EDGED CRYSTALLINE COHOMOLOGY

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In the memory of Pierre Berthelot

Abstract. We introduce a new family of ringed sites, called *edged crystalline sites*. These are all variants of the crystalline site and they are parametrised by certain functions $\tau : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, called *edge-types*. When τ is linear, the τ-edged crystalline sites exhibit connections to rigid cohomology and overconvergent F-isocrystals. Exponential edge-types instead give rise to the conjectured family of log-decay crystalline cohomology theories, parametrised by positive real numbers. In this case, the F-isocrystals are the log-decay F-isocrystals, which were previously constructed over smooth curves by Kramer-Miller.

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1. Introduction

The aim of this article is to construct a new family of Grothendieck sites, the *edged crystalline sites*. These sites are parameterized by non-zero superadditive maps of sets $\tau : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, called *edged-types*. We establish that if τ is linear, the cohomology of these sites recovers rigid cohomology of smooth varieties. In addition, for general τ, we prove an *affine de Rham comparison*, adapting the proof from [BdJ11] to our setting.

1.1. **Crystalline and rigid cohomology.** The realm of p-adic cohomology theories is vast, with two prominent players being rigid cohomology and crystalline cohomology. Rigid cohomology, constructed by Berthelot using rigid geometry, possesses desirable properties such as Poincaré duality, the Lefschetz trace formula, and the theory of weights. However, establishing finiteness results for its cohomology groups remains challenging. Notably, the coherence of relative rigid cohomology for smooth and proper morphisms, known as *Berthelot's conjecture*, remains an open problem [Laz16].

Conversely, crystalline cohomology, pioneered by Grothendieck and chiefly developed by Berthelot and Ogus, is not well behaved for non-proper or singular varieties. Nonetheless, the finiteness statements in the smooth and proper setting are facilitated by a comparison with algebraic de Rham cohomology. The crystalline analogue of Berthelot's conjecture has been resolved [Mor19, Appendix], [DTZ23], where the finiteness is established through the coherence of higher direct images of coherent sheaves under proper morphisms.

Given the strengths and weaknesses of rigid and crystalline cohomology, the motivation behind edged crystalline cohomology is to amalgamate the desirable aspects of both. Another work in the same direction has been done by Langer, [Lan23].

An interesting aspect of edged crystalline cohomology is also its interaction with *log-crystalline cohomology*. This might be used to reinterpret Kedlaya's semi-stable reduction theorem on the relation between overconvergent F -isocrystals and log-F-isocrystals.

1.2. **Marked algebraic geometry.** To define edged crystalline cohomology, we introduce the notion of *marked schemes*, a generalization of *modulus pairs* introduced in [KMSY21]. The theory of marked schemes provides a natural framework for studying regular functions on algebraic varieties with poles bounded at a designated marking. The τ-edged crystalline site consists of *marked PD-thickenings* that have poles of a certain type at the marking, as dictated by τ.

1.3. **Log-decay crystalline cohomology.** Wan proved in [Wan96] that the L-function of *log-decay F-isocrystals* with parameter $r \in \mathbb{R}_{>0}$ (*i.e.*, the growth of p-adic valuations of the coefficients is $\frac{1}{r}\log_p$) over $\mathbb{A}^n_{\mathbb{F}_{p^s}}$ is p-adic meromorphic in the open disk of radius $p^{1/r}$. On the other hand, Emerton and Kisin proved in [EK01] that the L-function of unit-root F-isocrystals (corresponding to the limit case $r \to \infty$) is p-adic meromorphic on the closed disk of radius 1. They employed *Katz's cohomological formula*, formulated with étale cohomology. Wan expected that there should be a variant of Katz's cohomological formula in his context which explains his result. In particular, he expected some *log-decay crystalline cohomology theories* with suitable Lefschetz trace formulas. Our θ^r-edged crystalline cohomology provides a candidate for such a theory and could be used to reinterpret and extend his result.

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2. Marked algebraic geometry

The first goal of this article is to construct categories of *marked rings* and *marked schemes*, which are generalisations of the category of *modulus pairs* introduced in [KMSY21]. We also define some topologies on the category of marked schemes, as the *Zariski topology*, *v-Zariski topology*, and the *v-étale topology*.

2.1. Marked rings.

Definition 2.1.1. A *simply marked ring* is a pair $(A; I_A)$ with A a ring[1] and I_A an ideal of A. A morphism of simply marked rings $(A; I_A) \to (B; I_B)$ is a morphism of rings $A \to B$ such that $I_B \subseteq I_A B$. We denote by $\underline{\mathrm{Ring}}^{\mathrm{sim}}$ the (big) category of simply marked rings.

Example 2.1.2. If $(A; (f))$ and $(B; (g))$ are simply marked rings with principal ideals, a ring morphism $\varphi\colon A \to B$ is¨ a morphism of simply marked rings if and only if $\varphi(f)|g$. In this case, φ extends naturally to a morphism $A_f \to B_g$ which sends $A.\frac{1}{f} \subseteq A_f$ to $B.\frac{1}{g} \subseteq B_g$.

Definition 2.1.3. There is a natural faithful functor $v\colon \underline{\mathrm{Ring}}^{\mathrm{sim}} \to \mathrm{Ring}$ obtained by forgetting the ideal. This functor admits both a left adjoint $u_1\colon \mathrm{Ring} \to \underline{\mathrm{Ring}}^{\mathrm{sim}}$, which sends a ring A to the simply marked ring $(A; (1))$, and a right adjoint $u_0\colon \mathrm{Ring} \to \underline{\mathrm{Ring}}^{\mathrm{sim}}$, which sends A to $(A; (0))$. Both u_1 and u_0 are fully faithful. If not said differently, u_1 will be the embedding we use to compare

[1]In this article the rings are all unital and commutative.

rings and simply marked rings. We will also say that an object in the essential image of u_1 is a *trivially marked ring*.

Lemma 2.1.4. *The category of simply marked rings does not admit equalisers.*

Proof. We consider $A := \mathbb{Z}[x,y]$, $B := A[z]/(x-y)z$, and we write $f\colon B \to B$ for the morphism of A-algebras which sends $z \mapsto 0$. The subring $A \subseteq B$ is the equaliser of $\mathrm{id}_B, f\colon B \to B$ in Ring. Therefore, if $\mathrm{id}_B, \alpha\colon (B;(xz)) \to (B;(0))$ had an equaliser it would be of the form $(A;I_A)$ with $A \subseteq B$ and $(xz) \subseteq I_A B$. Since both $(A;(x))$ and $(A;(y))$ equalise $\mathrm{id}_B, f\colon (B;(xz)) \to (B;(0))$, we would also have that $I_A \subseteq (x) \cap (y) = (xy)$. This would imply that $xz \in xyB$, which is not true as one can check after inverting z. □

Lemma 2.1.5. *The category of simply marked rings does not admit coproducts.*

Proof. Suppose that $(A;I_A)$ was the coproduct of $(\mathbb{Z}[x];(x))$ and $(\mathbb{Z}[y];(y))$ in $\underline{\mathrm{Ring}}^{\mathrm{sim}}$. Since v commutes with coproducts, then $A = \mathbb{Z}[x,y]$. In addition, $I_A \subseteq (xy)$ since $(A;I_A)$ admits morphisms from $(\mathbb{Z}[x];(x))$ and $(\mathbb{Z}[y];(y))$. On the other hand, there should be a morphism $(A;I_A) \to (\mathbb{Z}[t];(t))$ corresponding to the morphisms $(\mathbb{Z}[x];(x)) \to (\mathbb{Z}[t];(t))$ and $(\mathbb{Z}[y];(y)) \to (\mathbb{Z}[t];(t))$ which send $x \mapsto t$ and $y \mapsto t$ respectively. This would imply that $(t) \subseteq I_A\mathbb{Z}[t] \subseteq (t^2)$, which is a contradiction. □

The previous lemmas show that $\underline{\mathrm{Ring}}^{\mathrm{sim}}$ in general does not admit neither finite limits nor finite colimits. In order to make geometric operations, we want to enlarge the category to allow finite colimits.

Definition 2.1.6. Let A be a ring and let $\Sigma_A = \{I_{A,\ell}\}_{\ell \in L}$ be a finite set of ideals of A. The *marking of A associated to Σ_A* is the presheaf over $\underline{\mathrm{Ring}}^{\mathrm{sim}}$ which sends $\underline{R} \in \underline{\mathrm{Ring}}^{\mathrm{sim}}$ to the subset

$$\bigcap_{\ell \in L} \mathrm{Hom}((A;I_{A,\ell}), \underline{R}) \subseteq \mathrm{Hom}(A,R).$$

A *marked ring* is the datum of a ring A and a marking $h_{\underline{A}}$ associated to some finite set of ideals of A. A morphism $(A;h_{\underline{A}}) \to (B;h_{\underline{B}})$ of marked rings is a morphism $A \to B$ of rings such that the induced morphism $h_B \to h_A$ sends $h_{\underline{B}}(\underline{R})$ to $h_{\underline{A}}(\underline{R})$ for every $\underline{R} \in \underline{\mathrm{Ring}}^{\mathrm{sim}}$. We denote by $\underline{\mathrm{Ring}}$ the category of marked rings.

Notation 2.1.7. If $(A;h_{\underline{A}})$ is a marked ring associated to the set of ideals $\{I_{A,\ell}\}_{\ell\in L}$, we will also denote it by $(A;I_{A,\ell})_{\ell \in L}$. Nonetheless, note that $h_{\underline{A}}$ does not determine uniquely the family $\{I_{A,\ell}\}_{\ell \in L}$, so that $(A;I_{A,\ell})_{\ell\in L}$ is only one of many possible presentations of $(A;h_{\underline{A}})$.

Lemma 2.1.8. *If $\underline{A} = (A;I_{A,\ell})_{\ell \in L}$ is a marked ring and for some $\ell_1 \neq \ell_2 \in L$ we have $I_{A,\ell_1} \subseteq I_{A,\ell_2}$, then $(A;I_{A,\ell})_{\ell\in L} = (A;I_{A,\ell})_{\ell \in L\setminus\{\ell_2\}}$. In particular, if A is a valuation ring, then $\underline{A}$ is simply marked.*

Proof. This follows from the fact that for every marked ring $\underline{R} \in \underline{\mathrm{Ring}}^{\mathrm{sim}}$, we have that

$$\mathrm{Hom}((A;I_{A,\ell_1}), \underline{R}) \subseteq \mathrm{Hom}((A;I_{A,\ell_2}), \underline{R}).$$

□

Definition 2.1.9. Let $(A;h_{\underline{A}})$ be a marked ring and let $A \to B$ be a ring homomorphism. There is a canonical marking $h_{\underline{B}}$ on B defined as the fibre product

$$\begin{array}{ccc} h_{\underline{B}} & \longrightarrow & h_B \\ \downarrow & & \downarrow \\ h_{\underline{B}} & \longrightarrow & h_A. \end{array}$$

We say that $h_{\underline{B}}$ is the marking on B *induced* by $h_{\underline{A}}$. Concretely, if $(A; h_{\underline{A}}) = (A; I_{A,\ell})_{\ell \in L}$, then $(B; h_{\underline{B}}) = (B; I_{A,\ell}B)_{\ell \in L}$.

Definition 2.1.10. We say that a marked ring $\underline{A}$ is *principal* if it can be written in the form $(A; I_{A,\ell})_{\ell \in L}$ with each $I_{A,\ell}$ principal. For simplicity, if $\{f_\ell\}_{\ell \in L}$ are generators of $\{I_{A,\ell}\}_{\ell \in L}$ we will also write $(A; f_\ell)_{\ell \in L}$. If $\underline{A} = (A; f_\ell)_{\ell \in L}$ is a principal marked ring, we say that an element $f \in A$ is a *slicing element* of $\underline{A}$ if

$$\sqrt{(f)} = \sqrt{\prod_{\ell \in L} (f_\ell)}.$$

Lemma 2.1.11. *The category* $\underline{\mathrm{Ring}}$ *has finite colimits and finite products and their formation commutes with the forgetful functor* $v \colon \underline{\mathrm{Ring}} \to \mathrm{Ring}$. *The initial object is* $(\mathbb{Z}; 1)$ *and the final object is* $(0; 0)$.

Proof. If $(A; I_A)$ is a simply marked $(C; 1)$-algebra and $(B; 1)$ is trivially marked C-algebra, then the fibre coproduct $(A; I_A) \otimes_{(C;1)} (B; 1)$ is equal to $(A \otimes_C B; I_A \otimes B)$. Indeed, for every simply marked ring we have

$$\mathrm{Hom}((A \otimes_C B; I_A \otimes B), \underline{R}) = \mathrm{Hom}((A; I_A), \underline{R}) \times_{\mathrm{Hom}(C,R)} \mathrm{Hom}(B, R)$$

since for $(\varphi_1, \varphi_2) \in \mathrm{Hom}(A, R) \times_{\mathrm{Hom}(C,R)} \mathrm{Hom}(B, R)$ we have $(\varphi_1 \otimes \varphi_2)(I_A \otimes B)R = \varphi_1(I_A)R$. From this, we deduce that if $(A; I_{A,\ell})_{\ell \in L}$ and $(B; I_{B,m})_{m \in M}$ are marked rings over $\mathrm{Spec}(C; I_{A,n})_{n \in N}$, the fibre coproduct is the marked ring

$$(A \otimes_C B; I_{A,\ell}, I_{B,m})_{\ell \in L, m \in M}.$$

Moreover, if we have a double arrow $f, g \colon (A; I_{A,\ell})_{\ell \in L} \to (B; I_{B,m})_{m \in M}$, we can take the ring then the underlying ring of the coequaliser $\underline{C}$ is the coequaliser of the underlying rings and its marking is induced by the one of $(B; I_{B,m})_{m \in M}$ by taking the extensions of the ideals $I_{B,m}$. The product $\prod_{i=1}^n \underline{A}_i$ is the ring $\prod_{i=1}^n A_i$ endowed with the marking of $\underline{A}_i$ on A_i. □

Remark 2.1.12. Note that by 2.1.11, the category $\underline{\mathrm{Ring}}$ is the finite cocompletion of the category $\underline{\mathrm{Ring}}^{\mathrm{sim}}$. Note also that in this case the finite cocompletion is obtained by adding only fibred coproducts.

