Monodromy groups of F-isocrystals

Marco D'Addezio

FB Mathematik und Informatik, Freie Universität Berlin

8 February 2018

Setting

Let p be a prime, q a power of p and X_0 a smooth geometrically connected variety over $\mathbb{F}_q$. Moreover, let $\mathbb{Q}_q$ be $\mathrm{Frac}(W(\mathbb{F}_q))$.
We denote by $\mathbf{Isoc}^\dagger(X_0)$ the category of *overconvergent isocrystals* on X_0.

Lisse sheaf on $X_0 \rightsquigarrow$ continuous ℓ-adic representation of $\pi_1^{\text{ét}}(X_0)$

Thanks to the Tannakian formalism:

Overconvergent isocrystal $\rightsquigarrow$ representation of an *affine group scheme*

Definition

Let $\mathbb{K}$ be a field, a $\mathbb{K}$-linear *neutral Tannakian category* is an abelian $\mathbb{K}$-linear category $\mathcal{C}$ with the following additional properties:

1. It is endowed with a symmetric monoidal structure $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is $\mathbb{K}$-linear, bi-additive, associative, commutative, and it admits a unit object $\mathbb{1}$;
2. $\mathrm{End}(\mathbb{1}) \simeq \mathbb{K}$;
3. $\forall\ M \in \mathcal{C}$ there exist $M^\vee$, $\mathrm{ev} : M \otimes M^\vee \to \mathbb{1}$ and $\delta : \mathbb{1} \to M^\vee \otimes M$ such that the compositions
$$M \xrightarrow{\mathrm{id}_M \otimes \delta} M \otimes M^\vee \otimes M \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_M} M$$
$$M^\vee \xrightarrow{\delta \otimes \mathrm{id}_{M^\vee}} M^\vee \otimes M \otimes M^\vee \xrightarrow{\mathrm{id}_{M^\vee} \otimes \mathrm{ev}} M$$
are the identity maps;
4. There exists a faithful exact $\mathbb{K}$-linear functor $\omega : \mathcal{C} \to \mathbf{Vec}_{\mathbb{K}}$ that preserves the monoidal structure. We call such an ω a *fiber functor* for $\mathcal{C}$.

Reconstruction theorem

Theorem (Grothendieck, Saavedra-Rivano, Deligne)

Let $\mathcal{C}$ be a $\mathbb{K}$-linear neutral Tannakian category. Every fiber functor ω induces an equivalence of Tannakian categories

$$\mathcal{C} \simeq \mathbf{Rep}_{\mathbb{K}}(\underline{\mathrm{Aut}}^{\otimes}(\omega)).$$

The group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an affine group scheme over $\mathbb{K}$. It is called the *Tannakian group* of $\mathcal{C}$ with respect to ω, denoted by $G(\mathcal{C}, \omega)$.

Functoriality

Let $(\mathcal{C}, \omega_{\mathcal{C}})$ and $(\mathcal{D}, \omega_{\mathcal{D}})$ be two $\mathbb{K}$-linear Tannakian categories endowed with fiber functors $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{D}}$. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor of Tannakian categories commuting with the fiber functors. Then φ induces a natural morphism

$$\varphi^* : G(\mathcal{D}, \omega_{\mathcal{D}}) \to G(\mathcal{C}, \omega_{\mathcal{C}}).$$

Monodromy of isocrystals

Proposition (Ogus, Crew)

If $X_0(\mathbb{F}_q) \neq \emptyset$, the category $\mathbf{Isoc}^\dagger(X_0)$ is a $\mathbb{Q}_q$-linear neutral Tannakian category.

We will assume from now on that $X_0(\mathbb{F}_q) \neq \emptyset$. We introduce the following notation:

- $\pi_1^{\mathrm{Isoc}^\dagger}(X_0)$:= the Tannakian group of $\mathbf{Isoc}^\dagger(X_0)$, called the *isocrystal fundamental group*;
- $G(M)$:= the Tannakian group of $\langle M\rangle^\otimes \subseteq \mathbf{Isoc}^\dagger(X_0)$, called the *monodromy group of M*.

The affine group scheme $G(M)$ is of finite type over $\mathbb{Q}_q$ and it is a quotient of the isocrystal fundamental group.

$$\pi_1^{\mathrm{Isoc}^\dagger}(X_0) \twoheadrightarrow G(M).$$

Frobenius structure

$F_{X_0} : X_0 \to X_0$ the q-th power Frobenius endomorphism.

Definition

A *Frobenius structure* for M is an isomorphism $\Phi : F_{X_0}^* M \xrightarrow{\sim} M$. Such a pair (M, Φ) is called an *overconvergent F-isocrystal*.
An overconvergent F-isocrystal is said to be *unit-root* if $\forall x_0 \in |X_0|$ the roots of the Frobenius characteristic polynomial at x_0 are p-adic units.

Main theorem on unit-root F-isocrystals

Theorem (Katz, Crew, Tsuzuki, Kedlaya, Shiho)

There exists a canonical equivalence of $\mathbb{Q}_q$-linear neutral Tannakian categories

$$\left(\begin{array}{c}\textit{unit-root}\\ \textit{overconvergent}\\ F\textit{-isocrystals}\end{array}\right) \xrightarrow{\sim} \left(\begin{array}{c}\textit{continuous } \mathbb{Q}_q\textit{-linear}\\ \textit{representations of } \pi_1^{\acute{e}t}(X_0)\\ \textit{satisfying a certain}\\ \textit{condition at infinity}\end{array}\right).$$

We denote by $\rho_{(M,\Phi)}$ the representation associated to (M,Φ).

