Talk 2: Differential operators*

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1 Monodromy

We will start the talk briefly presenting some phenomena of monodromy on the solutions of differential equations.

Analytic

Let's work over $\mathbb{C}$, with coordinate z. Consider the following differential equation, also known as the *hypergeometric differential equation*

$$(z(1-z)\partial_z^2 + (c-(1+a+b)z)\partial_z - ab)f(z) = 0,$$

where $a, b, c \in \mathbb{C}$. Suppose $c \notin \mathbb{Z}_{\leqslant 0}$ then we can define, when $|z| < 1$, the solution

$$F(a,b,c|z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(k)_n := k \cdot \ldots \cdot (p+n-1)$. This function can be holomorphically continued outside the ray ρ in the real line such that $\Re(z) \geq 1$. Similarly if $1+a+b-c \notin \mathbb{Z}_{\leqslant 0}$, the solution $F(a,b,1+a+b-c|1-z)$ can be continued analytically outside the ray ρ' contained in the real line such that $\Re(z) \leq 0$. These two solutions are defined on $U := \mathbb{C} \setminus (\rho \cup \rho')$ and one can check they are linearly independent, thus they form a basis of the vector space of holomorphic solutions on U.

Now if you take the solution obtained from $F(a,b,c|z)$ the limits on $\rho \cup \rho'$ coming from above or from below don't match. This phenomenon can be described for any $x \in \mathbb{C} \setminus \{0,1\}$, saying that there

*I thank Marcin for giving me his notes

is a monodromy action of $\pi_1(\mathbb{C}\setminus\{0,1\},x)$ on the vector space of solutions in a small neighbourhood of x (if you want on the stalk of the sheaf of solutions).

Algebraic

A similar phenomenon appears in the étale site. For example if k is a field of characteristic different from 2 and we take on $\mathrm{Spec}(k[z]_z)$ the differential equation $(2z\partial_z - 1)f = 0$, we don't have any nontrivial solution (it would be something like $\sqrt{z}$). But étale locally you can find a solution! Indeed if you consider the Kummer cover $\mathrm{Spec}(k[t]_t) \to \mathrm{Spec}(k[z]_z)$ such that z goes to t^2, the operator $2z\partial_z - 1$ becomes on $\mathrm{Spec}(k[t]_t)$ the operator $t\partial_t - 1$, that has solutions ct where $c \in k$. Thus the étale-local solution t "doesn't match with itself" and this can be explained again saying that there is a nontrivial action of the étale fundamental group of $\mathrm{Spec}(k[z]_z)$ on the stalk of the étale sheaf of solutions.

2 Differential operators as universal envelopping algebra

Let X be a variety over a field of any characteristic, if $\mathcal{T}_{X/k}$ is the tangent bundle we can consider an universal envelopping $\mathcal{O}_X$-algebra of $\mathcal{T}_{X/k}$, $\psi : \mathcal{T}_{X/k} \to \mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k})$, i.e. $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k})$ is an associative, unital left $\mathcal{O}_X$-algebra such that for every other $\mathcal{A}$, unitial associative left $\mathcal{O}_X$-algebra and for every map $\phi : \mathcal{T}_{X/k} \to \mathcal{A}$ of left $\mathcal{O}_X$-modules that commutes with the bracket there exists a unique map η of unital left $\mathcal{O}_X$-algebras from $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k})$ to $\mathcal{A}$ s.t. $\eta \circ \psi = \phi$.

Now if $\mathcal{E}$ is a vector bundle over X, if we have a map $\mathcal{T}_{X/k} \to \mathcal{E}\mathrm{nd}_k(\mathcal{E})$ of left $\mathcal{O}_X$-modules that commutes with the bracket, by definition this gives a map $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k}) \to \mathcal{E}\mathrm{nd}_k(\mathcal{E})$. The sheaf $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k})$ can be thus thought as a sheaf of derivations of higher order. In characteristic zero this is isomorphic to the sheaf of differential operators we will define in the next sections, but in positive characteristic they are different.

Thus one can choose if to consider $\mathcal{D}$-modules with one or the other definition of differential operators and this will give different theories. During this seminar, in the case of positive characteristic the sheaf of differential operators *won't be* the one obtained with the universal envelopping algebra, but the one we will define later.

