Recall: $\Lambda \subseteq \mathbb{C}$ lattice $\quad E = \mathbb{C}/\Lambda$

$\wp : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ holomorphic Λ-invariant.
with a double in each $\lambda \in \Lambda$
without poles outside Λ.

$$(\wp, \wp') : \mathbb{C} \setminus \Lambda \longrightarrow E \setminus \{[0]\} \longrightarrow \mathbb{C}^2$$
$$z \longmapsto (\wp(z), \wp'(z)).$$

The image is contained in the set

$$\{(x,y) \in \mathbb{C}^2 : y^2 = 4x^3 - g_2 x - g_3\}$$

$$g_2 = 60\, G_4 \qquad G_k = \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^k} \quad (k \geq 3).$$
$$g_3 = 140\, G_6$$

Thm: The map $(\wp, \wp')$ induces a biholomorphism of E with the compact Riemann surface

$$C = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) : x_0 x_2^2 = 4x_1^3 - g_2 x_1 x_0^2 - g_3 x_0^3\}.$$

Proof: We already saw: the map is injective.
(Surjective) Let $(x,y) \in \mathbb{C}^2$ be st.

$$y^2 = 4x^3 - g_2 x - g_3.$$

Look at function $f(z) := \wp(z) - x$; it is an even elliptic function with a pole of order 2. Therefore it has two zeroes (with multiplicity) z_1, z_2,

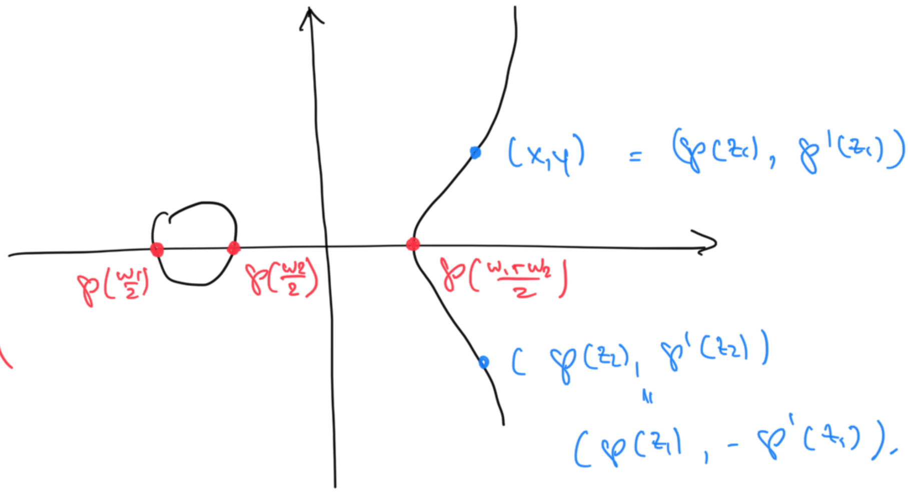
$$z_1 + z_2 \equiv 0 \pmod{\Lambda}.$$

We know that $y_i := \wp'(z_i)$ satisfies the equation

$$y^2 = 4x^3 - g_2 x - g_3$$

This equation has two zeroes : if $y \neq \wp(z_1)$, then $y = \wp'(z_2), = -\wp'(z_1)$.

$$y^2 = 4x^3 - g_2 x - g_3$$



for some $w_1, w_2 \in \Lambda$ basis of Λ.

This is the real locus of $y^2 = 4x^3 - g_2 x - g_3$.

This is the case where $4x^3 - g_2 x - g_3$ has 3 real zeroes.

In order to show that it is a biholomorphism, it suffices to see that C has no singular points. Indeed, a bijective holomorphic map between compact Riemann surfaces is a biholomorphism. Look at the equation

$$y^2 = 4x^3 - g_2 x - g_3$$

$$\begin{cases} \frac{\partial}{\partial x} : \; 12x^2 - g_2 = 0. \\ \frac{\partial}{\partial y} : \; 2y = 0 \longrightarrow y = 0. \end{cases}$$

Now singular points are of the form $(x, 0)$

with x a common zero of

$$P(x) = 4x^3 - g_2 x - g_3.$$

and $$P'(x) = 12x^2 - g_2.$$

But we saw that P has three pairwise distinct roots. There is no such a $x \in \mathbb{C}$.

