DIEUDONNÉ THEORY OVER $\mathcal{O}_C$

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1. Introduction

Today we will continue our study of the various classifications of p-divisible groups over different bases. In this talk, we will switch to mixed characteristic $(0, p)$. We will consider an algebraically closed nonarchimedean field $C/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_C$. The classification of p-divisible groups over $\mathcal{O}_C$ was obtained by Fargues and Scholze–Weinstein.

Theorem 1.1 (Fargues, Scholze–Weinstein)**.** *There exists an equivalence of categories*

$$\{\ p\text{-div groups}/\ \mathcal{O}_C\ \} \xrightarrow{\sim} \left\{ (T, W) \middle| \begin{array}{c} T \text{ is a free } \mathbb{Z}_p\text{-module of finite rank} \\ W \subseteq T \otimes C(-1) \text{ is a } C\text{-subvectorspace} \end{array} \right\},$$

Fargues proved the full faithfulness of the functor and Scholze–Weinstein the essential surjectivity. The result is pretty different from the other constructions of Dieudonné modules we have seen so far. It is in terms of linear algebra rather than semi-linear algebra. It reminds Riemann's classification of complex abelian varieties by their Hodge structures, thus using periods. I will come back to this difference later in the talk. I will start out by explaining how to construct this equivalence.

2. p-divisible groups over $\mathcal{O}_C$.

Let R be a ring.

Definition 2.1. A *p-divisible group* over R is a sheaf G of abelian groups over $(\mathrm{Sch}/R)_{\mathrm{fppf}}$, such that:

(1) $G = \varinjlim G[p^n]$ (p-torsion),
(2) $p : G \to G$ surjective (p-divisible),
(3) $G[p]$ is a finite locally-free group scheme.

We have a notion of *Cartier dual* of p-divisible groups $G \mapsto G^\vee := \varinjlim \mathcal{H}\mathrm{om}(G[p^n], \mathbb{G}_m)$ and the one of *Tate module* $T(G) := \varprojlim_p G[p^n] = \mathcal{H}\mathrm{om}(\mathbb{Q}_p/\mathbb{Z}_p, G)$ (not p-divisible of course). The Lie algebra of G is defined as the abelian group $\mathrm{Lie}(G) := \ker(G(R[\epsilon]) \to G(R))$.

Let us first consider the case when p is nilpotent in R. Let G be a p-divisible group over R.

Lemma 2.2 (Messing)**.** *The sheaf G is formally smooth. The completion of G along the zero section G^{inf} is representable by an affine formal scheme with finitely generated ideal of definition. Moreover, the module* $\mathrm{Lie}(G)$ *is projective and when it is free, $G^{\mathrm{inf}} \simeq \mathrm{Spf}(R[[x_1, \dots, x_d]])$. Finally, if G is connected, $G = G^{\mathrm{inf}}$.*

Thus G^{inf} can be thought p-adic analytically as a fibration in open balls.

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Construction 2.3 (Logarithm when p is nilpotent)**.** We have a group isomorphism $\log : \ker(G(R) \to G(R/p^2)) \xrightarrow{\sim} p^2\mathrm{Lie}(G)$ given by

$$\log(x) := \left(\sum_{i=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} \right) \epsilon$$

with inverse

$$\exp(x\epsilon) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To make things well defined we need that eventually $p^{2n}/n! = 0$ or, in other words, that (p^2) admits a nilpotent PD-structure. If p is odd, we could have replaced everywhere p^2 by p.

Suppose now that R is a p-adically complete $\mathbb{Z}_p$-algebra and let again G/R be a p-divisible group.

In general, G is not formally smooth. Even the basic case μ_{p^∞} is not formally smooth over R as the module of Kähler differentials is not projective. In this case, it is convenient to study the restriction to those algebras over R with p nilpotent. In other words, we can make p infinitesimal.

Definition 2.4. We define $\mathcal{G}(R)$ as the projective limit $\varprojlim_N G(R_N)$, where $R_N := R/p^{N+1}$. We also define $\mathrm{Lie}(\mathcal{G})$ as the $\mathbb{Z}_p$-module $\ker\left(\mathcal{G}(R[\epsilon]) \to \mathcal{G}(R)\right)$.