Example. If X is $k[x_1,\ldots,x_r]$ the global sections of the sheaf $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k})$ are just isomorphic to the Weyl algebra $k[x_1,\ldots,x_r,\partial_{x_1},\ldots,\partial_{x_r}]$. This (non commutative) algebra is obtained imposing relations $[x_i,x_j]=0$, $[\partial_{x_i},\partial_{x_j}]=0$ and $[\partial_{x_i},x_j]=\delta_{ij}$. The module structure is just given by the inclusion $k[x_1,\ldots,x_r] \hookrightarrow k[x_1,\ldots,x_r,\partial_{x_1},\ldots,\partial_{x_r}]$.

3 New operators

In characteristic p, in one variable, the composition of derivations will never be a differential operator of degree p or higher. Indeed ∂_x^p is the zero endomorphism. However there are other good candidates.

Take $f \in k[x]$, where k is a field of characteristic p, then $f(x+\epsilon) = f(x) + (\partial_x f)(x)\epsilon + \ldots + ?\epsilon^p + \ldots$, where ϵ is just a free variable and $? \in k[x]$. We would like writing $? = (\partial_x^p f)(x)/p!$ but the right term has no meaning. Anyway we can just define an endomorphism $\partial_x^{(p)}$ that associates $f \mapsto ?$, formally:

Definition 3.1. Let k be a field, for any $r \in \mathbb{N}$ and any $1 \le i \le r$ and for any $n \in \mathbb{N}$ we define $\partial_{x_i}^{(n)} : k[x_1,\ldots,x_r] \to k[x_1,\ldots,x_r]$ as the only k-linear map such that for any $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$, if $n \le \alpha_i$,

$$\partial_{x_i}^{(n)}(x_1^{\alpha_1}\ldots x_r^{\alpha_r}) = \binom{\alpha_i}{n} x_1^{\alpha_1}\ldots x_i^{\alpha_i-n}\ldots x_r^{\alpha_r},$$

and when $n > \alpha_i$

$$\partial_{x_i}^{(n)}(x_1^{\alpha_1} \dots x_r^{\alpha_r}) = 0.$$

One can check on monomials that for any $1 \le i, j \le r$ and any n, m:

- If $n!$ is invertible in k, $\partial_{x_i}^{(n)} = \frac{1}{n!}\partial_{x_i}^n$;
- $[\partial_{x_i}^{(n)}, \partial_{x_j}^{(m)}] = 0$;
- $\partial_{x_i}^{(n)} \circ \partial_{x_i}^{(m)} = \binom{n+m}{n}\partial_{x_i}^{(n+m)}$;
- For any $f, g \in k[x_1, \dots, x_r]$, we have $\partial_{x_i}^{(n)}(fg) = \sum_{m=0}^n \partial_{x_i}^{(n-m)}(f)\partial_{x_i}^{(m)}(g)$.

4 Principal parts

The differential operators we have defined in the previous section make sense only in $\mathbb{A}_k^n$, but we would like to work with a larger class of varieties: all the smooth varieties. Thus we want to find an intrinsic definition, as the one of universal enveloping algebra, but that allows us to work with operators as $\partial_x^{(n)}$. A nice way to define them, due to Grothendieck, is the one using *principal parts*. At the same way to study derivations one can introduce Kahler differentials, to study differential operators one introduce principal parts.

Let's work locally, let A be a k-algebra, $I := \ker(A \otimes_k A \xrightarrow{m} A)$, where m is the multiplication of A, $\iota : A \to A \otimes_k A$ that sends $a \mapsto a \otimes 1$ and $d^\infty : a \mapsto 1 \otimes a$, moreover let's call $d = d^\infty - \iota$. We will consider on $A \otimes_k A$ the left A-module structure induced by ι and the right A-module structure given by d^∞.

Definition 4.1. We will denote P_A^n the left and right A-module $A \otimes_k A/I^{n+1}$ and we will call it the *module of principal parts of A of order n*.

Examples. If $n = 1$, P_A^1 is just $A \oplus \Omega_{A/k}^1$.