Remark 2.1.13. There is also a nice ∞-category of *marked animated rings*. One first starts with the ∞-category $\mathcal{C} = \underline{\mathrm{AnRing}}^{\mathrm{sim}}$ of *simply marked animated rings*. The objects of the homotopy category $\mathrm{h}\mathcal{C}$ are defined to be surjections (at the level of π_0) of animated rings $(A \twoheadrightarrow A_0)$ and the morphisms $(A \twoheadrightarrow A_0) \to (B \twoheadrightarrow B_0)$ are homotopy classes of morphisms of animated rings $A \to B$ such that $B \to B \otimes_A^L A_0$ factors through B_0 (up to homotopy). This category is endowed with a natural functor $v : \mathrm{h}\mathcal{C} \to \mathrm{hAnRing}$ which sends $(A \twoheadrightarrow A_0)$ to A. Then $\mathcal{C}$ is defined as the fibre product $\mathrm{AnRing} \times_{\mathrm{hAnRing}} \mathrm{h}\mathcal{C}$ as ∞-categories.

2.2. **Marked schemes.**

Definition 2.2.1. A *premarked scheme* is a pair $\underline{X} = (X; h_{\underline{X}})$, where X is a scheme and

$$h_{\underline{X}} \colon \underline{\mathrm{Ring}}^{\mathrm{sim,op}} \to \mathrm{Set}$$

is a subfunctor of the composition $h_X \circ v$. We say that X is the *underlying scheme* of $\underline{X}$. A morphism of premarked schemes $\underline{X} \to \underline{Y}$ is a morphism of schemes $X \to Y$ such that $h_X \to h_Y$ sends $h_{\underline{X}}(\underline{R})$ to $h_{\underline{Y}}(\underline{R})$ for every $\underline{R} \in \underline{\mathrm{Ring}}^{\mathrm{sim}}$. We denote by $\underline{\mathrm{Sch}}^{\sim}$ the (big) category of premarked schemes.

Definition 2.2.2. The category of premarked schemes is naturally fibred over Sch via the faithful functor $v \colon \underline{\mathrm{Sch}}^{\sim} \to \mathrm{Sch}$ which sends $(X; h_{\underline{X}}) \mapsto X$. If $(X; h_{\underline{X}})$ is a premarked scheme and $T \to X$ is a morphism of schemes, we write $h_{\underline{X}}|_T$ for $h_{\underline{X}} \times_{h_X} h_T$ and we say that a morphism of premarked schemes is *strict* if it is isomorphic to a morphism of the form $(T; h_{\underline{X}}|_T) \to (X; h_{\underline{X}})$.

Definition 2.2.3. There is a canonical fully faithful functor $\underline{\mathrm{Ring}}^{\mathrm{op}} \to \underline{\mathrm{Sch}}^{\sim}$ which sends $\underline{A} = (A, h_{\underline{A}})$ to $\mathrm{Spec}(\underline{A}) := (\mathrm{Spec}(A); h_{\underline{A}})$. We say that a premarked scheme isomorphic to $\mathrm{Spec}(\underline{A})$ for some marked ring $\underline{A}$ is an *affine marked scheme*. A *marked scheme* is a premarked scheme $(X; h_{\underline{X}})$ such that for some Zariski covering $\{U_i \to X\}_{i \in I}$ all the restrictions $(U_i; h_{\underline{X}}|_{U_i})$ are affine marked schemes. We denote by $\underline{\mathrm{Sch}} \subseteq \underline{\mathrm{Sch}}^{\sim}$ the full (big) subcategory of marked schemes. A pair $(X; \mathcal{I}_X)$ where X is a scheme and $\mathcal{I}_X$ is a quasi-coherent sheaf of ideals of X defines naturally a marked scheme. We say that a marked scheme of this form is a *simply marked scheme*. A *principal marked affine scheme* is the spectrum of a principal marked ring.

Remark 2.2.4. The notion of modulus pair in [KM21] coincides with the one of a simply marked scheme $(X; \mathcal{I}_X)$ with $\mathcal{I}_X$ invertible.

We have functors $u_0, u_1 \colon \mathrm{Sch} \to \underline{\mathrm{Sch}}$ sending X to $(X; 0)$ and $(X; \mathcal{O}_X)$ respectively.

Lemma 2.2.5. *The composition $h_{\underline{X}} \circ u_1$ is represented by an open subscheme $j : X^\circ \hookrightarrow X$.*

Proof. This can be checked Zariski-locally on X. When $\underline{X} = \mathrm{Spec}(A; I_{A,\ell})_{\ell \in L}$, we have that $X^\circ = \mathrm{Spec}(A) \setminus \cup_{\ell \in L} V(I_{A,\ell})$. □

Remark 2.2.6. Note that if $(X; h_{\underline{X}})$ is a marked scheme, one can recover h_X from $h_{\underline{X}}$ by taking the composition $h_{\underline{X}} \circ u_0$. Therefore, the functor $h_{\underline{X}}$ is enough to determine the marked scheme.

Remark 2.2.7. If LogSch is the category of log-schemes, there is a faithful functor $w \colon \underline{\mathrm{Sch}} \to \mathrm{LogSch}$ which sends a marked scheme $\underline{X}$ to $(X, j_* \mathcal{O}^*_{X^\circ} \cap \mathcal{O}_X)$.

Example 2.2.8. We want to briefly mention here other examples of premarked schemes such that $h_{\underline{X}} \circ u_1$ is represented by an open subscheme. Consider $X = \mathrm{Spec}(A)$ with $A = \mathbb{Z}[x, y]$ and set

$$h_{\underline{X}}(R; I_R) := \{\varphi \colon A \to R \mid I_R \subseteq (\varphi(x)) \vee I_R \subseteq (\varphi(y))\}.$$

In this case, the composition $h_{\underline{X}} \circ u_1$ is represented by $\mathbb{A}^2_{\mathbb{Z}} \setminus \{0\}$. On the other hand, it is easy to check this is not a marked scheme. Another example is the limit $\underline{X} = \varprojlim_n \mathrm{Spec}(\mathbb{Z}[x]; x^n)$ in $\underline{\mathrm{Sch}}^{\sim}$, where $h_{\underline{X}} \circ u_1$ is represented by $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\}$.

Definition 2.2.9. We say that $\underline{Y} \to \underline{X}$ is an *open* (resp. *closed*) *immersion* if $Y \to X$ is an open (resp. closed) immersion.

Lemma 2.2.10. *The category* $\underline{\mathrm{Sch}}$ *has finite limits and arbitrary coproducts and their formation commutes with the forgetful functor* $v\colon \underline{\mathrm{Sch}} \to \mathrm{Sch}$*. The final object is* $\mathrm{Spec}(\mathbb{Z};1)$ *and the initial object is* $\underline{\emptyset} := \mathrm{Spec}(0;0)$.

Proof. This follows from the local computation done in Lemma 2.1.11. □

2.3. Cartier schemes.

Definition 2.3.1. A *locally principal marked scheme* is a marked scheme which admits a Zariski covering of principal marked affine schemes. A locally principal marked scheme is said to be a *Cartier scheme* if the marking is locally a finite set of Cartier divisors. We denote by $\underline{\mathrm{Sch}}^{\mathrm{Car}}$ the category of Cartier schemes. We also say that a marked ring is a *locally principal marked ring* (resp. a *Cartier ring*) if its spectrum is a locally principal marked scheme (resp. a Cartier scheme).

2.3.2. The natural fully faithful functor $\underline{\mathrm{Sch}}^{\mathrm{Car}} \hookrightarrow \underline{\mathrm{Sch}}$ admits a right adjoint $\underline{X} \mapsto \mathrm{Bl}(\underline{X})$. Locally this functor sends $\mathrm{Spec}(A; I_{A,\ell})_{\ell\in L}$ to the blowing up of $\mathrm{Spec}(A)$ with respect to each $I_{A,\ell}$. If $\underline{X}$ is locally principal, then $\mathrm{Bl}(\underline{X})$ is a closed marked subscheme of $\underline{X}$ which corresponds to the scheme-theoretic closure of $X^\circ \subseteq X$. Locally, the closed subscheme $\mathrm{Bl}(\underline{X}) \subseteq \underline{X}$ has the following description. Suppose that $\underline{X} = \mathrm{Spec}(\underline{A})$ with $\underline{A}$ principal and let f be a slicing element of $\underline{A}$. The underlying scheme of $\mathrm{Bl}(\underline{X})$ is then the marked affine scheme $\mathrm{Spec}(B)$ where $B := A/(0\colon f^\infty)$. Alternatively, B is the image of $\mathrm{im}(A \to A_f)$.

2.4. Generalities on marked schemes.

Lemma 2.4.1. *Let* $(A; h_{\underline{A}})$ *be a marked ring and let* $(B; I_B)$ *be a simply marked ring. A morphism* $A \to B$ *of rings induces a morphism of marked rings if and only if for every maximal ideal* $\mathfrak{m} \lhd B$*, the induced morphisms* $(A; h_{\underline{A}}) \to (B_{\mathfrak{m}}, I_B B_{\mathfrak{m}})$ *are morphisms of marked rings. In particular,* $h_{\underline{A}}$ *is determined by the value on the marked rings* $\underline{R} \in \underline{\mathrm{Ring}}^{\mathrm{sim}}$ *with* R *a local ring.*

Proof. It is enough to prove that for every ideal $J \lhd B$ and $b \in B$ we have that $b \in J$ if and only if $\frac{b}{1} \in JB_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \lhd B$. By replacing B with B/J we may assume $J = (0)$. Then the result is well-known. □

Definition 2.4.2. Let $\underline{A}$ be a marked ring and write A° for $\Gamma(\mathrm{Spec}(\underline{A}), \mathcal{O}^\circ)$. We say that an element in the kernel of $A \to A^\circ$ is a *torsion element* of $\underline{A}$ and we denote by $\mathrm{Tors}(\underline{A})$ the set of torsion elements of $\underline{A}$. We denote by $\underline{\tilde{A}}$ the quotient $A/\mathrm{Tors}(\underline{A}) \subseteq A^\circ$ endowed with the marking induced by the one of $\underline{A}$. We say that $\underline{A}$ is torsion-free if $A = \tilde{A}$. Note that a morphism $\underline{A} \to \underline{B}$ of marked rings, induces a unique morphism $\underline{\tilde{A}} \to \underline{\tilde{B}}$ of marked rings.

Definition 2.4.3. For every $n \geq 1$, we denote by $\underline{\mathbb{A}}^n$ the marked scheme

$$\mathrm{Spec}(\mathbb{Z}[t_1, \dots, t_n]; t_1, \dots, t_n).$$

For a marked ring $\underline{A}$, there is a natural identification of $\underline{\mathbb{A}}^n(\underline{A})$ with a certain subset of $\mathbb{A}^n(\underline{A}) = A^n$.

Lemma 2.4.4. *For a marked ring* $\underline{A}$ *and an element* $f \in \underline{\mathbb{A}}^1(\underline{A}) \subseteq A$*, there exists a unique* $g \in A^\circ$ *such that* $fg = 1$ *in* A°*. Therefore, if* $\underline{A}$ *is torsion-free,* $\underline{\mathbb{A}}^1(\underline{A}) \subseteq A \cap (A^\circ)^*$.

Proof. A morphism of marked rings $\varphi : (\mathbb{Z}[t]; t) \to \underline{A}$ which sends $t \mapsto f$ induces a morphism $\tilde{\varphi}\colon \mathbb{Z}[t]_t \to A^\circ$. We can then take $g := \tilde{\varphi}(\frac{1}{t})$. □

Definition 2.4.5. Let $\underline{A}$ be a marked ring. A *principal simplification* of $\underline{A}$ is a principal simply marked ring $\underline{B}$ such that $\underline{A} = \underline{B} \times_{(B;0)} (A;0)$.

Construction 2.4.6. Let $\underline{A} = (A; I_{A,\ell})_{\ell\in L}$ be a marked ring and for every $\ell \in L$, let $\{f_{\ell,m}\}_{m\in M_\ell}$ be a set of generators of I_ℓ. We associate to this datum the A-algebra

$$B := \frac{A[x, y_{\ell,m}]_{\ell\in L, m\in M_\ell}}{(x - \sum_{m\in M_\ell} f_{\ell,m} y_{\ell,m})_{\ell\in L}}.$$

We write $\underline{B}$ for $(B; x)$.

