surjective by construction, the second one is surjective because the composition of the two is surjective. Note that even though $Q^\dagger \subseteq N^\vee$, the second map needs not to be injective. Dualizing with respect to $\mathcal{E}$ we get inclusions $N \subseteq Q^\vee \subseteq M$.

Lemma 1.3.4. *The (φ, ∇)-module $Q^\vee$ is the $\dagger$-hull of N in M. In other words, $\overline{N}$ is the unique submodule of M which contains N and comes from some $\overline{N}^\dagger \subseteq M^\dagger$ such that $(\overline{N}^\dagger)^\vee \to N^\vee$ is injective.*

[3]Actually we could also assume that $C = \mathbb{A}^1_k$ and remove only the point ∞

Proof. By construction, $Q^\vee$ is †-extendable and it contains N, so that $\overline{N} \subseteq Q^\vee$. On the other hand, if P is †-extendable and contains N, then we have morphisms $(M^\dagger)^\vee \twoheadrightarrow (P^\dagger)^\vee \to N^\vee$, with the first one surjective. Therefore, the morphism $(P^\dagger)^\vee \to N^\vee$ factors through $Q^\dagger$. This implies $\overline{N} \supseteq Q^\vee$. □

Let us now recall the reverse filtration. For this we need to introduce two other discrete valuation fields which lift $k((t))^{\mathrm{alg}}$.

Definition 1.3.5. We consider the ring of Witt vectors $k((t))^{\mathrm{alg}}$ and we denote it by $\mathcal{O}_{\widetilde{\mathcal{E}}}$. Every element of $\widetilde{\mathcal{E}}$ can be written uniquely as $\sum_{n=0}^{\infty}[f_n]p^n$ where $[f_n]$ is the Teichmüller lift of some $f_n \in k((t))^{\mathrm{alg}}$. Consider the subring $\mathcal{O}_{\widetilde{\mathcal{E}}^\dagger} \subseteq \mathcal{O}_{\widetilde{\mathcal{E}}}$ of those series such that the t-adic valuations of f_n is bounded below by some linear function in n. This condition can be also rephrased as a convergence condition on p-adic annuli, by interpreting these series as Hahn series. The subring $\mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ is preserved by the Frobenius of $\mathcal{O}_{\widetilde{\mathcal{E}}}$. We write $\widetilde{\mathcal{E}}^\dagger$ and $\widetilde{\mathcal{E}}$ for the fraction fields. We will not talk about derivations in this context.

The main ingredient in the proof of Theorem 1.1.3 is the following result due to de Jong.

Proposition 1.3.6 (de Jong, Proposition 5.5)**.** *If $\widetilde{M}^\dagger$ is a φ-module over $\widetilde{\mathcal{E}}^\dagger$, then the following statements are true.*

(i) *$\widetilde{M}^\dagger$ admits the opposite slope filtration, i.e. there exists a filtration*

$$0 = S_{-1}^{\mathrm{rev}}(\widetilde{M}^\dagger) \subsetneq S_0^{\mathrm{rev}}(\widetilde{M}^\dagger) \subsetneq \cdots \subsetneq S_n^{\mathrm{rev}}(\widetilde{M}^\dagger) = \widetilde{M}^\dagger$$

of φ-modules over $\widetilde{\mathcal{E}}^\dagger$ such that $(S_i^{\mathrm{rev}}(\widetilde{M}^\dagger)/S_{i-1}^{\mathrm{rev}}(\widetilde{M}^\dagger)) \otimes_{\widetilde{\mathcal{E}}^\dagger} \widetilde{\mathcal{E}} \simeq S_{n-i}(\widetilde{M})/S_{n-i-1}(\widetilde{M})$.

(ii) *If $M^\dagger$ is isoclinic of slope s/r, $\widetilde{M}^\dagger[p^{1/r}]$ admits a basis of vectors $\{v_1, \dots, v_m\}$ such that $\varphi(v_i) = p^{s/r}v_i$.*

Idea of the proof. The result follows from a splitting property for the non-commutative ring $\widetilde{\mathcal{E}}^\dagger[F]$, where $Fa = \varphi(a)F$ for $a \in \widetilde{\mathcal{E}}^\dagger$. De Jong proves that for every polynomial $P(F) = F^n + a_{n-1}F^{n-1} + \cdots + a_0 \in \widetilde{\mathcal{E}}^\dagger[F]$ there exist $r > 0$ and $\lambda_1, \dots, \lambda_n \in \widetilde{\mathcal{E}}^\dagger[p^{1/r}]$ with non-increasing p-adic valuations such that $P(F) = (F - \lambda_1)(F - \lambda_2) \dots (F - \lambda_n)$. Applying this to the minimal polynomial $P(F)$ which annihilates $m \in \widetilde{M}^\dagger$, then $F((F - \lambda_2) \dots (F - \lambda_n)m) = \lambda_1 m$. □

Lemma 1.3.7. *The multiplication morphism $\mathcal{E} \otimes_{\mathcal{E}^\dagger} \widetilde{\mathcal{E}}^\dagger \to \widetilde{\mathcal{E}}$ is injective.*

Lemma 1.3.8. *Let $M^\dagger$ be a φ-module over $\mathcal{E}^\dagger$, N an isoclinic φ-module over $\mathcal{E}$ of slope s/r and $\psi : M \to N$ is a morphism of φ-modules. If the restriction of ψ to $M^\dagger$ is injective, then the maximal slope of M is s/r and the rank of $S_n(M)/S_{n-1}(M)$ is smaller than the rank of N.*

Proof. Since $\widetilde{\mathcal{E}}^\dagger$ is flat over $\mathcal{E}^\dagger$ and $\mathcal{E} \otimes_{\mathcal{E}^\dagger} \widetilde{\mathcal{E}}^\dagger \hookrightarrow \widetilde{\mathcal{E}}$, then $\psi|_{M^\dagger}$ induces an injective morphism

$$\psi' : \widetilde{M}^\dagger = M^\dagger \otimes_{\mathcal{E}^\dagger} \widetilde{\mathcal{E}}^\dagger \to N \otimes_{\mathcal{E}^\dagger} \widetilde{\mathcal{E}}^\dagger \to N \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}.$$

