Let k be a perfect field.

<u>def.</u> A function field of a curve over k is a finite type field over k with tr. deg. 1.

<u>Past week:</u>

$$K \rightsquigarrow (C, \mathcal{O}_C)$$

K: function field of a curve

$(C, \mathcal{O}_C)$: k-locally ringed space, the projective smooth curve associated with k.

The construction is functorial: let K, K' be function fields of curves$/k$, let $\varphi : K \to K'$ homomorphism of k-algebras making K' a finite extension of K.

Let $f \in K$ be transcendental.

$$\begin{array}{ccl} & K' & \\ & \uparrow \varphi & \Big] \text{ finite} \\ f & K & \\ \uparrow & \uparrow & \Big] \text{ finite} \\ t & k(t) & \end{array}$$

$A =$ normalization of $k[t]$ in K

$A' =$ ——————————— K'

$=$ normalization of A in K

$$U = \{ v \in C : v(f) \geq 0 \} = \mathrm{Max}(A)$$

$$U' = \{ v' \in C' : v'(\varphi(f)) \geq 0 \} = \mathrm{Max}(A')$$

Want to construct a map $\pi : (C', \mathcal{O}_{C'}) \longrightarrow (C, \mathcal{O}_C)$.

Set-theoretically:

discrete

$C' = \{$ equivalence classes of valuations on " $\}$ that are trivial on k

$$\pi : C' \longrightarrow C$$

$v' : K' \to \mathbb{Z} \cup \{\infty\} \mapsto v' \circ \varphi =: v : K \to \mathbb{Z} \cup \{\infty\}$ valuation trivial on k.

(not surjective : divide by index of ramification to have so)

Claim: $\pi^{-1}(U) = U'$.

Proof: $(\subseteq)$ $v' \in \pi^{-1}(U) \longrightarrow \underset{\overset{\vee\!/}{0}}{\pi(v')(f)} = v'(\varphi(f))$.

$(\supseteq)$ $v' \in U' \longleftrightarrow v(f) = v'(\varphi(f)) \geq 0$. $\square$

Therefore : 1) π is continuous.

2) want to define

$$\pi^{\#}_U : \underset{\overset{\shortparallel}{A}}{\mathcal{O}_C(U)} \longrightarrow \underset{\overset{\shortparallel}{A'}}{\mathcal{O}_{C'}(U')} = \underset{\overset{\shortparallel}{(\pi_* \mathcal{O}_{C'})(U)}}{\mathcal{O}_{C'}(\pi^{-1}(U))}.$$

$$A \xrightarrow{\varphi} A'$$

Rmk: For $g \in A$ transc. over k,

$$V' = \{v' \in U' : v'(\varphi(g)) = 0\} = \mathrm{Max}(A'[\varphi(g)^{-1}])$$

$$V = \{v \in U : v(g) = 0\} = \mathrm{Max}(A[g^{-1}]).$$

Claim: $\pi^{-1}(V) = V'$.

Since taking integral closure commutes with localization

$$A'\left[\frac{1}{\varphi(g)}\right] = \text{normalization of } A\left[\tfrac{1}{g}\right] \text{ in } K'.$$

$$A = \mathcal{O}_C(U) \longrightarrow \mathcal{O}_{C'}(U') = A'$$

$$\begin{array}{ccc} \downarrow & \circlearrowright & \downarrow \\ A\left[\frac{1}{g}\right] = \mathcal{O}_C(V) & \longrightarrow & \mathcal{O}_{C'}(V') = A'\left[\frac{1}{\varphi(g)}\right] \end{array}$$

so this contruction extends to the morphism of k-loc. ringed spaces

$$(U', \mathcal{O}_{C'}|_{U'}) \longrightarrow (U, \mathcal{O}_C|_U)$$

induce by the inclusion $\varphi: A \to A'$. $\square$

Rmk: The map $\pi: C' \to C$ is surjective: valuations can always be extended to a finite extension.

