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Counting integral points of bounded height on varieties with large fundamental group

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Abstract. The main result of this paper states the subpolynomial growth of the number of integral points with bounded height of a variety over a number field whose fundamental group is large. This generalizes a recent paper of Ellenberg, Lawrence and Venkatesh and replies to two questions asked therein.

1. Introduction

In order to state our theorem and cast it in a general framework, let us review some background material.

1.1. Statement. Let K be a number field with ring of integers $\mathcal{O}_K$, X a projective variety over K, L an ample line bundle on X, and h an (absolute logarithmic) height function relative to L; see Section 4.1. For an open subset U of X, a subring R of K containing $\mathcal{O}_K$, a projective flat $\mathcal{O}_K$-scheme $\mathcal{X}$ with generic fiber X, and a real number c, set

$$\nu(\mathcal{X}, L, U, h, R; c) := \log^+ \#\{x \in \mathcal{U}(R) : h(x) \leqslant c\},$$

where $\mathcal{U}$ is the complement of the Zariski closure of $X \smallsetminus U$ in $\mathcal{X}$, and $\log^+ t = \log \max\{1, t\}$ for a real number t. When $R = K$, the number $\nu(\mathcal{X}, L, U, h, R; c)$ only depends on the generic fiber, so we will write X instead of $\mathcal{X}$. When $U = X$, the redundant specification of the open subset will be discarded. Two height functions relative to L differ by a bounded term, thus the corresponding counting functions do as well.

The precise value of the function $c \mapsto \nu(\mathcal{X}, L, U, h, K; c)$ is not much of interest and here we will focus on its slope

$$\mathrm{gr.rat}_K(X, L, U) := \limsup_{c \to \infty} \frac{\nu(\mathcal{X}, L, U, h, K; c)}{c},$$

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henceforth called the *(linear) growth rate of rational points*. Of course, as the notation suggests, the real number $\mathrm{gr.rat}_K(X, L, U)$ does not depend on the choice of the model $\mathcal{X}$ nor on that of the height function h. Similarly, for a finite set S of places of K including the Archimedean ones, the real number

$$\limsup_{c\to\infty} \frac{\nu(\mathcal{X}, L, U, h, \mathcal{O}_{K,S}; c)}{c}$$

measures the presence of the S-integral points on U. Again this does not depend on $\mathcal{X}$ and h, but *a priori* does depend on the set S. Taking the supremum ranging over all such finite sets of places S allows to get rid of such a dependence and gives rise to an invariant

$$\mathrm{gr.int}_K(X, L, U)$$

called the *(linear) growth rate of integral points* of U with respect to X and L. When $U = X$, rational and integral points coincide, thus their growth rate do. As above, in such a case the redundant repetition of X is discarded from notation.

When U is geometrically integral and normal, we say that U has large geometric étale fundamental group if $\bar{U}$ has large algebraic fundamental group in the sense of Kollár, where $\bar{U}$ is the base-change of U to an algebraic closure of K; see Section 2. For the sake of brevity we sometimes omit the adjective 'geometric'. Time has come to state the main result of the present note:

Main theorem. *Let K be a number field, X a projective variety over K, L an ample line bundle over X, and U an open subset of X which is geometrically integral, normal, and has large geometric étale fundamental group. Then*

$$\mathrm{gr.int}_K(X, L, U) = 0.$$

In other terms, upon fixing a projective flat model $\mathcal{X}$ of X and a height function h relative to L, for each $\varepsilon > 0$ and each finite subset of places S of K there is $C_{S,\varepsilon} > 0$ such that

$$\nu(\mathcal{X}, L, U, h, \mathcal{O}_{K,S}; c) \leqslant \varepsilon c + C_{S,\varepsilon} \quad \text{for all } c \in \mathbb{R}.$$

This statement thus generalizes the main theorem of [17]. It also replies positively to the questions raised in [17] whether the main theorem therein would hold for varieties with large étale fundamental group and mixed Shimura varieties; see Section 1.3.

1.2. Heuristics. To put our result in perspective, recall that the asymptotic behaviour of the function $c \mapsto \nu(\mathcal{X}, L, U, h, K; c)$ for $c \to \infty$ ought to reflect the usual "trichotomy" of algebraic varieties into the Fano, the Calabi–Yau, and the general type classes (and their logarithmic variants).[1)] Discussing this we omit the involved height function, as its choice will not influence the result:

- When X is smooth Fano and L is the anti-canonical bundle $\omega_X^\vee$, Manin conjectured (assuming the Zariski density of $X(K)$) the existence of an open subset U of X for which

$$\nu(X, \omega_X^\vee, U, K; c) = [K : \mathbb{Q}]c + (\rho - 1)\log c + O(1),$$

1) The word "trichotomy" here is improper – rather, according to the Minimal Model Program, any integral variety should be obtained as iterated fibration with generic fibre belonging to one of the preceding three classes.