Construction 2.5. Starting from the logarithm isomorphism we have seen when p is nilpotent and going to the limit, we get an isomorphism of $\mathbb{Z}_p$-modules $\log : \ker(\mathcal{G}(R) \to G(R_1)) \xrightarrow{\sim} p^2\mathrm{Lie}(\mathcal{G})$. As for every element $x \in \mathcal{G}(R)$ there exists $n > 0$ such that $p^n x \in \ker(\mathcal{G}(R) \to G(R_1))$, we can extend the logarithm uniquely to a morphism $\log : \mathcal{G}(R) \to p^2\mathrm{Lie}(\mathcal{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{Lie}(\mathcal{G})[1/p]$. We also get the exact sequence

$$0 \to G(R) \to \mathcal{G}(R) \xrightarrow{\log} \mathrm{Lie}(\mathcal{G})[1/p]. \tag{2.5.1}$$

Example 2.6. Suppose now $G = \mu_{p^\infty}$ and $R = \mathcal{O}_C$. We have for every $N > 0$, $G(R_N) = (1+\mathfrak{m})/p^{N+1}$. Indeed every $x \in \mathfrak{m}$ is topologically nilpotent, so that $(1+x)^{p^n} \to 1$. Thus we have $\mathcal{G}(R) = 1 + \mathfrak{m}$. Similarly we can compute $\mathcal{G}(R[\epsilon]) = (1+\mathfrak{m}) \oplus \mathcal{O}_C$ so that $\mathrm{Lie}(\mathcal{G}) = \mathcal{O}_C$. On the other hand, $G(R[\epsilon]) = G(R) = \mu_{p^\infty}(C)$ thus $\mathrm{Lie}(G) = 0$. Then (2.5.1) becomes

$$0 \to \mu_{p^\infty}(C) \to 1 + \mathfrak{m} \xrightarrow{\log} C.$$

Notice that this exact sequence admits also a realization with adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$, namely

$$0 \to \mu_{p^\infty}(C) \to \mathring{B}(1,1) \xrightarrow{\log} \mathbb{G}_a^{\mathrm{an}}$$

where $\mathbb{G}_a^{\mathrm{an}} := \bigcup_{n=0}^{\infty} \mathrm{Spa}(C\langle p^n T\rangle, \mathcal{O}_C\langle p^n T\rangle)$ and $\mathring{B}(1,1)$ is the open ball centered in 1 with radius 1.

In general, for every p-divisible group over R, we can construct an exact sequence of adic groups

$$0 \to \mathcal{G}_\eta^{\mathrm{ad}}[p^\infty] \to \mathcal{G}_\eta^{\mathrm{ad}} \to \mathrm{Lie}(\mathcal{G}) \otimes_{\mathbb{Z}_p} \mathbb{G}_a^{\mathrm{an}} \tag{2.6.1}$$

over $\mathrm{Spa}(R[\frac{1}{p}], R)$ as in the example we discussed.

Construction 2.7. (Construction of $\beta_{\mathcal{G}}$) If $T := T(G)(C)$, then applying $T(-)(C)$ to the natural pairing $G \times G^\vee \to \mu_{p^\infty}$ we get $T(G^\vee)(C) = T^\vee(1)$. Therefore, there is a canonical morphism $T^\vee(1) \to \mathcal{H}\mathrm{om}(\mathbb{Q}_p/\mathbb{Z}_p, G^\vee)$. By adjunction, we get a map $T^\vee(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to G^\vee$. Dualizing we get $G \to T(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$. Passing to Lie algebras we obtain a morphism $\mathrm{Lie}(\mathcal{G}) \to T(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$ which we denote by $\beta_{\mathcal{G}}$.

