Algebraic curves

Regular rings of dimension 1

def: 1) Let A be an integral domain. We say that A is normal if it is integrally closed in $\mathrm{Frac}(A)$, i.e. if $x \in \mathrm{Frac}(A)$ is such that

$$x^d + a_{d-1}x^{d-1} + \dots + a_0 = 0 \qquad a_0, \dots, a_{d-1} \in A$$

then $x \in A$.

2) Let A be a ring. The (Krull) dimension of A is

$$\dim(A) = \sup \left\{ n \in \mathbb{N} : \begin{array}{l} \text{there exists a chain of} \\ \text{prime ideals} \\ \quad \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \end{array} \right\}$$

(e.g. fields are of dimension 0).

⚠ There are Noetherian rings of infinite dimension (Nagata).

Prop. Let $(A, \underset{\text{max. ideal}}{\mathfrak{m}})$ be a Noetherian local ring. Then,

$$\dim(A) \leq \dim_k \left(\mathfrak{m}/\mathfrak{m}^2 \right). \qquad k = A/\mathfrak{m} \text{ residue}$$

$\uparrow$ k-vector space of finite dimension because $\mathfrak{m}$ finitely generated

Proof: [Stacks, Proposition 10.59.8]. $\square$

Example. $A = k[\varepsilon] = k[t]/(t^2)$

It is local and the maximal ideal $\mathfrak{m} = (t)$ is nilpotent.

$$\dim(A) = 0 \quad \text{but} \quad m/m^2 \simeq k\varepsilon$$
$$\Downarrow$$
$$\dim_k(m/m^2) = 1.$$

<u>def:</u> A local Noetherian ring (A, m) is <u>regular</u> if

$$\dim(A) = \dim_k(m/m^2).$$

<u>Prop.</u> Let (A, m) be a local Noetherian ring of dimension 1. The following are equivalent:

1) A is normal;
2) A is regular;
3) A is a discrete valuation ring (DVR).

<u>def:</u> An integral domain A is a DVR if there exists a map $v : \mathrm{Frac}(A) \longrightarrow \mathbb{Z} \cup \{+\infty\}$ s.t.

1) $v(a) = \infty \iff a = 0$
2) $0 \neq a \in \mathrm{Frac}(A) \implies a \in A$ or $\frac{1}{a} \in A$.
3) $v(ab) = v(a) + v(b)$
4) $v(a+b) \geq \min(v(a), v(b))$.

<u>Recall:</u> A DVR A is local Noetherian and

$$A = \{a \in \mathrm{Frac}(A) : v(a) \geq 0\}$$

$$m = \{a \in A : v(a) > 0\} = (\pi)$$

$\uparrow$ a generator is called a uniformizer.

<u>Examples:</u> 1) $\mathbb{Z}(p) = \left\{ \frac{a}{p^i} : a \in \mathbb{Z} \right\} \subseteq \mathbb{Q}.$

$$v\left(\frac{a}{b}\right) = v\left(\frac{p^i a'}{p^j b'}\right) = i - j.$$

with $p \nmid a'b'$

a, b integers with ...

2) k field, $A = k[[t]]$ = formal power series w/ coefficients in k

$$v(f) = \mathrm{ord}_0(f) \qquad f \in k[[t]]$$

$$\mathfrak{m} = (t)$$

Proof: 2) $\Rightarrow$ 3) Since A is regular of dimension 1, the maximal ideal $\mathfrak{m} \subseteq A$ is principal. By Nakayama, it is generated by any element $\in \mathfrak{m} \setminus \mathfrak{m}^2$.
Let $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$. Again by Nakayama

$$\mathfrak{m}^i \setminus \mathfrak{m}^{i+1} = A^\times \pi^i$$

$$\Longrightarrow \quad \mathrm{Frac}(A) = A\left[\tfrac{1}{\pi}\right].$$

$$v : \mathrm{Frac}(A) \longrightarrow \mathbb{Z} \cup \{\infty\}$$
$$x \longmapsto \sup\{n \in \mathbb{Z} : x \in \mathfrak{m}^n\}$$

(for $n < 0$, $\mathfrak{m}^n = A \cdot \frac{1}{\pi^i} \subseteq \mathrm{Frac}(A)$.).

Exercise: it is a valuation. and A is DVR with v.

