Monodromy conjecture and proof of Veys' conjecture

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Notation

Today F will be a number field, Ω_F^{fin} the set of finite places and $\mathfrak{P}$ an element of Ω_F^{fin}. We will consider, as usual, a morphism $f : \mathbb{A}_F^d \to \mathbb{A}_F^1$ and we will study $X_0 := f^{-1}(0)$. We denote by $|X_0|$ the set of closed points of X_0. We will suppose fixed a log-resolution $h : (Y, E) \to (\mathbb{A}_F^d, X_0)$.

We recall the notation we are using in our seminar:

- $E = \bigcup_{i \in I} E_i$, with each E_i irreducible;
- N_i is the multiplicity of $f \circ h$ along E_i;
- $\nu_i - 1$ is the multiplicity of Jac_h, the jacobian ideal of h, along E_i;
- $\mathrm{lct}_x := \min\{\nu_i/N_i \mid x \in h(E_i)\}$ and $\mathrm{lct} := \min_{x \in |X_0|} \mathrm{lct}_x$;
- For every $\emptyset \neq J \subseteq I$, we have $E_J := \bigcap_{j \in J} E_j$ and $\mathring{E}_J = E_J \setminus \bigcup_{j \notin J} E_j$.

1 Analytic results

1.1 Bernstein polynomials

We start recalling the definition of the analytic zeta function attached to f. For every $\psi \in C_0^\infty(\mathbb{C}^d, \mathbb{R})$ we consider

$$Z_\psi^\infty(s) := \int_{\mathbb{C}^d} |f(z)|^{2s} \psi(z) \mathrm{d}z \mathrm{d}\overline{z}$$

defined for every $s \in \mathbb{C}$ such that $\Re s > 0$.

Theorem 1.1 (Bernstein). *$Z_\psi^\infty(s)$ admits a meromorphic continuition to $\mathbb{C}$. The poles are negative rational numbers.*

We will see how this zeta function encodes informations of the singularities of X_0. Before doing this we present an important tool for the study of this function, which is used, for example, to prove the previous theorem.

Let $\mathcal{O} := \mathbb{C}[z_1, \dots, z_n]$. For $f \in \mathcal{O}$, we consider $\mathcal{O}[s, f^{-1}]f^s$, a rank 1 free $\mathcal{O}[s, f^{-1}]$-module with signpost f^s. Let $D := \mathbb{C}[z_1, \dots, z_n, \partial_{z_1}, \dots, \partial_{z_n}]$ and $D[s]$ the ring of polynomials in the variable s with coefficients in D. We put a $D[s]$-action on $\mathcal{O}[s, f^{-1}]f^s$ by setting $\partial_{z_i}(gf^s) := (\partial_{z_i} g + g\frac{s\partial_{z_i} f}{f})f^s$ for every $g \in \mathcal{O}[s, f^{-1}]$. We denote by $D[s]f^s$ the sub-$D[s]$-module generated by f^s and $D[s]f^{s+1}$ the one generated by ff^s.

Definition 1.2 (Bernstein polynomial)**.** We define $b_f(s)$ as the minimal polynomial of the endomorphism of the $D[s]$-module $D[s]f^s/D[s]f^{s+1}$ given by the multiplication by s on the left. Equivalently $b_f(s)$ is the monic polynomial of minimal degree, such that there exists a differential operator $P \in D[s]$, satisfying $b_f(s)f^s = Pf^{s+1}$. We call such a polynomial, the *Bernstein polynomial* of f.

Theorem 1.3 (Bernstein)**.** *Every $f \in \mathcal{O}$ admits a Bernstein polynomial $b_f(s)$.*

You can verifying by setting $s = -1$ that $b_f(-1) = 0$. In general, Kashiwara has proven that all the roots of $b_f(s)$ are rational numbers.

Example 1.4. If $f = z_1^{N_1} z_2^{N_2}$, with $N_1, N_2 \in \mathbb{N}$ it is easy to show that

$$b_f(s) = \prod_{i=1}^{2} \prod_{j=0}^{N_i-1} \left(s + 1 - \frac{j}{N_i} \right)$$

and it satisfies the differential equation

$$b_f(s)f^s = \frac{1}{N_1^{N_1} N_2^{N_2}} \partial_{z_1}^{N_1} \partial_{z_2}^{N_2} f^{s+1}.$$

Remark 1.5. The computation of Bernstein polynomials in general is instead very difficult. Toshinori Oaku found an algorithm [Oak97] which computes $b_f(s)$ for every f using an analogue of Grobner basis for differential operators.