Let's look at infinity (the only point at infinity is $[0:1:0]$):

$$u = \frac{x}{y} \qquad v = \frac{1}{y} \qquad v = 4u^3 - g_2 u v^2 - g_3 v^3$$

We're interested in the point $(0,0)$.

$\frac{\partial}{\partial v}$: $1 - 2g_2 uv - 3g_3 v^2$ does not vanish at $(0,0)$

So the curve is non singular.

<u>Another proof</u>: Consider the meromorphic 1-form on C given by $\frac{dx}{y} = \omega$.

This form is holomorphic every on C: by deriving the equation of C, we obtain

$$2y\,dy = (12x^2 - g_2)\,dx$$

$$\frac{dx}{y} = 2\frac{dy}{12x^2 - g_2}.$$

Since $y=0$ and $12x^2 - g_2 = 0$ have no common solution in C, then ω is holomorphic everywhere (maybe except infinity?)

We have to check at infinity.

$$u = \frac{x}{y} \qquad x = uy = \frac{u}{v} \qquad dx = \frac{v\,du - u\,dv}{v^2}$$

$$v = \frac{1}{y} \qquad dv = -\frac{dy}{y^2} \qquad dy = -\frac{dv}{v^2}.$$

$$\omega = 2\left(-\frac{dv}{v^2}\right)\cdot \frac{1}{12\left(\frac{u}{v}\right)^2 - g_2}$$

$$= -2\,\frac{dv}{12u^2 - g_2 v^2} \overset{(*)}{=} \longleftarrow$$ this does not permit to conclude.

$$\frac{dx}{y} = \cancel{v}\cdot \frac{v\,du - u\,dv}{\cancel{v^2}} = du - \frac{u}{v}\,dv \overset{(**)}{=}$$

Derive the equation of C at infinity.

$$\boxed{v = 4u^3 - g_2 uv^2 - g_3 v^3}$$

$$dv\left(1 + 2g_2 uv + 3g_3 v^2\right) = du\left(12u^2 - g_2 v^2\right)$$

$$\frac{dv}{12u^2 - g_2 v^2} = \frac{du}{1 + 2g_2 uv + g_3 v^2}$$

Now we can conclude in two ways:

$$\overset{(*)}{=} \frac{du}{1 + 2g_2 uv + g_3 v^3}$$ which is holomorphic at $(u,v) = (0,0)$.

$$\overset{(**)}{=} du\left(1 - \frac{u}{v}\cdot\frac{1}{1 + 2g_2 uv + g_3 v^3}\right) = \ldots$$

Rule: $C_0 \subseteq \mathbb{C}^2 \xrightarrow{\;y\;} \mathbb{C}$ holomorphic

$(x,y) \longmapsto y$

$\{y^2 = 4x^3 - g_2 x - g_3\}$ $\Rightarrow$ $\frac{1}{y}$ meromorphic function on C, and it has a triple zero at infinity.
(it has three poles on the points with $y=0$).

$$(\wp, \wp')^* \omega = \frac{d(\wp(z))}{\wp'(z)} = \frac{\cancel{\wp'(z)}}{\cancel{\wp'(z)}} dz = dz.$$

So the map $(\wp, \wp')$ is injective on the tangent spaces. !