where ρ is the Picard rank of X.[2] Note that the factor $[K : \mathbb{Q}]$ appears because of the normalization of height adopted in this paper. As an instance of the conjecture, earlier Schanuel [35] proved

$$\nu(\mathbb{P}^N_K, \mathcal{O}(1), K; c) = [K : \mathbb{Q}](N+1)c + O(1).$$

- Rather vaguely,[3] logarithmic growth is expected for Calabi–Yau varieties (or rather some suitable non-empty open subset, as K3 surfaces contain rational curves). For example, let A be an abelian variety and $r := \dim_{\mathbb{Q}} A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ its Mordell-Weil rank. Then, for any ample line bundle L on A,

  $$\nu(A, L, K; c) = \tfrac{r}{2} \log c + O(1);$$

  see [27, Theorem B.6.3].
- As soon as one enters the general type realm, the Lang–Vojta conjecture in the projective case predicts the existence of a non-empty open subset for which there are only finitely many K'-rational points for any finite extension K' of K – in other words, the counting function of rational points for such an open subset is eventually constant. This is the case for curves and subvarieties of abelian varieties by celebrated theorems of Faltings [18–20]. Reformulating the Lang–Vojta conjecture in the non-compact case in terms of counting functions is less eloquent. Nonetheless, recall the finiteness of S-integral points for affine curves – a theorem of Siegel [38] – and for (fine) moduli spaces of curves and of abelian varieties – the Shafarevič conjecture, proved by Faltings in [18, 19].

The class of varieties with large étale fundamental group lies somewhere in between the Calabi–Yau and general type classes, as smooth Fano varieties are simply connected. More precisely, Kollár conjectures that a smooth projective variety with large étale fundamental group admits a finite étale cover which is a smooth family of abelian varieties over a projective variety of general type [28, Conjecture 1.10]; see [12, Theorem 6.5] for a partial result in this direction and [10, Theorem 3.16]. In view of this, logarithmic growth is expected for smooth projective varieties with large étale fundamental group.

Linear growth rates furnish invariants way rougher than the asymptotic behaviour. To stress this, note that the following holds:

- $\mathrm{gr.rat}_K(\mathbb{P}^N_K, \mathcal{O}(1)) = (N+1)[K : \mathbb{Q}]$ for an integer $N \geqslant 1$,
- $\mathrm{gr.rat}_K(A, L) = 0$ for an abelian variety A over K and an ample line bundle L on A,
- $\mathrm{gr.int}_K(X, L, U) = 0$ for a smooth projective curve X over K, an ample line bundle L on X and an open subset U of X, provided that U is not isomorphic to the projective or the affine line over K.

In particular, linear growth rates in general cannot distinguish Calabi–Yau varieties from those of general type. Needless to say, zero linear growth rate is a much weaker condition than logarithmic growth or no growth at all.

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2) As of nowadays, Manin's conjecture is known not to hold in such a form [3] and alternative hypothetical statements have been suggested. This does not affect the main theme of our paper, and the interested reader may consult for example [32].

3) To the extent of our knowledge there is no precise conjecture on the behaviour of the growth of rational points for Calabi–Yau varieties.

1.3. Examples. The main theorem can be applied in the following situations:

(1) Smooth curves (not isomorphic to the affine or projective line), abelian varieties and their moduli spaces have large étale fundamental group. However, as just recalled, results way more precise than the main theorem are known in these cases.

(2) Fine moduli spaces of Calabi–Yau varieties and complete intersections (save a finite list of exceptions in low degree) have large étale fundamental group, thus the main theorem applies. These moduli spaces carry a geometric variation of pure Hodge structures, thus [17] applies as well.

(3) Pure or mixed Shimura varieties (associated with a torsion-free congruence subgroup) have large étale fundamental group. When they are not pure, they are the prototypical examples in which the results in [17] cannot be applied, whereas the main theorem here can – this replies to a question raised in [17]. Even in the pure case the main theorem here has a wider range of application than the one in [17]. Indeed, pure Shimura varieties of exceptional type (associated with a torsion-free congruence subgroup) carry a canonical variation of pure Hodge structures whose underlying local system is large. It is not known whether this variation is of geometric origin [23]. Since these varieties are hyperbolic, the Lang–Vojta conjecture implies that they should only have finitely many integral points. To our knowledge, although much weaker than what expected, our result is the first pointing in this direction.

1.4. Strategy. The main theorem is the combination of two results of independent interest. The first one is of arithmetic nature:

Theorem A. *Let X be an integral projective variety of dimension n over a number field K, and L an ample line bundle over X. Assume there is a subvariety Z of X and an integer $d \geqslant 1$ such that any positive-dimensional integral subvariety Y of X not contained in Z satisfies* $\deg(Y, L_{|Y}) \geqslant d^{\dim Y}$. *Then*

$$\mathrm{gr.rat}_K(X, L, X \smallsetminus Z) \leqslant \frac{n(n+3)}{2d}[K:\mathbb{Q}].$$

Its proof builds on uniform bounds on the number of hypersurfaces needed to cover the rational points of a subvariety of the projective space. These bounds were initiated by Heath-Brown [26] via the determinant method [7], and pursued by many authors including Broberg [8], Salberger [34], and Chen [13, 14]. The crucial remark here is that these bounds become stronger as the degree increases. The hypothesis that all subvarieties have large degree allows to work by induction and bound the number of points.