The relation between Bernstein polynomials and the analytic zeta function is explained in the following result.

Proposition 1.6. *For every $\psi \in C_c^\infty(\mathbb{C}^n, \mathbb{R})$ and $m \in \mathbb{N}$, if s_0 is a pole of $Z_\psi^\infty(s)$ with $\mathrm{Re}(s_0) \geq -m$, then $s_0 + j$ is a root of $b_f(s)$ for some integer $0 \leq j \leq m$.*

Proof. The proof proceed by induction on m. If $m = 0$ it holds emptily because the analytic zeta function has no poles in the half-plane $\mathrm{Re}(s) \geq 0$.

For the inductive step we use the Berstein polynomial of f. We know here exists $P \in D[s]$ such that

$$b_f(s)f^s = Pf^{s+1}.$$

Applying the conjugation we also get

$$\overline{b_f(s)}\overline{f}^s = \overline{P}\overline{f}^{s+1}.$$

Hence

$$|b_f(s)|^2 Z_\psi^\infty(s) = |b_f(s)|^2 \int_{\mathbb{C}^n} |f(z)|^{2s} \psi(z)\, \mathrm{d}z\mathrm{d}\overline{z} = \int_{\mathbb{C}^n} P\overline{P} \left(|f(z)|^{2(s+1)} \right) \psi(z)\, \mathrm{d}z\mathrm{d}\overline{z}.$$

Thanks to the partial integration formula [1] the RHS is equal to $Z_{P\overline{P}(\psi)}^\infty(s)$, thus the partial differential equation defining $b_f(s)$ translates to

$$|b_f(s)|^2 Z_\psi^\infty(s) = Z_{P\overline{P}(\psi)}^\infty(s+1).$$

If s_0 is a pole of $Z_\psi^\infty(s)$ which is not a root of $b_f(s)$, then $s_0 + 1$ is a pole of $Z_{P\overline{P}(\psi)}^\infty(s)$. Hence we can use the inductive hypothesis on $Z_{P\overline{P}(\psi)}^\infty(s)$ getting the final result. ☐

[1] Here we are strongly using the fact we are working with analytic zeta functions. Indeed this formula has no analogue for $\mathfrak{P}$-adic and motivic zeta functions.

Proposition 1.7. $\mathrm{lct}_x = \sup\{s|\ |f|^{-2s}$ *integrable around* $x\}$

Proof. Exercise: Take a log-resolution $h : (Y, E) \to (\mathbb{A}^d, X_0)$, then use the change of variables formula. □

Corollary 1.8. *For every* $x \in X_0$, $-\mathrm{lct}_x$ *is a zero of* b_f.

Proof. Exercise: The reasoning is analogue to the proof of Proposition 1.6. □

1.2 Monodromy

Milnor showed that if $f : \mathbb{C}^d \to \mathbb{C}$ is an algebraic morphism then for every $x \in \mathbb{C}^d$ such that $f(x) = 0$, there exists a ball $B \subseteq \mathbb{C}^d$ centered at x and a punctured ball $A \subseteq \mathbb{C} \setminus \{0\}$ centered at 0 such that $A \subseteq f(B)$ and $f|_B$ is a locally trivial C^∞-fibration over A with fiber $F_x := f^{-1}(t) \cap B$ where t is a certain point in A. If we choose a generator of the topological fundamental group of A, it induces an endomorphism T_x on $\bigoplus_{i=0}^{2d} H^i_{sing}(F_x, \mathbb{Z})$. The eigenvalues of T_x are called the *monodromy eigenvalues* at x.

Theorem 1.9 (Malgrange [Mal83], Barlet [Bar84])**.** *For every* $\alpha \in \mathbb{R}$, *the class* $[\alpha] \in \mathbb{R}/\mathbb{Z}$ *is represented by a root of the Bernstein polynomial if and only if* $\exp(2\pi i\alpha)$ *is a monodromy eigenvalue for a certain* $x \in X_0$.

Hence as a consequence we obtain the main result of this section.

Theorem 1.10. *If for some* $\psi \in C_c^\infty(\mathbb{C}^n, \mathbb{R})$, *a complex number* s_0 *is a pole of* $Z_\psi^\infty(s_0)$, *then* $\exp(2\pi i s_0)$ *is a monodromy eigenvalue for some* $x \in X_0$.