The restriction of ψ' to $S_0^{\mathrm{rev}}(\widetilde{M}^\dagger)$ induces a non-trivial morphism $S_0^{\mathrm{rev}}(\widetilde{M}^\dagger) \otimes_{\widetilde{\mathcal{E}}^\dagger} \widetilde{\mathcal{E}} \to N \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}$. This implies that the slope of $S_0^{\mathrm{rev}}(\widetilde{M}^\dagger)$, which is the maximal slope of M, is s/r. Moreover, by Proposition 1.3.6.(ii), $(S_0^{\mathrm{rev}}(\widetilde{M}^\dagger)[p^{1/r}])^{\varphi = p^{s/r}}$ is a $\mathbb{Q}_p(p^{1/r})$-vector space of dimension equal to the rank of $S_n(M)/S_{n-1}(M)$. Similarly, by the Dieudonné–Manin decomposition, $(N\otimes_{\mathcal{E}}\widetilde{\mathcal{E}}[p^{1/r}])^{\varphi=p^{s/r}}$ is a $\mathbb{Q}_p(p^{1/r})$-vector space of dimension equal to the rank of N. We then obtain the inequality of ranks thanks to the injectivity of ψ'. □

Proof of Proposition 1.3.2. By Lemma 1.2.1, it is enough to prove the result when N is isoclinic of slope s/r and $\overline{N}=M$. In that case we have to check that $\overline{N}$ has minimal slope s/r and that the inclusion $N\subseteq S_0(\overline{N})$ is an equality. By Lemma 1.3.4, the morphism $\overline{N}^{\vee}\to N^{\vee}$ satisfies the assumptions of Lemma 1.3.8, thus $\overline{N}$ has minimal slope s/r. Moreover, since

$$\mathrm{rk}(N)=\mathrm{rk}(N^{\vee})\geq S_n(\overline{N}^{\vee})/S_{n-1}(\overline{N}^{\vee})=\mathrm{rk}(S_0(\overline{N})),$$

we get $N=S_0(\overline{N})$. □

1.4. **Local to global.** To pass from the local situation to the global situation it is convenient to construct partial †-hulls. For every $0\leq i\leq s$ let $\overline{\mathcal{N}}_{Z_i}$ be the smallest sub-F-isocrystal of $\mathcal{M}$ over C which contains $\mathcal{N}$ and comes from an F-isocrystal overconvergent along $Z_i:=\{a_1,\dots,a_i\}$. Note that $Z_0=\emptyset$ so that $\overline{\mathcal{N}}_{Z_0}=\mathcal{N}$. We have a chain $\mathcal{N}=\overline{\mathcal{N}}_{Z_0}\subseteq\overline{\mathcal{N}}_{Z_1}\subseteq\cdots\subseteq\overline{\mathcal{N}}_{Z_s}=\overline{\mathcal{N}}$. Write $\overline{N}_{Z_i}$ for the (φ,∇)-module over $\hat{A}_{C,K}$ associated to $\overline{\mathcal{N}}_{Z_i}$. For every i, let $\mathcal{E}_{a_i}$ be $\mathcal{E}$ endowed with the morphism $\hat{A}_{C,K}\to\mathcal{E}$ induced by a_i and $\mathcal{M}_{a_i}$ the base change of $\mathcal{M}$ to $\mathcal{E}_{a_i}$. We define in an analogous way $\mathcal{E}^{\dagger}_{a_i}$.

Proposition 1.4.1 (Tsuzuki)**.** *For every $1\leq i\leq s$, the (φ,∇)-module $\overline{N}_{Z_i}\otimes\mathcal{E}_{a_i}$ is the †-hull of $\overline{N}_{Z_{i-1}}\otimes\mathcal{E}_{a_i}$ in $\mathcal{M}_{a_i}$.*

Proof. Let $C_i:=C\cup Z_i$ and $\hat{A}^{\dagger}_{C,C_i,K}:=\Gamma(]C_i[_{\mathbb{P}^1_K},j^{\dagger}_C\mathcal{O}_{\mathbb{P}^1_K})$. The $\hat{A}_{C,K}$ modules $\overline{N}_{Z_i}$ associated to $\overline{\mathcal{N}}_{Z_i}$ can be constructed as in §1.3.3 by replacing $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ with the rings $\hat{A}_{C,K}$ and $\hat{A}^{\dagger}_{C,C_i,K}$. By an analogue of Lemma 1.3.4, for every i the (φ,∇)-module $\overline{N}_{Z_i}$ comes from a (φ,∇)-module $\overline{N}^{\dagger}_{Z_i}$ over $\hat{A}^{\dagger}_{C,C_i,K}$ such that $(\overline{N}^{\dagger}_{Z_i})^{\vee}\to N^{\vee}$ is injective. From this it follows that even the natural maps $(\overline{N}^{\dagger}_{Z_i})^{\vee}\to(\overline{N}_{Z_{i-1}})^{\vee}$ are injective for every i. By [Tsu19, Lemma 3.1], $\hat{A}^{\dagger}_{C,C_i,K}\to\mathcal{E}^{\dagger}_{a_i}$ is flat and the natural morphism $\hat{A}^{\dagger}_{C,C_{i-1}}\otimes_{\hat{A}^{\dagger}_{C,C_i,K}}\mathcal{E}^{\dagger}_{a_i}\to\mathcal{E}_{a_i}$ is injective. Therefore, the morphism $(\overline{N}^{\dagger}_{Z_i}\otimes_{\hat{A}^{\dagger}_{C,C_i,K}}\mathcal{E}^{\dagger}_{a_i})^{\vee}\to(\overline{N}_{Z_{i-1}}\otimes_{\hat{A}_{C,K}}\mathcal{E}_{a_i})^{\vee}$ obtained as the composition of

$$(\overline{N}^{\dagger}_{Z_i})^{\vee}\otimes_{\hat{A}^{\dagger}_{C,C_i,K}}\mathcal{E}^{\dagger}_{a_i}\hookrightarrow(\overline{N}_{Z_{i-1}})^{\vee}\otimes_{\hat{A}^{\dagger}_{C,C_i,K}}\mathcal{E}^{\dagger}_{a_i}\hookrightarrow(\overline{N}_{Z_{i-1}})^{\vee}\otimes_{\hat{A}_{C,K}}\mathcal{E}_{a_i}$$

is injective. This proves what we wanted by Lemma 1.3.4. □

Since $S_0(\overline{N}_{Z_i})\otimes\mathcal{E}_{a_i}=S_0(\overline{N}_{Z_i}\otimes\mathcal{E}_{a_i})$, by Proposition 1.3.2 and Proposition 1.4.1 we have

$$S_0(\mathcal{N})=S_0(\overline{\mathcal{N}}_{Z_0})=S_0(\overline{\mathcal{N}}_{Z_1})=\cdots=S_0(\overline{\mathcal{N}}_{Z_s})=S_0(\overline{\mathcal{N}}).$$

This concludes the proof of Theorem 1.1.3.