Theorem A will not be applied directly to variety itself, but rather to a well-chosen cover:

Theorem B. *Let X be a normal integral projective variety over an algebraically closed field of characteristic* 0*, L an ample line bundle over X, and U a non-empty open subset of X whose étale fundamental group is large. Then, given an integer $d \geqslant 1$, there is a finite surjective map $\pi\colon X' \to X$ with X' normal integral such that π is étale over U and, for each positive-dimensional integral subvariety Y' of X' meeting $\pi^{-1}(U)$,*

$$\deg(Y', \pi^* L_{|Y'}) \geqslant d.$$

See Section 3.1.1 for a reminder on the degree of a line bundle. The construction of the cover in the statement appears in previous work of the first-named author [9] and relies on boundedness of families of normal cycles in Kollár's terminology (see Section 3.2).

The strategy of applying uniform bounds to well-chosen unramified covers stems back to [17]. Roughly, Theorems A and B play the role respectively of Lemmas 4.2 and 4.1 therein, although here the arithmetic and geometric ingredients are completely disjoint thus leading to a simpler proof. Let us point out some other differences: first, contrarily to [17], the construction of the cover in Theorem B involves no period mapping; second, we apply directly the bound on the number of points on the whole cover, rather than bounding the number of hypersurfaces covering integral points in the fiber of the period mapping.

Organization of the paper. Introduction left aside, the paper has four sections, respectively dealing with large fundamental groups, the proof of Theorem A, Theorem B and the Main theorem.

Conventions. A *variety* over a field k is a separated finite type k-scheme. A *subvariety* is always understood to be closed.

2. Large fundamental groups

Let X be a normal integral variety over an algebraically closed field k of characteristic 0.

2.1. Definition. The variety X *has large étale fundamental group* if, for any positive-dimensional integral subvariety $Y \subset X$ with normalization $\widetilde{Y} \to Y$, we have

$$\#\operatorname{Im}(\pi_1^{\text{ét}}(\widetilde{Y}) \to \pi_1^{\text{ét}}(X)) = \infty,$$

where we abusively drop the choice of a base-point for étale fundamental groups. Over the complex numbers this is what Kollár calls having large algebraic fundamental group [28, 29]; see also [11].

Remark 2.1. Thanks to [28, Proposition 2.9.1], the étale fundamental group of X is large if and only if, for any non-constant morphism $f : Y \to X$ with Y integral normal, the image of the induced map $\pi_1^{\text{ét}}(Y) \to \pi_1^{\text{ét}}(X)$ is infinite. Furthermore, it is sufficient (thus equivalent) to test the latter condition only on smooth connected curves Y.

2.2. Basic properties. The following are direct consequences of the definition and the above remark:

(1) The class of (integral, normal) algebraic varieties with large étale fundamental group is closed under products.

(2) Given a quasi-finite morphism $f : X' \to X$ between integral normal algebraic varieties, if X has a large étale fundamental group, then X' has a large étale fundamental group too. The converse holds as soon as f is finite étale.

(3) Let $f : X' \to X$ be a morphism between integral normal algebraic varieties with connected normal fibers and satisfying the following property: the sequence

$$1 \to \pi_1^{\text{ét}}(f^{-1}(x)) \to \pi_1^{\text{ét}}(X') \to \pi_1^{\text{ét}}(X) \to 1$$

is short exact for all $x \in X(k)$. If X and all the fibers (at closed points) of f have large étale fundamental group, then so does X'. This is the case for isotrivial fibrations whose base and fiber have large étale fundamental group.

(4) Let k' be an algebraically closed field extension of k. If the étale fundamental group of the variety X' deduced from X by extending scalars to k' is large, then so is the one of X. Applying the specialization map of étale fundamental groups to an exhaustive family of normal cycles (see Section 3.2) lets one to show that the converse holds when X is proper. In the non-compact case we ignore whether having large étale fundamental group is a property compatible with extension of scalars.

(5) All smooth connected curves, except the affine and the projective line, have large étale fundamental group.

Remark 2.2. Taking products of a suitable number of copies of an elliptic and of a curve of genus $\geqslant 2$ yields examples of smooth projective (connected) varieties X with large étale fundamental group and any possible Kodaira dimension between 0 and $\dim X$.

2.3. Comparison with the topological fundamental group. Over the complex numbers, the property of having large étale fundamental group can be read off the usual topological fundamental group. For, upon fixing a point $x \in X(\mathbb{C})$, the natural group homomorphism

$$i_X \colon \pi_1^{\mathrm{top}}(X(\mathbb{C}), x) \to \pi_1^{\text{ét}}(X, x)$$

identifies the étale fundamental group with the profinite completion of the topological one. Let $\widehat{X}$ be the topological cover of $X(\mathbb{C})$ corresponding to $\operatorname{Ker} i_X \subset \pi_1^{\mathrm{top}}(X(\mathbb{C}), x)$.

Proposition 2.3 ([28, Proposition 2.12.3]). *Suppose X proper. Then the étale fundamental group of X is large if and only if the complex analytic space $\widehat{X}$ does not contain positive-dimensional compact complex analytic subspaces.*

The latter condition is satisfied, for instance, if the complex space $\widehat{X}$ is holomorphically separable, that is, given two distinct points x, y, there is a global holomorphic function f such that $f(x) \neq f(y)$. Examples of holomorphically separable spaces are open subsets of $\mathbb{C}^n$ (or, more generally, open subsets of Stein spaces). The properness assumption is missing in the statement of [28, Proposition 2.12.3], although crucially invoked in the proof. In the non-compact case the statement is false as one can see by considering a positive-dimensional simply connected affine variety, e.g., the affine space.