<u>Point of view with Riemann-Roch:</u>

X compact Riemann surface

$X \supseteq D$ divisor $\qquad D = \sum_{i=1}^{n} \underset{\mathbb{Z}}{\underset{\cap}{\overset{\mathrm{mult}_{x_i}(D)}{m_i}}} [x_i] \qquad x_1, \dots, x_n \in X.$

$$\underset{\mathcal{L}(D)}{\overset{\|}{H^0(X, \mathcal{O}(D))}} = \{ f : X \to \mathbb{P}^1(\mathbb{C}) : \underset{\sum_{x \in X} \mathrm{ord}_x(f)\,[x]}{\overset{\|}{\mathrm{div}(f)}} + D \geq 0 \}$$

$$\mathrm{ord}_x(f) + \mathrm{mult}_x(D) \geq 0.$$

<u>e.g.</u> $H^0(X, \mathcal{O}(n[x])) = \{ f : X \to \mathbb{P}^1(\mathbb{C}) : \mathrm{ord}_x(f) \geq -n \}$

f is holomorphic outside $\{x\}$ and it has a pole at worst of order n at x.

$$H^0(X, \Omega^1(D)) = \{ \omega \text{ meromorphic form} : \underset{\sum_{x \in X} \mathrm{ord}_x(\omega)\,[x]}{\overset{\|}{\mathrm{div}(\omega)}} + D \geq 0 \}.$$

<u>Th</u> (Riemann-Roch). Let D be a divisor on X. Then $H^0(X, \mathcal{O}(D))$ and $H^0(X, \Omega^1(-D))$ are finite dimensional

complex vector space and

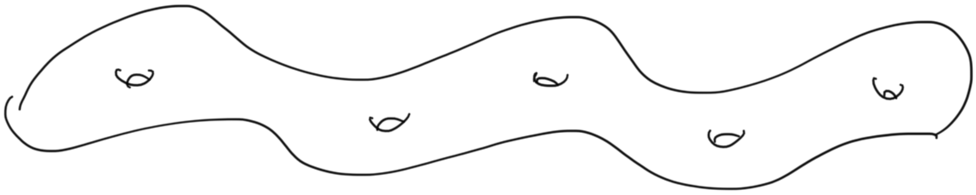
$$h^0(X, \mathcal{O}(D)) - h^0(X, \Omega^1(-D)) = \deg D + 1 - g.$$

$$\underset{\dim H^0(X, \mathcal{O}(D))}{\shortparallel} \qquad \underset{\dim H^0(X, \Omega^1(-D))}{\shortparallel}$$

$$D = \sum_{i=1}^{n} m_i [x_i] \qquad \deg D = \sum_{i=1}^{n} m_i.$$

g = genus of X = $h^0(X, \Omega^1_X)$

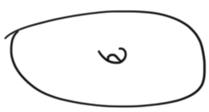
= # holes of X



genus 5

<u>Example:</u> $E = \mathbb{C}/\Lambda$ elliptic curve.

$\underset{\text{homeomorphically}}{\simeq} (\mathbb{R}/\mathbb{Z})^2$



So E is a compact Riemann surface of genus 1.

$O_E = [0] \in E$. Consider $D = [0]$.

$H^0(E, \mathcal{O}(0 \cdot D))$ = { meromorphic function on E such that $\operatorname{div}(D) \geq 0$ }

= { holomorphic functions E }

= { elliptic functions without poles } $= \mathbb{C}$

$H^0(E, \mathcal{O}(D))$ = { meromorphic functions on E

with at most a simple pole in 0}.

$= \left\{ \begin{array}{l} \text{elliptic functions with at most} \\ \text{a simple pole} \end{array} \right\} = \mathbb{C}$

$$H^0(E, \mathcal{O}(2D)) = \left\{ \begin{array}{l} \text{elliptic functions with at most} \\ \text{a double pole in } 0 \end{array} \right\}$$

$$= \mathbb{C} \oplus \mathbb{C}\wp = \{a\wp + b : a, b \in \mathbb{C}\}.$$

$$H^0(E, \mathcal{O}(3D)) = \langle 1, \wp, \wp' \rangle$$

$$H^0(E, \mathcal{O}(4D)) = \langle 1, \wp, \wp^2, \wp' \rangle.$$

$$H^0(E, \mathcal{O}(5D)) = \langle 1, \wp, \wp^2, \wp', \wp\wp' \rangle.$$

$$H^0(E, \mathcal{O}(6D)) = \langle \overset{1}{1}, \overset{2}{\wp}, \overset{3}{\wp^2}, \overset{4}{\wp'}, \overset{5}{\wp\wp'}, \overset{6}{\wp'^2}, \overset{7}{\wp^3} \rangle \quad (*)$$