2. Parabolicity conjecture of F-isocrystals

2.1. **Introduction.** Retain notation as in the previous talk. We saw a proof of $\mathrm{MS}(X)$ when X has dimension 1. Today we talk about another related conjecture on the *monodromy groups* of F-isocrystals, the *parabolicity conjecture* and we will prove $\mathrm{MS}(X)$ in any dimension. We will assume today that k is an algebraically closed. Let X be a smooth connected variety over k and let $x \in X(k)$ be a rational point. The category $\mathbf{Isoc}(X)$ of convergent isocrystals admits a fibre functor $\omega_x : \mathbf{Isoc}(X) \to \mathbf{Vec}_K$, namely a faithful exact K-linear $\otimes$-functor. This makes $\mathbf{Isoc}(X)$ a *neutral Tannakian categories.* We will not recall the definition of a neutral Tannakian category, it is enough to know that ω_x induces an equivalence $\mathbf{Isoc}(X) \xrightarrow{\sim} \mathbf{Rep}_K(\pi_1^{\mathbf{Isoc}}(X))$ where $\pi_1^{\mathbf{Isoc}}(X) := \underline{\mathrm{Aut}}(\omega_x)$ is the affine group scheme over K of $\otimes$-automorphisms of the fibre functor ω_x. The group $\pi_1^{\mathbf{Isoc}}(X)$ is very big! If we are simply interested in studying the behaviour of one object $\mathcal{M} \in \mathbf{Isoc}(X)$, we can consider the image $G(\mathcal{M})$ of the tautological representation of $\pi_1^{\mathbf{Isoc}}(X) \to \mathrm{GL}(\omega_x(\mathcal{M}))$. The linear algebraic group $G(\mathcal{M})$ is the *monodromy group* attached to $\mathcal{M}$. If $\langle \mathcal{M} \rangle \subseteq \mathbf{Isoc}(X)$ is the smallest strictly full abelian $\otimes$-subcategory which contains direct sums of subquotients of $\mathcal{M}^{\otimes n} \otimes (\mathcal{M}^\vee)^{\otimes m}$, then $\langle \mathcal{M} \rangle$ is equivalent via ω_x to the category $\mathbf{Rep}_K(G(\mathcal{M}))$. Note that by construction, in $\langle \mathcal{M} \rangle$ we also have, for example, the wedge products $\bigwedge^r \mathcal{M}$. A similar story holds for $\mathbf{Isoc}^\dagger(X)$. We write $G(\mathcal{M}^\dagger)$ for the monodromy group of $\mathcal{M}^\dagger$. We have that $G(\mathcal{M}) \subseteq G(\mathcal{M}^\dagger) \subseteq \mathrm{GL}(\omega_x(\mathcal{M}^\dagger))$. If $\mathcal{M}$ admits the slope filtration and $\mathrm{Stab}(S_\bullet) \subseteq G(\mathcal{M}^\dagger)$ is the stabiliser of the slope filtration of $\omega_x(\mathcal{M})$, then $G(\mathcal{M}) \subseteq \mathrm{Stab}(S_\bullet)$. Indeed, the slope filtration of the F-isocrystal $(\mathcal{M}, \Psi)$ induces a filtration of the isocrystal $\mathcal{M}$ by simply forgetting the F-structure. A natural question, initially asked (in a weaker form) by Crew, is whether this inclusion is an equality.

Conjecture 2.1.1 ($\mathrm{P}(X)$)**.** *Let X be a smooth connected variety and $\mathcal{M}^\dagger$ an overconvergent F-isocrystal over X. If $\mathcal{M}$ admits the slope filtration, then $G(\mathcal{M}) = \mathrm{Stab}(S_\bullet)$.*

Crew proved the conjecture for some overconvergent F-isocrystals coming from geometry, namely for the higher image of generic abelian schemes and for the F-isocrystals associated to Kloosterman sums. In this talk, we will we will present the proof of $\mathrm{MS}(X)$ and $\mathrm{P}(X)$ for smooth connected varieties of any dimension. Here the logical path of the proof

$$\mathrm{MS}(X) \text{ for curves} \Rightarrow \mathrm{P}(X) \text{ for curves} \Rightarrow \mathrm{MS}(X) \Rightarrow \mathrm{P}(X).$$

In the previous talk we saw a variant of Tsuzuki's proof of $\mathrm{MS}(X)$ for curves. Let us continue from there.

2.2. **Proof of $\mathrm{MS}(X) \Rightarrow \mathrm{P}(X)$.** Write σ for the lift of the Frobenius of K. If $\mathcal{M}$ is an F-isocrystal, then the monodromy group $G(\mathcal{M})$ is endowed with an extra structure, namely an isomorphism $\Phi : G(\mathcal{M})^{(\sigma)} \xrightarrow{\sim} G(\mathcal{M})$. We write F-$\langle \mathcal{M} \rangle$ for the category

$$\left\{ (\mathcal{N}, \Psi) \;\middle|\; \mathcal{N} \in \langle \mathcal{M} \rangle \text{ and } \Psi : F^*\mathcal{N} \xrightarrow{\sim} \mathcal{N} \right\}.$$

By [Cre92], Proposition 2.4, this is equivalent to the category

$$F\text{-}\langle \omega_x(\mathcal{M}) \rangle := \left\{ (V, \rho) \;\middle|\; V \in \mathbf{F\text{-}Isoc}(k) \text{ and } \rho : G(\mathcal{M}) \to \mathrm{GL}(V) \text{ is a } \Phi\text{-equivariant representation} \right\}.$$

Let us also consider $\langle \mathcal{M} \rangle_F \subseteq F$-$\langle \mathcal{M} \rangle$, the smallest Tannakian subcategory containing $\mathcal{M}$ and the F-isocrystals coming from k. Write $\langle \omega_x(\mathcal{M}) \rangle_F$ for the essential image of $\langle \mathcal{M} \rangle_F$ in F-$\langle \omega_x(\mathcal{M}) \rangle$.

Again, we give analogous definitions for overconvergent F-isocrystals. We will also use the analogue notation for Φ-equivariant representations of algebraic groups over K endowed with an isomorphism $\Phi : G^{(\sigma)} \xrightarrow{\sim} G$. If $\mathcal{M}$ is a †-extendable F-isocrystal admitting the slope filtration, for every F-isocrysal $\mathcal{N}^\dagger \in \langle \mathcal{M}^\dagger \rangle_F$ the subgroup $\mathrm{Stab}(S_\bullet) \subseteq G(\mathcal{M}^\dagger)$ stabilises the slope filtration of $\omega_x(\mathcal{N}^\dagger)$. This follows from the fact that the property is preserved by direct sum, tensor product, dual, subquotients and twist by F-isocrystals coming from k.

We prove a Φ-equivariant version of Chevalley theorem.