Theorem 2.8 (Fargues)**.** *There is a natural short exact sequence*

$$0 \to \mathrm{Lie}(\mathcal{G}) \otimes_{\mathcal{O}_C} C(1) \xrightarrow{\beta_{\mathcal{G}}(1)} T \otimes_{\mathbb{Z}_p} C \xrightarrow{\beta_{\mathcal{G}^\vee}^\vee} (\mathrm{Lie}(\mathcal{G}^\vee) \otimes_{\mathcal{O}_C} C)^\vee \to 0.$$

This is analogous to the Hodge filtration on the H^1 of a complex abelian variety (or even a complex torus) in terms of Lie algebras of the abelian variety and its dual.

Construction 2.9. We are now ready to define the functor of Theorem A. We send a p-divisible group G to the pair $\Big(T,\ \mathrm{Lie}(\mathcal{G}) \otimes_{\mathcal{O}_C} C \xrightarrow{\beta_{\mathcal{G}}} T(G)(C) \otimes_{\mathbb{Z}_p} C(-1)\Big)$.

Proof of the full faithfulness in Theorem 1.1.

Lemma 2.10. *Let G be a p-divisible group over $\mathcal{O}_C$, then $G \simeq G^\circ \oplus G^{\acute{e}t}$ with G° connected and $G^{\acute{e}t}$ constant.*

Proof. As $\mathcal{O}_C$ is henselian, for every n, the subscheme $(G[p^n])^\circ \subseteq G[p^n]$ is a (finite flat) subgroup[1]. Therefore, we have an exact sequence

$$0 \to (G[p^n])^\circ \to G[p^n] \to G[p^n]^{\text{ét}} \to 0.$$

Since $\mathcal{O}_C$ is strictly henselian, $G[p^n]^{\text{ét}}$ is constant. Finally, as $\mathcal{O}_C$ is a valutative ring (normal local ring would have been enough) with algebraically closed fraction field, $H^1_{\mathrm{fppf}}(\mathcal{O}_C, (G[p^n])^\circ) = 0$[2]. This implies that the exact sequence splits. We get the result then passing to the limit for every n. □

Remark 2.11. The same result is true for p-divisible rigid-analytic groups[3] over $\mathcal{O}_C$. This was proven by Fargues.

For every connected p-divisible group $G/\mathcal{O}_C$, let H be the p-divisible group $T(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ and consider the morphism $G \to H$ of Construction 2.7. We have an isomorphism on the C-points of the associated Tate modules. On Lie algebras, the induced map is the one we denoted by $\beta_{\mathcal{G}}$. We consider the exact sequence (2.6.1) for the groups involved. We have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G}_\eta^{\mathrm{ad}}[p^\infty] & \longrightarrow & \mathcal{G}_\eta^{\mathrm{ad}} & \xrightarrow{\log} & \mathrm{Lie}(\mathcal{G}) \otimes \mathbb{G}_a^{\mathrm{an}} \\
 & & \downarrow \wr & & \downarrow & & \downarrow \beta_{\mathcal{G}} \\
0 & \longrightarrow & \mathcal{H}_\eta^{\mathrm{ad}}[p^\infty] & \longrightarrow & \mathcal{H}_\eta^{\mathrm{ad}} & \xrightarrow{\log} & \mathrm{Lie}(\mathcal{H}) \otimes \mathbb{G}_a^{\mathrm{an}}.
\end{array}$$

[1] Use that there is an injection from the π_0 of the special fiber to the π_0 of the generic fiber.

[2] Every torsor under this group has generically a rational point which then is defined over $\mathcal{O}_C$ by the properness of $(G[p^n])^\circ$.

[3] Rigid-analytic groups with finite, locally free, faithfully flat, topologically nilpotent multiplication by p.

We check that the right square is cartesian. If (R, R^+) is a complete affinoid ring and $x \in \mathcal{H}_\eta^{\mathrm{ad}}(R, R^+)$ goes to $(\mathrm{Lie}(\mathcal{G}) \otimes \mathbb{G}_a^{\mathrm{an}})(R, R^+)$, then there exists an n such that $p^n x$ comes from $\mathcal{G}_\eta^{\mathrm{ad}}(R, R^+)$, because log is locally an isomorphism. As $\mathcal{G}_\eta^{\mathrm{ad}}$ and $\mathcal{H}_\eta^{\mathrm{ad}}$ are p-divisible rigid-analytic group with the same p-torsion points, then x comes from $\mathcal{G}_\eta^{\mathrm{ad}}(R, R^+)$ as well, so we are done.