Lemma 2.4.7. *The marked ring $\underline{B}$ of Construction 2.4.6 is a principal simplification of $\underline{A}$.*

Proof. Let $\underline{C} = (C; I_{C,m})_{m\in M}$ be a marked ring and let $\varphi\colon C \to A$ be a morphism of rings. Write $\varphi_B\colon C \to B$ for the morphism of rings obtained by composing φ with the natural morphism $A \to B$. Suppose that $\varphi_B\colon \underline{C} \to \underline{B}$ is a morphism of marked rings. We have to prove that the same is true for $\varphi\colon \underline{C} \to \underline{A}$. Let R be a ring and let $\psi : A \to R$ be a morphism of rings. We want to show that for every $m \in M$, we have that $I_{C,m}R \supseteq \bigcap_{\ell\in L} I_{A,\ell}R$. For an element $g \in \bigcap_{\ell\in L} I_{A,\ell}R$, there exists a multiset $\{h_{\ell,m}\}_{\ell\in L, m\in M}$ with $h_{\ell,m} \in R$, such that $g = \sum_{m\in M} \psi(f_{\ell,m})h_{\ell,m}$ for every $\ell \in L$. Therefore, there exists a morphism $\psi_g\colon B \to R$ extending $\psi\colon A \to R$ and such that $\psi_g(x) = g$. The morphism ψ_g upgrades to a morphism $\underline{B} \to (R; g)$ of marked rings. Since $\varphi_B\colon \underline{C} \to \underline{B}$ is a morphism of marked rings, we deduce that $g \in I_{C,m}R$ for every m. By the arbitrariness of $g \in \bigcap_{\ell\in L} I_{A,\ell}R$, we deduce the desired result. □

Notation 2.4.8. When $(A; \Sigma)$ is a torsion-free principal marked ring with $\Sigma = \{f_1, \dots, f_n\}$, we write $A[\frac{x}{f_1}, \dots, \frac{x}{f_n}]$ or $A[\frac{x}{\Sigma}]$ for the principal simplification of Construction 2.4.6 with respect to the set $\{f_1, \dots, f_n\}$.

Definition 2.4.9. An *open $\star$-immersion* $\underline{Y} \to \underline{X}$ of marked schemes is a morphism such that $Y^\circ \to X^\circ$ is an open immersion.

Lemma 2.4.10. *Let $\varphi : \underline{A} \to \underline{B}$ be a morphism of principal marked rings. The induced morphism* $\mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ *is an open $\star$-immersion if and only if there exist $f_1, \dots, f_n \in A$ such that $\varphi(f_1), \dots, \varphi(f_n)$ generate the unit ideal in B° and such that $\varphi : A^\circ_{f_i} \to B^\circ_{\varphi(f_i)}$ is an isomorphism for every i.*

2.5. **Strong completion of marked rings.**

Definition 2.5.1. If $\underline{A}$ is a principal marked ring with slicing element f and J is an ideal of A, then the *strong J-completion* of $\underline{A}$ is the ring

$$A^{\curlywedge}_J := \varprojlim_e A/(J^e : f^\infty)$$

endowed with the marking induced by the natural morphism $A \to A^{\curlywedge}_J$. If $A = A^{\curlywedge}_J$, we say that A is *strongly J-complete*. If J is not specified we assume that $J = (p)$.

Remark 2.5.2. Note that for every $e > 0$ we have that $A/(J^e : f^\infty) \twoheadrightarrow A/(J : f^\infty)$ is a thickening.

Lemma 2.5.3. *The ring $A^{\curlywedge}_J$ is the closure of $\tilde{A} \subseteq A_f$ with respect to the J-adic topology.*

3. Topologies on marked schemes

3.1. Zariski topology.

Definition 3.1.1. A *Zariski covering* of $\underline{X}$ is a family $\{\underline{U}_i \to \underline{X}\}_{i\in I}$ of strict open immersions such that $\{U_i \to X\}_{i\in I}$ is a Zariski covering.

For the moment, I have put the rest of Chapter §3 as an appendix as it can be skipped at a first reading.

4. Edged crystalline site

4.1. Principal edged localisation.

Definition 4.1.1. An *edge-type* is a map of sets $\tau\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which is not constantly 0 and such that for every $e_1, e_2 \in \mathbb{Z}_{>0}$ we have that

$$\tau(e_1 + e_2) \geq \tau(e_1) + \tau(e_2).$$

For an edge-type τ and a positive integer n, we denote by $\tau_n\colon : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the *n-th scaling* of τ, defined by $\tau_n(e) = n\tau(e)$.

Definition 4.1.2. Let A be a ring, $\Sigma = \{f_1, \dots, f_n\}$ a finite subset of A with $f = \prod_{i=1}^n f_i$, and τ an edge-type. The *(crystalline) principal τ-localisation* of A with respect to Σ is the A-subalgebra $A^\tau_\Sigma \subseteq A_f$ generated by A and all the fractions $\frac{p^e}{f_1^{u_1}\cdots f_n^{u_n}}$ with $e \geq 1$ and $\tau(e) \geq \sum_{i=1}^n u_i$. We denote by $\underline{A}^\tau_\Sigma$ the marked ring $(A^\tau_\Sigma; \Sigma)$. For an A-module M, the principal τ-localisation of M with respect to Σ, denoted by M^τ_Σ, is the A^τ_Σ-submodule of M_f generated by M.

Lemma 4.1.3. *The A-module A^τ_Σ is generated by 1 and the fractions $\frac{p^e}{f_1^{u_1}\cdots f_n^{u_n}}$ with $e \geq 1$ and $\tau(e) \geq \sum_{i=1}^n u_i$*

Proof. This follows from the fact that

$$\frac{p^{e_1}}{f_1^{u_{1,1}}\cdots f_n^{u_{n,1}}} \cdot \frac{p^{e_2}}{f_1^{u_{1,2}}\cdots f_n^{u_{n,2}}} = \frac{p^{e_1+e_2}}{f_1^{u_{1,1}+u_{1,2}}\cdots f_n^{u_{n,1}+u_{n,2}}}$$

and $\tau(e_1 + e_2) \geq \tau(e_1) + \tau(e_2) \geq \sum_{i=1}^n (u_{i,1} + u_{i,2})$. □

Example 4.1.4. The edge-type $\lambda\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined by $\lambda(e) = e$ corresponds to the overconvergent theory. In this case we have that

$$\mathbb{Z}[t]^{\lambda_n}_t = \mathbb{Z}\left[t, \frac{p}{t^n}\right].$$

The edge-type $\kappa_\infty(e) \equiv \infty$ gives

$$\mathbb{Z}[t]^{\kappa_\infty}_t = \varinjlim_e \mathbb{Z}\left[t, \frac{p}{t^e}\right].$$

Construction 4.1.5. Let $f : \mathbb{Z}_{>0} \to \mathbb{R}$ be a map of sets. We define $\tau^f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ inductively by

$$\tau^f(e) := \max\{\sup_{f(u)\leq e}\{u\}, \max_{1\leq i\leq e-1}\{\tau(i) + \tau(e-i)\}\}.$$

We say that τ^f is the edge-type associated to the *decay function* f.

Definition 4.1.6. Let $f_r : \mathbb{Z}_{>0} \to \mathbb{R}$ be the decay function $f_r(u) := \frac{\log_p(u)}{r}$, where r is a positive real number. We denote by θ^r the edge-type τ^{f_r}. This type corresponds to the r-log-decay theory, [K-M16], [K-M21], [K-M22].

Example 4.1.7. When $h = 1$ we have that $\theta^1(e) = p^e$ and

$$\mathbb{Z}[t]_t^{\theta_n^1} = \varinjlim_e \mathbb{Z}\left[t, \frac{p}{t^{np}}, \dots, \frac{p^e}{t^{np^e}}\right].$$

Remark 4.1.8. Despite the name, a principal τ-localisation of A is not a localisation of A in the classical sense. The A-algebra A_f^τ is not even flat as it happens already for the basic example $A = \mathbb{Z}[t]$ and $\tau = \lambda_1$. In this case, $\mathbb{F}_p[t] \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t]_t^{\lambda_1}$ has some non-trivial t-torsion.

Remark 4.1.9. For every edge-type τ there exists a morphism $\mathbb{Z}[t]_t^\tau \to \mathbb{Z}[t]_t^{\tau_p}$ which sends t to t^p. Thus $\varinjlim_n \mathbb{Z}[t]_t^{\tau_n}$ admits a lift of the Frobenius endomorphism of $\mathbb{F}_p[t]$.

Lemma 4.1.10. *For an edge-type τ, a ring A, and a finite subset $\Sigma = \{f_1, \dots, f_n\} \subseteq A$, we have that*

$$A_\Sigma^\tau = \tilde{A}[\tfrac{x}{\Sigma}]_x^\tau \cap A_f \subseteq A_f[x, x^{-1}]$$

where $f = \prod_{i=1}^n f_i$ and $\tilde{A} = A/(0 : f^\infty)$.

Proof. This follows from the identity

$$\frac{p^e}{f_1^{u_1} \cdot \dots \cdot f_n^{u_n}} = \frac{p^e}{x^{u_1 + \dots + u_n}} \cdot \frac{x^{u_1}}{f^{u_1}} \cdot \dots \cdot \frac{x^{u_n}}{f^{u_n}}$$

for every $e \geq 1$ and $\tau(e) \geq \sum_{i=1}^n u_i$. □

Lemma 4.1.11. *Let A be a ring, $\Sigma = \{f_1, \dots, f_n\} \subseteq A$ a finite subset and τ an edge-type. For an A-module M, the principal τ-localisation M_Σ^τ is canonically isomorphic to*

$$(M \otimes_{A[t_1,\dots,t_n]} A[t_1, \dots, t_n]_{t_1,\dots,t_n}^\tau)/f^\infty\text{-torsion},$$

where $t_1, \dots, t_n$ act on M via the multiplication by $f_1, \dots, f_n$ and $f := \prod_{i=1}^n f_i$.

Proof. The natural morphism $M \otimes_{A[t_1,\dots,t_n]} A[t_1, \dots, t_n]_{t_1,\dots,t_n}^\tau \to M_f$ corresponds to the localisation with respect to f. This implies that the kernel is the f^∞-torsion of the source. The result then follows from the definition of principal τ-localisation. □

Lemma 4.1.12. *If M is an A module, then for $i \geq 1$ we have that $\mathrm{Tor}_i^A(M, A_\Sigma^\tau)$ is of f^∞-torsion.*

Proof. We know that $\mathrm{Tor}_i^A(M, A_f)$ vanishes for $i \geq 1$, so that it is enough to prove that $\mathrm{Tor}_{i+1}^A(M, A_f/A_\Sigma^\tau)$ is of f^∞-torsion. This follows from the fact that A_f/A_Σ^τ itself is of f^∞-torsion. □

Lemma 4.1.13. *The principal τ-localisation of A-modules commutes with localisation.*

Proof. By definition it is enough to prove the statement for A itself and the localisation with respect to $g \in A$. In this case both $(A_\Sigma^\tau)_g$ and $(A_g)_\Sigma^\tau$ are A_g-subalgebras of A_{fg} generated by the fractions $\frac{p^e}{f_i^u}$ satisfying the conditions given by τ. □

4.2. **Edged schemes.** If $\underline{A} = (A;\Sigma)$ is a principal marked ring, the principal τ-localisation A^τ_Σ might depend a priori on the choice of Σ. We want to prove here that this is not the case, thanks to the following more intrinsic description.

Lemma 4.2.1. *Let A be a ring, $\Sigma = \{f_1, \dots, f_n\}$ be a finite subset of A, and $f := \prod_{i=1}^n f_i$. Write $\Lambda^\tau(A)$ for the A-subalgebra of A_f generated by the fractions $\frac{p^e}{g_1^{u_1}\cdot\ldots\cdot g_m^{u_m}}$ with $e, m \geq 1$, $(g_1, \dots, g_m) \in \underline{\mathbb{A}}^m(A;\Sigma)$, and $\tau(e) \geq \sum_{i=1}^m u_i$. We have that $\Lambda^\tau(A) = A^\tau_\Sigma$.*

Proof. We first note that we may assume $(A;\Sigma)$ torsion-free. The inclusion $A^\tau_\Sigma \subseteq \Lambda^\tau(A)$ follows by definition since $(f_1, \dots, f_n)$ is an element of $\underline{\mathbb{A}}^n(A;\Sigma)$. For the other containment let us first assume that $n = 1$. In this case, for every $(g_1, \dots, g_m) \in \underline{\mathbb{A}}^m(A;f)$ there exist $h_1, \dots, h_m \in A$ such that $f = g_ih_i$ for every $1 \leq i \leq m$. Therefore,

$$\frac{p^e}{g_1^{u_1}\cdot\ldots\cdot g_m^{u_m}} = \frac{p^e}{f^{u_1+\cdots+u_m}} \cdot h_1^{u_1}\cdot\ldots\cdot h_n^{u_n}$$

in A_f for every $e, u_1, \dots, u_m \geq 0$. This shows that $\Lambda^\tau(A) \subseteq A^\tau_f$.

When $n > 1$, we want to deduce the result from the case $n = 1$ by considering the principal marked ring $(A[\frac{x}{f_1}, \dots, \frac{x}{f_n}]^\tau_x; x)$. By the previous step, we know that

$$\Lambda^\tau(A) \subseteq \Lambda^\tau(A[\tfrac{x}{f_1}, \dots, \tfrac{x}{f_n}]) = A[\tfrac{x}{f_1}, \dots, \tfrac{x}{f_n}]^\tau_x.$$

On the other hand, by Lemma 4.1.10, we have that $A^\tau_\Sigma = A[\frac{x}{f_1}, \dots, \frac{x}{f_n}]^\tau_x \cap A_f \subseteq A_f[x, x^{-1}]$. This shows that $\Lambda^\tau(A) \subseteq A^\tau_\Sigma$, as we wanted. $\square$

Definition 4.2.2. For every $n \geq 1$, let $\underline{\mathbb{A}}^{n,\tau}$ be the marked scheme

$$\mathrm{Spec}(\mathbb{Z}[t_1, \dots, t_n]^\tau_{t_1,\dots,t_n}; t_1, \dots, t_n).$$

We say that a Cartier local ring $\underline{A}$ is a *τ-edged local ring* if the natural maps $\underline{\mathbb{A}}^{n,\tau}(\underline{A}) \to \underline{\mathbb{A}}^n(\underline{A})$ are bijections for every $n \geq 1$. We say that a Cartier scheme $\underline{X}$ is τ-edged if all the marked local rings $\mathcal{O}_{\underline{X},x}$ are τ-edged for every point $x \in X$. We write $\underline{\mathrm{Sch}}^\tau$ for the full subcategory of $\underline{\mathrm{Sch}}$ with objects τ-edged schemes. Similarly, we say that a marked ring $\underline{A}$ is *τ-edged* if $\mathrm{Spec}(\underline{A})$ is a τ-edged scheme.

