Recall: C, C' be smooth projective curves associated with function fields K, K'

$\pi : C' \to C$ a non-constant morphism of alg. varieties

$$\#\{x' \in C' : \mathrm{ord}_{x'}(\pi) > 1\} < +\infty.$$

Def: The ramification divisor of π is

$$\mathrm{Ram}(\pi) = R_\pi = \sum_{x' \in C'} \mathrm{length}_{\mathcal{O}_{C',x'}} \left(\Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}} \right) [x']$$

$x = \pi(x')$

$$R_\pi \geq \sum_{x' \in C'} \left(\mathrm{ord}_{x'}(\pi) - 1 \right) [x']$$

with equality if and only if $\mathrm{ord}_{x'}(\pi) \neq 0$ in k for each $x' \in C'$.

Prop-Def: C smooth proj. curve associated w/ a function field K.

$\Omega_{K/k}$ = space of meromorphic differentials.

1) For $x \in C$, $\omega \in \Omega_{K/k}$,

$$\mathrm{ord}_x(\omega) := \mathrm{ord}_x\left(\frac{\omega}{dt}\right) \quad \text{for a uniformizer } t \in \mathcal{O}_{C,x}$$

does not depend on the choice of the uniformizer.

2) For $\omega \in \Omega_{K/k}$,

$$\{x \in C : \mathrm{ord}_x(\omega) \neq 0\} \text{ is a finite set.}$$

Def: $\mathrm{div}(\omega) = \sum_{x \in C} \mathrm{ord}_x(\omega)[x]$.

3) The image of $\mathrm{div}(\omega)$ in $\mathrm{Pic}(X) = \mathrm{Div}(X) / \langle \mathrm{div}(f) : f \in K^* \rangle$ is independent of ω and called the canonical class K_C.

Def: The genus g of C is $h^0(K_C) = \dim_k H^0(\mathrm{div}(\omega))$ for some $\omega \in \Omega_{K/k}$.

Rmk: Let D, D' be divisors such that $D' = D + \mathrm{div}(f)$.

$$\begin{aligned} H^0(D') &\longrightarrow H^0(D) \\ g &\longmapsto fg \end{aligned} \quad \text{is an isomorphism.}$$

$$\Longrightarrow \quad h^0(D) = h^0(D')$$

Proof: 1) If $t, t' \in \mathcal{O}_{C,x}$ are uniformizer, then $dt, dt' \in \Omega_{C,x}$

Proof. 1) ... $\Omega_{\mathcal{O}_{C,x}/k}$

are generators, so that

$$\mathrm{ord}_x\left(\frac{dt'}{dt}\right) = 0.$$

2) In order to show, it suffices to show it for one differential form ω because then we will have

$$\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega).$$

So let $t \in \mathcal{O}_{C,x}$ be a uniformizer.

$$\begin{array}{ccc} K & \supseteq & A \\ \cup & & \cup \\ k(t) & - & k[t] \end{array}$$

A normalization.

Trick: $K/k(t)$ is separable because $dt \in \Omega_{C,x}$ is a generator $\Rightarrow dt \neq 0$.

$$\Longrightarrow \quad K \underset{k(t)}{\otimes} \Omega_{k(t)/k} \longrightarrow \Omega_{K/k} \qquad (*)$$

is an isomorphism of K-vector spaces.

I want to prove the finiteness for $\omega = dt$.

1) $\{x' \in C : \mathrm{ord}_{x'}(dt) < 0\} \subseteq \{x' \in C : \mathrm{ord}_{x'}(t) < 0\}$ finite.

$dt \in \Omega_{A/k[t]}$

If $\mathrm{ord}_{x'}(t) \geq 0$ and t' is a uniformizer at x'

$$\frac{dt}{dt'} \in \mathcal{O}_{C,x'} \Rightarrow \mathrm{ord}_{x'}(dt) \geq 0.$$

2) $\{x' \in C : \mathrm{ord}_{x'}(dt) > 0\} = \{m \in A : \Omega_{A/k[t]} \underset{A}{\otimes} A/m \neq 0\}$

last time

Because of the isomorphism $(*)$, we know that $\Omega_{A/k[t]}$ is torsion

$$\begin{array}{ccccccc} A \underset{k[t]}{\otimes} \Omega_{k[t]/k} & \longrightarrow & \Omega_{A/k} & \rightarrow & \Omega_{A/k[t]} & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K \underset{k(t)}{\otimes} \Omega_{K/k} & \xrightarrow{\sim} & \Omega_{K/k} & \longrightarrow & 0 = \Omega_{A/k[t]} \underset{A}{\otimes} K & & \end{array}$$

$- \otimes K$

$\Longrightarrow$ RHS is finite.