3) $\Rightarrow$ 1) Let $x \in \mathrm{Frac}(A)^\times$ t.q.

$$x^d + a_{d-1}x^{d-1} + \dots + a_0 = 0 \qquad a_0, \dots, a_{d-1} \in A$$

So, if $v(x) \geq 0$, then we're done. Suppose by contradiction, $v(x) < 0$.

$$a_0 = -\left(\sum_{i=1}^{d-1} a_i x^i + x^d\right).$$

$$v(a_i x^i) = v(a_i) + i\,v(x) \geq i v(x) > d\,v(x)$$

$$v(\ldots) \qquad \underbrace{}_{a_i \in A} \ \geq 0 \qquad\qquad i < d$$

$$v\left(\sum_{i=1}^{d-1} a_i x^i + x^d\right) \underset{\uparrow}{=} v(x^d) = d\, v(x) < 0$$

$$v(a_0) \geq 0 \qquad\qquad v(x) \neq v(y) \Rightarrow v(x+y) = \min(v(x), v(y)).$$

1) $\Rightarrow$ 2) We have to show that $\mathfrak{m}$ is principal: it suffices (Contradiction!) to find $x \in \mathrm{Frac}(A)$ s.t. $x\mathfrak{m} = A$. ($\Rightarrow \mathfrak{m} = (x^{-1})$.)

<u>Notation</u>: if $I \subseteq A$ ideal, then

$$I^{-1} = \{x \in \mathrm{Frac}(A) : xI \subseteq A\} \supseteq A.$$

Take $x \in \mathfrak{m}^{-1}$. There are two possibilities:

$$x\mathfrak{m} \subseteq \mathfrak{m} \qquad \text{or} \qquad \underbrace{x\mathfrak{m} = A}_{\text{this is the thing we want}}.$$

Suppose $x\mathfrak{m} \subseteq \mathfrak{m}$. By looking at $\mathfrak{m}$ as a finitely generated A-module (A Noetherian) by Cayley-Hamilton there is a monic polynomial $P \in A[t]$ s.t.

$$P(x) = 0.$$

Since A is normal by hypothesis, we have $x \in A$.

In order to obtain a contradiction, we have to show that

$$A \subsetneq \mathfrak{m}^{-1}.$$

Consider the following set of ideals:

$$\mathfrak{I} = \{ I \subseteq A \text{ ideal s.t. } A \subsetneq I^{-1} \}$$

$\mathcal{F}$ is non-empty because it contains the principal ideals and A is not a field.

$\underset{A \text{ Noeth.}}{\Longrightarrow}$ there is a maximal element $I \in \mathcal{F}$.

<u>Claim:</u> The ideal I is prime. ($\Rightarrow I = m$).

<u>Proof of the Claim.</u> Let $x, y \in A$ s.t. $xy \in I$ and $x \notin I$.

$$(x) + I \overset{?}{\neq} I \underset{\text{maximality}}{\Longrightarrow} (I + (x))^{-1} = A.$$

Let $z \in I^{-1} \setminus A$.

$$zy\left((x)+I\right) = z\Big(\underset{\substack{\cup\!\mathsf{I} \\ I}}{(xy)} + yI\Big)$$

$$\subseteq I^{-1}I + I^{-1}I \subseteq A.$$

$$\Longrightarrow zy \in (I + (x))^{-1} = A.$$

$$\Longrightarrow z \in (I + (y))^{-1} \quad \text{because } yz \in A \text{ and } z \in I^{-1}.$$

$$\underset{\substack{\cup\!\!\!\!\not\;\;\\ A}}{I^{-1} \subseteq (I + (y))^{-1}} \ni z \text{ not in } A \underset{\text{max}}{\Longrightarrow} I = I + (y)$$

$$\Downarrow$$

$$y \in I. \quad \square$$

<u>def:</u> A ring A is a Dedekind ring if it is Noetherian, <u>normal</u> and of dimension 1.

$\Downarrow$ integral domain.

<u>Rmq:</u> Let A be a Noetherian integral domain of dimension 1. Then the following are equivalent:

1) A ~~ext~~ is normal (i.e. A is a Dedekind ring);

2) for all max ideal $m \subseteq A$, the localization A_m at m is a DVR;

3) for all max ideal $m \subseteq A$,

$$\dim_{A/m} (m/m^2) = 1.$$

[Proof: Normal is a local property.].