Lemma 4.2.3. *For a marked ring, a multiplicative subset $S \subseteq A$, and $n \geq 1$, we have that $\underline{\mathbb{A}}^n(S^{-1}\underline{A}) = \varinjlim_{s\in S} \underline{\mathbb{A}}^n(\underline{A}_s)$.*

Proof. It is enough to treat the case when $n = 1$. We first suppose in addition that $\underline{A} = (A; f)$. For this case, we note that a morphism $\varphi\colon (\mathbb{Z}[t]; t) \to (S^{-1}A; f)$ corresponds to the choice of a divisor of f in $S^{-1}A$. In other words, it corresponds to the choice of an element $\frac{g}{s} \in S^{-1}A$ such that there exist $h \in A$ and $t, u \in S$ satisfying $(gh - fst)u = 0$. Thus φ restricts to a morphism $(\mathbb{Z}[t]; t) \to (A_{stu}; f)$, as we wanted. For the general case, we choose a principal simplification $\underline{B}$ of $\underline{A}$. The result then follows from the previous part thanks to the fact that

$$S^{-1}\underline{B} \times_{(S^{-1}B;0)} (S^{-1}A; 0) = \varinjlim_{s\in S}(\underline{B}_s \times_{(B_s;0)} (A_s; 0)).$$

$\square$

Lemma 4.2.4. *A Cartier scheme $\underline{X}$ is τ-edged if and only if for every $n \geq 1$ and for a basis of Zariski opens $\underline{U} \subseteq \underline{X}$ we have that $\underline{\mathbb{A}}^{n,\tau}(\underline{U}) \to \underline{\mathbb{A}}^n(\underline{U})$ is a bijection.*

Lemma 4.2.5. *Let $\underline{X}$ be a locally principal marked scheme and τ an edge-type. There is a (unique) sheaf $\mathcal{O}^{\tau}_{\underline{X}}$ over $(\underline{X})_{\mathrm{Zar}}$ such that $\mathcal{O}^{\tau}_{\underline{X}}(\mathrm{Spec}(A;\Sigma)) = A^{\tau}_{\Sigma}$ for every $\mathrm{Spec}(A;\Sigma) \in (\underline{X})_{\mathrm{Zar}}$.*

Proof. By Lemma 4.2.1, the assignment $(A;\Sigma) \mapsto A^{\tau}_{\Sigma}$ does not depend on the choice of the presentation of the principal marked ring. By standard arguments it is then enough to check that for every $\{g_1,\dots,g_n\} \subseteq A$ generating the unit ideal we have that A^{τ}_{Σ} is the equaliser of

$$\prod_i (A_{g_i})^{\tau}_{\Sigma} \rightrightarrows \prod_{i,j} (A_{g_i g_j})^{\tau}_{\Sigma}.$$

This follows from Lemma 4.1.13 and Zariski descent. □

Definition 4.2.6. Let $\underline{X}$ be a locally principal marked scheme and for an edge-type τ, let $\mathcal{O}^{\tau}_{\underline{X}}$ be the sheaf of Lemma 4.2.5. We denote by $\Lambda^{\tau}(\underline{X}) \to \underline{X}$ the marked scheme obtained by taking the relative spectrum of $\mathcal{O}^{\tau}_{\underline{X}}$. We say that $\Lambda^{\tau}(\underline{X}) \to \underline{X}$ is the *τ-localisation* of $\underline{X}$.

Lemma 4.2.7. *The τ-localisation of a locally principal marked scheme is τ-edged.*

Lemma 4.2.8. *Write $\underline{\mathrm{Sch}}^{\mathrm{lp}}$ for the category of locally principal marked schemes. For every τ, the τ-localisation functor is a right adjoint of the inclusion functor $\underline{\mathrm{Sch}}^{\tau} \to \underline{\mathrm{Sch}}^{\mathrm{lp}}$.*

Proof. It is enough to prove that for $\underline{A}$ and $\underline{B}$ torsion-free principal marked rings with $\underline{B}$ τ-edged, the natural map $\alpha\colon \mathrm{Hom}(\underline{A}^{\tau},\underline{B}) \to \mathrm{Hom}(\underline{A},\underline{B})$ is a bijection. For the injectivity, note that both $\mathrm{Hom}(\underline{A}^{\tau},\underline{B})$ and $\mathrm{Hom}(\underline{A},\underline{B})$ are subsets of $\mathrm{Hom}(A^{\circ},B^{\circ})$. For the surjectivity we note that the τ-localisation functor induces a map

$$\mathrm{Hom}(\underline{A},\underline{B}) \to \mathrm{Hom}(\underline{A}^{\tau},\underline{B}^{\tau}) = \mathrm{Hom}(\underline{A}^{\tau},\underline{B}),$$

which is the right inverse of α. □

4.3. More on edged localisation.

Definition 4.3.1. There is a natural descending filtration of ideals on A^{τ}_{Σ} defined by

$$\mathrm{Fil}^e_p(A^{\tau}_{\Sigma}) := (p^e \colon f^{\infty}).$$

We call it the *refined p-adic filtration*. The natural A^{τ}_{Σ}-module structure on the graded pieces $\mathrm{gr}^e_p(A^{\tau}_{\Sigma}) := \mathrm{Fil}^e_p(A^{\tau}_{\Sigma})/\mathrm{Fil}^{e+1}_p(A^{\tau}_{\Sigma})$ induces an $\overline{A}$-module structure where $\overline{A}$ is the ring $\mathrm{gr}^0_p(A^{\tau}_{\Sigma}) = A^{\tau}_{\Sigma}/(p\colon f^{\infty}) = A/p$.

Example 4.3.2. When $A = \mathbb{Z}[t]$ and $\Sigma = \{t\}$, we have that $\overline{A} = \mathbb{F}_p[t]$ and for $e \geq 1$, the module $\mathrm{gr}^e(A^{\tau}_{\Sigma})$ is the $\mathbb{F}_p[t]$-module $\frac{1}{t^{\tau(e)}}\mathbb{F}_p[t]$.

Lemma 4.3.3. *We have that*

$$A^{\tau}_f/\mathrm{Fil}^e_p(A^{\tau}_f) = (A/p^e)^{\tau}_f.$$

Proof. The morphism $A^{\tau}_f \to (A/p^e)^{\tau}_f$ corresponds to

$$(A \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t]^{\tau}_t)/f^{\infty}\text{-torsion} \to (A/p^e \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t]^{\tau}_t)/f^{\infty}\text{-torsion},$$

where $\mathbb{Z}[t] \to A$ sends t to f. From this description it is clear that the morphism is surjective and the kernel is $\mathrm{Fil}^e_p(A^{\tau}_f)$.

□

Lemma 4.3.4. *If R is a local ring with maximal ideal (p), then $(R[[t]]^{\tau}_{t})^{\curlywedge}_{p}$ (see Definition 2.5.1) is a local ring with maximal ideal* $\mathfrak{m} := (p : t^{\infty}) + (t)$.

Proof. We have that $(R[[t]]^{\tau}_{t})^{\curlywedge}_{p}/\mathfrak{m} = R[[t]]/(p,t) = R/p$, which is a field. It remains to prove that all the elements not in $\mathfrak{m}$ are invertible. This can be checked on each quotient $R[[t]]^{\tau}_{t}/\mathrm{Fil}^{e}_{p}(R[[t]]^{\tau}_{t})$. For this we note that $R[[t]]^{\tau}_{t}/\mathrm{Fil}^{e}_{p}(R[[t]]^{\tau}_{t}) \twoheadrightarrow R[[t]]^{\tau}_{t}/\mathrm{Fil}^{1}_{p}(R[[t]]^{\tau}_{t}) = R/p[[t]]$ is a thickening so the result follows from the fact that $R/p[[t]]$ is a local ring with maximal ideal (t). □

4.4. **Edged crystalline site.**

Definition 4.4.1. A *(locally principal) marked* PD-*scheme* is a PD-scheme (U,T,γ) endowed with the choice of a locally principal[2] marking on U. We denote it by $(\underline{U},T,\gamma)$. A *marked* PD-*thickening* is a marked PD-scheme $(\underline{U},T,\gamma)$ such that $U \hookrightarrow T$ is a thickening. We denote by $\underline{\mathrm{CRIS}}$ the category of marked PD-thickenings. If (S_0,S,γ) is a PD-thickening, we denote by $\underline{\mathrm{CRIS}}_S$ the category of $(\underline{U},T,\delta) \in \underline{\mathrm{CRIS}}$ endowed with a morphism $(\underline{U},T,\delta) \to (S_0,S,\gamma)$. For a marked scheme $\underline{X}$ over S_0 we denote by $\mathrm{CRIS}(\underline{X}/S)$ the category of $(\underline{U},T,\delta) \in \underline{\mathrm{CRIS}}_S$ endowed with a morphism of marked S_0-schemes $\underline{U} \to \underline{X}$ and by $\underline{\mathrm{Cris}}(\underline{X}/S) \subseteq \mathrm{CRIS}(\underline{X}/S)$ the full subcategory of those $(\underline{U},T,\delta)$ such that $\underline{U} \to \underline{X}$ is a strict open immersion.

Hypothesis 4.4.2. In what follows we assume that (S_0,S,γ) is a PD-ring with S a $\mathbb{Z}_{(p)}$-scheme and $\gamma_n(p) = \frac{p^n}{n!}$.

Definition 4.4.3. A family of morphisms of marked PD-thickenings $\{(\underline{U}_i,T_i,\gamma_i) \to (\underline{U},T,\gamma)\}_{i\in I}$ is a *Zariski covering* if the following conditions are satisfied.

1. $\underline{U}_i = \underline{U} \times_T T_i$ for every i.
2. $\{T_i \to T\}_{i\in I}$ is a Zariski covering.

We endow $\underline{\mathrm{CRIS}}_S$ with the Zariski topology. We write $\mathcal{O}_{\mathrm{cris}}$ for the sheaf over $\underline{\mathrm{CRIS}}_S$ given by $(\underline{U},T,\gamma) \mapsto \Gamma(T,\mathcal{O})$. This is represented by $(\mathbb{A}^1_{S_0},\mathbb{A}^1_S,\gamma) \in \underline{\mathrm{CRIS}}_S$. We also write $\underline{\mathcal{O}}_{\mathrm{cris}} \subseteq \mathcal{O}_{\mathrm{cris}}$ for the subsheaf of sets over $\underline{\mathrm{CRIS}}_S$ represented by $(\underline{\mathbb{A}}^1_{S_0},\mathbb{A}^1_S,\gamma) \in \underline{\mathrm{CRIS}}_S$. When $\underline{X}$ is an S_0-scheme, the sheaves $\mathcal{O}_{\mathrm{cris}}$ and $\underline{\mathcal{O}}_{\mathrm{cris}}$ naturally define sheaves $\mathcal{O}_{\underline{X}/S}$ and $\underline{\mathcal{O}}_{\underline{X}/S}$ over $\mathrm{CRIS}(\underline{X}/S)$.

Construction 4.4.4. Given a marked PD-thickening $(\underline{U},T,\gamma)$ we denote by $T^{\circ} \subseteq T$ the open subscheme which corresponds topologically to the open subscheme $U^{\circ} \subseteq U$. The assignment $(\underline{U},T,\gamma) \mapsto \Gamma(T^{\circ},\mathcal{O})$ defines a quasi-coherent sheaf of $\mathcal{O}_{\mathrm{cris}}$-algebras on $\underline{\mathrm{CRIS}}_S$ that we denote by $\mathcal{O}^{\circ}_{\mathrm{cris}}$. For an edge-type τ, we denote by $\mathcal{O}^{\tau,1}_{\mathrm{cris}} \subseteq \mathcal{O}^{\circ}_{\mathrm{cris}}$ the subsheaf of $\mathcal{O}_{\mathrm{cris}}$-algebras generated Zariski-locally by the fractions

$$\frac{p^e}{e!\cdot g_1^{u_1}\cdot\ldots\cdot g_m^{u_m}}$$

with $e,m \geq 1$, $\tau(e) \geq \sum_{i=1}^{m} u_i$, and $g_1,\ldots,g_m$ local sections of $\underline{\mathcal{O}}_{\mathrm{cris}}$.

Definition 4.4.5. A marked PD-thickening $(\underline{U},T,\gamma)$ over S is a *τ-edged* PD-*thickening over S* if the restriction of $\mathcal{O}^{\tau,1}_{\mathrm{cris}}$ to T_{Zar} coincides with $\mathcal{O}_{\mathrm{cris}}$. We denote by $\underline{\mathrm{CRIS}}^{\tau}_S \hookrightarrow \underline{\mathrm{CRIS}}_S$ the full subcategory of τ-edged PD-thickenings and by $\mathcal{O}^{\tau}_{\mathrm{cris}}$ the restriction of $\mathcal{O}_{\mathrm{cris}}$ to $\underline{\mathrm{CRIS}}^{\tau}_S$. If $\underline{X}$ is a marked scheme over S_0 we also denote by $\mathrm{CRIS}^{\tau}(\underline{X}/S) \subseteq \mathrm{CRIS}(\underline{X}/S)$ and $\mathrm{Cris}^{\tau}(\underline{X}/S) \subseteq \mathrm{Cris}(\underline{X}/S)$ the full subcategories of τ-edged PD-thickenings and by $\mathcal{O}^{\tau}_{\underline{X}/S}$ the restriction of $\mathcal{O}_{\underline{X}/S}$ to $\mathrm{CRIS}^{\tau}(\underline{X}/S)$.

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[2]In order to simplify the discussion we will work here with locally principal markings.