Proposition 2.2.1. *Let G be an algebraic group over K endowed with an isomorphism $\Phi : G^{(\sigma)} \xrightarrow{\sim} G$, $H \subseteq G$ subgroup stable under Φ, $V \in \mathbf{F\text{-}Isoc}(k)$ and $\rho : G \hookrightarrow \mathrm{GL}(V)$ a faithful Φ-equivariant representation. There exists $V' \in \langle\ (V,\rho)\ \rangle_F$ and a line $L \subseteq V'$ stable under the Frobenius structure such that H is the stabiliser of $L \subseteq V'$ in G.*

Proof. We may assume that $G = \mathrm{GL}(V)$. By the K-linear Chevalley theorem, [Del82, Proposition 3.1.(b)], there exists a K-linear representation $\rho' : \mathrm{GL}(V) \to \mathrm{GL}(V')$ of $\mathrm{GL}(V)$ such that H is the stabiliser of a line $L \subseteq V'$. The K-linear representation ρ' is semi-simple. For each irreducible summand $V'_0 \subseteq V'$ there exist $n > 0$, $m \geq 0$ and an irreducible K-linear representation τ of the symmetric group S_n, such that V'_0 is isomorphic to $\mathrm{Hom}_{S_n}(\tau, V^{\otimes n}) \otimes (\det(V)^\vee)^{\otimes m}$ as a representation of $\mathrm{GL}(V)$, where S_n acts on $V^{\otimes n}$ by permutations. The representation $\mathrm{Hom}_{S_n}(\tau, V^{\otimes n}) \otimes (\det(V)^\vee)^{\otimes m}$ is naturally endowed with a Frobenius structure induced by the Frobenius structure of V. Therefore, we can endow V' with a Frobenius structure and upgrade ρ' to a Φ-equivariant representation in $\langle\ (V, \mathrm{id})\ \rangle_F$. Since H is the stabiliser of $L \subseteq V'$ and it is stable under the action of Φ on $\mathrm{GL}(V)$, the line $L \subseteq V'$ is stable under the Frobenius structure. This ends the proof. □

Theorem 2.2.2. *For every smooth connected variety X the statement* $\mathrm{MS}(X)$ *implies* $\mathrm{P}(X)$.

Proof. We have to prove that $\mathrm{Stab}(S_\bullet) \subseteq G(\mathcal{M})$. By Proposition 2.2.1, there exists an overconvergent F-isocrystal $\mathcal{N}^\dagger \in \langle \mathcal{M}^\dagger \rangle_F$ and a rank 1 sub-F-isocrystal $\mathcal{L} \subseteq \mathcal{N}$, such that $G(\mathcal{M})$ is the stabiliser of the line $L := \omega_x(\mathcal{L}) \subseteq \omega_x(\mathcal{N}) := V$. To prove the result it is enough to show that $\mathrm{Stab}(S_\bullet)$ stabilizes L. Let $\overline{\mathcal{L}}$ be the †-hull of $\mathcal{L}$ in $\mathcal{N}$ and write $\overline{L}$ for $\omega_x(\overline{\mathcal{L}})$. Let s be the slope of $\mathcal{L}$ and $V^{\leq s} \subseteq V$ the subspace of slope smaller or equal than s. Assuming $\mathrm{MS}(X)$, we know that $L = \overline{L} \cap V^{\leq s}$. Since $\overline{\mathcal{L}} \subseteq \mathcal{N}$ admits by definition a †-extension, $\mathrm{Stab}(S_\bullet) \subseteq G(\mathcal{M}^\dagger)$ stabilizes $\overline{L}$. On the other hand, $\mathrm{Stab}(S_\bullet)$ stabilizes $V^{\leq s}$ because $\mathcal{N}$ is an object in $\langle \mathcal{M}^\dagger \rangle_F$. This implies that $\mathrm{Stab}(S_\bullet)$ stabilizes L, thus $G(\mathcal{M}) = \mathrm{Stab}(S_\bullet)$ as we wanted. □

Corollary 2.2.3. *Let X be a smooth connected variety such that* $\mathrm{MS}(U)$ *is true for every open $U \subseteq X$. If $\mathcal{M}^\dagger$ is an overconvergent F-isocrystal over X, then $\pi_0(G(\mathcal{M})) = \pi_0(G(\mathcal{M}^\dagger))$.*

Proof. (1) We first prove that the map $\pi_0(G(\mathcal{M})) \to \pi_0(G(\mathcal{M}^\dagger))$ is surjective. For this we do not need the previous theorem. Write $\langle \mathcal{M}^\dagger \rangle_{\mathrm{fin}}$ and $\langle \mathcal{M} \rangle_{\mathrm{fin}}$ for the Tannakian subcategories of objects with finite monodromy groups. We have to show that the functor $\langle \mathcal{M}^\dagger \rangle_{\mathrm{fin}} \to \langle \mathcal{M} \rangle_{\mathrm{fin}}$ is fully faithful[4] and closed under subquotients. Note that $\langle \mathcal{M}^\dagger \rangle_{\mathrm{fin}}$ and $\langle \mathcal{M} \rangle_{\mathrm{fin}}$ are semi-simple categories so that the full faithfulness already implies that the essential image is closed under subquotients, because irreducible objects are sent to irreducible objects. Let

[4]One can actually prove that the entire functor $\langle \mathcal{M}^\dagger \rangle \to \langle \mathcal{M} \rangle$ is fully faithful. The speaker will eventually write down the general proof somewhere.

Γ be a (finite) set of representatives of the isomorphism classes of the irreducible objects in $\langle \mathcal{M}^\dagger \rangle_{\text{fin}}$. Write $\mathcal{N}^\dagger$ for the direct sum of the objects in Γ. Looking at the internal homs it is enough to show that $\mathrm{Hom}_{\mathbf{Isoc}(X)}(\mathbb{1}, \mathcal{N})$ has dimension 1. Write $\mathcal{P} \subseteq \mathcal{N}$ for the greatest subobject of $\mathcal{N}$ isomorphic to $\mathbb{1}^{\oplus n}$ for some n. Since $\langle \mathcal{M}^\dagger \rangle_{\text{fin}}$ is stable under F^*, we have that $F^*\mathcal{N}^\dagger \simeq \mathcal{N}^\dagger$ so that $\mathcal{N}^\dagger$ can be endowed with some Frobenius structure Ψ. By the maximality of $\mathcal{P}$, the Frobenius structure Ψ preserves $\mathcal{P} \subseteq \mathcal{N}$. Thus the inclusion is also an inclusion $(\mathcal{P}, \Psi|_{\mathcal{P}}) \subseteq (\mathcal{N}, \Psi)$ of F-isocrystals. Note that the isocrystal $(\mathcal{P}, \Psi|_{\mathcal{P}})$ is †-extendable. By the full faithfulness theorem, the inclusion is induced by an inclusion $(\mathcal{P}^\dagger, \Psi|_{\mathcal{P}^\dagger}) \subseteq (\mathcal{N}^\dagger, \Psi)$. By the construction of $\mathcal{N}^\dagger$, then $\mathbb{1}^{\oplus n} = \mathcal{P}^\dagger \subseteq \mathcal{N}^\dagger$ has rank 1. This concludes the proof.