3) Let $\omega, \omega' \in \Omega_{K/k}$. Then there is $f \in K$ s.t.

$$\omega' = f\omega \Rightarrow \mathrm{div}(\omega') = \mathrm{div}(f) + \mathrm{div}(\omega). \qquad \square$$

Example: 1) $\mathbb{P}^1_k$ with coordinate t. $\omega = dt$.

$k = \bar{k}$ • $a \in k \Rightarrow$ uniformizer is $t-a$.

• uniformizer at ∞ : $\frac{1}{t} = u$.

• $a \in k \rightarrow d(t-a) = dt \Rightarrow \frac{dt}{d(t-a)} = 1$

$\Rightarrow \mathrm{ord}_a(dt) = 0.$

• $du = -\frac{1}{t^2}\, dt \Rightarrow dt = -\frac{du}{u^2}$

$\frac{dt}{du} = -\frac{1}{u^2} \Longrightarrow \mathrm{ord}_\infty(dt) = -2.$

$\Longrightarrow \mathrm{div}(dt) = -2[\infty]$

$\deg(\mathrm{div}(dt)) < 0 \Longrightarrow h^0(K_{\mathbb{P}^1}) = 0$

$\Longrightarrow \mathbb{P}^1$ is of genus 0

For non-alg. closed k, points in $\mathbb{A}^1_k$ are irreducible polynomials (up to scalar factors).

If $P \in k[t]$ is irreducible

$\Rightarrow dP = P'(t)\, dt$ We know that $P'(t)$ in $k[t]/P(t)$ is $\neq 0$
(k perfect) residue field at $m = (P)$.

$\Longrightarrow \frac{dt}{dP} = P'(t)^{-1}$

i.e. P' is invertible in $\mathcal{O}_{\mathbb{P}^1, m}$.

$\Longrightarrow \mathrm{ord}_m(dt) = 0.$

2) $k = \bar{k}$ $y^2 = p(x)$ with $p(x) \in k[x]$ of degree 3 without multiple roots.

$A = k[x,y]/(y^2 - p(x))$ $K = \mathrm{Frac}(A)$

$\Omega_{A/k} = \frac{A\,dx \oplus A\,dy}{(2y\,dy - p'(x)\,dx)}$ $\omega = \frac{dx}{y} \in \Omega_{K/k}$.

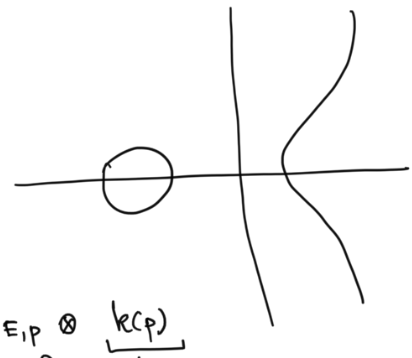
Since all the points of $y^2 = p(x)$ are non-singular as well as the point $[0:0:1] \in \mathbb{P}^2$?, the smooth proj curve associated with is given by the closure E of $\{y^2 = p(x)\}$ in $\mathbb{P}^2$.

[illegible] but the computation over k is the

(We saw this over $\mathbb{C}$, but the computation ...
same.)

I want to show that $\mathrm{ord}_p(\omega) \geq 0$ for each $p \in E$.

$$\Omega_{A/k} = \frac{A\,dx \oplus A\,dy}{(2y\,dy - p'(x)\,dx)}.$$



If $p = (x_0, y_0)$ is such that $y_0 \neq 0$

then

$$\frac{k\,dx \oplus k\,dy}{(2y_0\,dy - p'(x_0)\,dx)} = \Omega_{E,p} \otimes_{\mathcal{O}_{E,p}} \underbrace{k(p)}_{k}$$

is generated by dx. Therefore

$$\frac{\omega}{dx} = \frac{1}{y_0} \neq 0 \text{ as an element of}$$

$\Rightarrow \mathrm{ord}_p(\omega) = 0.$

If not, $\frac{1}{2}\omega = \frac{dx}{2y} = \frac{dy}{p'(x)}$. If $y_0 = 0$, then we know that $p'(x_0) \neq 0$ (because the point $(x_0, 0)$ is non-singular) otherwise $p(x_0) = p'(x_0) = 0$.

and the k-vector space

$$\frac{k\,dx \oplus k\,dy}{(2y_0\,dy - p'(x_0)\,dx)} \text{ is generated by } dy$$

$$\Longrightarrow \mathrm{ord}_p(\omega) = \mathrm{ord}_p\left(\tfrac{1}{2}\omega\right) = \mathrm{ord}_p\left(\frac{dy}{dy} \cdot \frac{1}{p'(x_0)}\right) = 0.$$