If $A = k[x]$, for any n

$$P_A^n = k[x, y]/((y-x)^{n+1}) \overset{\epsilon := y-x}{=} k[x][\epsilon]/(\epsilon^{n+1}) = \oplus_{i=0}^n k[x]\epsilon^i.$$

We also have $d^\infty(f) = f(x+\epsilon)$ [1] and $dx = \epsilon$.

Consider $\varphi_i : P_A^n \to A$ the left $k[x]$-linear map such that $\varphi_i(\epsilon^j) = \delta_{i,j}$ (φ_i is the functional dual to ϵ^j). Let's call $D_i := \varphi_i \circ d^\infty : A \to A$, we have

$$D_i(x^\alpha) = (\varphi_i \circ d^\infty)(x^\alpha) = \varphi_i((x+\epsilon)^\alpha) = \varphi_i\left(\sum_{j=0}^n \binom{\alpha}{j} x^{\alpha-j}\epsilon^j\right) = \binom{\alpha}{i}x^{\alpha-i}.$$

Thus D_i is exactly $\partial_x^{(i)}$! Similarly, if you have more variables, $\partial_{x_1}^{(\alpha_1)} \dots \partial_{x_r}^{(\alpha_r)}$ can been seen as the functional dual to $(dx_1)^{\alpha_1} \dots (dx_r)^{\alpha_r}$.

Thus we are interested in studying in general the map $\mathrm{Hom}_A(P_A^n, A) \xrightarrow{\Psi_n} \mathrm{End}_k(A)$, $\varphi \mapsto \varphi \circ d^\infty$.

Remark. The map Ψ_n is injective because $d^\infty(A)$ generates $P_{A/k}^n$ as a left A-module.

Let's try to understand the image.

[1] The map d^∞ will also denote the map induced by d^∞ from A to $P_{A/k}^n$

Definition 4.2. We define inductively the submodules $D^{\leqslant n}_{A/k}$ of $\mathrm{End}_k(A)$: $D^{\leqslant -1}_{A/k} = 0$ and

$$D^{\leqslant n+1}_{A/k} := \{D \in \mathrm{End}_k(A) \mid \forall f \in A,\ [D, f] \in D^{\leqslant n}_{A/k}\}$$

Proposition 4.3. *For any $D \in \mathrm{End}_k(A)$ TFAE:*

i) $D \in \mathrm{Im}(\Psi_n)$;

ii) $D \in D^{\leqslant n}_{A/k}$;

iii) $\forall f_1, \ldots, f_{n+1}, g \in A$ *we have*

$$\sum_{H \subseteq I_{n+1}} (-1)^{|H|} \prod_{i \in H} f_i \cdot D\left(\prod_{j \notin H} f_j g\right) = 0,$$

where $I_{n+1} = \{1, \ldots, n+1\}$.

Proof. We start proving $i) \leftrightarrow iii)$. For any element $D \in \mathrm{End}(A)$ we can define a map $\varphi_D : A \otimes_k A \to A$ that sends $a \otimes b \mapsto aD(b)$ thus such that $D = \varphi_D \circ d^\infty$. Thus $D \in \mathrm{Im}(\Psi_n)$ if and only if $\varphi(I^{n+1}) = 0$. As I is generated as a right (even left) A-module by the elements of the form $1 \otimes f - f \otimes 1$[2], the last condition is equivalent to $\Big(\forall f_1, \ldots f_{n+1}, g \in A,\ \varphi(\prod_i (1 \otimes f_i - f_i \otimes 1).g) = 0\Big)$. By induction one can show that

$$\varphi_D\left(\prod_i (1 \otimes f_i - f_i \otimes 1).g\right) = \sum_{H \subseteq I_{n+1}} (-1)^{|H|} \prod_{i \in H} f_i \cdot \varphi_D\left(\prod_{j \notin H} (1 \otimes f_j).g\right).$$

Thus $\Big(\forall f_1, \ldots f_{n+1}, g \in A,\ \varphi(\prod_i (1 \otimes f_i - f_i \otimes 1).g) = 0\Big) \leftrightarrow iii)$ as we wanted.