(2) To prove that $\pi_0(G(\mathcal{M})) \to \pi_0(G(\mathcal{M}^\dagger))$ is an isomorphism knowing (1), it is harmless to shrink X to some dense open U. This follows from two observations. First, by Theorem 1.2.3, $\pi_0(G(\mathcal{M}^\dagger)) \to \pi_0(G(\mathcal{M}^\dagger|_U))$ is an isomorphism. The second observation is that $\pi_0(G(\mathcal{M})) \to \pi_0(G(\mathcal{M}|_U))$ is surjective. To prove this, as in (1), it is enough to prove that $\langle \mathcal{M} \rangle_{\text{fin}} \to \langle \mathcal{M}|_U \rangle_{\text{fin}}$ is fully faithful. In addition, it is enough to prove the full faithfulness for objects in $\langle \mathcal{M} \rangle_{\text{fin}}$ endowed with Frobenius structure. Then we can apply Katz–Crew and deduce the result from the analogous result for p-adic representations of the étale fundamental group of normal varieties.

(3) We assume that $\mathcal{M}$ admits the slope filtration and $\mathcal{M}^\dagger$ is semi-simple. By Theorem 2.2.2, $G(\mathcal{M})$ is a parabolic subgroup of $G(\mathcal{M}^\dagger)$. The subgroup $G(\mathcal{M}^\dagger)^\circ \cap G(\mathcal{M}) \subseteq G(\mathcal{M}^\dagger)^\circ$ is then connected because it is a parabolic subgroup of a connected group. This shows that the map $\pi_0(G(\mathcal{M})) \to \pi_0(G(\mathcal{M}^\dagger))$ is injective and hence it is an isomorphism.

□

2.3. **A tame Lefschetz theorem.** The main issue to reduce $\mathrm{MS}(X)$ to the curves case is due to the wild ramification. We would like to find a smooth connected curve $C \subseteq X$ such that $\pi_1^{\mathbf{Isoc}}(C) \twoheadrightarrow \pi_1^{\mathbf{Isoc}}(X)$. This is for example possible for the étale fundamental group in characteristic 0 or for the tame étale fundamental group in positive characteristic. The failure of the existence of such a nice curve for the entire étale fundamental group in positive characteristic is already clear for $\mathbb{A}^2_k$. Suppose that we found a curve $C \subseteq \mathbb{A}^2_k$ such that $\pi_1^{\text{ét}}(C) \to \pi_1^{\text{ét}}(\mathbb{A}^2_k)$ is surjective. Suppose that C is cut by the equation $f(x, y) = 0$. Then we can consider the connected finite étale Artin–Schreier cover $X \to \mathbb{A}^2_k$ defined by $t^p - t - f(x, y) = 0$. The fibre product $X \times_C \mathbb{A}^2_k$ is a trivial cover of C, so that the image of $\pi_1^{\text{ét}}(C)$ is contained in $\pi_1^{\text{ét}}(X) \subsetneq \pi_1^{\text{ét}}(\mathbb{A}^2_k)$, contradiction.

Definition 2.3.1. If Y is smooth and proper variety, $D \subseteq Y$ is a simple normal crossing and $X := Y \setminus D$, we say that an overconvergent isocrystal over X is *docile* if it admits a log-extension with respect to D with nilpotent residues. We denote the category of docile overconvergent isocrystals by $\mathbf{Isoc}(X)^{\dagger,\text{doc}}$.

Abe and Esnault proved the following Lefschetz theorem for docile overconvergent isocrystals over perfect fields. This corresponds to the "tame setting".

Theorem 2.3.2 ([AE18]). *Let $Y \subseteq \mathbb{P}^n_k$ be a smooth connected projective variety of dimension ≥ 2 and D a simple normal crossing divisor. For every smooth curve $\overline{C} \subseteq Y$ which is a complete intersection of hypersurfaces intersecting transversally D, if we write C for $\overline{C} \setminus D$, the functor*

$$\mathbf{Isoc}(X)^{\dagger,\mathrm{doc}} \to \mathbf{Isoc}(C)^{\dagger,\mathrm{doc}}$$

is fully faithful.

Unluckily this is not sufficient to prove $\mathrm{MS}(X)$ in higher dimension. We will explain how to refine their result and prove the following theorem.

Theorem 2.3.3. *Let $Y \subseteq \mathbb{P}^n_k$ be a smooth connected projective variety of dimension ≥ 2 and D a simple normal crossing divisor. If $\mathcal{M}^\dagger$ is a docile overconvergent F-isocrystal over $X := Y \setminus D$ with respect to D, then there exists a smooth connected curve $C \subseteq X$ such that the restriction functor*

$$\langle \mathcal{M}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$$

is an equivalence of categories.

To prove Theorem 2.3.3 we will also need $\mathrm{P}(X)$ for smooth connected curves, which follows from Theorem 1.1.3 and Theorem 2.2.2. A first result in the direction of Theorem 2.3.3 is the following well-known isoclinic Lefschetz theorem.

Proposition 2.3.4 (after Pink–Serre). *Let X be a smooth connected variety and $\mathcal{N}$ a finite direct sum of isoclinic F-isocrystal over X. There exists a connected finite étale cover $\widetilde{X} \to X$ with the property that for every smooth connected curve $C \subseteq X$ such that $\widetilde{X} \times_X C$ is connected, the restriction functor*

$$\langle \mathcal{N} \rangle_F \to \langle \mathcal{N}|_C \rangle_F$$

is an equivalence of categories.

Proof. Since $\langle \mathcal{N} \rangle_F$ contains the F-isocrystals coming from k, we reduce to the case when $\mathcal{N}$ is unit-root. Thanks to Katz–Crew theorem, it is enough to prove the analogous statement for p-adic representation of the étale fundamental group. In this case we can apply Pink–Serre result [Ser89, §10.6], [Kat90, Key Lemma 8.18.3]. The finiteness of the étale cover follows from the fact that the image of the p-adic representation is a p-adic Lie group and its Frattini is an open subgroup. □

The next subtle is a consequence of the property $\mathrm{P}(X)$ and it helps to describe the subgroup $X^*(G(\mathcal{M}^\dagger)) \subseteq X^*(G(\mathcal{M}))$ in the docile situation.