Proposition 4.4.6. *The inclusion functor* $\underline{\mathrm{CRIS}}^{\tau}_S \hookrightarrow \underline{\mathrm{CRIS}}_S$ *admits a right adjoint* $\Lambda^{\tau}\colon \underline{\mathrm{CRIS}}_S \to \underline{\mathrm{CRIS}}^{\tau}_S$.

Proof. Given a marked PD-thickening $(\underline{U}, T, \gamma)$, we denote by $\Lambda^{\tau,1}(T) \to T$ the T-scheme obtained by taking the relative spectrum of $\mathcal{O}^{\tau,1}_{\mathrm{cris}}$ over T_{Zar}. There is a natural PD-structure on $\Lambda^{\tau,1}(\mathcal{I})$, the pullback of $\mathcal{I}$ to $\Lambda^{\tau,1}(T)$, and a natural marking on $\Lambda^{\tau,1}(U) := V(\Lambda^{\tau,1}(\mathcal{I}))$. This assignment defines an endofunctor $\Lambda^{\tau,1}\colon \underline{\mathrm{CRIS}}_S \to \underline{\mathrm{CRIS}}_S$. We write $\Lambda^{\tau,n}$ for the composition $\Lambda^{\tau,1} \circ \dots \circ \Lambda^{\tau,1}$, where $\Lambda^{\tau,1}$ is repeated n-times. We then define $\Lambda^{\tau}(\underline{U}, T, \gamma) := \varprojlim_{n\geq 1} \Lambda^{\tau,n}(\underline{U}, T, \gamma)$. One can check that $\Lambda^{\tau}(\underline{U}, T, \gamma)$ is τ-edged. □

Lemma 4.4.7. *The category* $\underline{\mathrm{CRIS}}^{\tau}_S$ *has all finite coproducts and non-empty finite limits.*

4.5. **Edged crystals and quasi-crystals.**

Definition 4.5.1. A *τ-edged crystal* over $\underline{X}/S$ is a crystal of quasi-coherent $\mathcal{O}^{\tau}_{\underline{X}/S}$-modules over $\underline{\mathrm{CRIS}}^{\tau}(\underline{X}/S)$. We write $\mathbf{Cr}^{\tau}(\underline{X}/S)$ the category they form. We also say that a locally quasi-coherent sheaf of $\mathcal{O}^{\tau}_{\underline{X}/S}$-modules $\mathcal{F}$ is a *τ-edged quasi-crystal* if for every morphism $f : (\underline{U}, T, \delta) \to (\underline{U}', T', \delta')$ with $T \to T'$ a closed immersion, the comparison morphism $c_f\colon f^*\mathcal{F}|_{T'} \to \mathcal{F}|_T$ is surjective.

4.5.2. Let k be a perfect field, W its ring of Witt vectors, and K the fraction field of W. For a separated scheme X of finite type over k and an edge-type τ, the τ-edged crystalline complex of X is the complex

$$R\Gamma_{\tau\text{-cris}}(X/W) := \varinjlim_{X\subseteq \underline{Y}} R\Gamma(\mathrm{Cris}^{\tau}(\underline{Y}/W), \mathcal{O}^{\tau}_{\underline{Y}/W})$$

with $\underline{Y}$ proper Cartier. The category of *coherent τ-edged isocrystals* over X, denoted by $\mathbf{Isoc}^{\tau}(X/K)$, is the 2-colimit of the isogeny category of crystals in coherent $\mathcal{O}^{\tau}_{\underline{Y}/W}$-modules over $\mathrm{Cris}^{\tau}(\underline{Y}/W)$ for different embeddings $X \subseteq \underline{Y}$ with $\underline{Y}$ proper Cartier.

4.6. **Edged localisation of marked** PD**-rings.**

Definition 4.6.1. A PD-*ring* is a ring A endowed with an ideal I and a PD-structure γ on I. We denote such a datum by $(A, A/I, \gamma)$. We denote by PD-Ring the category they form. A *marked* PD-*ring* is a PD-ring $(A, A/I, \gamma)$ endowed with a locally principal marking $h_{A/I}$ on A/I. We write $(A, \underline{A/I}, \gamma)$ do indicate such a datum and we denote by $\underline{\text{PD-Ring}}$ the category they form.

The functor Λ^{τ} of Proposition 4.4.6 naturally defines an endofunctor

$$\Lambda^{\tau}\colon \underline{\text{PD-Ring}}^{\wedge}_p \to \underline{\text{PD-Ring}}^{\wedge}_p,$$

where $\underline{\text{PD-Ring}}^{\wedge}_p \subseteq \underline{\text{PD-Ring}}$ is the full subcategory of p-adically complete PD-rings. We say that a p-complete marked PD-ring is a *τ-edged* PD-*ring* if $\Lambda^{\tau}(A, \underline{A/I}, \gamma) = (A, \underline{A/I}, \gamma)$.

Definition 4.6.2. Let $(A, A/I, \gamma)$ be a PD-ring. For every $f, g \in A$ which differ by an element in I and $s \geq e \geq 0$, we define

$$\Psi_{e,s}(f, g) := \sum_{i=1}^{p^s} \frac{p^s!}{p^e(p^s - i)!} f^{p^s - i}\gamma_i(g - f) \in A$$

$$u_s(f, g) := 1 + \tfrac{p^s}{f^{p^s}}\Psi_{s,s}(f, g) \in A_f.$$

Lemma 4.6.3. *For every $s \geq e \geq 0$, the following identities hold.*

(1) $p^e \Psi_{e,s}(f,g) = (g^{p^s} - f^{p^s})$,
(2) $f^{p^s} u_s(f,g) = g^{p^s}$,
(3) $u_s(f,g) = 1 + \frac{p^e}{f^{p^s}} \Psi_{e,s}(f,g)$,
(4) $u_1(f,g)^{p^s} = u_s(f,g)$.

Proof. The first identity can be checked for $\Psi_{e,s}(x, x+y)$ in $\mathbb{Z}[x]\{y\}$ and then deduced for every A by sending $x \mapsto f$ and $y \mapsto g - f$. The other three identities follow from the first one. □

Definition 4.6.4. For an edge-type τ we write e_τ for $\min_{\tau(e) \neq 0}\{e\}$ and q_τ for p^{e_τ}. We say that τ is p-small if $q_\tau | \tau(e)$ for every $e \geq 1$. Note that τ_{q_τ} is p-small for every edge-type τ.

Lemma 4.6.5. *Let $(A, A/I, \gamma)$ be a p-complete* PD*-ring. If $f, g \in A$ are elements with the same reduction modulo I, then $(A_f)^\wedge_p = (A_g)^\wedge_p$ and $(A^\tau_{f^{q_\tau}})^\curlywedge_p = (A^\tau_{g^{q_\tau}})^\curlywedge_p$.*

Proof. We note that we may further assume that $p^n A = 0$ for some $n \geq 1$. Then, by Lemma 4.6.3, we have that $g^{p^n} - f^{p^n} = p^n \Psi_{a,a}(f,g) = 0$, so that $A_f = A_g$. For the other part, we note that by definition $\frac{q_\tau}{f^{q_\tau}}$ is an element of $A^\tau_{f^{q_\tau}}$, so that $u_{e_\tau}(f^v, g^v)$ is a unit of $A^\tau_{f^{q_\tau}}$ for every $v \geq 0$. Thanks to Lemma 4.6.3.(2), for every $e \geq 0$, we have that

$$\frac{p^e}{g^{q_\tau v}} u_{e_\tau}(f^v, g^v) = \frac{p^e}{f^{q_\tau v}}$$

in A_f. Since $u_{e_\tau}(f^v, g^v)$ is invertible in $A^\tau_{f^{q_\tau}}$, if $\frac{p^e}{f^{q_\tau v}} \in A^\tau_f$ then $\frac{p^e}{g^{q_\tau v}} \in A^\tau_{f^{q_\tau}}$ as well. This shows that $A^\tau_{g^{q_\tau}} \subseteq A^\tau_{f^{q_\tau}}$. By symmetry we deduce that $A^\tau_{f^{q_\tau}} = A^\tau_{g^{q_\tau}}$, as we wanted. □

Lemma 4.6.6. *Let $(A, A/I, \gamma)$ be a p-complete* PD*-ring. If $f, g \in A$ are elements such that their reductions in A/I generate the same ideal, then $(A^\tau_{f^{q_\tau}})^\curlywedge_p = (A^\tau_{g^{q_\tau}})^\curlywedge_p$.*

Proof. We may assume that there exists $n \geq 1$ such that $p^n A = 0$. By the assumption, there are $u, v \in A$ such that $[g^{q_\tau}] = [u f^{q_\tau}]$ and $[f^{q_\tau}] = [v g^{q_\tau}]$ in A/I. On the one hand, there are canonical morphisms $A^\tau_{f^{q_\tau}} \to A^\tau_{uf^{q_\tau}} \to A^\tau_{uvf^{q_\tau}}$. On the other hand, by Lemma 4.6.5, we have that $A^\tau_{uf^{q_\tau}} = A^\tau_{g^{q_\tau}}$ and $A^\tau_{uvf^{q_\tau}} = A^\tau_{f^{q_\tau}}$. This yields the desired result. □

Corollary 4.6.7. *Let $(A, \underline{A/I}, \gamma)$ be a p-complete marked* PD*-ring with $\underline{A/I} = (A/I; f)$ and let $g \in A$ be an element which generates (f) modulo I. Write I^τ_e for* $\ker((A/p^e)^\tau_g \to A/(I, p^e))$. *If τ is p-small, then*

$$\Lambda^\tau(A) = \varprojlim_e D_{I^\tau_e}((A/p^e)^\tau_g).$$

5. Main results

5.1. First cohomological results.

Lemma 5.1.1. *If $\mathcal{F}$ is a locally quasi-coherent sheaf of $\mathcal{O}^\tau_{\underline{X}/S}$-modules over* $\mathrm{Cris}^\tau(\underline{X}/S)$, *then for every $(\underline{U}, T, \delta) \in \mathrm{Cris}^\tau(\underline{X}/S)$ with U affine we have that*

$$H^i((\underline{U}, T, \delta), \mathcal{F}) = 0$$

for $i > 0$.

Proof. This is an analogue of [Stacks, Lemma 07JJ]. ☐

Hypothesis 5.1.2. From now on, suppose that $(S_0, S, \gamma) = (\mathrm{Spec}(A/I), \mathrm{Spec}(A), \gamma)$ with A a $\mathbb{Z}_{(p)}$-algebra and let $\underline{X}$ be a p-nilpotent τ-edged scheme over S_0 of the form $\mathrm{Spec}(C; f)$. We further assume τ p-small.

Definition 5.1.3. We write $\mathrm{Cris}^\tau(\underline{C}/A)$ for the category of p-nilpotent τ-edged PD-rings $(B, \underline{C}, \delta)$ over A. Note that $\mathrm{Cris}^\tau(\underline{C}/A)$ admits a natural embedding into $\mathrm{Cris}(C/A)$.

Lemma 5.1.4. *If we denote by $\mathcal{C}^\tau$ the category $\mathrm{Cris}^\tau(\underline{C}/A)$ endowed with the chaotic topology, then for every locally quasi-coherent sheaf of $\mathcal{O}_{\underline{X}/S}$-modules $\mathcal{F}$ over $\mathrm{Cris}^\tau(\underline{X}/S)$, we have*

$$R\Gamma(\mathcal{C}^\tau, \mathcal{F}|_{\mathcal{C}^\tau}) = R\Gamma(\mathrm{Cris}^\tau(\underline{X}/S), \mathcal{F}).$$

Proof. This follows from Lemma 5.1.1 as in [Stacks, Lemma 07JK]. ☐

5.2. **A weakly initial object.**

Definition 5.2.1. We define $\mathrm{Cris}^{\tau,\curlywedge}(\underline{C}/A)$ to be the category of p-complete τ-edged PD-rings $(B, \underline{C}, \delta)$ over A.

Construction 5.2.2. Choose a polynomial A-algebra P which admits a surjection $P \twoheadrightarrow C$ and choose a lift $g \in P$ of $f \in C$. Since C is p-nilpotent, for $e \gg 0$ the quotient $P \twoheadrightarrow C$ factors through $P_e := P/p^e$. We define $J_e := \ker(P_e \to C)$, $D_e := D_{J_e}(P_e)$, and $D := \varprojlim_e (D_e)$. We also write $D_e^\tau := (D_e)_g^\tau$ and $D^\tau := \varprojlim_e D_e^\tau$. The ring D (resp. D^τ) has a natural PD-structure on $J := \ker(D \to C)$ (resp. $J^\tau := \ker(D^\tau \to C)$) that we denote by δ_D (resp. δ_{D^τ}).

Lemma 5.2.3. *The marked p-complete* PD-*ring $(D^\tau, \underline{C}, \delta_{D^\tau})$ is a weakly initial object of* $\mathrm{Cris}^{\tau,\curlywedge}(\underline{C}/A)$.