RR: $h^0(E, \mathcal{O}(6D)) - \overbrace{h^0(E, \underbrace{\Omega^1(-6D)}_{\deg < 0})}^{0} = 6$

Recall: $\deg D < 0 \longrightarrow h^0(X, \mathcal{O}(-D)) = 0.$

$\deg \Omega^1_X(-D) = 2g - 2 - \deg D \underset{g=1}{=} -\deg D$

$$\Longrightarrow \quad \boxed{h^0(E, \mathcal{O}(6D)) = 6.}$$

(*) We have the linear relation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3.$$

Remark: On a compact Riemann surface X of genus g, a divisor D of degree $\geq 2g+1$ is <u>very ample</u>, that is the map

$$X \longrightarrow \mathbb{P}(H^0(X,D)^*)$$ dual

$$x \longmapsto H_x = \{f \in H^0(X,D) : \mathrm{ord}_x(f) > -\mathrm{mult}_x(D)\}$$

↳ this has dimension one less than $H^0(X,D)$

is a biholomorphism of X with its image.

<u>Example:</u> $E = \mathbb{C}/\Lambda$, $g=1$, $2g+1 = 3$

$$E \longrightarrow \mathbb{P}(H^0(E, \mathcal{O}(3[0]))^*) \cong \mathbb{P}^2(\mathbb{C})$$

basis $1, \wp, \wp'$

This is the map that we already defined.

<u>Maps between elliptic curves</u>

Let $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$ be lattices. If $\alpha \in \mathbb{C}$ is such that $\alpha\Lambda_1 \subseteq \Lambda_2$ then we obtain a holomorphic map

$$\begin{aligned} \varphi_\alpha : \mathbb{C}/\Lambda_1 &\longrightarrow \mathbb{C}/\Lambda_2 \\ [z] &\longmapsto [\alpha z]. \\ [0] &\longmapsto [0]. \end{aligned}$$

<u>Th:</u> The map

$$\{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subseteq \Lambda_2\} \longrightarrow \{f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2 : f \text{ is holomorphic and } f(0)=0\}$$

$$\alpha \longmapsto \varphi_\alpha$$

is a bijection.

<u>Proof:</u> (Injective). Let $\alpha, \beta \in \mathbb{C}$ s.t. $\alpha\Lambda_1 \subseteq \Lambda_2$ and $\beta\Lambda_1 \subseteq \Lambda_2$. Suppose $\varphi_\alpha = \varphi_\beta$, which means

$$\alpha z \equiv \beta z \pmod{\Lambda_2} \quad \text{for all } z \in \mathbb{C}.$$

$\Longleftrightarrow \quad (\alpha-\beta)z \in \Lambda_2$ but $(\alpha-\beta)z$ is a holomorphic function: if non constant the image is open; but $\Lambda_2 \subseteq \mathbb{C}$ is closed. So $(\alpha-\beta)z$ is constant.

$\Longrightarrow \quad \alpha-\beta = 0.$

(Surjective) From the topological point of view

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C}/\Lambda \\ \cong & & \cong \\ \mathbb{R}^2 & \longrightarrow & (\mathbb{R}/\mathbb{Z})^2 \end{array}$$

$\mathbb{R}^2$ is contractible $\Downarrow$ $\mathbb{C}$ is the universal cover of $\mathbb{C}/\Lambda$.

Let $f: \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a holomorphic mapping.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Lambda_1 \arrow[r, "f"] & \mathbb{C}/\Lambda_2 \end{tikzcd}$$

the map f lifts to a continuous map $\tilde{f}: \mathbb{C} \to \mathbb{C}$. s.t. $\tilde{f}(0) = 0$. because $f(0)=0$.