As the right square is cartesian, we can reconstruct $\mathcal{G}_\eta^{\mathrm{ad}}$ starting from T and $\beta_{\mathcal{G}}$. Moreover, thanks to Lemma 2.10, we can reconstruct the p-divisible G as $\mathrm{Spf}(H^0(Y, \mathcal{O}_Y^+)) \oplus \pi_0(\mathcal{G}_\eta^{\mathrm{ad}})$ where Y is the neutral component of $\mathcal{G}_\eta^{\mathrm{ad}}$ [4]. □

Let k be the residue field of $\mathcal{O}_C$, one has a pullback functor,

$$\{\ p\text{-div groups}/\ \mathcal{O}_C\ \} \to \{\ p\text{-div groups}/\ k\ \},$$

which then induces a mysterious functor

$$\left\{ (T, W) \middle| \begin{array}{c} T \text{ is a free } \mathbb{Z}_p\text{-module of finite rank} \\ W \subseteq T \otimes C(-1) \text{ is a } C\text{-subvectorspace.} \end{array} \right\} \xrightarrow{?} \{\ \text{Dieudonné modules}/\ W(k)\ \}.$$

Notice that the datum of a Dieudonné module is the same as the datum of a finite free $W(k)$-modules M endowed with a φ-linear isomorphism $\varphi_M : M[\frac{1}{p}] \xrightarrow{\sim} M[\frac{1}{p}]$ such that [5] $M \subseteq \varphi_M(M) \subseteq p^{-1}M$. Our next goal will be to give a better explanation of this functor using a ring which admits both $\mathcal{O}_C$ and $W(k)$ as quotients.

3. Breuil–Kisin–Fargues module

Definition 3.1 (Fontaine)**.**
(1) $\mathcal{O}_C^\flat := \varprojlim_{x \to x^p} \mathcal{O}_C/p$, $C^\flat := \mathrm{Frac}(\mathcal{O}_C^\flat)$;
(2) $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$.

Fix $p^\flat \in \mathcal{O}_C^\flat$ a compatible system $(p, p^{1/p}, p^{1/p^2}, ...)$ and take $[p^\flat] \in A_{\mathrm{inf}}$ the Teichmüller lift. There exists a quotient map $\theta : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C$ which deforms the quotient $\mathcal{O}_C^\flat \twoheadrightarrow \mathcal{O}_C/p$. The kernel of θ is the ideal generated by $\xi := (p - [p^\flat])$. Notice that by the functoriality of the Witt vectors we also have a quotient $A_{\mathrm{inf}} \twoheadrightarrow W(k)$ with kernel $([p^\flat])$. Finally, as $\mathcal{O}_C^\flat$ is perfect, we have $A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C^\flat$ which kills p.

(Beautiful picture of $\mathrm{Spa}(A_{\mathrm{inf}})$ with the points x_k, x_C, $x_{C^\flat}$ and $\kappa(x) := \frac{\log|[p^\flat](\widetilde{x})|}{\log|p(\widetilde{x})|}$).

Consider $\mathcal{Y} := \mathrm{Spa}(A_{\mathrm{inf}}) \setminus \{x_k\}$. It is an analytic adic space. We will use $\mathcal{Y}$ to interpret the pairs (T, W) (classifying p-divisible groups) in a geometric way, namely as *modifications* of trivial vector bundles. Ultimately, we want to attach to such a pair a *Breuil–Kisin–Fargues module*. We recall the definition.

Definition 3.2. A *BKF-module* is a pair (M, φ_M), where M is a finite free A_{inf}-module and $\varphi_M : M[\frac{1}{\xi}] \xrightarrow{\sim} M[\frac{1}{\varphi(\xi)}]$ is a φ-linear isomorphism.

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[4] One should actually check that the adic group law gives a formal group law. This is an easy computation on morphisms between open balls.

[5] We consider the covariant Dieudonné theory.

We will spend the rest of the talk explaining next theorem.