At the same time $ii) \leftrightarrow iii)$, indeed by definition $D \in D^{\leqslant n}_{A/k}$ if and only if $\forall f_1, \ldots f_{n+1}, g \in A$, $[\ldots[D, f_1], \ldots], f_{n+1}](g) = 0$. By induction one can show that this last identity is the same as the one of $iii)$, thus we are done. □

Definition 4.4. We denote $D_{A/k} := \bigcup_{n \in \mathbb{N}} D^{\leqslant n}_{A/k}$, this will be the *module of differential operators of A*. Any element $D \in D^{\leqslant n}_{A/k} \setminus D^{\leqslant n-1}_{A/k}$ will be a *differential operator of order n*.

Remark. The module $D_{A/k}$ inherits from $\mathrm{End}_k(A)$ a structure of unital, associative ring as well as a left and right A-module structure. Moreover we can define a bracket $[-,-]$ that endows $D_{A/k}$ with the k-Lie algebra structure.

In general for any scheme X over k the construction we have done locally glues to the sheaves of differential operators, denoted $\mathcal{D}_{X/k}$ with filtration $\bigcup \mathcal{D}^{\leqslant n}_{X/k}$.

[2]To see this if $\sum a_i(f_i \otimes g_i) \in I$ with $a_i \in k$ and $f_i, g_i \in A$ then $\sum a_i f_i g_i = 0$, thus $\sum a_i(f_i \otimes g_i) = -\sum(1 \otimes a_i f_i - a_i f_i \otimes 1).g_i$

5 Coordinates on $\mathcal{D}_{A/k}$ and relations with the universal enveloping algebra

Now we want to show the following proposition, we have put in the Appendix the details on regular sequences.

Proposition 5.1. *If X is a smooth variety, for any point $x \in X$, we can find an affine open* Spec(A) *that contains x and we can find $x_1, \dots, x_r \in A$ such that for any n,*

$$P^n_{A/k} = \bigoplus_{\alpha_1 + \ldots + \alpha_r \leq n} A \ (dx_1)^{\alpha_1} \ldots (dx_r)^{\alpha_r}.$$

Proof. Since X is separated the diagonal map is a closed immersion and we also know that both X and $X \times_k X$ are smooth. Thus we can apply Proposition A.5 and Lemma A.6 to find an affine open Spec(A) in X, such that I, the kernel of the multiplication $A \otimes_k A \to A$ is generated by a regular sequence. Let's call it $a_1, \dots, a_r$, then by Proposition A.2 we obtain that

$$P^n_{A/k} = \bigoplus_{\alpha_1 + \ldots + \alpha_r \leq n} A \ a_1^{\alpha_1} \ldots a_r^{\alpha_r}.$$

Now up to take a localisation of A, as X is smooth, we can find $x_1, \dots, x_r$ in A such that $dx_1, \dots, dx_r$ form a basis of I/I^2. Thanks to the A-linear isomorphism that sends a_i to dx_i we obtain the final result. $\square$

Remark. The decomposition of $P^n_{A/k}$ of the previous proposition implies passing to the dual to a decomposition

$$D^{\leqslant n}_{A/k} = \bigoplus_{\alpha_1 + \ldots + \alpha_r \leq n} A \ \partial_{x_1}^{(\alpha_1)} \ldots \partial_{x_r}^{(\alpha_r)}.$$

Thus for any smooth variety locally we can just work with compositions of the operators $\partial_{x_j}^{(\alpha_i)}$!

Finally let's use the coordinates to understand the relations between $\mathcal{D}_{X/k}$ and $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k})$. In any characteristic we have a canonical map $\mathcal{T}_{X/k} \to \mathcal{D}_{X/k}$ of left $\mathcal{O}_X$-modules that commutes with bracket, thus, by definition, we have an unique map $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k}) \to \mathcal{D}_{X/k}$ that commutes with the inclusions of $\mathcal{T}_{X/k}$.

In characteristic 0, as $\partial_{x_j}^{(\alpha_i)} = \frac{1}{\alpha_i!} \partial_{x_j}^{\alpha_i}$, locally

$$D^{\leqslant n}_{A/k} = \bigoplus_{\alpha_1 + \ldots + \alpha_r \leq n} A \ \partial_{x_1}^{\alpha_1} \ldots \partial_{x_r}^{\alpha_r},$$

then $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}_{X/k}) \xrightarrow{\sim} \mathcal{D}_{X/k}$.