Main theorem on unit-root F-isocrystals

If (M, Φ) is unit-root, M is controlled by the restriction of $\rho_{(M,\Phi)}$ to $\pi_1^{\text{ét}}(X_0 \otimes \overline{\mathbb{F}}_q)$. For example, we have the following fact.

Lemma

The subgroup $\rho_{(M,\Phi)}(\pi_1^{\text{ét}}(X_0 \otimes \overline{\mathbb{F}}_q)) \subseteq G(M)(\mathbb{Q}_q)$ is Zariski dense.

Goal

Theorem (MD'A)

Let A_0 be an abelian variety over $\mathbb{F}_q$ and M be a semi-simple overconvergent isocrystal such that $F_{A_0}^* M \simeq M$. Then there exists a finite étale cover $f_0 : Y_0 \to A_0$ such that $f_0^* M$ is trivial on Y_0.

Corollary (Tsuzuki)

For every F-isocrystal on A_0, the Newton polygon of the Frobenius characteristic polynomials at closed points is independent of the point.

The global monodromy theorem

Proposition (Crew, Abe)

Let M be an overconvergent isocrystal of rank 1 such that $F_{X_0}^* M \simeq M$, then $G(M)$ is finite.

Sketch of the proof.

1. M is a rank 1 overconvergent isocrystal M that admits a Frobenius structure, thus it also admits a Frobenius structure Φ such that (M, Φ) is a unit-root overconvergent F-isocrystal.
2. As the representation $\rho_{(M,\Phi)}$ is of rank 1, its image is commutative.
3. (2) and the condition at infinity on $\rho_{(M,\Phi)}$ imply, by class field theory, that $\rho(\pi_1^{\text{ét}}(X_0 \otimes \overline{\mathbb{F}}_q))$ is finite.
4. As $\rho(\pi_1^{\text{ét}}(X_0 \otimes \overline{\mathbb{F}}_q))$ is dense in $G(M)(\mathbb{Q}_q)$ we conclude.

□

The global monodromy theorem

Theorem (The global monodromy theorem; Crew)

Let M be an overconvergent isocrystal such that $F_{X_0}^* M \simeq M$. The radical subgroup of $G(M)$ (i.e. the greatest connected normal solvable subgroup) is unipotent.

The case of abelian varieties

A lemma

Let A_0 be an abelian variety over $\mathbb{F}_q$.

Lemma

For every overconvergent isocrystal M on A_0, the algebraic group $G(M)$ is commutative.

Proof of the lemma.

Let $m_0 : A_0 \times A_0 \to A_0$ be the multiplication map of A_0, we take

$$\widetilde{m_*} : \pi_1^{\mathrm{Isoc}^\dagger}(A_0) \times \pi_1^{\mathrm{Isoc}^\dagger}(A_0) \simeq \pi_1^{\mathrm{Isoc}^\dagger}(A_0 \times A_0) \xrightarrow{m_*} \pi_1^{\mathrm{Isoc}^\dagger}(A_0).$$

It endows $\pi_1^{\mathrm{Isoc}^\dagger}(A_0)$ with a second group structure compatible with the structural one. By an Eckmann–Hilton argument, $\pi_1^{\mathrm{Isoc}^\dagger}(A_0)$ is commutative. Hence the same is true for its quotient $G(M)$. □

The case of abelian varieties

Theorem (MD'A)

Let A_0 be an abelian variety over $\mathbb{F}_q$ and M be a semi-simple overconvergent isocrystal such that $F_{A_0}^* M \simeq M$. Then there exists a finite étale cover $f_0 : Y_0 \to A_0$ such that $f_0^* M$ is trivial on Y_0.

Proof of the theorem

1. M semi-simple $\Rightarrow G(M)$ is a reductive group.
2. Previous lemma + (1) $\Rightarrow G(M) \simeq$ torus $\times$ commutative finite group. In particular, the radical of $G(M)$ is $G(M)^\circ$.
3. Global monodromy theorem $\Rightarrow G(M)^\circ$ is unipotent, hence trivial. Thus $G(M)$ is finite.

Theorem (MD'A)

Let A_0 be an abelian variety over $\mathbb{F}_q$ and M be a semi-simple overconvergent isocrystal such that $F_{A_0}^* M \simeq M$. Then there exists a finite étale cover $f_0 : Y_0 \to A_0$ such that $f_0^* M$ is trivial on Y_0.

Proof of the theorem.

4 An overconvergent isocrystal with finite monodromy admits a unit-root Frobenius structure. We denote by Φ one of these Frobenius structures of M.

5 $\rho_{(M,\Phi)}(\pi_1^{\text{ét}}(A_0 \otimes \overline{\mathbb{F}}_q))$ is finite, thus there exits $f_0 : Y_0 \to A_0$ finite étale such that $\rho_{(M,\Phi)}(\pi_1^{\text{ét}}(Y_0 \otimes \overline{\mathbb{F}}_q)) = 1$. Hence $f_0^* M$ is trivial.

□

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