Towards the Langlands Correspondence

1) On the Automorphic and Galois side
2) Statement of the correspondence (for GL_r)
3) Converse Theorem

$X/_{\mathbb{F}_q}$ smooth projective geometrically integral curve

$S \subseteq X$ reduced finite subscheme

F field of functions on X

$\mathbb{A}$ adele ring of F, $\mathbb{A}^\times$ idele ring

$\forall\, x \in |X|$, F_x is the completion at x

$\mathcal{O}_x$ ring of integers

G connected split reductive group$/_F$

Automorphic representations

We have a space of <u>automorphic forms</u> of G.

$$\mathcal{A}^{aut}_G := \left\{ f : G(F) \backslash G(\mathbb{A}) \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \exists\, K \text{ copen in } G(\mathbb{A}) \\ \text{s.t. } f \text{ is } K \text{ invariant} \\ \text{on the right} \\ + \text{ admissibility} \\ \text{condition.} \end{array} \right\}$$

$G(\mathbb{A})$ acts on $\mathcal{A}_G^{Aut}$ on the right.

The irr. sub-quotients of the repr. of $G(\mathbb{A})$ on $\mathcal{A}_G^{aut}$ are the so-called <u>automorphic representations</u> of G.

Inside we have $\mathcal{A}_G^{cusp} \subseteq \mathcal{A}_G^{aut}$, the subspace of <u>cusp forms</u>, namely $f \in \mathcal{A}_G^{aut}$ s.t.

| $\forall$ parabolic subgroup $P \subseteq G$ and every $g \in G(\mathbb{A}_F)$

$$\int_{N(F) \backslash N(\mathbb{A}_F)} f(gn)\, dn = 0$$

| where $N = R_u(P)$.

The irr. sub-representation of $G(\mathbb{A})$ in $\mathcal{A}_G^{cusp}$ are the <u>cuspidal automorphic representations</u>. You can think them as the "building blocks" of the automorphic representations.

<u>Flath's Theorem</u>

Any irreducible admissible representation π of $G(\mathbb{A})$ can be written uniquely as $\bigotimes'_{x \in |X|} \pi_x$ where π_x are admissible irr. representations of $G(F_x)$, <u>unramified</u> outside a certain $S \subseteq X$.

We recall that an irr. adm. repr of $G(F_x)$ is <u>unramified</u> iff $\pi_x^{G(\mathcal{O}_x)} \neq \{0\}$

$$\left\{\begin{array}{l}\text{unramified repr}\\ \text{of } G(F_x)\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{simple}\\ \mathcal{H}(G(F_x), G(\mathcal{O}_x))-\\ \text{modules}\end{array}\right\}$$

Satake isomorphism : $S : \mathcal{H}(G(F_x), G(\mathcal{O}_x)) \xrightarrow{\sim} \mathbb{C}[X_\bullet]^W$

Ex : If $G = GL_n$, the maximal torus is $\mathbb{G}_m^n$ and the Weyl group is S_n.

$$\overset{S.\,iso}{\Longrightarrow} \mathcal{H}(GL_n(F_x), GL_n(\mathcal{O}_x)) \xrightarrow{\sim} \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{S_n},$$

thus :

$$\left\{\begin{array}{l}\text{unr. repr}\\ \text{of } GL_n(F_x)\end{array}\right\} \longleftrightarrow \left\{ \{\xi_1, \dots, \xi_n\} \in (\mathbb{C}^*)^n / S_n \right\}$$

$$\pi_x \longmapsto \{z_1(\pi_x), \dots, z_n(\pi_x)\}$$

Hecke eigenvalues

You can think this bijection as an unramified local Langlands correspondence

Langlands dual

$$G \rightsquigarrow G^\vee \text{ Langlands dual}$$

$$(X^\bullet, \Phi, X_\bullet, \check{\Phi}) \qquad\qquad (X_\bullet, \check{\Phi}, X^\bullet, \Phi)$$

$G^\vee$ is related to the representation theory of G.

Ex:

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
Sp_{2n}	SO_{2n+1}

Looking again at the Satake isomorphism -

$$\{\text{unramified reps of } G(F_x)\} \xleftrightarrow{\sim} \mathrm{Hom}(X_*(T), \mathbb{C}^\times)/W$$

RHS $= \mathrm{Hom}(X^*(\check{T}), \mathbb{C}^\times)/W = \check{T}(\mathbb{C})/W = \check{G}(\mathbb{C})^{ss}/\text{conj}$ (Set of Satake parameters)

$$\pi_x \mapsto s(\pi_x) \in \check{G}(\mathbb{C})^{ss}/\text{conj}$$

Local L-factors: If we fix $\rho : \check{G} \to GL_r$,

we can associate to $\pi_x \mapsto L_x^\rho(\pi, t) := \det_\rho(1 - s(\pi_x)t)^{-1}$ $\in \mathbb{C}[[t]]$

⚠ The choice of a "good" ρ is in general a difficult problem. We would like to choose it in such a way to guarantee the functoriality of the Langlands Correspond. when G varies.