Remark 2.4. In view of the above characterization, it is useful to be able to determine $\widehat{X}$. One idle (but useful) case is when $\widehat{X}$ is itself a universal cover of $X(\mathbb{C})$, which boils to down to saying that i_X is injective (this is not always the case as pointed out by Toledo [40]). Recall that a group Γ is said to be:

- *linear* if it admits a faithful representation $\rho \colon \Gamma \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over some field,
- *residually finite* if the natural map $\Gamma \to \widehat{\Gamma}$, where $\widehat{\Gamma}$ is its profinite completion, is injective.

A classical result of Malcev [30] states that a finitely generated linear group is residually finite. Gathering the above considerations, if the topological fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)$ is linear, then $\widehat{X}$ is a universal cover of $X(\mathbb{C})$.

Example 2.5. The above remark can be applied to say that the étale fundamental group of X is large when X is

- an abelian variety, for it is the quotient of $\mathbb{C}^n$ by a lattice,
- a quotient of a bounded symmetric domain in $\mathbb{C}^n$ by a torsion-free co-compact lattice of its biholomorphism group.

It follows that semi-abelian varieties, being a fibration over an abelian variety by a torus, have large étale fundamental group.

2.4. The role of local systems. Still under the assumption $k = \mathbb{C}$, a local system $\mathcal{L}$ on $X(\mathbb{C})$ with coefficients in some field is *large* if, given a non-constant morphism $f : Y \to X$ with Y a normal irreducible complex variety, the local system $f^*\mathcal{L}$ has infinite monodromy. The étale fundamental group of a complex variety carrying a large local system is large. Local systems underlying variations of Hodge structures are the prominent example of large local systems. More precisely:

Proposition 2.6. *Let $\mathcal{L}$ be an integral local system on $X(\mathbb{C})$ underlying an admissible graded-polarizable variation of mixed Hodge structures. Then:*

(1) *The monodromy of $\mathcal{L}$ is finite if and only if the associated period mapping on the universal cover of $X(\mathbb{C})$ is constant.*

(2) *If the associated period mapping on the universal cover of $X(\mathbb{C})$ has discrete fibers then the local system $\mathcal{L}$ is large.*

Proof. (1) First of all we reduce to the case where X is smooth by considering the smooth locus $U \subset X$. Indeed, the constancy of a holomorphic mapping can be tested on a non-empty open subset; the finiteness of the monodromy of $\mathcal{L}$ can be checked on U because the natural map $\pi_1^{\mathrm{top}}(U(\mathbb{C})) \to \pi_1^{\mathrm{top}}(X(\mathbb{C}))$ is surjective by normality of X.

Now, if the monodromy of $\mathcal{L}$ is finite, then up to replacing X by a finite étale cover we may assume that the monodromy of $\mathcal{L}$ is trivial. In this case, the local system $\mathcal{L}$ is a constant sheaf on $X(\mathbb{C})$. The theorem of the fixed part (see [22], [16], [36] in the pure case, and [39] in the mixed case) states that there is a unique mixed Hodge structure on $\Gamma(X(\mathbb{C}), \mathcal{L})$ such that the adjunction morphism

$$\eta \colon \Gamma(X(\mathbb{C}), \mathcal{L}) \otimes \mathbb{Z}_{X(\mathbb{C})} \to \mathcal{L}$$

is a morphism of variations of mixed Hodge structures. If $\mathcal{L}$ is constant, then η is necessarily an isomorphism and the associated period mapping is constant.

Conversely, suppose that the associated period mapping is constant of value p. The period mapping on the universal cover is equivariant with respect to the action of $\pi_1^{\mathrm{top}}(X(\mathbb{C}))$, thus the monodromy of $\mathcal{L}$ fixes p. On the other hand $\mathcal{L}$ is supposed to be integral, hence the action of the monodromy on the period domain is proper [1, Corollary 3.8].

(2) Follows from (1). □

In fact, using that fibers of the period mapping are algebraic [2] one can see that $\mathscr{L}$ is large if and only if the period mapping on the universal cover has discrete fibers.

Example 2.7. Mixed Shimura varieties associated with a torsion-free congruence subgroup carry a large local system coming from their interpretation as period spaces for (graded polarized) integral mixed Hodge structures. In the exceptional case this local system is not known to be of geometric origin; see [23].

Example 2.8. A full range of (fine) moduli spaces of polarized varieties (e.g., those of smooth projective curves, Calabi–Yau varieties, most complete intersections) admit a large local system – namely, the one whose fiber at a point is the middle cohomology of the corresponding variety. Indeed, such a local a system is large when the "infinitesimal Torelli theorem" is satisfied, whence the above list; see [4].

3. Geometry

3.1. Degree of the singular locus.

3.1.1. Degree. For a proper variety X over a field k together with an ample line bundle L, let

$$\deg(X, L) := (\dim X)! \lim_{i\to\infty} \frac{\dim_k \Gamma(X, L^{\otimes i})}{i^{\dim X}}.$$

The theory of Hilbert polynomials shows that such a limit exists and is a positive rational number. When X is integral, the asymptotic version of Riemann–Roch states

$$\deg(X, L) = L^n,$$

where $n = \dim X$ and L^n is the top self-intersection of L. In particular, $\deg(X, L)$ is a positive integer.