Lemma 2.3.5. *Let X be a smooth connected variety such that every open U satisfies $\mathrm{P}(U)$ and $\mathcal{M}$ a $\dagger$-extendable F-isocrystal. If $\mathcal{L} \in \langle \mathcal{M} \rangle$ is $\dagger$-extendable, then $\mathcal{L}^\dagger \in \langle \mathcal{M}^\dagger \rangle$. In particular, if $\mathcal{M}^\dagger$ is docile along D then every rank 1 $\dagger$-extendable isocrystal $\mathcal{L} \in \langle \mathcal{M} \rangle$ is unramified along D.*

Proof. By Theorem 1.2.3 we may shrink X and assume that $\mathcal{M}$ admits the slope filtration. Moreover, we may assume without loss of generality that $\mathcal{M}^\dagger$ is semi-simple. Let us write G for $G(\mathcal{M}^\dagger \oplus \mathcal{L}^\dagger)$ and G' for $G(\mathcal{M}^\dagger)$. The inclusion $\langle \mathcal{M}^\dagger \rangle \subseteq \langle \mathcal{M}^\dagger \oplus \mathcal{L}^\dagger \rangle$ induces a surjective morphism $f : G \twoheadrightarrow G'$. We want to prove that $N := \ker(f) = 1$. Let $P \subseteq G$ be the subgroup associated to $\langle \mathcal{M} \oplus \mathcal{L} \rangle$. By $\mathrm{P}(X)$ the subgroup P is a parabolic subgroup of G, thus in particular it contains a maximal torus T of G. Since $\langle \mathcal{M} \rangle = \langle \mathcal{M} \oplus \mathcal{L} \rangle$, the morphism f is an isomorphism when restricted to P, so that $N \cap P = 1$. Therefore N is a finite group, because it is a normal subgroup of the reductive group G which intersects trivially the maximal torus T. The morphism $\pi_0(f)$ is an isomorphism by Corollary 2.2.3, because both $\pi_0(G)$ and $\pi_0(G')$ are equal to $\pi_0(P)$. This implies that N is contained in G°, the neutral component of G. Moreover, since N is finite, it is contained in $Z(G^\circ) \subseteq T$. This shows that $N = 1$, as we wanted.

The consequence follows from two facts. First, the property of being docile is preserved under the operations of taking direct sum, tensor product, dual and subquotients in $\mathbf{Isoc}^\dagger(X)$. The second one is that rank 1 docile overconvergent isocrystals have 0 residues, thus they are unramified.

□

Lemma 2.3.6. *Let X be a smooth connected variety over k, $D \subseteq X$ a divisor and $\mathcal{N}$ a direct sum of isoclinic F-isocrystals over X. There exists a dense open $V \subseteq D$ and a conic closed subscheme $Z \subseteq TX \times_X V$ of codimension at least 1 at every fibre which satisfies the following property.*

R$(\mathcal{N}, D, Z)$ *Let $C \subseteq X$ be a smooth curve intersecting V in a non-empty 0-dimensional subscheme and such that $TC \times_X V$ is not entirely contained in Z. For every $\mathcal{N}' \in \langle \mathcal{N} \rangle_F$ ramified at D, $\mathcal{N}'|_C$ is ramified at $D \cap C$.*

Proof. We may assume that D is irreducible and $\mathcal{N}$ is unit-root, so that we can associate to $\mathcal{N}$ a p-adic representation of the étale fundamental group of X. Since the image of the inertia subgroup of D is topological of finite type and virtually pro-p it admits finitely many cyclic quotients of prime order. Then the final result follows from the (proof of) [Dri12, Lemma 5.1]. □

Theorem 2.3.7 (Bertini's theorem)**.** *Let $Y \subseteq \mathbb{P}^n_k$ be a smooth connected projective variety over k, D a divisor of Y, V an open of D, Z a conic closed subset of $TY \times_Y V$ which has codimension 1 at each fibre, $U \subseteq Y$ a dense open and $\widetilde{U} \to U$ a connected finite étale cover. There exists a curve $C \subseteq Y$ over k satisfying the following conditions.*

1. *C is a smooth scheme theoretic complete intersection of hyperplanes intersecting D transversally.*
2. *$C \times_U \widetilde{U}$ is connected and non-empty.*
3. *C intersects each irreducible component W of V and the image of $TC \times_Y W \to TY \times_Y W$ is not entirely contained in $Z \times_V W$.*

Proof. By Bertini's theorem, in the form proven in Jouanoulou Theorem 6.3, the three conditions correspond to dense open subspaces in the Grassmannian of hyperplanes in $\mathbb{P}^n_k$. The result then follows from an induction on the dimension of Y. □

Proof of Theorem 2.3.3. We want to apply Theorem 2.3.7. During the proof we will impose three conditions on $\overline{C} \subseteq Y$. These conditions will ensure that if $C := \overline{C} \setminus D$, the restriction functor $\langle \mathcal{M}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence.

We first assume that $\overline{C} \subseteq Y$ is a smooth complete intersection of hyperplanes intersecting D transversally (Condition (1)). By Theorem 2.3.2, the restriction functor

$$\mathbf{Isoc}(X)^{\dagger,\mathrm{doc}} \to \mathbf{Isoc}(C)^{\dagger,\mathrm{doc}}$$

is fully faithful.

Let $U \subseteq X$ be a dense open where $\mathcal{M}$ acquires the slope filtration and such that $D' := Y \setminus U$ is a divisor. Write C_U for $C \cap U$ and $\mathcal{N}$ for $\mathrm{Gr}^{S_\bullet}(\mathcal{M}|_U)$. By Proposition 2.3.4, there exists a connected finite étale cover $\widetilde{U} \to U$, with the property that if $\widetilde{U} \times_U C_U$ is connected and non-empty then the functor $\langle \mathcal{N} \rangle_F \to \langle \mathcal{N}|_{C_U} \rangle_F$ is an equivalence of categories. Let us assume that C_U satisfies this condition (Condition (2)).

Consider the subgroup $X^*(G(\mathcal{M}|_U))_F \subseteq X^*(G(\mathcal{M}|_U))$ of rank 1 isocrystals which come from some rank 1 F-isocrystal in $\langle \mathcal{M}|_U \rangle_F$.

