Boundedness of the p-primary torsion of the Brauer group of an abelian variety

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Brauer group

Let X be a scheme,

$$\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_{m,X}) = H^2_{\mathrm{fppf}}(X, \mathbb{G}_{m,X}).$$

If X is a regular $\mathrm{Br}(X)$ is a *torsion group*.

Let ℓ be a prime number,

$$1 \to \mu_{\ell^n,X} \to \mathbb{G}_{m,X} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,X} \to 1.$$

$$0 \to \mathrm{Pic}(X)/\ell^n \xrightarrow{c_1} H^2_{\mathrm{fppf}}(X, \mu_{\ell^n,X}) \to \mathrm{Br}(X)[\ell^n] \to 0.$$

Tate conjecture

k finitely generated field $\mathrm{char}(k) = p \geqslant 0$ (e.g. $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{F}_p(t)$,...).

k_a/k algebraic closure, Γ_k absolute Galois group, X/k smooth proper variety, $\ell \neq p$.

Conjecture (Tate '66)

The cycle class map

$$\mathrm{NS}(X_{k_a})^{\Gamma_k}_{\mathbb{Q}_\ell} \xrightarrow{c_1} H^2_{\text{ét}}(X_{k_a}, \mathbb{Q}_\ell(1))^{\Gamma_k}$$

is an isomorphism.

Tate conjecture for abelian varieties

Theorem (Tate, Zarhin, Faltings)

If A is an abelian variety

$$\mathrm{NS}(A_{k_a})^{\Gamma_k}_{\mathbb{Z}_\ell} \xrightarrow{c_1} H^2_{\text{ét}}(A_{k_a}, \mathbb{Z}_\ell(1))^{\Gamma_k}$$

is an isomorphism.

Corollary

The group

$$\mathrm{Br}(A_{k_a})[\ell^\infty]^{\Gamma_k}$$

is finite for every $\ell \neq p$.

Tate conjecture for abelian varieties

Proof of corollary

$$\mathrm{Br}(A_{k_a})[\ell^\infty] \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus(b_2-\rho)}$$

$$\mathrm{T}_\ell(\mathrm{Br}(A_{k_a})) := \varprojlim_n \mathrm{Br}(A_{k_a})[\ell^n]$$

$$0 \to \mathrm{NS}(A_{k_a})^{\Gamma_k}_{\mathbb{Q}_\ell} \xrightarrow{c_1} H^2_{\text{ét}}(A_{k_a}, \mathbb{Q}_\ell(1))^{\Gamma_k} \to \mathrm{T}_\ell(\mathrm{Br}(A_{k_a}))^{\Gamma_k}[\tfrac{1}{\ell}] \to 0$$

is exact.

c_1 surjective $\Rightarrow \mathrm{T}_\ell(\mathrm{Br}(A_{k_a}))^{\Gamma_k} = 0 \Rightarrow$ no infinitely ℓ-divisible classes in $\mathrm{Br}(A_{k_a})[\ell^\infty]^{\Gamma_k} \Rightarrow \mathrm{Br}(A_{k_a})[\ell^\infty]^{\Gamma_k}$ is finite. □

Transcendental Brauer group

Definition

The *transcendental Brauer group* of X is

$$\mathrm{Br}(X_{k_a})^k := \mathrm{im}(\mathrm{Br}(X) \to \mathrm{Br}(X_{k_a})) \subseteq \mathrm{Br}(X_{k_a})^{\Gamma_k}.$$

Remark

By the previous corollary, for every $\ell \neq p$, $\mathrm{Br}(A_{k_a})^k[\ell^\infty]$ is finite.

What does it happen when $\ell = p > 0$?

Theorem

There exist abelian varieties such that

$$\mathrm{NS}(A_{k_a})^{\Gamma_k}_{\mathbb{Q}_p} \xrightarrow{c_1} H^2_{\mathrm{fppf}}(A_{k_a}, \mathbb{Q}_p(1))^{\Gamma_k}$$

is not surjective and $\mathrm{T}_p(\mathrm{Br}(A_{k_a}))^{\Gamma_k} \neq 0$.

We suppose

$$A = B \times B$$

with B an abelian variety.

Leray spectral sequence

We consider $E_2^{i,j} := H^i_{\mathrm{fppf}}(B_{k_a}, R^j\pi_{2*}\mu_{p^n}) \Rightarrow H^{i+j}_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n})$

$$R^1\pi_{2*}\mu_{p^n} = \mathrm{Pic}_{B_{k_a}/k_a}[p^n] = B^\vee_{k_a}[p^n]$$

$$E_2^{1,1} = H^1(B_{k_a}, B^\vee_{k_a}[p^n]) \xrightarrow{h} \mathrm{Hom}(B_{k_a}[p^n], B^\vee_{k_a}[p^n])$$

Construction of h

$$B_{k_a}[p^n] = \mathrm{Hom}(B^\vee_{k_a}[p^n], \mathbb{G}_{m,k_a})$$

$\forall [T] \in H^1(B_{k_a}, B^\vee_{k_a}[p^n])$ and S/k_a scheme,

$$h([T])(S) : \mathrm{Hom}(B^\vee_{k_a}[p^n], \mathbb{G}_{m,k_a})(S) \to B^\vee_{k_a}[p^n](S)$$

$$\tau \mapsto \tau_*(T_S) \in H^1(B_S, \mathbb{G}_{m,B_S})[p^n]$$

.