Lemma 3.1. *Let $\pi\colon Y \to X$ be a finite morphism between proper varieties over k and L an ample line bundle over X. If there exists a scheme-theoretically dense open subset U of X such that $\pi^{-1}(U) \to U$ is an isomorphism, then*

$$\deg(Y, \pi^* L) = \deg(X, L).$$

Proof. The natural map $\varphi\colon \mathcal{O}_X \to \pi_* \mathcal{O}_Y$ is injective because the open subset U is scheme-theoretically dense. The support of the coherent $\mathcal{O}_X$-module $F = \operatorname{Coker}\varphi$ is contained in the closed subset $X \smallsetminus U$, thus in particular of dimension $< \dim X$. For $i \geqslant 1$ big enough, the cohomology group $\mathrm{H}^1(X, L^{\otimes i})$ vanishes, yielding a short exact sequence

$$0 \to \Gamma(X, L^{\otimes i}) \to \Gamma(Y, \pi^* L^{\otimes i}) \to \Gamma(X, F \otimes L^{\otimes i}) \to 0.$$

Since the support of F has dimension $< \dim X$,

$$\lim_{i\to\infty} \frac{\dim \Gamma(X, F \otimes L^{\otimes i})}{i^{\dim X}} = 0.$$

The dimensions of X and Y being the same, the result follows. □

Lemma 3.2. *Let X be an equidimensional proper variety over k, L an ample line bundle. Then*

$$\deg(X, L) \geqslant \sum_{X'} \deg(X', L_{X'}),$$

the sum ranging on the irreducible components of X endowed with the reduced structure. Moreover, equality holds if X is reduced.

Proof. One may assume that X is reduced because, for $i \geqslant 0$ big enough, the restriction map $\Gamma(X, L^{\otimes i}) \to \Gamma(X_{\mathrm{red}}, L^{\otimes i})$ is surjective. In this case let $X_1, \dots, X_n$ be the irreducible components of X. Lemma 3.1, applied to the finite morphism $X_1 \sqcup \cdots \sqcup X_n \to X$, gives the desired result. □

3.1.2. Degree in families. Let $f : X \to S$ be a proper morphism between algebraic varieties over a field k and L a relatively ample line bundle on X. Consider:

- the function $\delta_{X/S,L} : S \to \mathbb{N}$,

  $$s \mapsto \max\{\deg(Z, L_{|Z}) : Z \text{ irreducible component of } X_s\},$$

  where Z is endowed with its reduced structure, and
- the function $\iota_{X/S} : S \to \mathbb{N}$ associating to $s \in S$ the number of irreducible components of $X_{\bar{s}}$ where $\bar{s}$ is a geometric point over s.

Lemma 3.3. *Let $f : X \to S$ be a proper morphism between varieties over k and L relatively ample line bundle on X. Then the functions $\delta_{X/S,L}, \iota_{X/S} : S \to \mathbb{N}$ are bounded above.*

Proof. The statement only depends on the reduced structure of S. Moreover, by treating separately each irreducible component, the scheme S may be supposed to be integral. By Noetherian induction, it suffices to show the statement on a non-empty open subset of S. Let η be the generic point of S and $X_{\eta,1}, \dots, X_{\eta,r}$ the irreducible components of X_η. Let X_i be the closure of $X_{\eta,i}$ in X for $i = 1, \dots, r$. According to [41, Lemma 054Y], there is an open subset S' of S such that

$$f^{-1}(S') \subset X_1 \cup \cdots \cup X_r.$$

Up to treating each of the X_i separately and up to replacing S by S', one may assume X to be integral. Upon shrinking S, the morphism f may be assumed to be flat [41, Proposition 052B]. By flatness the fibers of f are then pure of dimension $d := \dim X - \dim S$ [41, Lemma 02JS]. The relative ampleness of L implies the existence of $i_0 \geqslant 1$ such that, for $q \geqslant 1$ and $i \geqslant i_0$, the higher direct image $\mathrm{R}^q f_* L^{\otimes i}$ vanishes. For integers $i, q \geqslant 0$ the function

$$s \mapsto \dim_{\kappa(s)} \mathrm{H}^q(X_s, L^{\otimes i})$$

is then upper semi-continuous on S by [25, Theorem III.12.8]. Thus, up to replacing L by $L^{\otimes i_0}$ and S by a non-empty open subset, one may assume, for $i \geqslant 1$, $q \geqslant 1$ and $s \in S$,

$$\mathrm{H}^q(X_s, L^{\otimes i}) = 0.$$

The Hilbert polynomial of $L_{|X_s}$ does not depend on s, because f is flat [21, Theorem 5.10]. Combined with the vanishing of higher cohomology, this implies that $s \mapsto \dim_{\kappa(s)} \Gamma(X_s, L^{\otimes i})$

is constant for all $i \in \mathbb{N}$. As a consequence, so is the function

$$s \mapsto \deg(X_s, L_{|X_s}) = d! \lim_{i\to\infty} \frac{\dim_{\kappa(s)} \Gamma(X_s, L^{\otimes i})}{i^d}.$$