2 $\mathfrak{P}$-dic monodromy conjecture

We now switch to the $\mathfrak{P}$-adic zeta function defined in Tanya's talk. For simplicity we will only work with

$$Z^{\mathfrak{P}}(s) := \int_{\mathcal{O}_{\mathfrak{P}}} |f|^s_{\mathfrak{P}} \mathrm{d}x.$$

We have seen the following theorem due to Igusa.

Theorem 2.1 (Igusa)**.** $Z^{\mathfrak{P}}(s)$ *is rational in the variable* $t = q^{-s}$. *If* s_0 *is a pole of* $Z^{\mathfrak{P}}(s)$, *then*

$$s_0 \in -\frac{\nu_i}{N_i} + \frac{2\pi i}{\ln q}\mathbb{Z}$$

for some $i \in I$.

In general it is not the case that for every $i \in I$ there exists a pole of $Z^{\mathfrak{P}}(s)$ with real part equal to $-\nu_i/N_i$. Some numerical computations suggest a behavior of the poles of $Z^{\mathfrak{P}}(s)$ as the one in Theorem 1.10.

Conjecture 2.2 ($\mathfrak{P}$-adic monodromy conjecture)**.** For almost every $\mathfrak{P} \in \Omega_F^{fin}$, if s_0 is a pole of $Z^{\mathfrak{P}}(s)$, then $\exp(2\pi i\mathrm{Re}(s_0))$ is a monodromy eigenvalue for some x.

Let's try to understand how to attack this conjecture. We define the *monodromy zeta function* at x as

$$\zeta_x(t) := \prod_{n=0}^{2d} \det(1 - tT_x | H^n_{sing}(F_x, \mathbb{Z}))^{(-1)^{n+1}}.$$

There is an explicit formula of this function using the log-resoultion of $(\mathbb{A}^d, X_0)$.

Theorem 2.3 (A' Campo's formula [A'C75])**.**

$$\zeta_x(t) = \prod_{i \in I} (1 - t^{N_i})^{-\chi_{top}(\mathring{E}_i \cap h^{-1}(x))}$$

It may happen that a monodromy eigenvalue at x is not a zero or a pole of $\zeta_x(t)$, because of some unlucky cancellation. Nevertheless, Denef [Den93] has proven, thanks to the perversity of the nearby cycles complex, that every eigenvalue of the monodromy operator at x is a zero or a pole of $\zeta_y(t)$ for some point y, maybe different from x.

Denef has also proven that the $\mathfrak{P}$-adic zeta function admits an explicit formula using the log-resolution of $(\mathbb{A}^d_F, X_0)$. Take a model $\mathfrak{Y}$ of the log-resolution Y and for every $\emptyset \neq J \subseteq I$, let $\mathring{\mathfrak{E}}_{\mathfrak{J}}$ be the closure of $\mathring{E}_J$ in $\mathfrak{Y}$. Denote by $k_{\mathfrak{P}}$ the residue field at $\mathfrak{P}$.

Theorem 2.4 (Denef's formula)**.** *For almost every* $\mathfrak{P}$*,*

$$Z^{\mathfrak{P}}(s) = q^{-d} \sum_{\emptyset \neq J \subseteq I} |\mathring{\mathfrak{E}}_{\mathfrak{J}}(k_{\mathfrak{P}})| \prod_{j \in J} \frac{(q-1)q^{-N_j s - \nu_j}}{1 - q^{-N_j s - \nu_j}}$$

where q *is the cardinality of* $k_{\mathfrak{P}}$ *and* $|\mathring{\mathfrak{E}}_{\mathfrak{J}}(k_{\mathfrak{P}})|$ *is the cardinality of the* $k_{\mathfrak{P}}$*-points of* $\mathring{\mathfrak{E}}_{\mathfrak{J}}$*.*

Proof. [Den87, Theorem 3.1] □

We have now explicit formulas for the monodromy eigenvalues and for the poles of the $\mathfrak{P}$-adic zeta function. The main difficulty from here to prove the monodromy conjecture is to understand the configuration of the irreducible components E_i in Y.

The conjecture is known in the following cases:

– n = 2 ;

– n = 3 and f homogeneous ;

– some nice classes of singularities.

For references and some other facts about the $\mathfrak{P}$-adic monodromy conjecture you can look at [Nic09, Section 3].

