Parabolicity conjecture of F-isocrystals

k perfect field, $W \circlearrowleft \sigma$ ring of Witt vectors of k

$K := W[\frac{1}{p}]$, X/k smooth algebraic variety

Crystalline cohomology : $H^\bullet_{cris}(X/W)$

If X lifts to a formal scheme $\mathfrak{X}/\mathrm{Spf}\, W$, then

$$H^\bullet_{cris}(X/W) = H^\bullet_{dR}(\mathfrak{X}/W)$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad \mathbb{H}^\bullet(\mathcal{O} \to \Omega^1_{\mathfrak{X}/W} \to \Omega^2_{\mathfrak{X}/W} \to \cdots)$$

Example

$$H^{2n}_{cris}(\mathbb{P}^n_k/W) = H^{2n}_{dR}(\hat{\mathbb{P}}^n_W/W) = H^n(\mathbb{P}^n_W, \Omega^n_{\mathbb{P}^n_W/W}) =$$

$$\underset{\substack{\uparrow \\ \check{C}ech}}{=} W\cdot[\omega] \qquad \omega := \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \cdot \ldots \cdot x_n}$$

$$F : X \to X \quad \text{abs. Frobenius} \quad F^* \circlearrowleft H^\bullet_{cris}(X/W)$$
$$x \mapsto x^p$$

In the example: $F^*(a\omega) = \sigma(a) \dfrac{dx_1^p \wedge \cdots \wedge dx_n^p}{x_1^p \cdot \ldots \cdot x_n^p} \underset{\uparrow}{=} p^n \sigma(a)\,\omega$

$(dx_i^p = p x_i^{p-1} dx_i)$

Thm (Dieudonné-Manin) If $k=\bar{k}$ and X is proper, for each $i \geq 0$

$$H^i_{cris}(X/W)[\tfrac{1}{p}] = \bigoplus_{s/r \in S} H_{s/r}$$

where
- $S \subseteq \mathbb{Q}$ finite subset
- $H_{s/r} \neq 0$ K-vector space gen. by vectors v s.t. $(F^r)^* v = p^s v$.

S is the set of slopes of $H^i_{cris}(X/W)[\tfrac{1}{p}]$.

Examples

- $H^{2n}_{cris}(\mathbb{P}^m_k/W)$ is of slope n
- If E is an elliptic curve $S = \begin{cases} 0,1 & E \text{ is ord.} \\ \frac{1}{2} & E \text{ is s.s.} \end{cases}$

$$\begin{array}{c} Y \\ {\scriptstyle f}\big\downarrow {\scriptstyle \text{smooth proper}} \\ X \end{array}$$

$R^i f_{cris*} \mathcal{O}_{Y/W} \circlearrowleft \Phi$ is an <u>F-isocrystal</u> over X describing the variation of the crystalline cohomology groups of the fibres.

__Thm__ (Grothendieck-Katz) Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be an F-isocrystal over X. Up to shrinking X, there exists a unique filtration

$$0 = S_0(\mathcal{M}) \subsetneq S_1(\mathcal{M}) \subsetneq \dots \subsetneq S_m(\mathcal{M}) = \mathcal{M}$$

s.t. each $S_i(\mathcal{M})/S_{i-1}(\mathcal{M})$ has unique slope s_i and $s_0 < \dots < s_m$.

__Example__ $X(N)/\mathbb{F}_p$ modular curve of level $N \geq 3$ prime to p.

$$\begin{array}{l} \mathcal{E} \\ f \downarrow \text{ univ. ell. curve} \\ X(N) \end{array} \qquad \mathcal{M} := R^1 f_{cris*} \mathcal{O}_{\mathcal{E}/W} \circlearrowleft \Phi_{\mathcal{M}}$$

$U := X(N)^{ord}$ ordinary locus

$$0 \to \underbrace{S_1(\mathcal{M}|_U)}_{\substack{\text{rank } 1 \\ \text{slope } 0}} \to \mathcal{M}|_U \to \underbrace{S_1(\mathcal{M}|_U)^\vee(-1)}_{\substack{\text{rank } 1 \\ \text{slope } 1}} \to 0$$

while $R^1 f_{\acute{e}t*} \underline{\mathbb{Q}_\ell}$ is irreducible ...

$$\text{F-Isoc}^{\dagger}(X) \hookrightarrow \text{F-Isoc}(X)$$
$$(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}) \longmapsto (\mathcal{M}, \Phi_{\mathcal{M}})$$

<u>Overconvergent F-isocrystals</u>

"p-adic analogue of ℓ-adic lisse sheaves"

- Finite dimensional cohomology
- Theory of weights

From now on suppose X connected and endowed with a rational point x.