Proof. We first note that, thanks to Corollary 4.6.7 and the fact that $\underline{C}$ is τ-edged, the PD-ring $(D^\tau, \underline{C}, \delta_{D^\tau})$ is the τ-localisation of $(D, \underline{C}, \delta_D)$. In particular, $(D^\tau, \underline{C}, \delta_{D^\tau})$ is in $\mathrm{Cris}^{\tau,\curlywedge}(\underline{C}/A)$. To prove that it is in addition a weakly final let $(B, \underline{C}, \delta_B)$ be another object in $\mathrm{Cris}^{\tau,\curlywedge}(\underline{C}/A)$. We choose a morphism $P \to B$ lifting $P \to C$. This morphism sends g to some $g_B \in B$ lifting f and for $e \gg 0$ it induces a morphism $P_e \to B/(p^e : g_B^\infty)$. Since $B/(p^e : g_B^\infty), \underline{C}, \delta_B$ is a p-nilpotent τ-edged PD-ring, we also get natural morphisms $D_e^\tau \to B/(p^e : g_B^\infty)$ for $e \gg 0$. Since $B = \varprojlim_e B/(p^e : g_B^\infty)$, we get a morphism $D^\tau \to B$ compatible with the extra structures. This shows the desired result. ☐

Construction 5.2.4. We denote by $(D(\bullet)_e^\tau, \underline{C}, \delta_{D^\tau})$ the Čech nerve of $(D_e^\tau, \underline{C}, \delta_{D^\tau}) \twoheadrightarrow (C, \underline{C}, 0)$ in $\mathrm{Cris}^\tau(\underline{C}/A)$ and we write $D(\bullet)^\tau := \varprojlim_e D(\bullet)_e^\tau$. We also write $T(\bullet)_e^\tau$ for $\mathrm{Spec}(D(\bullet)_e^\tau)$. For $n \geq 0$ and $e \geq 1$, let $E(n)_e$ be the tensor product of $(n+1)$-copies of D_e^τ over A. Write $\alpha_{i,n}$ for $0 \leq i \leq n$ for the inclusion of D_e^τ in $E(n)_e$ corresponding to the i-th factor of $E(n)_e$. For $e \gg 1$, the ring $E(n)_e$ has a natural PD-structure on the ideals $\alpha_{i,n}(J_e^\tau)E(n)_e$, where J_e^τ is the image of J^τ in D_e^τ and there is a natural quotient $E(n)_e \twoheadrightarrow C$. We denote by $K(n)_e$ its kernel.

Lemma 5.2.5. *For every $n \geq 0$ and $e \gg 1$, we have*

$$D(n)_e^\tau = D_{K(n)_e}(E(n)_e)/(p^e : \alpha_{0,n}(g)^\infty).$$

In addition, for every $\alpha_{i,n} \colon D_e^\tau \to D(n)_e^\tau$, the algebra $D(n)_e^\tau$ is isomorphic to a PD-*polynomial ring $D_e^\tau\{x_i\}$ with* PD-*ideal $J_e^\tau D(n)_e^\tau + D_e^\tau\{x_i\}_+$.*

Lemma 5.2.6. *The cosimplicial module $\Omega^1_{E(\bullet)_e/A}$ is homotopic to 0.*

Proof. The module $\Omega^1_{E(n)_e/A}$ decomposes as the direct sum

$$\bigoplus_{\alpha_{i,n}\colon D^\tau_e \to E(n)_e} \Omega^1_{D^\tau_e/A} \otimes_{D^\tau_e,\alpha_{i,n}} E(n)_e.$$

We deduce that $\Omega^1_{E(n)_e/A}$ is homotopic to 0 thanks to the computation in [Bha12, Lem. 2.5]. ☐

Lemma 5.2.7. *For every cosimplicial module $M(\bullet)_e$ over $D(\bullet)^\tau_e$ and every $i > 0$, the cosimplicial module $M(\bullet)_e \otimes_{D(\bullet)^\tau_e} \Omega^i_{D(\bullet)^\tau_e/A}$ is homotopic to zero.*

Proof. Use Lemma 5.2.6 and Lemma 5.2.5. ☐

5.3. **Affine de Rham comparison.**

Construction 5.3.1. Let $\mathcal{F}$ be a τ-edged quasi-crystal. We denote by $M_e(\bullet)$ the cosimplical module

$$\mathcal{F}((\underline{X}, T(\bullet)^\tau_e, \delta_{D(\bullet)^\tau_e}))$$

and by $M(\bullet)$ the limit $\varprojlim_e M_e(\bullet)$. We also write M for $M(0)$ and for every $i, n \geq 0$ we denote by $(M \otimes_{D^\tau} \Omega^i_{D(n)^\tau/A})^\curlywedge$ the projective limit $\varprojlim_e (M_e \otimes_{D^\tau_e} \Omega^i_{D(n)^\tau_e/A})$. If $\mathcal{F}$ is a τ-edged crystal the two projections $\mathrm{pr}_i : T(1)^\tau_e \to T(0)^\tau_e$ with $i = 1, 2$ induce isomorphisms

$$\mathrm{pr}_1^*(M_e) \xrightarrow{\sim} M_e(1) \xleftarrow{\sim} \mathrm{pr}_2^*(M_e),$$

which define a flat topologically quasi-nilpotent connection

$$M \to (M \otimes_{D^\tau} \Omega^1_{D^\tau/A})^\curlywedge.$$

In turn, this gives a de Rham complex $(M \otimes_{D^\tau} \Omega^\bullet_{D^\tau/A})^\curlywedge$.

Proposition 5.3.2. *If $\mathcal{F}$ is a τ-edged quasi-crystal, the complex $M(\bullet)$ computes $R\Gamma(\mathrm{Cris}^\tau(\underline{X}/S), \mathcal{F})$.*

Proof. The result follows from Lemma 5.1.4 and Lemma 5.2.3 as in [Stacks, Prop. 07JN]. ☐

Lemma 5.3.3. *If $\mathcal{F}$ is a τ-edged quasi-crystal, then*

$$H^j(\mathrm{Cris}^\tau(\underline{X}/S), \mathcal{F} \otimes_{\mathcal{O}^\tau_{\underline{X}/S}} \Omega^i_{\underline{X}/S}) = 0$$

for all $i > 0$ and $j \geq 0$.

Proof. [Stacks, Lem. 07LF]. ☐

Theorem 5.3.4. *For every τ-edged crystal $\mathcal{F}$ there exists a quasi-isomorphism*

$$R\Gamma_{\tau\text{-cris}}(\underline{X}/S, \mathcal{F}) \xrightarrow{\sim} (M \otimes_{D^\tau} \Omega^\bullet_{D^\tau/A})^\curlywedge.$$

Proof. The proof is as in [Stacks, Prop. 07LG]. We consider the double complex $K^{\bullet,\bullet}$ defined by

$$K^{a,b} := (M \otimes_{D^\tau} \Omega^a_{D(b)^\tau/A})^\curlywedge.$$

By Lemma 5.2.7, the columns $K^{a,\bullet}$ are acyclic when $a > 0$ and $K^{0,\bullet}$ is quasi-isomorphic to $R\Gamma(\mathrm{Cris}(\underline{X}/S), \mathcal{F})$ thanks to Proposition 5.3.2. Combining Lemma 5.2.5 and [Stacks, Lem. 07LD] we deduce that for every $b \geq 0$ and every morphism $\alpha_{i,b} : D^\tau \to D(b)^\tau$ the morphisms $\alpha_{i,b,*} : (M \otimes_{D^\tau} \Omega^\bullet_{D^\tau/A})^\curlywedge \to (M \otimes_{D(b)^\tau} \Omega^\bullet_{D(b)^\tau/A})^\curlywedge = K^{\bullet,b}$ are quasi-isomorphisms. Note that even in this case, for a fixed b, the morphisms $\alpha_{i,b,*}$ induce the same morphism on cohomology. ☐

5.4. **Comparison with rigid cohomology.** The algebra $A_n := (\mathbb{Z}[t]_t^{\lambda_n})_p^\curlywedge$ is by definition the projective limit

$$\varprojlim_e \frac{\mathbb{Z}[t,\frac{p}{t^n}]}{(\frac{p}{t^n})^e} \subseteq \varprojlim_e \mathbb{Z}/p^e[t,t^{-1}] = \mathbb{Z}_p\langle t,t^{-1}\rangle.$$

Lemma 5.4.1. *If p is odd, the ideal $\frac{p}{t^n}A_{np}$ admits a (unique)* PD*-structure.*

We denote by $\mathbb{Z}_p\langle t,t^{-1}\rangle^-$ the $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p\langle t,t^{-1}\rangle$ of series of the form $\sum_{i=1}^\infty a_i t^{-i}$ with $a_i \in \mathbb{Z}_p$ and by A_n^- the intersection $A_n \cap \mathbb{Z}_p\langle t,t^{-1}\rangle^-$.

Lemma 5.4.2. *Every element of A_n^- can be written uniquely in the form*

$$\sum_{i=1}^\infty b_i(t)\left(\tfrac{p}{t^n}\right)^i$$

where each $b_i(t) \in \mathbb{Z}_p[t]$ is a polynomial of degree at most $n-1$.

Proof. This follows from the analogous result modulo $(\frac{p}{t^n})^e$ for every e. □

Write B_n for $A_n[\frac{1}{p}]$ and C_n for the subalgebra of $\mathbb{Z}_p\langle t,t^{-1}\rangle[\frac{1}{p}]$ of series $\sum_{j=-\infty}^\infty a_j t^j$ such that for j small enough $v_p(a_j) \geq \lceil -\frac{j}{n}\rceil$ (or in other words the series which converge for $p^{-1/n} < |t| \leq 1$).

Lemma 5.4.3. *The algebra C_n coincides with the subset of $f \in \mathbb{Z}_p\langle t,t^{-1}\rangle[\frac{1}{p}]$ which can be written as a sum $f = f_+ + f_-$ with $f_+ \in \mathbb{Z}_p\langle t\rangle[\frac{1}{pt}]$ and $f_- \in A_n^-$.*

Proof. It is enough to prove the result for those $f = \sum_{j=-\infty}^\infty a_j t^j \in C_n$ with $a_j = 0$ for $j \geq 0$. Write e_k for $\lceil \frac{k}{n}\rceil$ and suppose that $v_p(a_{-k}) \geq e_k$ for $m \gg 0$ and $k > mn$, then

$$f = \sum_{k=1}^{mn} a_{-k}t^{-k} + \sum_{k=mn+1}^\infty p^{e_k}a'_{-k}t^{-k}$$

where $a'_{-k} = \frac{a_k}{p^{e_k}} \in \mathbb{Z}_p$. Since

$$\sum_{k=mn+1}^\infty p^{e_k}a'_{-k}t^{-k} = \sum_{i=m+1}^\infty \sum_{k=(i-1)n+1}^{in} p^i a'_{-k}t^{in-k}t^{-in} = \sum_{i=m+1}^\infty b_i(t)\left(\tfrac{p}{t^n}\right)^i$$

with

$$b_i(t) := \sum_{k=(i-1)n+1}^{in} a'_{-k}t^{in-k} \in \mathbb{Z}_p[t]$$

of degree at most $n-1$, we conclude by Lemma 5.4.2. □

Lemma 5.4.4. $B_n = C_n$

Lemma 5.4.5. *There exists a natural isomorphism*

$$\varinjlim_i B_i \xrightarrow{\sim} \mathbb{Q}_p\langle t\rangle(\dagger 0),$$

where $\mathbb{Q}_p\langle t\rangle(\dagger 0) \subseteq \mathbb{Q}_p\langle t,t^{-1}\rangle$ is the subring of series which are overconvergent at 0.

Appendix A. The v-Zariski and v-étale topologies

A.1. Valuations on marked rings.

Definition A.1.1. A *multiplicative valuation* on a ring A is a map $|.|_v : A \to \Gamma \cup \{0\}$, where $(\Gamma, \times)$ is a totally ordered abelian group such that

(1) $|0|_v = 0$ and $|1|_v = 1$.
(2) $|xy|_v = |x|_v|y|_v$ for every $x, y \in A$.
(3) $|x + y|_v \leq \max\{|x|_v, |y|_v\}$ for every $x, y \in A$.

The *kernel* of $|.|_v$, denoted by $\mathfrak{p}_v$, is the preimage of 0. If $|.|_v$ is a valuation of A we write K_v for the fraction field of $A/\mathfrak{p}_v$ and by R_v the valuation subring of K_v of elements $x \in K_v$ such that $|x|_v \leq 1$. We say that two valuations $|.|_v, |.|_w$ are *equivalent*, if $\mathfrak{p}_v = \mathfrak{p}_w$ and $R_v = R_w$. If A has a marking $h_{\underline{A}}$, then it induces a simple marking on every ring R_v.

Definition A.1.2. We write by $\mathrm{Spv}(A)$ the set of equivalence classes of valuations of A and if $\varphi : B \to A$ is a ring homomorphism, we denote by $\mathrm{Spa}(A, B)$ the subset of $\mathrm{Spv}(A)$ of those valuations $|.|_v$ such that $|\varphi(B)|_v \leq 1$. This construction naturally globalise to a morphism $\varphi : X \to Y$ of schemes, and we write $\mathrm{Spa}(X, Y)$ for the set of valuations bounded over Y. If $\underline{X}$ is a marked scheme, then we define $\mathrm{Spa}(\underline{X}) := \mathrm{Spa}(X^\circ, X)$. We also write $\mathrm{Spa}(\underline{A})$ for $\mathrm{Spa}(\mathrm{Spec}(\underline{A}))$.

A.2. The v-Zariski topology.

Definition A.2.1. A *v-Zariski covering* of $\underline{X}$ is the datum of a set of morphisms of marked schemes $\{\varphi_i \colon \underline{Y}_i \to \underline{X}\}_{i\in I}$ such that the following conditions are satisfied.

(1) $\{\varphi_i^\circ \colon Y_i^\circ \to X^\circ\}_{i\in I}$ is a Zariski covering
(2) For every open immersion $\underline{U} \to \underline{X}$ with $\underline{U}$ principal affine, there exists a finite set J, a map $\mathbf{i} \colon J \to I$, and open immersions $\underline{V}_j \to \underline{Y}_{\mathbf{i}(j)}$ with $\underline{V}_j$ principal affine such that $\bigcup_i \varphi_j(\underline{V}_j(\underline{R})) = \underline{U}(\underline{R})$ for every Cartier valuation ring $\underline{R}$.

Note that Condition (2) is the variant of the following condition.

(2') $\bigcup_i \varphi_i(\underline{Y}_i(\underline{R})) = \underline{X}(\underline{R})$ for every Cartier valuation ring $\underline{R}$.

The difference, is an additional finiteness assumption that is simply the analogue of the finiteness assumption for fpqc coverings. Note that Condition (2) is strictly stronger than Condition (2') as explained by the following example.