Ex : $G = GL_r$, $\rho = id$, $L_x(\pi, t) = \prod_{i=1}^{r} (1 - z_i(\pi_x))^{-1}$

"Strong multiplicity one" Theorem (Piatetski-Shapiro)

The isomorphism class of a cuspidal aut. repr. π of GL_r is determined by the datum of the Hecke eigenvalues for almost every point $x \in |X|$.

Galois ℓ-adic representations

$\bar{x}$ geom. pt., $S \subseteq X$ reduced finite, $\overline{X} := X \otimes \overline{\mathbb{F}_q}$, $\overline{X-S} := (X-S) \otimes \overline{\mathbb{F}_q}$

We have an exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\acute{e}t}(\overline{X-S}, \bar{x}) \arrow[r] & \pi_1^{\acute{e}t}(X-S, \bar{x}) \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1^{\acute{e}t}(\overline{X-S}, \bar{x}) \arrow[r] \arrow[u, equal] & W(X-S, \bar{x}) \arrow[r] \arrow[u, hook] & W(\overline{\mathbb{F}_q}/\mathbb{F}_q) \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

$W(X-S,\bar{x})$ = Weil group of $X-S$

$W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ = Cyclic group generated by the q-Frobenius.

The topology on $W(X-S, \bar{x})$ is s.t. $\pi_1^{\acute{e}t}(\overline{X-S}, \bar{x}) \hookrightarrow W(X-S, \bar{x})$ is an open embedding

Frobenii: Let $x \in |X-S|$ a closed point, $x = \mathrm{Spec}\, \mathbb{F}_{q^s}$. By the functoriality of the Weil group it induces a family of morphisms $W(\overline{\mathbb{F}_q}/\mathbb{F}_{q^s}) \to W(X-S, \bar{x})$. The set of images of the inverse of the q^s-Frobenius (geometric Frobenius) gives a conjugacy class $\mathrm{Frob}_x \subseteq W(X-S, \bar{x})$.

Čebotarev Theorem The set $\bigcup_{x \in |X-S|} \mathrm{Frob}_x$ is dense in $\pi_1^{\acute{e}t}(X-S, \bar{x})$.

Galois homomorphisms: For every $\ell \neq p$, we will be interested in the continuous Galois homomorphisms unramified over S:

$$W(X-S, \bar{x}) \to G(\overline{\mathbb{Q}_\ell})$$

If $G = GL_r$ we will call these homomorphisms (ℓ-adic) Galois representations of rank r.

Local factors: For every reductive group $G^\vee$, σ a Galois hom. of $G^\vee$, after the choice of $\rho : G^\vee \to GL_r$ we can define for every $x \in |X - S|$,

$$L^\rho_x(\sigma, t) := \det_\rho\left(1 - t\,\sigma(\mathrm{Frob}_x)\right)^{-1}$$

Consequence of Čeb: Two semisimple Galois representations σ_2 and σ_2 are isomorphic iff they have the same L-factors for almost every point.

Langlands correspondence

Let G be a split connected reductive group$/_F$, $\rho : G^\vee \to GL_r$, $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Cuspidal aut. repr.}\\ \text{of } G \text{ unramified}\\ \text{over } S\end{matrix}\right\}\Big/\text{iso} \arrow[rr, dashed, shift left] \arrow[dr, "\text{many} \mapsto 1"'] & & \left\{\begin{matrix}\text{Galois hom}\\ W(X-S,\bar{x}) \to G^\vee(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\}\Big/\text{iso} \arrow[ll, dashed, shift left] \arrow[dl, "\text{many} \mapsto 1"] \\
& \begin{matrix}\text{Local}\\ \text{L-factors}\end{matrix} &
\end{tikzcd}$$

Compatibility π cuspidal and σ Galois are said *compatible* if for amost every $x \in |X|$,

$$L^\rho_x(\pi, t) = \iota\left(L^\rho_x(\sigma, t)\right).$$

From now on we will focus on GL_r. Let

$$\mathcal{A}_{r,S} := \left\{\begin{matrix}\text{cuspidal repr}\\ \text{of } GL_r \text{ unramified}\\ \text{outside } S\end{matrix}\right\} \quad \mathcal{A}_r := \bigcup_{S \subseteq X} \mathcal{A}_{r,S}$$

and

$$\mathcal{G}_{r,S} := \left\{ \begin{matrix} \text{irr. Galois repr.} \\ \text{of } W(X-S,\bar{x}) \end{matrix} \right\}, \quad \mathcal{G}_r := \bigcup_{\substack{S \subseteq X \\ \text{fin}}} \mathcal{G}_{r,S}.$$

Theorem (Drinfeld, L. Lafforgue). We fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. For every $r \geq 1$ there exists a unique bijection

$$\begin{aligned} \mathcal{A}_r &\xrightarrow{\sim} \mathcal{G}_r . \\ \pi &\longmapsto \sigma_\pi \end{aligned}$$