Rmk: $K \xrightarrow{\varphi} K' \xrightarrow{\varphi'} K''$ so that K''/K is finite

$$\Rightarrow \quad \pi_{\varphi' \circ \varphi} = \pi_\varphi \circ \pi_{\varphi'}$$

Summing up we constructed a functor

$$\mathcal{C} = \{ \text{function fields of a curve}/k \}$$

$$\downarrow \qquad \mathrm{Hom}_{\mathcal{C}}(K, K') = \{ \varphi: K \to K' \ k\text{-alg hom s.t. } [K' : \varphi(K)] < +\infty \}$$

always true

$$\{ {}^{k}\text{locally ringed spaces} \}$$

Rmk: Not fully faithful, but "almost"!

Let $\pi: (C', \mathcal{O}_{C'}) \longrightarrow (C, \mathcal{O}_C)$ be a morphism of ^{k}locally ringed spaces. Then (exercises session)

* either $\pi(C')$ is a point; we say that π is constant
* it comes from a $\varphi \in \mathrm{Hom}_{\mathcal{C}}(K, K')$.

Rule: let $\pi : C' \to C$ be a surjective morphism of alg. varieties

$x' \in C'$ $\quad \mathrm{ord}_{x'}(\pi)$ = ramification index of x' over $\pi(x')$

$\deg(\pi) = [K : K']$ $\quad K'$ = function field of C'

K = ———— C.

$$\Rightarrow \deg(\pi) = \sum_{\pi(x')=x} \mathrm{ord}_x(\pi)\,[k(x') : k(x)]$$

$x \in C$

$\downarrow$

ord_x is the unique surj. valuation in the class of x.

Prop ($\frac{1}{2}$ "Valuative Criterion of Properness")
Let C be the proj. smooth curve associated with a function field K. Let $U \subseteq C$ be a non-empty open subset, $x \in U$, $\varphi : U \setminus \{x\} \longrightarrow \mathbb{P}^n_k$ a morphism of alg. var. Then φ extends uniquely to U.

Pf: By induction on n, we may assume that the image of φ meets

$$\mathbb{A}^n \subseteq \mathbb{P}^n_k$$
$$t_1, \dots, t_n \leftrightarrow [1 : t_1 : \dots : t_n].$$

Up to shrinking U, we may suppose $U = \mathrm{Max}(A)$ affine and $\varphi(U \setminus \{x\}) \subseteq \mathbb{A}^n_k$.
Let v_0 be the valuation corresponding to x.

$$\mathcal{O}_C(U \setminus \{x\}) = \bigcap_{y \in U \setminus \{x\}} \mathcal{O}_{C,y}.$$

The morphism $\varphi : U \setminus \{x\} \to \mathbb{A}^n_k$ corresponds to

$a_1, \dots, a_n \in \mathcal{O}_C(U \setminus \{x\})$.

Let $i_0 \in \{1, \dots, n\}$ be such that

$$v_0(a_{i_0}) = \min v_0(a_i).$$

Up to reordering, we may assume $i_0 = n$. and

$$v(a_n) = 0 \quad \text{for all } v \in U \setminus \{x\}$$

$$\begin{array}{ccc}
k[t_1, \dots, t_n] & \longrightarrow & \bigcap_{v \neq v_0} \mathcal{O}_{C,v} \\
t_i & \longmapsto & a_i. \\
t_i = \frac{x_i}{x_0} \quad k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right] & & \Vert \\
\downarrow & & \\
k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]\left[\frac{x_0}{x_n}\right] & \longrightarrow & \bigcap_{v \neq v_0} \mathcal{O}_{C,v} \\
\downarrow\wr & & \uparrow \\
k\left[\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right]\left[\frac{x_n}{x_0}\right] & & \\
\cup\!\!\mid & & \\
k\left[\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right] & \longrightarrow & \mathcal{O}(U) = \bigcap_{v \in U} \mathcal{O}_{C,v} \\
\frac{x_i}{x_n} & \longmapsto & \frac{a_i}{a_n}
\end{array}$$

because a_n is invertible on $\mathcal{O}(U \setminus \{x\})$

$$v_0\left(\frac{a_i}{a_n}\right) \geq 0$$

$$v\left(\frac{a_i}{a_n}\right) \geq 0 \quad \text{for all } v \neq v_0$$

This is the morphism extending φ.

up to shrinking

$$U \setminus \{x\} \;=\; U \setminus \{x\} \;\subseteq\; U$$

$|$ $\qquad$ $|$ $\qquad$ $\vdots$ it extends.