Example A.2.2. Let R be a DVR with uniformiser π, fraction field K, and residue field k. Consider the ring $A := \prod_{n=0}^{\infty} R$ and write $\pi^{(n)} \in A$ for the element which is π on the n-th entry and 1 otherwise, $\pi \in A$ for the image of $\pi \in R$ via the diagonal embedding $R \to A$, $S_i \subseteq A$ for the multiplicative subset of A generated by $\pi^{(i)}, \pi^{(i+1)}, \cdots$, and $B_i := S_i^{-1}A$. Write $\underline{X} := \mathrm{Spec}(A; \pi)$ and $\underline{Y}_i := \mathrm{Spec}(B_i; \pi)$. Since $A_\pi = (B_i)_\pi$ for every $i \geq 0$, we have that the morphisms $Y_i^\circ \to X^\circ$ are all isomorphisms, so that (1) is satisfied. Note also that this implies that $\underline{Y}_i(\underline{K}) = \underline{X}(\underline{K})$ for every $i \geq 0$, where $\underline{K} := (K; 1)$. We want to show now that the family satisfies (2') as well but does not satisfy (2).

We have that both A and B_i have Krull dimension 1 and the minimal prime ideals of A are in bijection with the set of ultrafilters of $\mathbb{N}$. If $\mathfrak{p}$ is a minimal prime ideal of A associated to a non-principal ultrafilter, then $A/\mathfrak{p} \to B_i/\mathfrak{p}B_i$ is an isomorphism because $A/(e_j)_{j\geq 0} = B_i/(e_j)_{j\geq 0}$. On the other hand, if $\mathfrak{p}_j$ is the kernel of the j-th projection, then $A/\mathfrak{p}_j \to B_i/\mathfrak{p}_j B_i$ is an isomorphism for $j < i$ and the embedding $R \hookrightarrow K$ for $j \geq i$. Thanks to this observations, we deduce that if $\underline{R} := (R;\pi)$, then $\bigcup_{0\leq i} \underline{Y}_i(\underline{R}) = \underline{X}(\underline{R})$, but for each $M \geq 0$, we have that $\bigcup_{0\leq i\leq M} \underline{Y}_i(\underline{R}) \subsetneq \underline{X}(\underline{R})$.

Remark A.2.3. For Noetherian schemes the v-Zariski topology can be compared with the $\underline{\mathrm{M}}$Zar topology in [KM21].

Lemma A.2.4. *If $\{\varphi_i \colon \underline{Y}_i \to \underline{X}\}_{i\in I}$ is a finite family of morphisms of principal marked schemes such that Condition (1) and (2') are satisfied, then it is a v-Zariski covering.*

Definition A.2.5. A *big v-Zariski site*, denoted by $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$, is any site defined as in [Stacks, Def. 020S]. We also denote by $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ the localisation of $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$ with respect to an $\underline{S} \in \underline{\mathrm{Sch}}$ and by $(\underline{S})_{\mathrm{vZar}}$ the small v-Zariski site of $\underline{S}$.

Lemma A.2.6. *Every Zariski covering $\{\underline{U}_i \to \underline{X}\}_{i\in I}$ is a v-Zariski covering.*

Proof. We may assume that X is affine, so that there exists a subcovering $\{\underline{U}_j \to \underline{X}\}_{i\in J}$ with $J \subseteq I$ finite. After this reduction the result follows from the fact that if $\underline{R}$ is a Cartier valuation ring, every R-point of X defines an R-point of U_j for some J. □

Lemma A.2.7. *Every principal affine marked scheme admits a v-Zariski covering of principal affine simply marked scheme.*

Proof. Let $\underline{A} = (A; f_\ell)_{\ell\in L}$ be a principal marked ring with $L = \{1,\dots,n\}$. We consider the subring $B_1 \subseteq A_{f_n}$ (resp. $B_2 \subseteq A_{f_n}$) generated by the image of A and $\frac{f_{n-1}}{f_n}$ (resp. $\frac{f_n}{f_{n-1}}$) endowed with the marking $\{f_\ell\}_{\ell\in(L\setminus\{n\})}$ (resp. $\{f_\ell\}_{\ell\in(L\setminus\{n-1\})}$). We have that $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{i\in\{1,2\}}$ is a v-Zariski covering of $\mathrm{Spec}(\underline{A})$. The result then follows by induction on n. □

Lemma A.2.8. *If $(A; I_A)$ is a simply marked ring and I_A is generated by a set $\{f_\ell\}_{\ell\in L} \subseteq I_A$, then $\{\mathrm{Spec}(A; f_\ell) \to \mathrm{Spec}(A; I_A)\}_{\ell\in L}$ is a v-Zariski covering.*

Proof. To prove that $\{\mathrm{Spec}(A; f_\ell)^\circ \to \mathrm{Spec}(A; I_A)^\circ\}_{\ell\in L}$ is a Zariski covering it is enough to note that, by the assumption, for every prime ideal $\mathfrak{p}$ which does not contain I_A there exists an f_i which is not in $\mathfrak{p}$. By Lemma A.2.7, it is enough to prove the result after base change to $\varphi : (A; I_A) \to (B; g)$. Thus we have to show that $\{\mathrm{Spec}(B; \varphi(f_\ell), g) \to \mathrm{Spec}(B; g)\}_{\ell\in L}$ is a v-Zariski covering. For this purpose, we note that there exists a finite subset $L' \subseteq L$ such $g \in (\varphi(f_\ell))_{\ell\in L'}$. Since for every valuation $|.|_v \in \mathrm{Spa}(B; g)$, there exists $\ell \in L'$ such that $|g|_v \leq |f_\ell|_v$, we deduce that $\{\mathrm{Spec}(B; \varphi(f_\ell), g) \to \mathrm{Spec}(B; g)\}_{\ell\in L'}$ satisfies Condition (2). This concludes the proof. □

Lemma A.2.9. *Every marked scheme admits a v-Zariski covering of principal affine simply marked schemes.*

Proof. Thanks to Lemma A.2.6 we are reduced to the case of affine marked schemes. Since every affine marked scheme $\mathrm{Spec}(A; I_{A,1},\dots,I_{A,n})$ is the fibre product

$$\mathrm{Spec}(A; I_{A,1}) \times_{\mathrm{Spec}(A)} \cdots \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A; I_{A,n}),$$

it is enough to work with affine simply marked schemes. The result then follows from Lemma A.2.8. □

Lemma A.2.10. *Let $\underline{S}$ be a marked scheme with* $\mathrm{size}(S) = \kappa$. *For every v-Zariski covering* $\{\underline{U}_i \to \underline{S}\}_{i\in I}$ *there exists a refinement* $\{\underline{T}_j \to \underline{S}\}_{j\in J}$ *with* $\mathrm{size}(T_j) \leq \kappa$ *and* $|J| \leq \kappa$.

Definition A.2.11. A *big v-Zariski site*, denoted by $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$, is any site defined as in [Stacks, Def. 020S] using v-Zariski coverings. We also denote by $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ the localisation of $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$ with respect to a marked scheme $\underline{S} \in \underline{\mathrm{Sch}}$ and by $(\underline{S})_{\mathrm{vZar}}$ the subcategory of $\star$-open immersions $\underline{T} \to \underline{S}$.

A.3. **Quasi-compactness.**

Definition A.3.1. We say that a marked scheme is *v-quasi-compact* if every v-Zariski covering admits a finite subcovering. We also say that a marked scheme $\underline{X}$ is *retrocompact* if $X^\circ \to X$ is quasi-compact.

Lemma A.3.2. *A marked scheme $\underline{X}$ is retrocompact if and only if for every $x \in |X|$, there exists an affine open neighbourhood* $\mathrm{Spec}(A; I_{A,\ell})_{\ell\in L} \subseteq \underline{X}$ *and a finitely generated ideal J of A such that*

$$\sqrt{\prod_\ell I_{A,\ell}} = \sqrt{J}.$$

Lemma A.3.3. *A marked scheme is v-quasi-compact if and only if it admits a finite v-Zariski covering of v-refined principal marked schemes. In particular, affine marked schemes* $\mathrm{Spec}(A; I_{A,\ell})_{\ell\in L}$ *with each $I_{A,\ell}$ finitely generated are v-quasi-compact.*

Definition A.3.4. A morphism of schemes $\underline{Y} \to \underline{X}$ is *v-quasi-compact* if the preimage of a v-quasi-compact open is v-quasi-compact. We also say that a morphism $\underline{Y} \to \underline{X}$ is *v-separated* if the diagonal $\underline{Y} \to \underline{Y} \times_X \underline{Y}$ is v-quasi-compact. We denote by $\underline{\mathrm{Sch}}^{\mathrm{vqcqs}}$ the category of v-quasi-compact v-quasi-separated marked schemes.

A.4. **Structural sheaf and Serre vanishing.** While the presheaf $\mathcal{O}^\circ$ of $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ which sends $\underline{X} \mapsto \mathcal{O}(X^\circ)$ is clearly a sheaf, the presheaf $\mathcal{O}$ which sends $\underline{X} \mapsto \mathcal{O}(X)$ is not a sheaf. This can be seen, for example, noticing that $\emptyset$ covers every marked scheme of the form $(X; 0)$. We denote by $\mathcal{O}^+$ the sheafification of $\mathcal{O}$ with respect to the v-Zariski topology. To understand $\mathcal{O}^+$ it is useful to consider the presheaf $\tilde{\mathcal{O}} := \mathrm{im}(\mathcal{O} \to \mathcal{O}^\circ)$. Since $\mathcal{O}(\underline{X}) \to \mathcal{O}^\circ(\underline{X})$ is injective for Cartier schemes, Lemma A.4.5 implies that the v-Zariski sheafification of $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are the same. In addition, $\tilde{\mathcal{O}}$ is separated since $\mathcal{O}^\circ$ is a sheaf.

Definition A.4.1. The *v-refinement* of $\underline{A}$ is the marked ring $\underline{A}^+$ with A^+ the integral closure of $A \to A^\circ$ and the marking is induced by the marking of $\underline{A}$.

Definition A.4.2. We say that a morphism of marked scheme $\underline{Y} \to \underline{X}$ satisfies the *existence part* of the *marked valuative criterion*, if for every Cartier valuation ring $\underline{R}$ with fraction field K and every solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \underline{Y} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(\underline{R}) & \longrightarrow & \underline{X}, \end{array}$$

the dotted arrow exists.

Lemma A.4.3. *If $\underline{A}^+$ is the v-refinement of $\underline{A}$, then the square*

$$\begin{array}{ccc} A^+ & \longrightarrow & \prod_{v\in \mathrm{Spa}(\underline{A})} R_v \\ \downarrow & & \downarrow \\ A_f & \longrightarrow & \prod_{v\in \mathrm{Spa}(\underline{A})} K_v \end{array}$$

is cartesian.

Proof. Write D for the fibre product of A_f and $B := \prod_{v\in \mathrm{Spa}(\underline{A})} R_v$ over $C := \prod_{v\in \mathrm{Spa}(\underline{A})} K_v$. Since valuation rings are integrally closed, there is a natural morphism $A^+ \to D$. In addition, since $B \to C$ is injective, we deduce that $D \subseteq A_f$. It remains to prove that $A \to D$ is integral. Note that we may assume that f is a nonzerodivisor by replacing A by $A/(0 : f^\infty)$.

We follow the proof of Tag 01WM and Tag 01KE of [Stacks]. For an element $g \in D$ we write $J \subseteq A[t]$ for the kernel of the morphism $A[t] \to D_g$ which sends t to g^{-1}. We have that g is integral over A if and only if $1 \in J + (t)$. In turn, to check the last condition it is enough to prove that $\varphi : \mathrm{Spec}(D_g) \to \mathrm{Spec}(A[t]/J)$ is surjective. Note that φ is an isomorphism outside $V(f)$ so that every prime $\mathfrak{p} \subseteq A[t]/J$ which does not contain f is in the set-theoretic image of φ. In particular, since f is a nonzerodivisor in $A[t]/J$, every minimal prime of $A[t]/J$ is in the image. It remains to show that if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals in $A[t]/J$ with $f \notin \mathfrak{p}$ and $\mathfrak{p}$ is in the image of φ, then the same is true for $\mathfrak{q}$. Arguing as in Tag 01KE of [Stacks], this follows from the fact that $\mathrm{Spec}(D_g; f) \to \mathrm{Spec}(A[t]/J; f)$ satisfies the existence part of the marked valuative criterion. □

Lemma A.4.4. $\mathrm{Spec}(\underline{A}^+) \to \mathrm{Spec}(\underline{A})$ *is a v-Zariski covering.*

Proof. The natural map $\mathrm{Spa}(\underline{A}^+) \to \mathrm{Spa}(\underline{A})$ is a bijection and the valuation ring associated to a multiplicative valuation $v \in \mathrm{Spa}(\underline{A}^+)$ is canonically isomorphic to the one associated to the image in $\mathrm{Spa}(\underline{A})$. □

Lemma A.4.5. *Every marked scheme admits a v-Zariski covering of principal v-refined affine simply marked schemes.*

Proof. By Lemma A.2.9 every marked scheme admits a v-Zariski covering of principal affine marked schemes. By Lemma A.4.4 every principal affine marked scheme admits a v-refined v-Zariski covering. This concludes the proof. □

Definition A.4.6. Let $\underline{X} = \mathrm{Spec}(\underline{A})$ be an affine marked scheme. We say that $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{1\leq i\leq n}$ is a *standard v-Zariski covering* if all the $\underline{B}_i$ are principal v-refined marked rings.

Lemma A.4.7. *Let* $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{1\leq i\leq n}$ *be a v-Zariski covering of affine principal marked schemes. The sequence*

$$0 \to A^+ \to \prod_i (B_i)^+ \to \prod_{i,j} (B_i \otimes_A B_j)^+$$

is exact.