We showed that $\mathrm{ord}_p(\omega) = 0$ for all $p = (x_0, y_0)$ s.t.

$$y_0^2 = p(x_0).$$

We have to check what happens at infinity. Recall that

$$x_0 x_2^2 = p\left(\frac{x_1}{x_0}\right) x_0^3$$ is satisfied by only one point with $x_0 = 0$

This point is $[0:0:1]$.

$$u = \frac{x}{y}$$

$$v = \frac{1}{y} \qquad dv = -\frac{1}{y^2}\,dy \Rightarrow dy = -\frac{dv}{v^2}$$

$$\omega = \frac{dx}{y} = \frac{dy}{p'(x)} = -\frac{dv}{v^2}\cdot\frac{1}{p'\left(\frac{u}{v}\right)}$$

Here the point is that p' is a polynomial of degree 2 in x. So that

$$v^2 p'\left(\tfrac{u}{v}\right) \in k[u,v] \quad \text{homogeneous}$$

$\longrightarrow$ vanishes for $(u,v)=(0,0)$.

The equation at infinity of E is

$$v = P\left(\tfrac{u}{v}\right)\cdot v^3$$

Therefore

$$dv\left[1 - P'\left(\tfrac{u}{v}\right)\cdot\left(-\tfrac{u}{v^2}\right)v^3 - 3P\left(\tfrac{u}{v}\right)v^2\right] = \left[P'\left(\tfrac{u}{v}\right)\cdot\tfrac{1}{\cancel{v}}\cdot v^{\cancel{3}\,2}\right]du.$$

$$\Rightarrow \quad \frac{dv}{v^2 p'\left(\frac{u}{v}\right)} = \frac{du}{\underbrace{1 + p'\left(\frac{u}{v}\right)uv - 3P\left(\frac{u}{v}\right)v^2}_{\text{regular in } (0,0)}}$$

and does not vanish (check!)

$\mathrm{ord}_{(0,0)}(v) = 3 \qquad \mathrm{ord}_{(0,0)}(u) = 1$

u uniformizer while v is not.

<u>Def:</u> For an open subset $U \subseteq C$, set

$$H^0(U, \Omega_C) = \{\omega \in \Omega_{K/k} : \mathrm{ord}_x(\omega) \geq 0 \text{ for each } x \in U\}.$$

<u>Rmk:</u> $\omega \in \Omega_{K/k} \setminus \{0\}$

$$H^0(C, \Omega_C) = \{\underset{\omega'}{\underset{\parallel}{f\omega}} : f \in K \text{ s.t. } \underset{\operatorname{div}(f)+\operatorname{div}(\omega)}{\underset{\parallel}{\operatorname{div}(\omega')}} \geq 0\} = H^0(\operatorname{div}(\omega))$$

$$\dim_k H^0(C, \Omega_C) = h^0(\operatorname{div}(\omega)) =: \text{genus of } C.$$

<u>Thm</u> (Hurwitz formula): Let $\pi: C' \to C$ be a non-constant separable morphism (K'/K separable) between smooth projective curves. Then

$$\deg(K_{C'}) = \deg(\pi)\deg(K_C) + \deg(R_\pi).$$

and $\deg(R_\pi) \geq \sum_{x' \in C'} (\mathrm{ord}_{x'}(\pi) - 1)\deg(x')$ with equality iff $\mathrm{ord}_{x'}(\pi) \neq 0$ in k.

<u>Proof:</u> Let $\omega \in \Omega_{K/k}$. Since π is separable, the K'-linear map

$$K' \otimes_K \Omega_{K/k} \longrightarrow \Omega_{K'/k}$$
$$1 \otimes \omega \longmapsto \pi^*\omega$$

is an isomorphism.

It suffices to prove that

$$\operatorname{div}(\pi^*\omega) = \underbrace{\deg(\pi)\operatorname{div}(\omega)}_{\pi^*\operatorname{div}(\omega)} + R_\pi$$

$$\pi^*\operatorname{div}(\omega) = \pi^*\left(\sum_{x\in C} \operatorname{ord}_x(\omega)\,[x]\right)$$

$$= \sum_{x'\in C'} \operatorname{ord}_{x'}(\pi)\cdot \operatorname{ord}_x(\omega)\,[x'].$$

Fix $x'\in C'$. Let $t'\in \mathcal{O}_{C',x'}$ be a uniformizer. Let $x=\pi(x')$ and $t\in\mathcal{O}_{C,x}$ a uniformizer.