Leray spectral sequence

$R^2\pi_{2*}\mu_{p^n}$ is represented by a linear algebraic group G/k_a.

$$E_2^{0,2} = H^0_{\mathrm{fppf}}(B_{k_a}, R^2\pi_{2*}\mu_{p^n}) = \mathrm{Mor}(B_{k_a}, G) = \mathrm{Mor}(0_{B_{k_a}}, G) =$$
$$= H^2_{\mathrm{fppf}}(B_{k_a}, \mu_{p^n}).$$

$$\rightsquigarrow H^2_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n}) = H^2_{\mathrm{fppf}}(B_{k_a}, \mu_{p^n})^{\oplus 2} \oplus \mathrm{Hom}(B_{k_a}[p^n], B^\vee_{k_a}[p^n]).$$

On the other hand,

$$\mathrm{NS}(A_{k_a})/p^n = (\mathrm{NS}(B_{k_a})/p^n)^{\oplus 2} \oplus \mathrm{Hom}(B_{k_a}, B^\vee_{k_a})/p^n.$$

$\rightsquigarrow$ Enough to prove that $\mathrm{Hom}(B_{k_a}, B^\vee_{k_a})^{\Gamma_k}_{\mathbb{Q}_p} \to \mathrm{Hom}(B_{k_a}[p^\infty], B^\vee_{k_a}[p^\infty])[\frac{1}{p}]^{\Gamma_k}$ is not surjective in general.

Counterexample

If B is an elliptic curve with transcendental j-invariant.

$$\mathrm{Hom}(B_{k_a}, B_{k_a}^\vee)_{\mathbb{Q}_p}^{\Gamma_k} = \mathbb{Q}_p$$

$$\mathrm{Hom}(B_{k_a}[p^\infty], B_{k_a}^\vee[p^\infty])[\tfrac{1}{p}]^{\Gamma_k} = \mathrm{Hom}(B_{k_i}[p^\infty], B_{k_i}^\vee[p^\infty])[\tfrac{1}{p}] = \mathbb{Q}_p{}^{\oplus 2}$$

where $k_i \subseteq k_a$ is the purely inseparable closure.

Conclusion

There is an additional obstruction w.r.t. the purely inseparable extension $k \subseteq k_i$.

A flat Tate conjecture

$$H^2_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n})^k := \mathrm{im}(H^2_{\mathrm{fppf}}(A, \mu_{p^n}) \to H^2_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n})).$$

Theorem

The cycle class map

$$\mathrm{NS}(A)_{\mathbb{Z}_p} \xrightarrow{c_1} \varprojlim_n H^2_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n})^k$$

is an isomorphism and $\mathrm{Br}(A_{k_a})^k[p^\infty]$ *has finite exponent.*

Theorem (de Jong '98)

The morphism

$$\mathrm{Hom}(A, A^\vee)_{\mathbb{Z}_p} \to \mathrm{Hom}(A[p^\infty], A^\vee[p^\infty])$$

is an isomorphism.

$$\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}^{\mathrm{sym}}(A, A^\vee)$$
$$[\mathcal{L}] \mapsto \varphi_{\mathcal{L}}$$

$\varphi_{\mathcal{L}}$ is such that

$$(\mathrm{id} \times \varphi_{\mathcal{L}})^* \mathcal{P}_A = \Lambda(\mathcal{L})$$

where $\mathcal{P}_A$ is the Poincaré bundle and $\Lambda(\mathcal{L})$ is the Mumford bundle.

Constructing a morphism

$$\begin{array}{ccc} \mathrm{Pic}(A)^{\wedge} & \xrightarrow{\quad c_1 \quad} & H^2(A, \mathbb{Z}_p(1)) \\ \downarrow & & \vdots\, {\color{red}h} \\ \mathsf{Hom}^{\mathrm{sym}}(A, A^{\vee})_{\mathbb{Z}_p} & \xrightarrow{\sim} & \mathsf{Hom}^{\mathrm{sym}}(A[p^\infty], A^{\vee}[p^\infty]). \end{array}$$

$$h : H^2(A, \mathbb{Z}_p(1)) \xrightarrow{m^* - \pi_1^* - \pi_2^*} F^1 H^2(A \times A, \mathbb{Z}_p(1)) \to \mathsf{Hom}(A[p^\infty], A^{\vee}[p^\infty])$$

Consequence

Definition

$$H^2_{\mathrm{fppf}}(A_{k_a}, \mathbb{Z}_p(1))^k := \mathrm{im}(H^2_{\mathrm{fppf}}(A, \mathbb{Z}_p(1)) \to H^2_{\mathrm{fppf}}(A_{k_a}, \mathbb{Z}_p(1)))$$

$$\mathrm{T}_p(\mathrm{Br}(A_{k_a}))^k := \mathrm{im}(\mathrm{T}_p(\mathrm{Br}(A)) \to \mathrm{T}_p(\mathrm{Br}(A_{k_a}))).$$

Proposition

The cycle class map

$$\mathrm{NS}(A)_{\mathbb{Z}_p} \xrightarrow{c_1} H^2(A_{k_a}, \mathbb{Z}_p(1))^k$$

is surjective and $\mathrm{T}_p(\mathrm{Br}(A_{k_a}))^k = 0$.