3 Motivic zeta function

We have defined in the previous talk the naive motivic zeta function.

$$Z^{naive}(s) := \int_{\mathcal{L}(X_0)} \mathbb{L}^{-\mathrm{ord}_t(f)s} \mathrm{d}\mu = \sum_{n=1}^{\infty} \mathbb{L}^{-ns-d(n+1)} [\mathcal{X}_n / X_0] \in \mathcal{M}_{X_0}[[\mathbb{L}^{-s}]]$$

where $\mathcal{X}_n := \mathcal{L}_n(X_0) \cap \mathrm{ord}_t^{-1}(n)$. We will be interested in the study of $Z_x^{naive}(s) := Z^{naive}(s) \times_{X_0} x \in \mathcal{M}_x[[\mathbb{L}^{-s}]]$ where the fiber product is done on the coefficients of the series. We also consider the *topological zeta function* $Z_x^{top}(s) := \chi_{top}(Z_x^{naive}(s))$, defined taking the Euler characteristic of the coefficients. We have seen the following formula without a proof.

Theorem 3.1 (Denef-Loeser's formula)**.**

$$Z^{naive}(s) = \mathbb{L}^{-n} \sum_{\emptyset \neq J \subseteq I} (\mathbb{L}-1)^{|J|} [\mathring{E}_J / X_0] \prod_{j \in J} \frac{\mathbb{L}^{-N_j s - \nu_j}}{1 - \mathbb{L}^{-N_j s - \nu_j}}$$

Sketch of the proof. One can reduce to the case when X_0 is an snc divisor, taking a log-resolution and using the change of variables formula. Then the computation becomes easier thanks to the local description of the divisor E as the zero locus of monomials. If you want to see how to do concretely this last computation I added in the Appendix A an example. □

As a consequence we also have a formula for the topological zeta function:

$$Z_x^{top}(s) = \sum_{\emptyset \neq J \subseteq I} \chi_{top}(\mathring{E}_J \times_{X_0} x) \prod_{j \in J} \frac{1}{N_j s + \nu_j}. \tag{3.1.1}$$

We finally have all the tools to prove Veys' conjecture.

Theorem 3.2 (Veys' conjecture)**.** *If s_0 is a pole of order d of $Z_x^{top}(s)$ then $s_0 = -\operatorname{lct}_x$.*

Proof. If for some J, $\mathring{E}_J \neq \emptyset$ then, by a dimension reasoning on E_J, using that E is an snc divisor, the cardinality of J is at most d. Hence, by the Formula (3.1.1), if s_0 is a pole of order d, there exists $J_0 \subseteq I$ such that $|J_0| = d$, $\mathring{E}_{J_0} \cap h^{-1}(x) \neq \emptyset$ and for every $j \in J_0$, $-\nu_j/N_j = s_0$. In particular, J_0 is maximal with this property. Therefore we can apply the Main Theorem of Michael's talk. Namely, by the maximality of J_0, $\nu_j/N_j = \operatorname{lct}_x$ for every $j \in J_0$. This proves the theorem. □

We can ask for the motivic zeta function an analogous of the $\mathfrak{P}$-adic monodromy conjecture. In this case, to talk about the poles of the function is more delicate because $\mathcal{M}_{X_0}$ is not a domain (see [Poo02]).

Conjecture 3.3 (Motivic monodromy conjecture)**.** There exists a finite subset $S \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that

$$Z^{naive}(s) \in \mathcal{M}_{X_0}\left[\mathbb{L}^{-s}, \frac{1}{1-\mathbb{L}^{-as-b}}\right]_{(a,b)\in S} \subseteq \mathcal{M}_{X_0}[[\mathbb{L}^{-s}]]$$

and such that $(a,b) \in S$ implies $\exp(-2\pi i b/a)$ is a monodromy eigenvalue for some $x \in X_0$.

Specialisation to $\mathfrak{P}$-adic world
Take the ring

$$\mathscr{Z}_{\mathfrak{P}} := \mathbb{Q}\left[\frac{|k_{\mathfrak{P}}|^{-as-b}}{1-|k_{\mathfrak{P}}|^{-as-b}}\right]_{(a,b)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}}$$

where $|k_{\mathfrak{P}}|$ is the cardinality of the residue field at $\mathfrak{P}$. We denote by $\mathscr{Z}$ the quotient of $\prod_{\mathfrak{P}\in\Omega_E^{fin}} \mathscr{Z}_{\mathfrak{P}}$ by the ideal $\bigoplus_{\mathfrak{P}\in\Omega_E^{fin}} \mathscr{Z}_{\mathfrak{P}}$. We define a morphism of rings