For $s \in S$, the inequality

$$\deg(X_s, L_{|X_s}) \geqslant \sum_Z \deg(Z, L_{|Z}),$$

where the sum ranges on the irreducible components of X_s endowed with the reduced structure, shows that the function $\delta_{X/S,L}$ is bounded above. On the other hand, the degree is invariant under extension of scalars, thus

$$\deg(X_s, L_{|X_s}) \geqslant \sum_{\bar{Z}} \deg(\bar{Z}, L_{|\bar{Z}}),$$

where the sum ranges on the irreducible components of $X_{\bar{s}}$ endowed with the reduced structure, where $\bar{s}$ is a geometric point over s. Since $\deg(\bar{Z}, L_{|\bar{Z}})$ is a positive integer, the right-hand side of the above inequality is bounded below by $\iota_{X/S}(s)$. This shows that the function $\iota_{X/S}$ is bounded above. □

Proposition 3.4. *For integers $N, D \geqslant 1$, there is an integer $R_D = R_D(N)$ such that, for a field k, a subvariety X of $\mathbb{P}^N_k$ of degree $\leqslant D$, the following statements hold:*

(1) *the number of irreducible components of X is $\leqslant R_D$,*

(2) *any irreducible component Z of its singular locus X^{sing} (endowed with the reduced structure) has degree* $\deg(Z, \mathcal{O}_{\mathbb{P}^N}(1)) \leqslant R_D$,

(3) *for an algebraic closure $\bar{k}$ of k, the number of irreducible components of $X^{\mathrm{sing}}_{\bar{k}}$ is $\leqslant R_D$.*

Proof. According to [5, Exp. XIII, Corollaire 6.11 (ii)], it suffices to prove the statement when the subvarieties in question have a fixed Hilbert polynomial $P \in \mathbb{Q}[z]$. For, consider the Hilbert scheme

$$S = \mathrm{Hilb}^P_{\mathbb{P}^N_{\mathbb{Z}}, \mathcal{O}(1)}$$

of the closed subschemes of $\mathbb{P}^N_{\mathbb{Z}}$ with Hilbert polynomial P with respect to $\mathcal{O}(1)$. Let $X \subseteq \mathbb{P}^N_S$ be the universal family. The morphism $\pi \colon X \to S$ induced by the second projection is proper and flat. Let n be its relative dimension. For (1), apply Lemma 3.3 to the morphism $\pi \colon X \to S$ and the relatively ample line bundle $L := \mathcal{O}_{\mathbb{P}^N_S}(1)_{|X}$. For (2) and (3), let X^{sing} be the closed subset in X where the coherent $\mathcal{O}_X$-module $\Omega_{X/S}$ has rank $\geqslant n+1$. One concludes by applying Lemma 3.3 to the morphism $\pi \colon X^{\mathrm{sing}} \to S$ and the line bundle $L := \mathcal{O}_{\mathbb{P}^N_S}(1)_{|X^{\mathrm{sing}}}$. □

3.2. Families of normal cycles. Let k be a field of characteristic 0 and X a geometrically integral normal variety over k.

3.2.1. Definition. Following Kollár [28], a *normal cycle* on X is a morphism $f \colon Z \to X$ which is finite and birational onto its image, where Z is a geometrically integral normal variety. A *family of normal cycles* on X is the datum of morphisms $\pi \colon \mathcal{Z} \to S$ and $f \colon \mathcal{Z} \to X$

of k-schemes, where

- S is reduced and a countable disjoint union of varieties over k,
- the morphism π is separated, of finite type, flat and with geometrically integral normal fibers,
- for $s \in S$, the map $f_s\colon \mathcal{Z}_s \to X_{\kappa(s)}$ is a normal cycle, where $\kappa(s)$ is the residue field at s.

3.2.2. Exhaustive families. A family of normal cycles $(\pi\colon \mathcal{Z} \to S, f\colon \mathcal{Z} \to X)$ on X is *exhaustive* (*weakly complete* in Kollár's terminology) if, given a field extension k' of k and a normal cycle $g\colon Z \to X_{k'}$, there is a unique $s \in S(k')$ such that $g = f_s$. Given an exhaustive family of normal cycles $(\pi\colon \mathcal{Z} \to S, f\colon \mathcal{Z} \to X)$ on X and an open immersion $j\colon U \to X$, the couple

$$(\pi\colon \mathcal{Z} \times_X U \to \pi(\mathcal{Z} \times_X U), f\colon \mathcal{Z} \times_X U \to U)$$

is an exhaustive family of normal cycles on U. The arguments in [28, Proposition 2.4] allow to show the following facts:

Proposition 3.5. *If X is projective, then there exists an exhaustive family of normal cycles $(\pi\colon \mathcal{Z} \to S, f\colon \mathcal{Z} \to X)$ on X such that, for any ample line bundle L on X and any integer $d \geqslant 1$, the set*

$$S_{L,d} := \{s \in S : \deg(\mathcal{Z}_s, f_s^* L) \leqslant d\}$$

is a finite union of connected components of S (thus of finite type).

Proposition 3.6. *Suppose $k = \mathbb{C}$ and X projective. Let $(\pi\colon \mathcal{Z} \to S, f\colon \mathcal{Z} \to X)$ be an exhaustive family of normal cycles on X. Then there is a surjective immersion $\iota\colon S' \to S$ of finite type such that, for any connected component T of S, the induced map*

$$(\mathcal{Z}' \times_{S'} T')(\mathbb{C}) \to T'(\mathbb{C})$$

is a topological fiber bundle, where $T' = \iota^{-1}(T)$ and $\mathcal{Z}' = \mathcal{Z} \times_S S'$.