Achtung!

$$\mathrm{T}_p(\mathrm{Br}(A_{k_a}))^k = \mathrm{T}_p(\mathrm{Br}(A_{k_a})^k) \quad ?$$

Consequence

Achtung!

$$\mathrm{T}_p(\mathrm{Br}(A_{k_a}))^k = \mathrm{T}_p(\mathrm{Br}(A_{k_a})^k) \quad ?$$

- $\mathrm{Br}(A)[p^n] \twoheadrightarrow \mathrm{Br}(A_{k_a})^k[p^n]$?
- If yes, is the surjectivity preserved by $\varprojlim_n$?

$$K_A := \ker(\mathrm{Br}(A) \to \mathrm{Br}(A_{k_a}))$$

$$0 \to \mathrm{Br}(k) \to K_A \to H^1_{\mathrm{fppf}}(k, \mathrm{Pic}_{A/k}) \to 0.$$

Solution

Theorem (Gabber, Katz)

There exists a smooth projective connected curve C with $C(k) \neq \emptyset$ endowed with $C \to A$ such that $B := \mathrm{Jac}(C) \twoheadrightarrow A$.

$B \sim A \times A' \rightsquigarrow$ it is enough to prove the result for B.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & J & \longrightarrow & \mathrm{Br}(B_{k_a})^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_B & \longrightarrow & \mathrm{Br}(B) & \longrightarrow & \mathrm{Br}(B_{k_a})^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_C & \xrightarrow{\sim} & \mathrm{Br}(C) & \longrightarrow & \mathrm{Br}(C_{k_a})^k = 0 & \longrightarrow & 0
\end{array}$$

F is finite because

$$F = \ker(H^1_{\mathrm{fppf}}(k, \mathrm{Pic}_{B/k}) \to H^1_{\mathrm{fppf}}(k, \mathrm{Pic}_{C/k}))$$

is a subgroup of $H^1_{\mathrm{fppf}}(k, \mathrm{Pic}_{B/k}/\mathrm{Pic}^0_{B/k})$. Thus $J \subseteq \mathrm{Br}(B)$ provides a splitting of $\mathrm{Br}(B) \to \mathrm{Br}(B_{k_a})^k$ up to multiplication by m for a fixed $m \geqslant 0$. □

Specialisation Néron–Severi group

Theorem (André, Ambrosi, Christensen)

Let K be an algebraically closed field $\neq \overline{\mathbb{F}}_p$, X a finite type K-scheme, and $\mathcal{Y} \to X$ a smooth and proper morphism. For every geometric point $\overline{\eta}$ of X there is an $x \in X(K)$ such that $\mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{Y}_{\overline{\eta}})) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{Y}_x))$.

Counterexample

If $\mathcal{A} = \mathcal{E} \times \mathcal{E} \to X$ where $\mathcal{E} \to X$ is a non-isotrivial elliptic scheme over $X/\overline{\mathbb{F}}_p$, then

$$2 + 2 = \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_x)) > \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\overline{\eta}})) = 2 + 1.$$

Specialisation Néron–Severi group

Theorem

Let X be a connected normal scheme of finite type over $\mathbb{F}_p$ with generic point $\eta = \mathrm{Spec}(k)$ and let $f : \mathcal{A} \to X$ be an abelian scheme over X with constant slopes. For every closed point $x = \mathrm{Spec}(\kappa)$ of X we have

$$\mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\overline{x}})^{\Gamma_\kappa}) - \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\overline{\eta}})^{\Gamma_k}) \geqslant \mathrm{rk}_{\mathbb{Z}_p}(\mathrm{T}_p(\mathrm{Br}(\mathcal{A}_{\overline{\eta}}))^{\Gamma_k}).$$

References

- M. D., Boundedness of the p-primary torsion of the Brauer group of an abelian variety, arXiv:2201.07526 (2022).
- A. J. de Jong, Homomorphisms of Barsotti–Tate groups and crystals in positive characteristic, *Invent. Math.* **134** (1998), 301–333.
- G. Faltings, G. Wustholz, *Rational Points*, Aspects of Mathematics, Friedr. Vieweg & Sohn, 1984.
- J. Tate, Endomorphisms of abelian varieties over finite fields, *Invent. Math.* **2** (1966), 134–144.
- J. Tate, Conjectures on algebraic cycles in ℓ-adic cohomology, in *Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math.* **55**, Amer. Math. Soc., 1994, 71–83.
- Y. G. Zarhin, Endomorphisms of abelian varieties and points of finite order in characteristic p, *Mat. Zametki* **21** (1977), 737–744.