$$\{x_0 \neq 0\} \supseteq \{x_0 \neq 0, x_n \neq 0\} \subseteq \{x_n \neq 0\} \qquad \square$$

Rmk: $\varphi : U \setminus \{x\} \longrightarrow \mathbb{P}^n_k$ s.t. $\underbrace{\varphi^{-1}(\mathbb{A}^n_k)}_{\text{open}} \neq \emptyset$

Then $\overset{V \shortparallel}{\varphi^{-1}(\mathbb{A}^n) \cup \{x\}}$ is open. There is an affine open neighbourhood of x contained in V.

Prop. Let C be the smooth proj. curve associated with a function field K. Then C admits a closed embedding in some projective space.

Pf: Cover C by two affine open subsets U, V

$\left(\text{e.g. } \begin{array}{l} U = \{x \in X : \mathrm{ord}_x(f) \geq 0\} \\ V = \{ \quad \text{———} \quad \leq 0\} \end{array} \quad \begin{array}{l} \text{for } f \in K \\ \text{transc.} \end{array} \right).$

$U = \mathrm{Max}(A) \qquad V = \mathrm{Max}(B)$

A, B finitely generated k-algebras

$a_1, \dots, a_m$ generators of A

$b_1, \dots, b_n$ ——— B.

$$\begin{array}{lll} k[t_1, \dots, t_m] \twoheadrightarrow A & \text{surjective} \longleftrightarrow & U \hookrightarrow \mathbb{A}^m_k \\ t_i \longmapsto a_i & & \text{closed embedding} \end{array}$$

$$\begin{array}{lll} k[u_1, \dots, u_n] \twoheadrightarrow B & \text{surj.} \longleftrightarrow & V \longrightarrow \mathbb{A}^n_k \\ u_i \longmapsto b_i & & \text{closed embedding.} \end{array}$$

Look at the morphism

$$U \cap V \xrightarrow{\varphi} \mathbb{A}^m_k \times_k \mathbb{A}^n_k$$

$$C \underset{\substack{\text{extends}\\ \text{uniquely}}}{\overset{\exists!\varphi}{\dashrightarrow}} \mathbb{P}^m_k \times \mathbb{P}^n_k \underset{\substack{\text{Segre}\\ \text{embedding}\\ (C \text{ closed})}}{\hookrightarrow} \mathbb{P}_k^{(m+1)(n+1)-1}$$

$$[x_0 : \dots : x_m], [y_0 : \dots : y_n] \mapsto [x_i y_j : \substack{i=0,\dots,m \\ j=0,\dots,n}]$$

Claim: φ is a closed embedding (exercise).

Example $\quad U = \mathbb{P}^1_k \setminus \{0, \infty\} = \mathrm{Max}(k[t, t^{-1}])$

$$U \hookrightarrow \mathbb{A}^2 \qquad \alpha : k[x,y] \to k[t, t^{-1}]$$
$$x \mapsto t$$
$$y \mapsto t^{-1}$$

$\ker(\alpha) = (xy - 1)$.

si $k =$ alg clos:

$\varphi : k^* \to \{(t, t^{-1}) \in k^2\}$

$\mathrm{Im}\, \varphi = \{(x,y) \in k^2 : xy = 1\}$.

Recall: $\mathbb{P}^n_k = \mathbb{P}(k^{n+1})$ $\quad (f_1, \dots, f_r)$ f_i are homogeneous

$= \{$ prime homogeneous ideal of $k[x_0, \dots, x_n]$ that does not contain $(x_0, \dots, x_n)$ and maximal w.r.t. this property $\}$

k is perfect, $\bar{k}$ an alg. closure $\quad \cong \left[\left(\bar{k}^{n+1} \setminus \{0\}\right) / \bar{k}^* \right] / \mathrm{Gal}(\bar{k}/k)$.

with respect to

$f \in k[x_0, \dots, x_n]$ homogeneous.

$D_+(f) = \{ x \in \mathbb{P}^n_k : f \notin m_x \}$ basis of open

$\mathcal{O}_{\mathbb{P}^n}(D_+(f)) = \left\{ \frac{g}{f^d} : g \text{ homogeneous of degree } d \cdot \deg(f) \right\}$.