$$\mathscr{N} : \mathcal{M}_{X_0}\left[\frac{\mathbb{L}^{-as-b}}{1-\mathbb{L}^{-as-b}}\right]_{(a,b)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}} \to \mathscr{Z}$$

in the following way: for every variety T we take a model $\mathfrak{T}$ over $\mathcal{O}_F$ and we send the class $[T/X_0] \in \mathcal{M}_{X_0}$ to the class $[(|\mathfrak{T}(k_{\mathfrak{P}})|)_{\mathfrak{P}\in\Omega_F^{fin}}]$ where $|\mathfrak{T}(k_{\mathfrak{P}})|$ is the number of $k_{\mathfrak{P}}$-points of the model $\mathfrak{T}$. The morphism $\mathscr{N}$ is a well defined morphism of rings because two models of T are isomorphic for almost every $\mathfrak{P}$. Putting together Denef and Loeser's formulas for $\mathfrak{P}$-adic and motivic zeta function we obtain the following.

Theorem 3.4 (Denef-Loeser)**.**

$$\mathscr{N}(Z^{naive}(s)) = \left[\left(Z^{\mathfrak{P}}(s)\right)_{\mathfrak{P}\in\Omega_F^{fin}}\right].$$

As a consequence of this, the motivic monodromy conjecture implies the $\mathfrak{P}$-adic monodromy conjecture for almost every $\mathfrak{P}$.

A An example

We want to compute the naive motivic zeta function

$$Z^{naive}(s) := \sum_{n=1}^{\infty} \mathbb{L}^{-ns-d(n+1)}[\mathcal{X}_n/X_0]$$

when $f = x^{N_1}y^{N_2}$. We can decompose X_0 as a disjoint union $\mathring{E}_1 \sqcup \mathring{E}_2 \sqcup \mathring{E}_{12}$ with $\mathring{E}_1 = \{x = 0, y \neq 0\}$, $\mathring{E}_2 = \{x \neq 0, y = 0\}$ and $\mathring{E}_{12} = \{x = 0, y = 0\}$.

To compute the motivic zeta function we need to understand $[\mathcal{X}_n/X_0]$ for every n. We recall that $\mathcal{X}_n$ is the subscheme of $\mathcal{L}_n(\mathbb{A}^d)$ with $\mathbb{C}$-points the n-jets with order n. The $\mathbb{C}$-points of $\mathcal{L}_n(\mathbb{A}^d)$ corresponds to $\mathrm{Hom}_{Ring}(\mathbb{C}[x,y], \mathbb{C}[t]/t^{n+1})$, hence they are determined by the images of x and y, namely a pair $(a_0 + a_1t + \cdots + a_nt^n, b_0 + b_1t + \cdots + b_nt^n)$ with a_i and b_i complex numbers. The order with respect to t of a certain $\gamma_n \in \mathrm{Hom}_{Ring}(\mathbb{C}[x,y], \mathbb{C}[t]/t^{n+1})$ is given by $v_t(\gamma_n(x^{N_1}y^{N_2}))$, where v_t is the standard valuation on $\mathbb{C}[t]/t^{n+1}$ with $v_t(t) = 1$. The previous decomposition translates in a decomposition $\mathcal{X}_n = \mathring{\mathcal{E}}_{1,n} \sqcup \mathring{\mathcal{E}}_{2,n} \sqcup \mathring{\mathcal{E}}_{12,n}$ where $\mathring{\mathcal{E}}_{J,n} := (\pi_0^n)^{-1}(E_J) \cap \mathcal{X}_n$.

Let's study one piece at a time. The variety $\mathring{\mathcal{E}}_{1,n}$ is a locally trivial fibration of $\mathring{E}_1$. We want to understand the fiber. We fix a point $(\overline{a_0}, \overline{b_0}) \in \mathring{E}_1$, hence $\overline{a_0} = 0$ and $\overline{b_0} \neq 0$. The points $\gamma_n = (\overline{a_0} + a_1t + \cdots + a_nt^n, \overline{b_0} + b_1t + \cdots + b_nt^n)$ over $(\overline{a_0}, \overline{b_0})$ are precisely given by the condition $v_t(\gamma_n(x^{N_1}y^{N_2})) = n$. The element $\gamma_n(y)$ is invertible as $\overline{b_0} \neq 0$, thus $v_t(\gamma_n(x))N_1 = n$. In particular $N_1|n$ and if we denote $\alpha_1 := n/N_1$, then $a_1 = \cdots = a_{\alpha_1-1} = 0$, $a_{\alpha_1} \neq 0$. There are no conditions on the other b_i, hence $\mathring{\mathcal{E}}_{1,n}$ is a $(\mathbb{G}_m \times \mathbb{A}^{n-\alpha_1} \times \mathbb{A}^n)$-bundle over $\mathring{E}_1$ when $N_1|n$ and it's empty if $N_1 \nmid n$.