The crucial property of exhaustive families needed later is the following boundedness:

Proposition 3.7. *Suppose k is an algebraically closed subfield of $\mathbb{C}$. Let L be an ample line bundle over X, U a non-empty open subset of X and $d \geqslant 1$ an integer. Then, up to conjugation, there are only finitely many subgroups of $\pi_1^{\text{ét}}(U)$ of the form*

$$\operatorname{Im}(\pi_1^{\text{ét}}(f^{-1}(U)) \to \pi_1^{\text{ét}}(U))$$

with $f\colon Z \to X$ a normal cycle such that $\deg(Z, f^* L) \leqslant d$.

Proof. Since the geometric étale fundamental group is insensible to extension of scalars, and since there are more normal cycles on $X_{\mathbb{C}}$ than on X, there is no loss of generality in assuming $k = \mathbb{C}$. Let $(\pi\colon \mathcal{Z} \to S, f\colon \mathcal{Z} \to X)$ be an exhaustive family of normal cycles satisfying the property in the statement of Proposition 3.5. According to Proposition 3.6, one may suppose that, for each connected component T of S, the induced map $\pi^{-1}(T)(\mathbb{C}) \to T(\mathbb{C})$ is a topological fiber bundle. Let S_U be the image of $\mathcal{Z}_U := f^{-1}(U)$ in S via π. Then

$$(\pi\colon \mathcal{Z}_U \to S_U, f\colon \mathcal{Z}_U \to U)$$

is an exhaustive family of normal cycles on U; moreover, for each connected component T of S_U, the induced map $\pi^{-1}(T)(\mathbb{C}) \to T(\mathbb{C})$ is again a topological fiber bundle. It follows that, for $t, t' \in T(\mathbb{C})$, the images of $\pi_1^{\mathrm{top}}(\mathcal{Z}_{U,t}(\mathbb{C}))$ and $\pi_1^{\mathrm{top}}(\mathcal{Z}_{U,t'}(\mathbb{C}))$ in $\pi_1^{\mathrm{top}}(U(\mathbb{C}))$ are conjugated subgroups. Passing to profinite completions, the corresponding affirmation holds for étale fundamental groups. Let $C_{L,d}$ be the set of conjugacy classes of subgroups of $\pi_1^{\text{ét}}(U)$ obtained as the image of $\pi_1^{\text{ét}}(g^{-1}(U)) \to \pi_1^{\text{ét}}(U)$ with $g\colon Z \to X$ a normal cycle such that

$$\deg(Z, g^*L) \leqslant d.$$

With the notation of Proposition 3.5, the argument above shows the inequality

$$\#C_{L,d} \leqslant \#(\text{connectected components of } S_{L,d})$$

and the right-hand side is $< \infty$ by the cited result. □

3.3. Proof of Theorem B. Let us begin with two easy facts:

Lemma 3.8. *Let $\pi\colon X' \to X$ be a finite surjective morphism between integral normal varieties over k which is Galois of group G and étale over an open subset U of X. Let Y' be an integral subvariety of X' meeting $\pi^{-1}(U)$ and $Y := \pi(Y')$. Then the map $\pi_{|Y'}\colon Y' \to Y$ has degree*

$$\#\operatorname{Im}(\pi_1^{\text{ét}}(\nu^{-1}(U \cap Y)) \to G),$$

where $\nu\colon \widetilde{Y} \to Y$ is the normalization.

Proof. Since the degree is computed on the generic point, there is no harm in replacing the varieties X and X' respectively by U and $\pi^{-1}(U)$. In this case the morphism π is finite étale thus so is the induced morphism $\widetilde{Y} \times_X X' \to \widetilde{Y}$. By [24, Exp. I, Théorème 9.5 (i)] it follows that the fibered product $\widetilde{Y} \times_X X'$ is normal. The connected components of $\widetilde{Y} \times_X X'$ are therefore integral [41, Lemma 033M] and the normalization $\widetilde{Y}'$ of Y' is one of them. The usual dictionary between connected covers and subgroups of the fundamental group implies that the degree of the induced map $\widetilde{\pi}\colon \widetilde{Y}' \to \widetilde{Y}$ is $\#\operatorname{Im}(\pi_1^{\text{ét}}(\widetilde{Y}) \to G)$. The normalization being a birational map, the degree of $\pi\colon Y' \to Y$ coincides with that of $\widetilde{\pi}$, whence the statement. □

Lemma 3.9. *Let Γ be a residually finite group and F a finite subset of Γ. Then there is a finite-index normal subgroup N of Γ such that the map $F \to \Gamma/N$ is injective.*

Proof. Consider the finite subset $F' := \{\gamma\delta^{-1} : \gamma, \delta \in F\} \smallsetminus \{e\}$, where $e \in \Gamma$ is the neutral element. Saying that Γ is residually finite amounts to the fact that, for each $\gamma \in \Gamma \smallsetminus \{e\}$, there is a finite-index normal subgroup N_γ to which γ does not belong. Then the finite-index normal subgroup $N := \bigcap_{\gamma \in F'} N_\gamma$ does the job. □

We are now in position to proceed with the proof.