Theorem 3.3 (Fargues)**.** *There exists an equivalence of categories*

$$\left\{ \textit{BKF-modules } (M, \varphi_M) \textit{ s.t. } M \subseteq \varphi_M(M) \subseteq \frac{1}{\varphi(\xi)} M \right\} \xrightarrow{\sim} \{(T, W)\}.$$

Remark 3.4. Using Bhatt–Morrow–Scholze integral p-adic Hodge theory one can also prove that if (M, φ_M) is the BKF-module associated to a p-divisible group $G/\mathcal{O}_C$, then $(M \otimes_{A_{\inf}} W(k), \varphi_M \otimes_{A_{\inf}} W(k))$ is the covariant Dieudonné module of G_k.

To construct the functor one introduces another category, which is related to both categories.

Definition 3.5. A *Shtuka* over $\mathrm{Spa} C^\flat$ (with one leg at $\varphi^{-1}(x_C)$) is a vector bundle $\mathcal{E}$ over $\mathcal{Y}_{[0,\infty)}$ together with an isomorphism $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E}|_{\mathcal{Y}_{[0,\infty)} \setminus \varphi^{-1}(x_C)} \to \mathcal{E}|_{\mathcal{Y}_{[0,\infty)} \setminus \varphi^{-1}(x_C)}$ meromorphic at $\varphi^{-1}(x_C)$.

We have the following result.

Theorem 3.6 (Kedlaya)**.** *The natural functor*

$$\{\textit{BKF-modules}\} \to \left\{ \textit{Shtukas over } \mathrm{Spa}(C^\flat) \right\}$$

is an equivalence.

This theorem is rather technical and we will not comment it too much. To construct a quasi-inverse one has to pass from a Shtuka, which is something defined over $\mathcal{Y}_{[0,\infty)}$, to a BKF-module, which is defined on the entire $\mathrm{Spa}(A_{\inf})$. Thus it is a result on the extension of vector bundles.

Another result on Shtukas we will need is the following one.

Theorem 3.7 (Kedlaya–Liu)**.** *If $\mathcal{E}$ is a Shtuka, then $\mathcal{E}_{x_{C^\flat}} \simeq^{\varphi^{-1}} T_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{Y}, x_{C^\flat}}$ where $T_{\mathcal{E}}$ is some finite free $\mathbb{Z}_p$-module (the φ^{-1}-action is on the right).*

The theorem is not too surprising. We know that $(\mathcal{O}_{\mathcal{Y}, x_{C^\flat}})^\wedge \simeq W(C^\flat)$ and, by a classical theorem of Katz, if R is a perfect ring, a φ-module over $W(R)$ is the same as an étale $\mathbb{Z}_p$-local system over $\mathrm{Spec}(R)$. With this theorem we have a way to associate to a Shtuka $\mathcal{E}$ a finite free $\mathbb{Z}_p$-module $T_{\mathcal{E}}$. We can also extend the previous isomorphism.

Corollary 3.8. *There is a unique φ^{-1}-equivariant isomorphism*

$$\iota_{\mathcal{E}} : \mathcal{E}_{\mathcal{Y}'} \simeq T_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{Y}'}$$

extending the isomorphism of the theorem, where $\mathcal{Y}' := \mathcal{O}_{\mathcal{Y}_{[0,\infty)} \setminus \bigcup_{n \geq 0} \varphi^n(x_C)}$. Moreover, the isomorphism is meromorphic at the points $\varphi^n(x_C)$ for $n \geq 0$.

Notice that we have chosen to work with φ^{-1} so that for every $r \geq 0$, we have that φ^{-1} is an endomorphism of $\mathcal{Y}_{[0,r]}$. In the proof of the corollary the idea is to extend $\iota_{\mathcal{E}}$ to a small $\mathcal{Y}_{[0,r]}$ first and then use the observation that every quasi-compact open of $\mathcal{Y}'$ is eventually in $\mathcal{Y}_{[0,r]}$ after apply φ^{-1} enough times.