Lemma 2.3.8. *The group $X^*(G(\mathcal{M}|_U))_F$ is canonically isomorphic to the group of unit-root rank 1 F-isocrystals in $\langle \mathcal{N} \rangle_F$.*

Proof. First, note that rank 1 F-isocrystals are isoclinic, thus the rank 1 F-isocrystals in $\langle \mathcal{M} \rangle_F$ are the same as the rank 1 F-isocrystals in $\langle \mathcal{N} \rangle_F$. Subsequently, we show that a rank 1 isocrystal admits at most one unit-root Frobenius structure. Let ψ_1 and ψ_2 be two unit-root Frobenius structure of a rank 1 isocrystals $\mathcal{L}$. The F-isocrystal $(\mathcal{L}, \psi_1) \otimes (\mathcal{L}, \psi_2)^\vee$ is an unit-root F-isocrystals coming from k. Thus, by the Dieudonné–Manin classification it is the unit object of $\mathbf{F}$-$\mathbf{Isoc}(U)$, as we wanted. Finally, we note that every rank 1 F-isocrystal has integral slope, thus after twist by a rank 1 F-isocrystal over k it becomes unit-root. □

We also define $X^*(G(\mathcal{M}|_{C_U}))_F \subseteq X^*(G(\mathcal{M}|_{C_U}))$, $X^*(G(\mathcal{M}^\dagger|_U))_F \subseteq X^*(G(\mathcal{M}^\dagger|_U))$,... in the analogous way and we have for $X^*(G(\mathcal{M}|_{C_U}))_F \subseteq X^*(G(\mathcal{M}|_{C_U}))$ the analogue of Lemma 2.3.8. Since $\langle \mathcal{N} \rangle_F \to \langle \mathcal{N}|_{C_U} \rangle_F$ is an equivalence, by Lemma 2.3.8, it follows that $X^*(G(\mathcal{M}|_U))_F = X^*(G(\mathcal{M}|_{C_U}))_F$. By Lemma 2.3.6, there exists a dense open $V \subseteq D'$ and a conic closed subscheme $Z \subseteq TX \times_X V$ of codimension at least 1 at every fibre which satisfies the property $\mathrm{R}(\mathcal{N}, D', Z)$. Suppose that $\overline{C}$ satisfies the assumptions in $\mathrm{R}(\mathcal{N}, D', Z)$ (Condition (3)), write $X^*(G(\mathcal{M}|_U))_F^{\mathrm{ur}} \subseteq X^*(G(\mathcal{M}|_U))_F$ and $X^*(G(\mathcal{M}|_{C_U}))_F^{\mathrm{ur}} \subseteq X^*(G(\mathcal{M}|_{C_U}))_F$ for the subgroups of unramified isocrystals. By $\mathrm{R}(\mathcal{N}, D', Z)$, we get $X^*(G(\mathcal{M}|_U))_F^{\mathrm{ur}} = X^*(G(\mathcal{M}|_{C_U}))_F^{\mathrm{ur}}$.

We know that $\mathrm{P}(C_U)$ is true thanks to Theorem 1.1.3 and Theorem 2.2.2. Thus by Lemma 2.3.5,

$$X^*(G(\mathcal{M}^\dagger|_C))_F = X^*(G(\mathcal{M}^\dagger|_{C_U}))_F \subseteq X^*(G(\mathcal{M}|_{C_U}))_F$$

is contained in $X^*(G(\mathcal{M}|_{C_U}))_F^{\mathrm{ur}} = X^*(G(\mathcal{M}|_U))_F^{\mathrm{ur}}$. Therefore, all the rank 1 F-isocrystals in $\langle \mathcal{M}^\dagger|_C \rangle_F$ are restriction of rank 1 F-isocrystals over Y. Let Γ be a finite set of unit-root F-isocrystals over C which generate $X^*(G(\mathcal{M}^\dagger|_C))_F$ and write $\mathcal{P}|_C$ for their direct sum. By the previous observation, $\mathcal{P}|_C$ comes from an unramified F-isocrystal $\mathcal{P}$ over X. Note that $\mathcal{P}$ is $\dagger$-extendable and $\mathcal{P}^\dagger$ is docile. The pullback functor $\langle \mathcal{M}^\dagger \oplus \mathcal{P}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$ is fully faithful because $\mathcal{M}^\dagger \oplus \mathcal{P}^\dagger$ is docile. Moreover, by construction, every rank 1 F-isocrystal in $\langle \mathcal{M}^\dagger|_C \rangle_F$ comes from a rank 1 F-isocrystal in $\langle \mathcal{M}^\dagger \oplus \mathcal{P}^\dagger \rangle_F$. By Proposition 2.2.1, $\langle \mathcal{M}^\dagger \oplus \mathcal{P}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence of categories.

On the other hand, we also know that by definition $\langle \mathcal{M}^\dagger \rangle \subseteq \left\langle \mathcal{M}^\dagger \oplus \mathcal{P}^\dagger \right\rangle$ is fully faithful and closed under the operation of taking subquotients. Thus the functor $\langle \mathcal{M}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$ is fully faithful and the essential image is closed under subquotients. This implies that it is an equivalence, as we wanted. It remains to show that a curve $\overline{C}$ with these three conditions exists. This is guaranteed by Theorem 2.3.7. □

2.4. **Some consequences.** Thanks to the reductions as in §1.2 and Kedlaya's semistable reduction theorem, Theorem 2.3.3 implies the following result.

Theorem 2.4.1 (Tsuzuki, D'Ad)**.** *For every smooth connected variety X over a perfect field of positive characteristic,* $\mathrm{MS}(X)$ *is true.*

As a consequence, by Theorem 2.2.2, we prove the parabolicity conjecture.

Theorem 2.4.2. *For every smooth connected variety X over an algebraically closed field of positive characteristic,* $\mathrm{P}(X)$ *is true.*

Corollary 2.4.3. *Let $\mathcal{M}^\dagger$ be a semi-simple overconvergent F-isocrystal which admits the slope filtration. The natural functor $\alpha : \langle\mathcal{M}^\dagger\rangle \to \langle\mathcal{M}\rangle$ admits a left adjoint β. Moreover β admits left derived functors $L^i\beta$ (without passing to the ind-categories) and they vanish for i greater than the dimension of $G(\mathcal{M}^\dagger)/G(\mathcal{M})$.*

Proof. This follows from $\mathrm{P}(X)$ thanks to the standard theory on the parabolic induction. □

Definition 2.4.4. We say that a $\dagger$-extendable F-isocrystal over X is PBS[5] (pure in bounded subobjects) if for every connected open $U \subseteq X$, the isoclinic subobjects of the restriction have minimal generic slope.

We prove the following generalization of Tsuzuki's result over curves.