Thus if $N_1|n$, $[\mathring{\mathcal{E}}_{1,n}/X_0] = (\mathbb{L}-1)\mathbb{L}^{2n-\alpha_1}[\mathring{E}_1/X_0]$ and

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{L}^{-ns}\mathbb{L}^{-2(n+1)}[\mathring{\mathcal{E}}_{1,n}/X_0] &= \sum_{N_1|n} \mathbb{L}^{-ns}\mathbb{L}^{-2(n+1)}(\mathbb{L}-1)\mathbb{L}^{2n-\alpha_1}[\mathring{E}_1/X_0] = \\ &= \mathbb{L}^{-2}(\mathbb{L}-1)[\mathring{E}_1/X_0] \sum_{\alpha_1=1}^{\infty} \mathbb{L}^{\alpha_1(-N_1s-1)} = \\ &= \mathbb{L}^{-2}(\mathbb{L}-1)[\mathring{E}_1/X_0] \frac{\mathbb{L}^{-N_1s-1}}{1-\mathbb{L}^{-N_1s-1}}.\end{aligned}$$

The same reasoning applies to $\mathring{E}_2$.

The case $\mathring{\mathcal{E}}_{12,n}$ is slightly different. The scheme $\mathring{E}_{12,n}$ consists only of one point $(0,0)$. The n-jets $\gamma_n = (a_1 t + \cdots + a_n t^n, b_1 t + \cdots + b_n t^n)$ over $(0,0)$ with order n are again given by the condition $\gamma_n(x^{N_1} y^{N_2}) = n$, thus we have $v_t(\gamma_n(x))N_1 + v_t(\gamma_n(y))N_2 = n$. For every choice of $(\alpha_1, \alpha_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $\alpha_1 N_1 + \alpha_2 N_2 = n$, the n-jets with $v_t(\gamma_n(x)) = \alpha_1$ and $v_t(\gamma_n(y)) = \alpha_2$ give a variety isomorphic to $(\mathbb{G}_m^2 \times \mathbb{A}^{n-\alpha_1} \times \mathbb{A}^{n-\alpha_2})$ over $\mathring{E}_{12}$. In other words, if $\alpha_1 N_1 + \alpha_2 N_2 = n$, $[\mathring{\mathcal{E}}_{12,n}/X_0] = (\mathbb{L}-1)^2 \mathbb{L}^{2n-\alpha_1-\alpha_2}[\mathring{E}_{12}/X_0]$ and

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{L}^{-ns}\mathbb{L}^{-2(n+1)}[\mathring{\mathcal{E}}_{12,n}/X_0] &= \sum_{n=1}^{\infty} \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{Z}_{>0} \\ \alpha_1 N_1 + \alpha_2 N_2 = n}} \mathbb{L}^{-ns}\mathbb{L}^{-2(n+1)}(\mathbb{L}-1)^2 \mathbb{L}^{2n-\alpha_1-\alpha_2}[\mathring{E}_{12}/X_0] = \\ &= \mathbb{L}^{-2}(\mathbb{L}-1)^2[\mathring{E}_{12}/X_0] \sum_{\alpha_1,\alpha_2 \in \mathbb{Z}_{>0}} \mathbb{L}^{\alpha_1(-N_1 s-1)}\mathbb{L}^{\alpha_2(-N_1 s-1)} = \\ &= \mathbb{L}^{-2}(\mathbb{L}-1)^2[\mathring{E}_{12}/X_0] \frac{\mathbb{L}^{-N_1 s-1}}{1-\mathbb{L}^{-N_1 s-1}} \frac{\mathbb{L}^{-N_2 s-1}}{1-\mathbb{L}^{-N_2 s-1}}.\end{aligned}$$

Now you can put the three pieces together and compare the result with Theorem 3.1. Recall that in our case $\nu_i = 1$.

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