Prop. Let A be a Dedekind ring with fraction field K. Let L be a finite separable extension of K. Then, the integral closure B of A in L,

$$B = \{ x \in L : P(x) = 0 \text{ for a monic polynomial } P \in A[t] \}$$

is a Dedekind ring and it is a f.g. A-module.

Proof: Suppose first of knowing that B is a f.g. A-module. Since A is Noetherian, B is a Noeth. A-module, thus a Noetherian ring. The ring B is normal by definition and B is of dimension 1.

(Let $q \subseteq B$ non zero, then $q \cap A$ is a prime ideal, which is non zero because it contains the norms of the elements of q. Therefore,

$$\underbrace{B/q}_{\text{integral domain}} = \text{finite dimensional } A/p\text{-vector space}$$

$\uparrow$ B is f.g.

$$\Longrightarrow B/q \text{ is a field.})$$

and B is a finitely generated

We're left with showing that B is a finitely generated A-module. Since L/K is separable, the trace map

$$\begin{aligned} \langle , \rangle : \quad L \times L &\to K \\ (x, y) &\mapsto Tr_{L/K}(xy) \end{aligned}$$

is a non degenerate bilinear form.

Pick a K-basis $b_1, \dots, b_n$ of L which lies in B.

Let $b_1^*, \dots, b_n^*$ the dual basis w.r.t. $\langle , \rangle$:

$$\langle b_i, b_j^* \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Then

$$x = \sum_{j=1}^{n} \langle x, b_j \rangle b_j^*$$

$$b \in B \implies \underset{L/K}{Tr}(b) \in A$$

$$b_j, x \in B \implies \langle x, b_j \rangle = Tr(x b_j) \in A$$

$$\implies B \subseteq \underbrace{Ab_1^* + \dots + Ab_n^*}_{\substack{\text{finitely generated } A\text{-module} \\ \Rightarrow \text{ Noetherian } A\text{-mod.}}}$$

$\Rightarrow B$ is f.g. A-module. □

Rmq: The statement is false in general if the extension L/K is not separable. But it remains true if A is a finitely generated algebra over a field.

Cor: Let k be a perfect field. Let $K/k(x)$ be a finite extension. Let A be the integral closure of $k[x]$ in K. ... and a Dedekind

Then, A is a finitely generated k-algebra ...

domain.

<u>Proof:</u> If $K/_{k(x)}$ is separable, then we're done. Suppose that this is not the case. Then $\mathrm{char}(k) = p > 0$ and there is a power $q = p^e$ of p s.t.

$$K^q = \{ x^q : x \in K \}$$

is contained in the separable closure K^{sep} of $k(x)$. In particular $k(x)K^q$ is a separable extension of $k(x)$. Consider the compositum field $L = k(x^{1/q})K$.

$$\begin{tikzcd}
L \arrow[r, "q\text{-power}", "\sim"'] & K^q k(x) \\
k(x^{1/q}) \arrow[u, "sep"] \arrow[r, "\sim"] & k(x) \arrow[u, "sep."'] \\
k \arrow[u] \arrow[r, "\sim", "q\text{ power}"'] & k \arrow[u]
\end{tikzcd}$$

k perfect.

The integral closure of $k[x]$ in $k[x^{1/q}]$ is $k[x^{1/q}]$. (which is of finite type). Thus the integral closure B of $k[x]$ in L is the integral closure of $k[x^{1/q}]$ in L, and by the separable case, a finitely generated $k[x]$-module. Therefore the integral closure A of $k[x]$ in K is contained in B, thus it is a $k[x]$-module of finite type. $\square$

<u>Algebraic varieties</u>

Let k be a field.