Corollary 2.4.5. *If $\mathcal{M}$ is a $\dagger$-extendable F-isocrystal over a smooth connected variety over a positive characteristic perfect field, then there exists a unique filtration $0 = P_{-1}(\mathcal{M}) \subsetneq P_0(\mathcal{M}) \subsetneq \cdots \subsetneq P_n(\mathcal{M}) = \mathcal{M}$ of $\dagger$-extendable F-isocrystals such that for every $i \geq 0$, the quotient $P_i(\mathcal{M})/P_{i-1}(\mathcal{M})$ is* PBS *with minimal generic slope $t_i \in \mathbb{Q}$ and $t_0 > t_1 > \cdots > t_n$.*

Proof. After shrinking the variety in order to get the slope filtration, we construct $P_{n-1}(\mathcal{M})$ as the $\dagger$-hull of the direct sum of the isoclinic subobjects of $\mathcal{M}$ which do not have minimal generic slope. By $\mathrm{MS}(X)$, we know that $S_0(\mathcal{M}) \cap P_{n-1}(\mathcal{M}) = 0$ so that if $\mathcal{M} \neq 0$ then $\mathcal{M}/P_{n-1}(\mathcal{M}) \neq 0$. One can easily check that $\mathcal{M}^\dagger/P_{n-1}(\mathcal{M}^\dagger) \neq 0$ is PBS by construction. By induction on the rank of $\mathcal{M}^\dagger$, we prove the result. □

Definition 2.4.6. A pair $(\mathcal{M}, \iota)$ is a *$\dagger$-compactification* of $\mathcal{N}$ if $\mathcal{M}$ is a $\dagger$-extendable F-isocrystal and $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ is an injective morphism with $\overline{\mathcal{N}} = \mathcal{M}$. A morphism $(\mathcal{M}_1, \iota_1) \to (\mathcal{M}_2, \iota_2)$ of $\dagger$-compactifications is a morphism $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ of F-isocrystals such that $\psi \circ \iota_1 = \iota_2$. We will often drop ι in the notation. We say that a weakly final object of the category of $\dagger$-compactifications of $\mathcal{N}$ is a *minimal $\dagger$-compactification* of $\mathcal{N}$.

We have the following stronger form of Kedlaya's conjecture.

Corollary 2.4.7. *If $\mathcal{N}$ is isoclinic and it embeds in a $\dagger$-extendable F-isocrystal, then it admits a minimal $\dagger$-compactification.*

Proof. Since the category of F-isocrystals is noetherian, it is enough to prove that for every pair $(\mathcal{M}_1, \iota_1), (\mathcal{M}_2, \iota_2)$ of $\dagger$-compactifications of $\mathcal{N}$, we can find isomorphic $\dagger$-compactifications $(\mathcal{M}'_1, \iota'_1) \simeq (\mathcal{M}'_2, \iota'_2)$ with (surjective) morphisms $(\mathcal{M}_i, \iota_i) \twoheadrightarrow (\mathcal{M}'_i, \iota'_i)$. Write $\mathcal{M}$ for $\mathcal{M}_1 \oplus \mathcal{M}_2$ and consider the inclusion of $\mathcal{N}$ in $\mathcal{M}$ induced by ι_1 and ι_2. Write $\overline{\mathcal{N}}$ for the $\dagger$-hull of $\mathcal{N}$ in $\mathcal{M}$. We claim that $\iota_1 : \mathcal{N} \to \mathcal{M}_1/(\mathcal{M}_1 \cap \overline{\mathcal{N}})$ and $\iota_2 : \mathcal{N} \to \mathcal{M}_2/(\mathcal{M}_2 \cap \overline{\mathcal{N}})$ are injective. Indeed, after shrinking X in

[5]This is the dual of Tsuzuki's notion of PBQ overconvergent F-isocrystals.

order to acquire slope filtrations, by MS(X) we have $\mathcal{M}_1 \cap \overline{\mathcal{N}} \cap \iota_1(\mathcal{N}) \subseteq \mathcal{M}_1 \cap S_0(\overline{\mathcal{N}}) = \mathcal{M}_1 \cap \mathcal{N} = 0$ and the same is true for ι_2. Finally we get

$$\mathcal{M}_1/(\mathcal{M}_1 \cap \overline{\mathcal{N}}) \simeq \mathcal{M}/\overline{\mathcal{N}} \simeq \mathcal{M}_2/(\mathcal{M}_2 \cap \overline{\mathcal{N}})$$

which concludes the proof.

□

Remark 2.4.8. If $\mathcal{N}$ is isoclinic and admits a †-compactification $\mathcal{N} \subseteq \overline{\mathcal{N}}$, then the minimal †-compactification is simply $\mathcal{N} \subseteq P_n(\overline{\mathcal{N}})/P_{n-1}(\overline{\mathcal{N}})$.

2.5. **An application to abelian varieties.** Let $F/\mathbb{F}_p$ be a finitely generated extension and let A be an abelian variety over F. We use next immediate consequence of MS(X) to prove a result on the separable torsion points of A.

Corollary 2.5.1. *Let $\mathcal{M}$ be a †-extendable F-isocrystal over X. If $\mathcal{N}$ is an isolinic subobject of maximal slope, then $\mathcal{N}$ is †-extendable.*

Theorem 2.5.2. *If $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a division algebra, then the group $A(E^{\mathrm{sep}})[p^\infty]$ is finite.*

Proof. The proof is by contradiction. Suppose that the group $A(E^{\mathrm{sep}})[p^\infty]$ is infinite, then the p-divisible group $A_{E^{\mathrm{sep}}}[p^\infty]/E^{\mathrm{sep}}$ admits $\mathbb{Q}_p/\mathbb{Z}_p$ as a subgroup. Let $\widetilde{\mathcal{H}}$ be the maximal p-divisible constant subgroup of $A_{E^{\mathrm{sep}}}[p^\infty]$. By Galois descent, $\widetilde{\mathcal{H}}$ descends to some étale p-divisible group $\mathcal{H}/E$. Let X be a smooth connected variety over $\mathbb{F}_p$ with function field E and such that A and $\mathcal{H}$ have good models over X. Choose an abelian scheme $\mathcal{A}/X$ which is a model of A.

Let $\mathcal{M}$ be the crystalline Dieudonné module of $\mathcal{A}[p^\infty]$ and $\mathcal{N}$ the crystalline Dieudonné module of the model of $\mathcal{H}$ in $\mathcal{A}[p^\infty]$. The F-isocrystal $\mathcal{N}$ is a quotient of $\mathcal{M}$. As $\mathcal{H}$ is étale, the F-isocrystal $\mathcal{N}$ is unit-root. On the other hand, $\mathcal{M}$ is †-extendable. Therefore, by Corollary 2.5.1, the same is true for $\mathcal{N}$. Thanks to the theory of weights, the overconvergent F-isocrystal $\mathcal{M}^\dagger$ is semi-simple in the category of overconvergent F-isocrystal. Thus $\mathcal{N} \subsetneq \mathcal{M}$ is actually a direct summand of $\mathcal{M}$. This implies that $\mathrm{End}(\mathcal{M})$ contains some non-trivial idempotent. Thanks to de Jong's theorem, $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p \simeq \mathrm{End}(\mathcal{M})$. This leads to a contradiction since $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a division algebra by assumption. □

Question 2.5.3. *Let $f : \mathcal{A} \to X$ be an abelian scheme over a smooth connected variety X defined over an algebraically closed field of characteristic p. If $\mathcal{A}$ is generically traceless, can $R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}$ have a †-extendable unit-root subquotient?*

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