Observations:

1) The unicity is given by Čebotarëv.
2) The injectivity is a consequence of the "strong multiplicity one" Theorem.
3) If $r=1$ the cuspidal automorphic repr. unr. outside S are the characters of $F^\times \backslash \mathbb{A}^\times / \prod_{x \notin |S|} \mathcal{O}_x^\times$. At the same time Class Field Theory is telling us that

$$F^\times \backslash \mathbb{A}^\times / \prod_{x \notin |S|} \mathcal{O}_x^\times \xrightarrow{\sim} W(X-S,\bar{x})^{ab}$$

hence, as a consequence, we obtain the bijection $\mathcal{A}_1 \xrightarrow{\sim} \mathcal{G}_1$.
4) Thanks to the previous theorem for any character $\chi \in \mathcal{A}_1 = \mathcal{A}_2$ we can twist at the same time a representation of $GL_r(\mathbb{A})$ and a Galois representation by χ. Hence we can reduce to the correspondence between cuspidal automorphic representation with finite order central character and Galois repr. with finite order (under $\otimes$) determinant
5) Surjectivity

"Reconstruction" Theorem (Piatetski-Shapiro): Let $r \geq 2$ be an integer, $\pi = \bigotimes'_{x \in |X|} \pi_x$ an admissible representation of $GL_r(\mathbb{A})$ that admits a Whittaker model. Suppose moreover $\exists\, S$ such that:

i) The central character factors through $F^\times$.

ii) $\forall r' < r$ $\forall \pi' \in A_{r'}$ unramified over S the L-functions $L(\pi \times \pi', t)$ and $L(\check{\pi} \times \check{\pi}', t)$ are polynomials and $L(\pi \times \pi', t) = \varepsilon(\pi \times \pi', t) \, L(\check{\pi} \times \check{\pi}', (qt)^{-1})$

<u>Then</u> there exists an automorphic representation (not forcely cuspidal) that is isomorphic to π_x $\forall x \notin |S|$.

<u>Converse Theorem</u>

<u>(Surjectivity)</u>: Suppose we have already constructed $\forall r' < r$ $A_{r'} \to G_{r'}$, then $\exists$ $G_r \to A_r$ (everything preserving local L-factors). ⚠ The proof uses many facts we have not seen during the previous talks.

<u>Proof</u>: Let $\sigma \in G_{r,S}$, $\forall x \notin |S|$ we can find π_x, an unram. representation having as Hecke eigenvalues the Frobenius eigenvalues of σ at x. We can then complete the family $\{\pi_x\}_{x \notin |S|}$ obtaining an admis. representation $\pi = \bigotimes'_{x \in |X|} \pi_x$ with a <u>Whittaker model</u> having as central character det σ. We want to apply now the previous theorem. We know $\forall r' < r$ and $\pi' \in A_{r',S}$ there exists by assumption a comp. $\sigma \in G_{r',S}$.
If we choose χ enough ramified on S, of finite order, $\forall x \in S$, $L_x(\chi\pi \times \pi', t) = L_x(\chi \otimes \sigma \otimes \sigma', t) = 1$ and the same holds for the contragredient repr. Hence

$$L(\chi\pi \times \pi', t) = L(\chi \otimes \sigma \otimes \sigma', t) \quad \text{and}$$
$$L(\chi^{-1}\check{\pi} \times \check{\pi}', t) = L(\chi^{-1} \otimes \sigma^\vee \otimes (\sigma')^\vee, t)$$

then the functional equation for the adm. repr is a consequence of the fuctional equation for Galois representation (Poincaré duality) once we know the (non-trivial) fact that ε-factors correspond (Laumon formula).

We have not yet finished, we need to show that π is cuspidal. Thanks to a result of Langlands, if π is not cuspidal $\exists$ $(\pi_1, \dots, \pi_m) \in A_{r_1,S} \times \dots \times A_{r_m,S}$ s.t. $r_i < r$ and s.t. the Hecke eigenvalues of π are disjoint union of the Hecke eigenvalues of $(\pi_1, \dots, \pi_m)$. By the assumption $\exists$ $(\sigma_1, \dots, \sigma_m)$ compatible Galois representations. Hence by Čebotarëv $\sigma \simeq \sigma_1 \oplus \_ \oplus \sigma_r$ and this is not possible because σ is assumed to be irreducible.

$\square$

What is still missing is the proof of a map $A_r \to G_r$ when $r \geq 2$.

In the next 4 talks we will work with the moduli space of Shtukas. Looking at the étale cohomology we can construct a $\overline{\mathbb{Q}}_\ell$-representation V_ℓ^{cusp} of the group $GL_2(\mathbb{A}) \times W(\bar{F}/F) \times W(\bar{F}/F)$.

We will show that $V_\ell^{cusp} \simeq \bigoplus_\pi \pi \otimes \rho_\pi \otimes \rho_\pi^\vee(-1)$. The compatibility between π and ρ_π is proven via a point-counting + trace formulas for both sides.