Rmk: $\tilde{f}$ is holomorphic (look at the definition of the atlas of $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$).

For $\omega \in \Lambda_1$, and $z \in \mathbb{C}$

$$\tilde{f}(z+\omega) \equiv \tilde{f}(z) \pmod{\Lambda_2}.$$

$$\Longleftrightarrow \quad \tilde{f}(z+\omega) - \tilde{f}(z) \in \Lambda_2$$

By the same argument that we did before, we obtain that

$g(z) = \tilde{f}(z+\omega) - \tilde{f}(z)$ is constant

$\longrightarrow \quad g'(z) = 0 \quad \longrightarrow \quad \tilde{f}'(z+\omega) = \tilde{f}'(z)$ for all $z \in \mathbb{C}$ and $\omega \in \Lambda_1$

$\Longrightarrow$ $\tilde{f}'$ is elliptic and holomorphic everywhere.

$\Longrightarrow$ $\tilde{f}'$ is constant. Call $\alpha \in \mathbb{C}$ its value.

Therefore we obtain that $\tilde{f}$ is of the form:

$$\tilde{f}(z) = \alpha z + \beta.$$

$$\longrightarrow \tilde{f}(0) = \beta = 0 \Rightarrow \tilde{f}(z) = \alpha z. \qquad \square$$

<u>Cor:</u> Let $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$ be lattices. Then $E_1 = \mathbb{C}/\Lambda_1$ and $E_2 = \mathbb{C}/\Lambda_2$ are biholomorphic if and only if $\exists a \in \mathbb{C}^*$ s.t. $a\Lambda_1 = \Lambda_2$.

<u>Isomorphism classes of elliptic curves.</u>

$\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$ Poincaré's upper half plane.

$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

<u>Lemma:</u> The following maps

$$\begin{array}{ccccc} \mathbb{H}/SL_2(\mathbb{Z}) & \xrightarrow{\alpha} & \{\text{lattices in } \mathbb{C}\}/\mathbb{C}^* & \xrightarrow{\beta} & \{(0,E) : E \text{ elliptic curve}, 0 \in E\} \\ [\tau] & \longmapsto & [\mathbb{Z} \oplus \mathbb{Z}\tau] & & \\ & & [\Lambda] & \longmapsto & [([0], \mathbb{C}/\Lambda)] \end{array}$$

by def $E = \mathbb{C}/\Lambda$

are well-defined and bijective.

<u>Proof:</u> We already studied β. Let's study α.

(Well-defined). $\tau, \tau' \in \mathbb{C}$ s.t. $\tau' = \dfrac{a\tau + b}{c\tau + d}$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Take $\lambda = (c\tau + d)$.

$$\begin{aligned}
\lambda : \quad \mathbb{Z} \oplus \mathbb{Z}\tau' &\longrightarrow \mathbb{Z} \oplus \mathbb{Z}\tau. \\
1 &\longmapsto \lambda = c\tau + d \\
\tau' &\longmapsto \lambda\tau' = a\tau + b \in \mathbb{Z} \oplus \mathbb{Z}\tau.
\end{aligned}$$

Moreover

$$\tau = \frac{d\tau' - b}{-c\tau' + a} \longrightarrow \text{multiplication by } \lambda$$

is a bijection $\mathbb{Z} \oplus \mathbb{Z}\tau \simeq \mathbb{Z} \oplus \mathbb{Z}\tau'$.

Therefore α is well-defined.

(Injective). Let $\lambda \in \mathbb{C}^*$ be such that

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \\
\cup & & \cup \\
\mathbb{Z} \oplus \mathbb{Z}\tau' & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}\tau \\
 & & \Downarrow \\
1 & \longmapsto & \lambda =: c\tau + d \qquad c, d \in \mathbb{Z}. \\
\tau' & \longmapsto & \lambda\tau' =: a\tau + b \qquad a, b \in \mathbb{Z} \\
 & & \Uparrow \\
 & & \mathbb{Z} \oplus \mathbb{Z}\tau
\end{array}$$

the fact that multiplication by λ gives an isomorphism
$\mathbb{Z} \oplus \mathbb{Z}\tau' \simeq \mathbb{Z} \oplus \mathbb{Z}\tau$ implies

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1. \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

(Surjective). $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

$$\operatorname{sign} \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) = - \operatorname{sign} \operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right).$$

We may suppose $\operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$.