Lemma A.4.8. *If* $\underline{X} = \mathrm{Spec}(\underline{A})$ *is a principal v-refined affine marked scheme, then* $\mathcal{O}^+(\underline{X}) = A$.

Proof. By the previous discussion, to compute $\mathcal{O}^+$ it is enough to sheafify $\tilde{\mathcal{O}}$. Since $\tilde{\mathcal{O}}$ is separated, we deduce that

$$\mathcal{O}^+(\underline{X}) = \varinjlim_{\underline{\mathcal{U}}} \check{H}^0(\underline{\mathcal{U}}, \tilde{\mathcal{O}})$$

where the colimit runs over the v-Zariski coverings of $\underline{X}$. Since $\underline{X}$ is v-quasi-compact, by Lemma A.4.5 it is enough to prove that for every v-refined covering $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{1\leq i\leq n}$, the sequence

$$0 \to A \to \prod_i \tilde{\mathcal{O}}(\mathrm{Spec}(\underline{B}_i)) \to \prod_{i,j} \tilde{\mathcal{O}}(\mathrm{Spec}(\underline{B}_i \otimes_A \underline{B}_j))$$

is exact. For every i choose a slicing element g_i of $\underline{B}_i$. Since each g_i is a nonzerodivisor, we have that $\tilde{\mathcal{O}}(\mathrm{Spec}(\underline{B}_i)) = B_i$ for every i. In addition, we have by definition that $\tilde{\mathcal{O}}(\mathrm{Spec}(\underline{B}_i \otimes_A \underline{B}_j)) \subseteq (B_i \otimes_A B_j)_{g_i\otimes g_j}$, so that $\tilde{\mathcal{O}}(\mathrm{Spec}(\underline{B}_i \otimes_A \underline{B}_j)) \subseteq (\underline{B}_i \otimes_A \underline{B}_j)^+$. Combining this with Lemma A.4.7 we deduce the desired result. □

Lemma A.4.9. *If $f : \underline{Y} \to \underline{X}$ be a separated morphism of marked schemes such that $Y^\circ \to X^\circ$ is an isomorphism, then for every $n \geq 1$, the diagonal closed immersion $\underline{Y} \hookrightarrow \underline{Y} \times_X \cdots \times_X \underline{Y}$ into the n-fold fibre product over X is a v-Zariski covering. In addition, if $\underline{Y} \to \underline{X}$ is a v-Zariski covering and $\mathcal{F}$ is a v-Zariski sheaf on $\underline{X}$, then $R\Gamma_{\mathrm{vZar}}(\underline{X}, \mathcal{F}) = R\Gamma_{\mathrm{vZar}}(\underline{Y}, f^*\mathcal{F})$.*

Corollary A.4.10. *If $\underline{Y} \to \underline{X}$ and $\{\underline{U}_i \to \underline{Y}\}_{i\in I}$ are v-Zariski covering of vqcqs marked schemes and $\underline{Y} \to \underline{X}$ is separated, then the v-refined hypercovering associated to $\{\underline{U}_i \to \underline{Y}\}_{i\in I}$ is canonically isomorphic to the one of $\{\underline{U}_i \to \underline{X}\}_{i\in I}$.*

Theorem A.4.11. *Let $\underline{X}$ be an affine principal marked scheme such that X is smooth over a field k. The cohomology groups $H^i_{\mathrm{vZar}}(\underline{X}, \mathcal{O}^+)$ vanish for $i > 0$.*

Proof. First note that if $\underline{Y} \to \underline{X}$ is a modification, $\mathcal{U} = \{\underline{U}_i \to \underline{Y}\}_{i\in I}$ is a covering of $\underline{Y}$, and $\mathcal{U}_X = \{\underline{U}_i \to \underline{X}\}_{i\in I}$ is the induced covering of $\underline{X}$, then by Lemma A.4.9 we have that $\check{H}^\bullet(\mathcal{U}, \mathcal{O}^+) = \check{H}^\bullet(\mathcal{U}_X, \mathcal{O}^+)$. Therefore, looking Zariski locally on X and after taking macaulayfication, it is enough to show that for every projective modification $\underline{Y} \to \underline{X}$ with Y Cohen–Macaulay and every Zariski covering $\mathcal{U} = \{\underline{U}_\ell \to \underline{Y}\}_{\ell\in L}$ and every class in $\check{H}^\bullet_{\mathrm{vZar}}(\mathcal{U}, \mathcal{O}^+)$, there exists a Zariski refinement $\mathcal{V} = \{\underline{V}_m \to \underline{Y}\}_{m\in M}$ which kills the class. This follows from Kovacs' vanishing. □

A.5. **Functions with bounded poles.**

Definition A.5.1. Let $(R; I_R)$ be a marked valuation ring. We write R°_n for the subgroup of R° of elements $q \in R^\circ$ such that $qI_R^n \subseteq R$. For a marked ring $\underline{A}$ we write A°_n for the subgroup of A° of elements such that for every $v \in \mathrm{Spa}(\underline{A})$ the image in R°_v is contained in $(R_v)^\circ_n$. The assignment $\underline{A} \mapsto A^\circ_n$ globalises to a v-Zariski sheaf $\mathcal{O}^\circ_n$ over $\underline{\mathrm{Sch}}_{\mathrm{vZar}}$.

A.6. **Comparison with the Zariski cohomology.**

If $\underline{S}$ is any marked scheme, we have a functor

$$v\colon (\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}} \to (\mathrm{Sch}/S)_{\mathrm{Zar}}$$

which forgets the marking. This functor admits as a right adjoint the functor

$$u_{1/\underline{S}}\colon (\mathrm{Sch}/S)_{\mathrm{Zar}} \to (\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$$

which sends $X \to S$ to $\underline{S} \times_{v_1(S)} u_1(X) \to \underline{S}$. The counit of the adjunction is an isomorphism.

Lemma A.6.1. *The functor v is cocontinuous and $u_{1/\underline{S}}$ is continuous.*

Proof. If $\{U_i \to v(\underline{X})\}$ is a covering in $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, then $\{u_{1,\underline{X}}(U_i) \to \underline{X}\}$ is a covering in $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ by Lemma A.2.6. We deduce that v is cocontinuous. □

We write

$$\alpha\colon \mathrm{Sh}((\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}) \to \mathrm{Sh}((\mathrm{Sch}/S)_{\mathrm{Zar}})$$

for the map of topoi induced by v (viewed as a cocontinuous functor). We have that $\alpha_*(\mathcal{F})(X) = \mathcal{F}(u_{1/\underline{S}}(X))$ and $\alpha^{-1}(\mathcal{G})(\underline{X}) = \mathcal{G}(X)$ for every $\underline{X} \to \underline{S}$. Moreover, the counit $\alpha^{-1}\alpha_* \to \mathrm{id}$ is given by

$$\alpha^{-1}(\alpha_*\mathcal{F})(\underline{X}) = \alpha_*\mathcal{F}(X) = \mathcal{F}(u_{1/\underline{S}}(X)) \to \mathcal{F}(\underline{X})$$

and the unit $\mathrm{id} \to \alpha_*\alpha^{-1}$ is the identity. By [Stacks, Lem. 09YX], α^{-1} admits also a left adjoint $\alpha_!$ such that $\alpha_! h_{\underline{X}} = h_X$.

Lemma A.6.2. *If $\underline{X}$ is a retrocompact marked scheme, then $\alpha_*\mathcal{O}^+_{\underline{X}}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Definition A.6.3. If $\underline{X}$ is a retrocompact marked scheme we denote by $\underline{X}^+$ the marked scheme with underlying scheme $X^+ := \mathrm{Spec}_X(\alpha_*\mathcal{O}_{\underline{X}})$ and marking induced by the one of $\underline{X}$ via the natural morphism $X^+ \to X$. We say that $\underline{X}$ is *v-refined* if $X^+ \to X$ is an isomorphism.

Lemma A.6.4. *If $\underline{X}$ is retrocompact we have that* $\mathrm{Sh}((\underline{X}^+)_{\mathrm{vZar}}) = \mathrm{Sh}(\underline{X}_{\mathrm{vZar}})$.

Lemma A.6.5. *For a retrocompact marked scheme $\underline{X}$ we have that the natural square*

$$\begin{array}{ccc} (X_{\mathrm{red}})^+ & \longrightarrow & X^+ \\ \downarrow & & \downarrow \\ X_{\mathrm{red}} & \longrightarrow & X. \end{array}$$

is cartesian.

A.7. **Stalks.** If $\underline{R}$ is a Cartier valuation ring, for every $\underline{x} \in \underline{X}(\underline{R})$ one associates the functor $p_{\underline{x}}\colon \underline{X}_{\mathrm{vZar}} \to \mathrm{Set}$ which sends $\underline{U} \to \underline{X}$ to the set of $\underline{R}$-points over $\underline{x}$. This defines a point of the site $\underline{X}_{\mathrm{vZar}}$ (cf. [Stacks, Tag 00Y5]).

Definition A.7.1. If $\mathcal{F}$ is a sheaf of $\underline{X}_{\mathrm{vZar}}$, the stalk of $\mathcal{F}$ at $\underline{x}$, denoted by $\mathcal{F}_{\underline{x}}$, is the inverse limit

$$\varinjlim_{\underline{T} \to \underline{U} \to \underline{X}} \mathcal{F}(\underline{U}),$$

where $\underline{T} := \mathrm{Spec}(\underline{R})$ and the composition $\underline{T} \to \underline{U} \to \underline{X}$ is $\underline{x}$.

The following lemma is related to [KM21, Prop. 4.25].

Lemma A.7.2. *The family of points of $\underline{X}_{\mathrm{vZar}}$ associated to trivially marked local rings and Cartier valuation rings is a conservative family.*

Definition A.7.3. We say that a morphism of marked schemes $f\colon \underline{Y} \to \underline{X}$ is a modification if $f\colon Y \to X$ is proper and finitely presented and $f\colon Y^\circ \to X^\circ$ is an isomorphism.

Proposition A.7.4. *Let $f\colon \underline{Y} \to \underline{X}$ be a strict modification. The v-Zariski higher direct image $Rf_*\mathcal{O}^+_{\underline{Y}}$ is quasi-isomorphic to $\mathcal{O}^+_{\underline{X}}$.*

Proof. Stein's factorisation [Stacks, Thm. 03H2] to show $f_*\mathcal{O}^+_{\underline{Y}} = \mathcal{O}^+_{\underline{X}}$. □

A.8. The v-étale topology.

Definition A.8.1. Let $\underline{X}$ be a principal affine marked scheme. A *standard v-étale covering* of $\underline{X}$ is the datum of a finite set of morphisms of principal affine marked schemes $\{\underline{Y}_i \to \underline{X}\}_{i\in I}$ such that the following conditions are satisfied.

(1) $\{Y_i^\circ \to X^\circ\}_{i\in I}$ is an étale covering.
(2) For every Cartier valuation ring $\underline{R}$ and every $T \in \underline{X}(\underline{R})$ there exists a strict extension $\underline{R} \subseteq \underline{R}'$, an $i \in I$, and $T' \in \underline{Y}_i(\underline{R}')$ lifting T.

The *v-étale topology* is the topology generated by the v-Zariski topology and the standard v-étale coverings.

Let k is a field of positive characteristic p, let $\underline{A} \subseteq \underline{B}$ be the strict extension of marked rings $(k[x];x) \subseteq (k[x,y]/(y^p - x^{p-1}y - x^{p-1});x)$. Consider the induced morphism $f : \underline{Y} \to \underline{X}$ where $\underline{X} := \mathrm{Spec}(\underline{A})$ and $\underline{Y} := \mathrm{Spec}(\underline{B})$. Note that f is a v-étale covering.

Lemma A.8.2. *The sheaf $\mathcal{O}^+$ does not satisfy v-Zariski cohomological descent with respect to f.*

Proof. Let G be the Galois group of f seen as a constant group over k and let $\underline{Y}_\bullet$ be the Čech nerve of $\underline{Y} \to \underline{X}$. We have that $\Gamma(\underline{Y}_\bullet, \mathcal{O}^+) \simeq \mathrm{Hom}_G(\mathbb{Z}[G^{\bullet+1}], B)$ where $\mathbb{Z}[G^{\bullet+1}]$ is the simplicial group associated to the bar resolution of G and B is endowed with the natural G-action. This implies that the spectral sequence $E_2^{i,j} := H^i(G, H^j_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+))$ converges to $H^{i+j}_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+)$, where the action of G on $H^j_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+)$ is the one induced by the action on $\underline{Y}$. We have that $H^0_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+) = B$ and $H^i_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+) = 0$ for $i > 0$, so that $H^i_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+) = H^i(G, B)$. Since $G = \mathbb{Z}/p$ as abstract groups, if $\sigma \in G$ is the automorphism which sends $y \mapsto y + x$ and $\mathrm{Tr} : B \to B$ is the endomorphism $1 + \sigma + \cdots + \sigma^{p-1}$, then $H^{2i+1}_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+) = B^{\mathrm{Tr}=0}/(\sigma - 1)B$. The unit $1 \in B$ has trivial trace, but it is not in the image of $\sigma - 1$, as one can check after reducing modulo x. We deduce that $H^{2i+1}_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+) \neq 0$ for every $i \geq 0$. $\square$

Similarly one can prove the following result.

Proposition A.8.3. *Let R be a rank 1 valuation ring with fraction field K and let R^{sep} be the integral closure of R in a separable closure of K. For every nonzero $f \in R$, we have that*

$$H^\bullet_{\mathrm{v\acute{e}t}}(\mathrm{Spec}(R;f), \mathcal{O}^+) = H^\bullet(K, R^{\mathrm{sep}})$$

where $H^\bullet(K, R^{\mathrm{sep}})$ is Galois cohomology.

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