In characteristic p, the map is neither injective nor surjective. Indeed $0 \neq \partial_{x_i}^p \mapsto 0$ and at the same time $\partial_{x_i}^{(p)}$ is not in the image. To see the last fact one can check that $\partial_{x_i}^{(p)}$ is an operator of order p, but all the differential operators coming from $\mathcal{U}_{\mathcal{O}_x}(\mathcal{T}_{X/k})$ have order less than p.

A Regular sequences

In this appendix the rings will be unital and commutative.

Definition A.1. If A' is a ring, $a_1, \dots, a_r$ is said to be a regular sequence of A' if $\forall i$, $A'/(a_1, \dots, a_i) \overset{\cdot a_{i+1}}{\to} A'/(a_1, \dots, a_i)$ is injective.

Proposition A.2. *If an ideal I of A' is generated by a regular sequence $a_1, \dots, a_r$, then*

$$I^n/I^{n+1} = \bigoplus_{\alpha_1+\ldots+\alpha_r=n} (A'/I)\ a_1^{\alpha_1} \cdot \ldots \cdot a_r^{\alpha_r}.$$

Proof. Let $B := \mathbb{Z}[x_1, \dots, x_r]$ and consider on A' the B-module structure induced by the map that sends $x_i \to a_i$. Call J the ideal of B generated by $x_1, \dots, x_r$. The $\mathbb{Z}$-modules J^n/J^{n+1} are isomorphic to $\bigoplus_{\alpha_1+\cdots+\alpha_r=n} \mathbb{Z}x_1^{\alpha_1} \dots x_r^{\alpha_r}$, moreover as a B-module they are isomorphic to

$$\bigoplus_{\alpha_1+\cdots+\alpha_r=n} (B/J)\ x_1^{\alpha_1} \dots x_r^{\alpha_r}.$$

To show the final result it's thus enough to prove that for any n, $J^n/J^{n+1} \otimes_{B/J} A'/I = I^n/I^{n+1}$. We start proving that $\mathrm{Tor}_1^B(A', B/J) = 0$ with the following lemma.

Lemma A.3. *Let C be a ring and $c_1, \dots c_r$ a regular sequence of C, then the Koszul complex $K_C(c_1, \dots, c_r)$ is quasi isomorphic to the complex concentrated in degree* 0

$$\cdots \to 0 \to C/(c_1, \dots, c_r) \to 0 \to \dots.$$

Proof. We recall how to construct the Koszul complex. For any i let $K_C(c_i)$ be the complex with C in degree -1 and 0 and the nontrivial differential equals to the multiplication by c_i. For any i we define $K_C(c_1, \dots, c_i) := K_C(c_1) \otimes \cdots \otimes K_C(c_i)$, it is a complex of free C-modules. We can check that

$$K_C(c_1, \dots, c_{i+1}) = \mathrm{cone}(K_C(c_1, \dots, c_i) \xrightarrow{\cdot c_{i+1}} K_C(c_1, \dots, c_i)),$$

thus $K_C(c_1, \dots, c_i) \xrightarrow{\cdot c_{i+1}} K_C(c_1, \dots, c_i) \to K_C(c_1, \dots, c_{i+1}) \xrightarrow{+1}$ is a distinguished triangle. Until this point we never used the fact that $c_1, \dots, c_r$ is a regular sequence.

Now to show the final statement we proceed by induction on r. If $r = 1$ it's a consequence on the hypothesis on c_1. For the inductive step we use the long exact sequence in cohomology induced by the previous distinguished triangle. We have by the inductive hypothesis that $H^{-1}(K(c_1, \dots, c_r)) = 0$, thus we have an exact sequence

$$0 \to H^{-1}(K(c_1, \dots, c_{r+1})) \to C/(c_1, \dots, c_r) \xrightarrow{\cdot c_{r+1}} C/(c_1, \dots, c_r) \to H^0(K(c_1, \dots, c_{r+1})) \to 0$$[3].