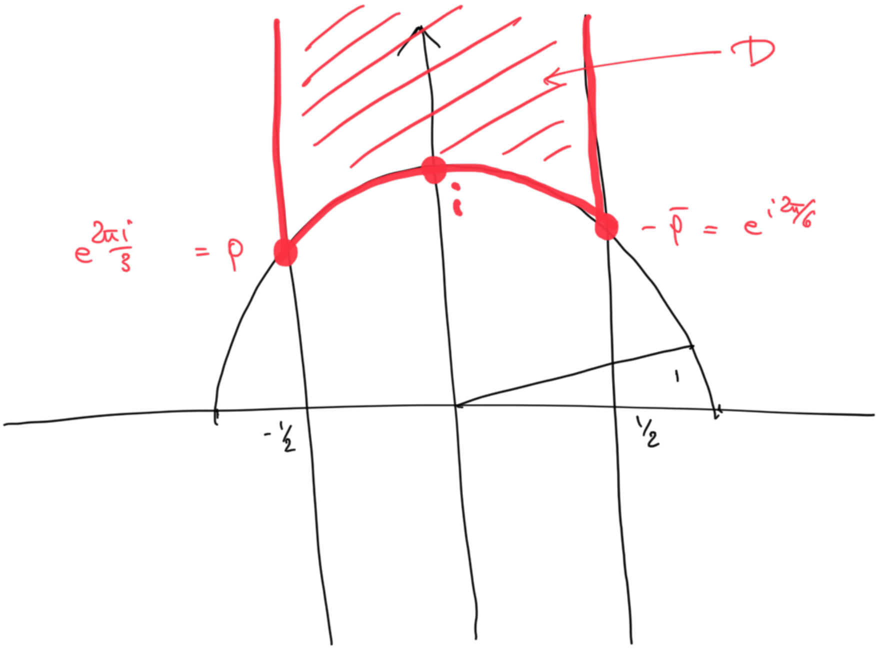
$$\frac{1}{\omega_2}\Lambda = \mathbb{Z}\underbrace{\left(\frac{\omega_1}{\omega_2}\right)}_{\tau} \oplus \mathbb{Z}. \Rightarrow \operatorname{Im}(\tau) > 0. \qquad \square$$

Fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$.

Consider the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

$$D = \{ z \in \mathbb{H} : |\mathrm{Re}(z)| \le \tfrac{1}{2}, \ |z| \ge 1 \}$$



Th: (1) For each $\tau \in \mathbb{H}$, there is $g \in SL_2(\mathbb{Z})$ s.t. $g\tau \in D$

(2) if $\tau, \tau' \in D$ lie in the same $SL_2(\mathbb{Z})$-orbit, then

$\mathrm{Re}(\tau) = \pm\frac{1}{2}$ and $\tau' = \tau \pm 1$

or $|\tau| = 1$ and $\tau' = -\frac{1}{\tau}$

(3) for $\tau \in D$

$$\mathrm{Stab}_{\underset{SL_2(\mathbb{Z})/\{\pm id\}}{\overset{\shortparallel}{PSL_2(\mathbb{Z})}}}(\tau) = \begin{cases} \text{trivial} & \text{if } \tau \ne i, \rho, -\bar{\rho} \\ \# \langle S \rangle = 2 & \tau = i \\ \# \langle ST \rangle = 3 & \tau = \rho \\ \# \langle TS \rangle = 3 & \tau = -\bar{\rho} \end{cases}$$

Proof: Attend "Modular forms" or [Serre, A course in arithmetic, Chap. VII, Th 1 and 2].