$$t = u t'^e \qquad e = \operatorname{ord}_{x'}(\pi).$$

$$\operatorname{ord}_x(\omega) = \operatorname{ord}_x\left(\frac{\omega}{dt}\right)$$

$$\operatorname{ord}_{x'}(\pi^*\omega) = \operatorname{ord}_{x'}\left(\frac{\pi^*\omega}{dt'}\right)$$

$$dt = u e t'^{e-1} dt' + t'^e du.$$

$$\frac{\omega}{dt'} = \frac{\omega}{dt}\cdot\frac{dt}{dt'} = \frac{\omega}{dt}\left[u e t'^{e-1} + t'^e\right]$$

$$\Longrightarrow \operatorname{ord}_{x'}\left(\frac{\omega}{dt'}\right) = \underbrace{\operatorname{ord}_{x'}\left(\frac{\omega}{dt}\right)}_{\substack{e\,\operatorname{ord}_x\left(\frac{\omega}{dt}\right) \\ \operatorname{ord}_{x'}(\pi)}} + \operatorname{mult}_x(\operatorname{Ram}(\pi)).$$

This concludes the proof. $\square$

Rmk: D divisor on C, then

$$H^0(\Omega_C(D)) := \{\omega\in\Omega_{K/k} : \operatorname{div}(\omega)+D\geq 0\}$$ this is a finite-dimensional k-vector space.

$$\| \\ H^0(K_C+D)$$

Thm (Riemann-Roch). Let C be a smooth proj. curve of genus g. Let D be a divisor on C. Then,

$$h^0(D) - h^0(K_C - D) = \deg D + 1 - g.$$

Pf: See Dat's notes. $\square$

Cor: $\deg K_C = 2g-2$.

Pf: $D = K_C$

$$\underbrace{h^0(K_C)}_{g} - h^0(\underbrace{K_C - K_C}_{0}) = \deg K_C + 1 - g \Rightarrow \deg K_C = 2g-2. \quad \square$$

Hurwitz: $$(2\underset{\uparrow \text{ genus of } C'}{g'} - 2) = (\deg \pi)(2\underset{\uparrow \text{ genus of } C}{g} - 2) + \deg R_\pi.$$

Cor: $\deg D \geq 2g-1 \Rightarrow h^0(D) = \deg D + 1 - g$.

Pf: $\deg D \geq 2g-1 \underset{\neq}{>} 2g-2 = \deg K_C$

$$\Longrightarrow \deg(K_C - D) < 0$$

$$\Longrightarrow h^0(K_C - D) = 0. \Rightarrow \text{Conclude by Riemann-Roch.} \quad \square$$

Projective embeddings of curves.

Let C be a smooth projective curve of genus g.

Def: A divisor D on C is globally generated if, for each $x \in C$, there is $f \in H^0(D)$ s.t.

$$\mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 0.$$

Suppose $k = \bar{k}$.

Lemma: If $\deg(D) \geq 2g$, then D is globally generated.

Pf: Take $x \in C$. Look at the divisor $D' = D - [x]$. Then,

$$\deg D' \geq 2g-1$$

$$\Rightarrow h^0(D') = \deg D - 1 + 1 - g$$

$$h^0(D) = \deg D + 1 - g = h^0(D') + 1.$$

Therefore there is $f \in H^0(D) \setminus H^0(D')$.

$$\Longrightarrow \mathrm{ord}_x(f) = -\mathrm{mult}_x(D).$$

Otherwise, f would belong to $H^0(D')$. $\square$

Kodaira map: Let D be a globally generated divisor. We want to define a morphism of algebraic varieties

$$i_D : C \longrightarrow \mathbb{P}(H^0(D)^*)$$

$\ulcorner$ Intuition: $(k=\bar{k})$ $\quad x \in C$

$$H_x = \{ f \in H^0(D) : \ \mathrm{mult}_x(D) + \mathrm{ord}_x(f) > 0 \} = H^0(D-[x])$$

$H_x \subseteq H^0(D)$. D globally generated $\Rightarrow \dim H_x = \dim H^0(D) - 1$.

$\Longrightarrow H_x$ is a hyperplane : it can been as a point of $\mathbb{P}(H^0(D)^*)$.

$f \in H^0(D) \longrightarrow$ think as a linear form on $H^0(D)^*$.

$\mathbb{P}(H^0(D)^*) \supseteq D_+(f)$

$$i_D^{-1}(D_+(f)) \ni x \iff f \notin H_x$$
$$\iff \mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 0.$$

$$\Longrightarrow i_D^{-1}(D_+(f)) = \{ x \in C : \ \mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 0 \}$$

this is an open subset of C.