For $\mathcal{M}^{\dagger} \in \text{Isoc}^{\dagger}(X)$, Crew constsucted two algebraic groups $/K$

$$G(\mathcal{M}) \subseteq G(\mathcal{M}^{\dagger})$$

called <u>monodromy groups</u>

These are the Tannaka groups of

$$\langle \mathcal{M} \rangle_{\text{Isoc}(X)} \quad \text{and} \quad \langle \mathcal{M}^{\dagger} \rangle_{\text{Isoc}^{\dagger}(X)} \quad .$$

<u>Thm</u>A (Crew's parabolicity conjecture).
Let $(\mathcal{M}^+, \Phi_{\mathcal{M}})$ be an overconvergent F-isocrystal over X with constant slopes. The subgroup $G(\mathcal{M}) \subseteq G(\mathcal{M}^+)$ is the stabiliser of the slope filtration.
Moreover, if $\mathcal{M}^+$ is semi-simple, $G(\mathcal{M})$ is a parabolic subgroup.

<u>Consequences</u>:

<u>Theorem</u>. Let k be a finite field and let $f: A \to X$ be an abelian scheme with constant slopes. If $\mathcal{M} := R^1 f_{cris*} \mathcal{O}_{A/W}$, then $Gr_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$ is semi-simple.
In particular, $R^1 f_{\acute{e}t*} \underline{\mathbb{Q}_p}$ is semi-simple.

<u>Theorem.</u> Suppose that $\dim X = 1$ and let $\mathbb{A}$ be its ring of adèles. For each $r > 0$, the isomorphism class of a $\overline{\mathbb{Q}}_p$-linear cuspidal automorphic representation of $GL_r(\mathbb{A})$ is determined by the datum of the Hecke eigenvalues of minimal slope at all but finitely many places.

<u>Def</u> (t-hull) For an inclusion $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ we write $(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$ for the smallest t-extendable F-isocrystal in $(\mathcal{M}, \Phi_{\mathcal{M}})$ which contains $(\mathcal{N}, \Phi_{\mathcal{N}})$. We say that $(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$ is the <u>t-hull</u> of $(\mathcal{N}, \Phi_{\mathcal{N}})$.

<u>Thm</u> B (Tsuzuki, D'A) If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is t-extendable and has constant slopes, then

$$S_1(\mathcal{N}) = S_1(\overline{\mathcal{N}}) \; .$$

Thm B $\Rightarrow$ Thm A Suppose k finite.

$G := G(\mathcal{M}^+, \Phi_{\mathcal{M}})$, $H := G(\mathcal{M}, \Phi_{\mathcal{M}})$,

$H_s := \mathrm{Stab}_G(S_\bullet)$

$$\mathrm{Rep}_K(G) \simeq \langle \mathcal{M}^+, \Phi_{\mathcal{M}} \rangle_{F\text{-}\mathrm{Isoc}^+(X)}$$

$$\mathrm{Rep}_K(H) \simeq \langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle_{F\text{-}\mathrm{Isoc}(X)}$$

We already know that $H \subseteq H_s$ because $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration.

It remains to prove that $H_s \subseteq H$.

By Chevalley's theorem, $\exists\ V \in \mathrm{Rep}(G)$ and $L \subseteq V$ a line s.t. $H = \mathrm{Stab}_G(L \subseteq V)$.

This inclusion corresponds to an inclusion

$$(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq (\mathcal{V}, \Phi_{\mathcal{V}})$$

in $F\text{-}\mathrm{Isoc}(X)$ with $(\mathcal{V}, \Phi_{\mathcal{V}})$ $+$-extendable.

Write s for the slope of $(\mathcal{L}, \Phi_{\mathcal{L}})$.

By Thm B, $\mathcal{L} = S_1(\mathcal{L}) \overset{B}{=} S_1(\overline{\mathcal{L}}) = \overline{\mathcal{L}} \cap \mathcal{V}^{\leq s}$.

Switching to representations again,

$$L = \bar{L} \cap V^{\leq s}$$

- H_s stabilises $\bar{L}$ because $H_s \subseteq G$ and $\bar{\mathcal{L}}$ is t-extendable.

- H_s stabilises $V^{\leq s}$ because it is the stabiliser of the slope filtration.

$\leadsto$ H_s stabilises L

$\leadsto$ $H_s \subseteq H$

$\boxtimes$