$f = x_0$:

$$\mathcal{O}_{\mathbb{P}^n}(D_f(x_0)) = \left\{ \frac{f}{x_0^d} : f \text{ is hom of degree } d \right\}$$

$$= k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right].$$

$$\frac{f(x_0, \dots, x_n)}{x_0^d} = f\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \qquad \square$$

<u>Divisors</u> Let k be a perfect field.

<u>Re-def</u> A function field of a curve is a finite type field K over k of tr. deg. 1 s.t. k is alg. closed in K. ← From now supplementary hypothesis

<u>Rmk:</u> This is the case iff $K \otimes_k \bar{k}$ is again a field ($\Leftrightarrow$ integral) for an alg. closure $\bar{k}$.

<u>Example:</u> • $K = \mathbb{C}(t)$ as a field over $\mathbb{R}$.

$$K \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}(t) \otimes_{\mathbb{R}} \mathbb{R}[x]/(x^2+1)$$

$$\cong \mathbb{C}(t) \times \mathbb{C}(t)$$ ← I don't want this kind of things.

• $K = \mathbb{R}(t)$ as a field over $\mathbb{R}$.

• $K = \operatorname{Frac}\left(k[x,y]/(y^2 - p(x)) \right)$

with $p(x) \in k[x]$ deg 3 without multiple zeroes char $(k) \neq 2$.

$$K \otimes_k \bar{k} = \operatorname{Frac}\left(\bar{k}[x,y]/(y^2 - p(x)) \right).$$

irreducible in $\bar{k}$

- $K = \mathrm{Frac}\left(k[x,y] / (f) \right)$ with f irreducible in $\bar{k}[x,y]$.

<u>Def:</u> A divisor D on a smooth projective curve is a finite sum $\sum_{i=1}^{n} m_i [x_i]$, $m_i \in \mathbb{Z}$, $x_i \in C$,

i.e. an element of the free abelian group generated by C.

$$\mathrm{mult}_{x_i}(D) := m_i.$$

$$D = \sum_{x \in C} \mathrm{mult}_x(D) [x]$$ where $\mathrm{mult}_x(D) \neq 0$ for only finitely many points

- $$\deg D = \sum_{x \in C} \mathrm{mult}_x(D) \deg(x)$$

$$\deg(x) = [k(x) : k]$$

- $D \leq D'$ if $\mathrm{mult}_x(D) \leq \mathrm{mult}_x(D')$ for all $x \in C$

- D <u>effective</u> if $D \geq 0 \iff \mathrm{mult}_x(D) \geq 0$ $\forall x \in C$.

<u>Def:</u> Let $f \in K$, K is the function field of C.

$$\mathrm{div}(f) := \sum_{x \in C} \mathrm{ord}_x(f) [x]$$

<u>Lemma:</u> $\deg(\mathrm{div}(f)) = 0$.

<u>Proof:</u> Of course we may suppose f transc.

$f : C \to \mathbb{P}^1_k$ corresponding to

$$k(t) \to K$$
$$t \mapsto f.$$

$[K : k(t)] < +\infty$

$\deg(f) = \sum_{f(x)=0} \boxed{\mathrm{ord}_x(f)} \deg(x)$

$\uparrow$ because the residue field at $0 \in \mathbb{P}^1_k$ is k.

$\mathrm{ord}_x(1/f)$

$= \sum_{f(x)=\infty} -\mathrm{ord}_x(f) \deg(x)$

because a local parameter at $\infty \in \mathbb{P}^1_k$ is $1/t$

$\Rightarrow \deg(\mathrm{div}(f)) = \deg(f) - \deg(f) = 0.$ $\square$

<u>Rmk:</u> Let $x \in C$ be such that $f(x) = 0$.

$$\begin{array}{ccccccc} \text{normalisation} & A & \subseteq & K & & f \\ & | & & | & & \uparrow \\ (t) \subseteq & k[t] & \subseteq & k(t) & & t \end{array}$$

$m \subseteq A$ the max ideal corresponding to x.