The next step is to construct a lattice $\Xi_{\mathcal{E}}$, which for the BKF-modules appearing in Theorem 3.3 will be related to the C-subvectorspace W (see Remark 3.14). To construct a lattice we need the following result.

Lemma 3.9 (Beauville–Laszlo)**.** *Let R be a ring, f a non-zero-divisor, $\hat{R}$ the f-adic completion of R. There is an equivalence of grupoids*

$$\mathbf{Vec}(R) \simeq \mathbf{Vec}(R[f^{-1}]) \times_{\mathbf{Vec}(\hat{R}[f^{-1}])} \mathbf{Vec}(\hat{R}).$$

Remark 3.10. In other words the lemma is saying that the datum of a vector bundle over R is the same as the one of vector bundles over $R[f^{-1}]$ and $\hat{R}$ together with an isomorphism of the two over $\hat{R}[f^{-1}]$.

Before passing to $\mathcal{Y}$ let us make explicit the classification of modifications of trivial vector bundles over a down-to-earth curve.

Corollary 3.11. *Let X/k be a smooth curve, $x \in X(k)$ and $U := X \setminus \{x\}$, $n > 0$, then there is an equivalence of categories*

$$\left\{ (\mathcal{V}, \alpha) \middle| \begin{array}{c} \mathcal{V} \textit{ vector bundle/ } X \\ \alpha : \mathcal{O}_U^{\oplus n} \to \mathcal{V}|_U \end{array} \right\} \xrightarrow{\sim} \{ \ k[[t]]\textit{-lattices of } k((t))^{\oplus n} \}.$$

The equivalence sends $(\mathcal{V}, \alpha) \mapsto \mathcal{V}|_{(\mathcal{O}_{X,x})^\wedge} \subseteq_\alpha k((t))^{\oplus n}$.

In our case, notice that the ring $(\mathcal{O}_{\mathcal{Y},x_C})^\wedge$ is the ξ-adic completion of $A_{\mathrm{inf}}[1/p]$, and it is denoted by B^+_{dR}. Its fraction field (obtained by inverting ξ) is denoted by B_{dR}. This ring has the property that n-th graded pieces $\xi^n B^+_{\mathrm{dR}}/\xi^{n+1} B^+_{\mathrm{dR}} \simeq C(n)$.

Construction 3.12. We start with a Shtuka $\mathcal{E}$. As $\iota_\mathcal{E}$ is meromorphic at x_C, we restrict $\iota_\mathcal{E}$ to an isomorphism over the punctured disk $(\mathcal{O}_{\mathcal{Y},x_C})^\wedge[\xi^{-1}]$, namely an isomorphism $\widehat{\mathcal{E}}_{x_C} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}} \simeq T_\mathcal{E} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$. We denote by $\Xi_\mathcal{E} \subseteq T_\mathcal{E} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ the B^+_{dR}-lattice given by the image of $\widehat{\mathcal{E}}_{x_C}$ in $\widehat{\mathcal{E}}_{x_C} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$.

Proposition 3.13. *The functor*

$$\left\{ \textit{Shtukas over } \mathrm{Spa}(C^\flat) \right\} \to \left\{ (T, W) \middle| \begin{array}{c} T \textit{ is a free } \mathbb{Z}_p\textit{-module of finite rank} \\ \Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \textit{ is a } B^+_{\mathrm{dR}}\textit{-lattice.} \end{array} \right\}$$

which sends $\mathcal{E}$ to $(T_\mathcal{E}, W_\mathcal{E})$ is an equivalence of categories.

Remark 3.14. There is a fully faithful functor $\{(T, W)\} \to \{(T, \Xi)\}$ which sends (T, W) to (T, Ξ_W) where Ξ_W is the unique B^+_{dR}-lattice such that

$$T \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}} \subseteq \Xi_W \subseteq \xi^{-1}(T \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}})$$

and the image in $\xi^{-1}(T \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}})/(T \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}}) = T \otimes C(-1)$ is W. Notice that the last identification holds because $\xi^{-1} B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} = C(-1)$.

End of the proof of Theorem 3.3. Putting together Theorem 3.6, Proposition 3.13 and Remark 3.14 we get Theorem 3.3.