F

def: Let X be a topological space. 1) A pre-sheaf of k-vector spaces on X is the datum for every open subset $U \subseteq X$ of a vector space $F(U)$ and, for each $V \subseteq U \subseteq X$ open subsets, of a k-linear map

$$\varphi_{UV} : F(U) \to F(V).$$

s.t. for $W \subseteq V \subseteq U$ we have

$$\varphi_{UW} = \varphi_{VW} \circ \varphi_{UV}.$$

2) A presheaf is a sheaf, if for all open subset $U \subseteq X$ and open cover $U = \bigcup_{i \in I} U_i$, the following sequence is exact

$$0 \to F(U) \to \prod_{i \in I} F(U_i) \to \prod_{i,j \in I} F(U_i \cap U_j)$$

$$f \mapsto (f|_{U_i})_{i \in I}$$

$$(g_i)_{i \in I} \mapsto (g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j}).$$

here $f|_{U_i} = \varphi_{UU_i}(f)$

def: 1) A k-locally ringed space is the datum of a topological space X and a sheaf in k-algebras s.t. for each $x \in X$, the ring (called "the local ring at x")

$$\mathcal{O}_{X,x} := \varinjlim_{\substack{x \ni U \\ U \subseteq X \\ \text{open}}} \mathcal{O}_X(U) \quad \text{is local.}$$

2) A morphism between k-locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the datum of a continuous map

$f: X \to Y$ and a homomorphism of sheaves in k-algebras $f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$, i.e. for any subset $V \subseteq Y$ a k-algebra homomorphism

$$\mathcal{O}_Y(V) \xrightarrow{f^{\#}_V} \mathcal{O}_X(f^{-1}(V))$$

s.t.

1) for $V' \subseteq V$ open, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "f^{\#}_V"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X(f^{-1}(V)) \arrow[d] \\
\mathcal{O}_Y(V') \arrow[r, "f^{\#}_{V'}"] & \mathcal{O}_X(f^{-1}(V'))
\end{tikzcd}$$

2) for all $x \in X$, the induced homomorphism

$$f^{\#}_x : \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$$
$$\mathcal{O}_{Y,f(x)} = \varinjlim_{V \ni f(x)} \mathcal{O}_Y(V), \qquad \mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U)$$

is a local homomorphism

$$f^{*}_x(m_Y) \subseteq m_X$$

m_Y: max ideal of $\mathcal{O}_{Y, f(x)}$

m_X: max idea in $\mathcal{O}_{X,x}$.

Affine varieties: let A be a k-algebra of f.t.

$\mathrm{Max}(A) = \{ m \subseteq A \text{ max ideals} \}$

$I \subseteq A$ ideal, $V(I) = \{ m \in \mathrm{Max}(A) : I \subseteq m \} = \mathrm{Max}(A/I)$

... $\mathrm{Max}(A)$: the closed subsets are the one of topology on Max(A). The closed subsets ... are of the form $V(I)$.

For $f \in A$:

$$V(f) = V(fA) = \{ m \in \mathrm{Max}(A) : f \in m \}$$
$$= \mathrm{Max}(A/fA)$$

principal open subsets

$$D(f) = \mathrm{Max}(A) \setminus V(f) = \{ m \in \mathrm{Max}(A) : f \notin m \}$$
$$= \mathrm{Max}(\underbrace{A[\tfrac{1}{f}]}_{A_f}) \quad \text{localization of } A \text{ at } f.$$

$$A_f = A[t]/(tf-1) \underset{\uparrow}{=} \left\{ \frac{a}{f^n} : n \in \mathbb{N} \right\}.$$

A integral domain.

The open subsets $D(f)$ form a basis for the topology of Max(A).

<u>Sheaf of k-algebras:</u> $X = \mathrm{Max}(A)$

$$\mathcal{O}_X(D(f)) := A_f.$$

This extends uniquely to a sheaf of k-algebras on Max(A).

In order to prove this, the key Lemma is the following:

<u>Lemma</u> (Partition of unity): Let $f_1, \dots, f_n \in A$ be s.t.

$$\mathrm{Max}(A) = \bigcup_{i=1}^n D(f_i) \longleftrightarrow (f_1, \dots, f_n) = A$$

Then there exists $a_1, \dots, a_n \in A$ s.t. $\sum a_i f_i = 1$. In particular, the sequence

$$0 \to \underset{\mathcal{O}(X)}{\overset{\shortparallel}{A}} \to \prod_{i=1}^n \underset{\mathcal{O}(D(f_i))}{\overset{\shortparallel}{A_{f_i}}} \to \prod_{i,j=1}^n \underset{\mathcal{O}(D(f_i) \cap D(f_j))}{\overset{\shortparallel}{A_{f_i f_j}}}$$

is exact.