$u \in m$ uniformizer

$t \mapsto f = u^e \qquad e =$ valuation of f at x

$=$ ramification index at x. $\square$

<u>def:</u> D divisor

$$H^0(D) = \{ f \in K^* : \mathrm{div}(f) + D \geq 0 \} \cup \{0\}$$

It is a k-vector space, of dim $h^0(D)$

<u>Prop:</u> 1) $D = 0 \Rightarrow H^0(D) = k$.

2) $\deg D < 0 \Rightarrow H^0(D) = 0$.

3) $H^0(D)$ is finite-dimensional and for $D \le D'$ $\quad 0 \le h^0(D') - h^0(D) \le \deg D' - \deg D$.

Proof: 1) $D = 0$ $\quad H^0(D) = \{ f \in K^*: \operatorname{div}(f) \ge 0 \} \cup \{0\}$
(transc.)

Suppose by contradiction that f is non constant.
Then it induces a morphism

$$f: C \to \mathbb{P}^1_k \quad \text{which is surjective.}$$

$\Rightarrow \exists x \in C$ s.t. $f(x) = \infty \iff \operatorname{ord}_x(f) < 0$. ↯.

2) $f \in H^0(D)$, $f \ne 0 \Rightarrow \operatorname{div}(f) + D \ge 0$

$$\underbrace{\deg(\operatorname{div}(f))}_{=0} + \deg D = \deg(\operatorname{div}(f) + D) \ge 0$$
$$= \deg D$$

3) We may suppose $\deg D \ge 0$. Arguing by induction on $\deg D' - \deg D$, we may suppose

$$D' = D + [x].$$

$$0 \to H^0(D) \to H^0(D') \to \underbrace{\mathfrak{m}_x^{-r-1} / \mathfrak{m}_x^{-r}}_{\cong k(x)}$$

$\mathfrak{m}_x$ = maximal ideal
$\cap$
$\mathcal{O}_{C,x}$ $\qquad r = \operatorname{mult}_x(D)$

$$\Longrightarrow \quad h^0(D') \le \underbrace{h^0(D) + \underbrace{\dim(\mathfrak{m}^{-r-1}/\mathfrak{m}^{-r})}_{[k(x):k] = \deg(x)}}$$

$\| \leftarrow D' = D + [x]$

$$h^0(D) + \deg D' - \deg D. \qquad \square$$

<u>Def:</u> Let C, C' be smooth proj. curves $/k$. Let $\pi: C' \to C$ be a surjective morphism of alg. varieties.

Let D be a divisor on C.

$$\pi^* D := \sum_{x' \in C'} \mathrm{ord}_{x'}(\pi)\, \mathrm{mult}_{\pi(x')}(D)\, [x']$$

<u>Lemma:</u> 1) $\deg(\pi^* D) = \deg(\pi) \deg(D)$

2) $f \in K(C)$, $\qquad \mathrm{div}(\pi^* f) = \pi^* \mathrm{div}(f)$.

the function field of C $\qquad$ the image of f in $K(C')$

<u>Proof:</u> 1)

$$\deg(\pi^* D) = \sum_{x'} \mathrm{ord}_{x'}(\pi)\, \mathrm{mult}_{\pi(x')}(D) \cdot \deg(x')$$

$$= \sum_{x \in C} \left(\sum_{\pi(x') = x} \mathrm{ord}_{x'}(\pi) \deg(x') \right) \mathrm{mult}_x(D)$$

$$\deg(x') = [k(x') : k]$$
$$= [k(x') : k(x)]\,[k(x) : k]$$

$$= \sum_{x \in C} \underbrace{\left(\sum_{\pi(x') = x} \mathrm{ord}_{x'}(\pi)\, [k(x') : k(x)] \right)}_{\deg(\pi)} \mathrm{mult}_x(D)\ \deg(\text{x})$$

$$= \deg(\pi) \deg(D).$$

2) $$\pi^* \mathrm{div}(f) = \sum_{x' \in C'} \underbrace{\mathrm{ord}_{x'}(\pi)\, \mathrm{ord}_x(f)}\ [x'] = \mathrm{div}(\pi^* f).$$

$\mathrm{ord}_{x'}(t)$

$$\mathrm{ord}_{x'}(f) = \mathrm{ord}_x(f) \cdot \mathrm{ord}_{x'}(\pi) \qquad \square$$