As $c_1, \dots, c_{r+1}$ is a regular sequence, the multiplication by c_{r+1} is injective. Moreover the cokernel is clearly $C/(c_1, \dots, c_{r+1})$, thus we are done. □

Considering the exact sequence of B-modules $0 \to J^{n+1} \to J^n \to J^n/J^{n+1} \to 0$, as J^n/J^{n+1} is a direct sum of B-modules isomorphic to B/J, by the previous lemma $\mathrm{Tor}_1^B(A', J^n/J^{n+1}) = 0$ for any n. Tensoring the exact sequence by A' we obtain by induction that $A' \otimes_B J^{n+1} = I^{n+1}$. By the right exactness of the tensor product, this implies that $J^n/J^{n+1} \otimes_B A' = I^n/I^{n+1}$ but

$$J^n/J^{n+1} \otimes_B A' = J^n/J^{n+1} \otimes_{B/J} B/J \otimes_B A' = J^n/J^{n+1} \otimes_{B/J} A'/I$$

and we are done.

□

[3]Notice that the positive cohomology of Koszul complex is 0 by construction

Definition A.4. Let $X \xrightarrow{f} Y$ be a closed immersion. It is a regular closed immersion if for any x in X, the kernel of $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is generated by a regular sequence.

Proposition A.5. *Let* (A, m_A) *and* (B, m_B) *two regular local rings, and* $f : A \to B$ *a surjective morphism, then* I*, the kernel of* f*, is generated by a regular sequence.*

Proof. During the proof we will use repeatedly that a regular local ring is a domain.

We can prove the result by induction on the dimension of A. If the dimension is zero, A is a field and the proposition is true. If the dimension is bigger than zero we have two cases: $I \subseteq m_A^2$ or $I \nsubseteq m_A^2$. In the first case the map induced on the cotangent spaces $m_A/m_A^2 \to m_B/m_B^2$ is an isomorphism, then $I = 0$ (generated by the empty regular sequence) if not we would have $\dim(B) < \dim(A)$ and this doesn't agree with the isomorphism of cotangent spaces.

If we are in the second situation we take $a_1 \in I \setminus (I \cap m_R^2)$, then $R/(a_1)$ is again a regular local ring. By the inductive hypothesis the surjective map induced by f from $A/(a_1)$ to B has kernel generated by a regular sequence $a_2, \ldots a_r$. Then $a_1, \ldots, a_r$ is a regular sequence of A. □

Lemma A.6. *Let* A *be a noetherian ring and* I *an ideal of* A*. If* p *is a prime ideal of* A *and* I_p *in* A_p *is generated by a regular sequence then there exists* $f \notin p$ *such that* I_f *in* A_f *is generated by a regular sequence.*

Proof. Let $a_1/s_1, \ldots, a_r/s_r$ be the regular sequence that generates I_p, then there exists $g_1 \notin p$ such that the sequence is defined in A_{g_1}. Let's take the map $A_{g_1}^r \to I_{g_1}$ that sends any element of the standard basis e_i to a_i/s_i and let's call M the cokernel of this map. We know that cokernels commute with localisation and the induced map $A_p^r \to I_p$ is surjective, thus $M_p = 0$. As M is finitely generated there exists $g_2 \notin p$ such that M_{g_2} is zero. Thus $a_1/s_1, \ldots, a_r/s_r$ generate the ideal $I_{g_1 g_2}$ of $A_{g_1 g_2}$.

With a similar reason we can find $g_3 \notin p$ such that $a_1/s_1, \ldots, a_r/s_r$ form a regular sequence in $A_{g_1 g_2 g_3}$. Indeed we know that kernels

$$N_{i+1} := \mathrm{Ker}\left(A_{g_1}/(a_1/s_1, \ldots, a_i/s_i) \xrightarrow{\cdot a_{i+1}/s_{i+1}} A_{g_1}/(a_1/s_1, \ldots, a_i/s_i)\right)$$

are zero when localized at p and they are finitely generated A_{g_1}-modules because they are submodules of finitely generated A_{g_1}-modules (A_{g_1} is noetherian). Then we can find $g_3 \notin p$ such that all the localisations $(N_i)_{g_3}$ are zero and this is exactly what we needed. □