$\Longrightarrow i_D$ is continuous.

$$\mathcal{O}(D_+(f)) = [\mathrm{Sym}\ H^0(D)]_{(f)} \longrightarrow \mathcal{O}(i_D^{-1}(D_+(f))).$$

$$\frac{g}{f^d} \longmapsto \frac{\mu(g)}{f^d}$$

with $g \in \mathrm{Sym}^d H^0(D)$

We have a map

$$\mathrm{Sym}^d H^0(D) \xrightarrow{\mu} H^0(dD)$$
$$[g_1] \cdots [g_d] \longmapsto g_1 \cdots g_d$$

$g_i \in H^0(D)$

Rmk: $i_D(x) \in D_+(f) \Rightarrow \mathrm{ord}_x\left(\frac{\mu(g)}{f^d}\right) = \underbrace{\mathrm{ord}_x(\mu(g))}_{\geq -\mathrm{mult}_x(D)} - \underbrace{\mathrm{ord}_x(f^d)}_{= -\mathrm{mult}_x(D)} \geq 0$

Claim: The map

$$\mathcal{O}(D_+(f)) \longrightarrow i_{D*}\mathcal{O}_C(D_+(f))$$

extends to a homomorphism of sheaves of k-algebras

$$\mathcal{O}_{\mathbb{P}(H^0(D)^*)} \longrightarrow i_{D*}\mathcal{O}_C.$$

$\ulcorner$ Sketch: In order to prove this, generalise the construction of the map for each $f \in \mathrm{Sym}^d H^0(D)$. $\lrcorner$

This will define a map of algebraic varieties $C \to \mathbb{P}(H^0(D))$.

Suppose k not necessarily alg. closed.

Fix a point $x \in C$. Let $m \subseteq \mathcal{O}_{C,x}$ be the max ideal. and $r = \mathrm{mult}_x(D)$.

$$\varepsilon_1 : \quad H^0(D) \longrightarrow m^{-r} / m^{-r+1}$$

kernel: $\{ f \in H^0(D) : \mathrm{mult}_x(D) + \mathrm{ord}_x(f) > 0 \}$.

$$\varepsilon_d : \quad H^0(dD) \longrightarrow m^{-rd} / m^{-rd+1}.$$

kernel $\{ f \in H^0(dD) : d\, \mathrm{mult}_x(D) + \mathrm{ord}_x(f) > 0 \}$.

$$\underbrace{\bigoplus_{d=0}^{\infty} \mathrm{Sym}^d H^0(D)}_{} \xrightarrow{\ \mu\ } \bigoplus_{d=0}^{\infty} H^0(dD) \longrightarrow \underbrace{\bigoplus_{d=0}^{\infty} m^{-rd} / m^{-rd+1}}_{}$$

$$\eta_x : \mathrm{Sym}\, H^0(D) \longrightarrow \bigoplus_{d=0}^{\infty} m^{-r} \otimes_{\mathcal{O}_{C,x}} k(x) = \mathrm{Sym}_{\mathcal{O}_{C,x}}(m^{-r}) \otimes_{\mathcal{O}_{C,x}} k(x)$$

Define $i_D(x) =$ kernel of η_x.

Rmk: The ideal $\ker(\eta_x) \subseteq \mathrm{Sym}\, H^0(D)$ is prime, homogeneous and maximal among those having these properties and not contain in $\bigoplus_{d=1}^{\infty} \mathrm{Sym}^d H^0(D)$

Indeed, $m^{-r} = t^{-r} \mathcal{O}_{C,x}$ (t uniformizer) $\quad \mathrm{Sym}_{\mathcal{O}_{C,x}}(m^{-r}) \cong \mathcal{O}_{C,x}[\lambda]$, $\lambda = t^{-r}$

$$\Longrightarrow \quad \mathrm{Sym}_{\mathcal{O}_{C,x}}(m^{-r}) \otimes_{\mathcal{O}_{C,x}} k(x) \simeq k(x)[\lambda]$$

Since $k(x)[\lambda]$ is an integral domain, $\ker(\eta_x)$ is a prime ideal. Since η_x respects the grading of these graded algebras, the kernel is homogeneous. Let $f \in H^0(D)$ be such that $\underbrace{\mathrm{mult}_x(D)}_{=r} + \mathrm{ord}_x(f) = 0$. Then the image in m^{-r}/m^{-r+1} is proportional to t^{-r}. So that

$$\eta_x(f) = \alpha \lambda \qquad \alpha \in k(x) \quad \text{and} \quad \alpha \neq 0.$$

This implies that $\ker(\eta_x)$ is maximal. $\square$