Proof of Theorem B. To begin, recall the setup: let X be a normal irreducible projective variety over an algebraically closed field k of characteristic 0, L an ample line bundle, U an non-empty open subset of X such that the étale fundamental group of U is large, and $d \geqslant 1$ an integer. By Lefschetz's principle, the varieties X, U and the line bundle L can be defined over a subfield of k finitely generated over $\mathbb{Q}$. Since having large étale fundamental group is a property that passes to algebraically closed subfields, there is no loss of generality in assuming that k is a subfield of $\mathbb{C}$. Then, according to Proposition 3.7, there are finitely many subgroups

$$H_1, \dots, H_r \subset \pi_1^{\text{ét}}(U)$$

such that, given a normal cycle $f: Z \to X$ such that $\deg(Z, f^*L) \leqslant d$, the image of the group homomorphism $\pi_1^{\text{ét}}(f^{-1}(U)) \to \pi_1^{\text{ét}}(U)$ is conjugated to some of the subgroups H_i. Needless to say, by definition of large étale fundamental group, each of the subgroups H_i is infinite. By design, the étale fundamental group is profinite (in particular, residually finite), thus each finite subset injects into some finite quotient: Lemma 3.9 implies that there exists a normal finite-index subgroup N of $\pi_1^{\text{ét}}(U)$ such that, for each $i = 1, \dots, r$,

$$\#\operatorname{Im}(H_i \to G := \pi_1^{\text{ét}}(U)/N) \geqslant d.$$

Let $U' \to U$ be the finite étale cover of U associated with the subgroup N. As argued in the proof of Lemma 3.8, the normality of U implies that of U', thus the variety U', *a priori* just connected, is integral. Let $\pi: X' \to X$ be the normalization of X in U'. By construction, the variety X' is integral normal, and the morphism π is Galois of group G and étale over U. To see that such a cover fulfills the requirements, let Y' be a positive-dimensional integral subvariety of X' meeting $\pi^{-1}(U)$ and set $Y := \pi(Y')$. The projection formula reads

$$\deg(Y', \pi^* L_{|Y'}) = \deg(\pi_{|Y'}) \deg(Y, L_{|Y}),$$

where $\deg(\pi_{|Y'})$ is the degree of the map $Y' \to Y$ induced by π. Of course, if $\deg(Y, L_{|Y}) > d$, then $\deg(Y', \pi^* L_{|Y'}) \geqslant d$. Suppose instead $\deg(Y, L_{|Y}) \leqslant d$ and let $\nu: \widetilde{Y} \to Y$ be the normalization. By the projection formula (or Lemma 3.1),

$$\deg(\widetilde{Y}, \nu^* L_{|\widetilde{Y}}) = \deg(Y, L_{|Y}) \leqslant d.$$

According to Lemma 3.8, the map $\pi_{|Y'}$ has degree $\#\operatorname{Im}(\pi_1^{\text{ét}}(\nu^{-1}(U \cap Y)) \to G)$. On the other hand, by construction, the image of $\pi_1^{\text{ét}}(\nu^{-1}(U \cap Y)) \to \pi_1^{\text{ét}}(U)$ is conjugated to some of the subgroups H_i, thus

$$\#\operatorname{Im}(\pi_1^{\text{ét}}(\nu^{-1}(U \cap Y)) \to G) = \#\operatorname{Im}(H_i \to G) \geqslant d.$$

By ampleness of L, the degree $\deg(Y, L_{|Y})$ is a positive integer, thus the projection formula implies $\deg(Y', \pi^* L_{|Y'}) \geqslant d$ as desired. □

4. Arithmetic

Let K be a number field and $\mathcal{O}_K$ its ring of integers.

4.1. Growth rates.

4.1.1. Hermitian line bundles. A *Hermitian line bundle* $\bar{\mathcal{L}}$ on a proper flat scheme $\mathcal{X}$ over $\mathcal{O}_K$ is the datum of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and, for every embedding $\sigma: K \to \mathbb{C}$, of a continuous metric $\|\cdot\|_\sigma$ on the holomorphic line bundle $\mathcal{L}_{|\mathcal{X}_\sigma(\mathbb{C})}$. The collection of metrics $\{\|\cdot\|_\sigma\}$ is supposed to be compatible with complex conjugation: for $\sigma: K \to \mathbb{C}$ the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{V}(\mathcal{L}_\sigma)(\mathbb{C}) & \xrightarrow{\|\cdot\|_\sigma} & \mathbb{R}_+ \\ {\scriptstyle z\mapsto\bar z}\downarrow & & \| \\ \mathbb{V}(\mathcal{L}_{\bar\sigma})(\mathbb{C}) & \xrightarrow{\|\cdot\|_{\bar\sigma}} & \mathbb{R}_+. \end{array}$$

Here $\mathbb{V}(\mathcal{L}_\sigma)$ and $\mathbb{V}(\mathcal{L}_{\bar\sigma})$ are the total space of the line bundles $\mathcal{L}_\sigma$ and $\mathcal{L}_{\bar\sigma}$, and metrics on a line bundle are seen as functions on the total space of the line bundle in question. For a morphism of $\mathcal{O}_K$-schemes $f\colon \mathcal{Y} \to \mathcal{X}$, where the $\mathcal{O}_K$-scheme $\mathcal{Y}$ is also proper and flat, the pull-back $f^*\bar{\mathcal{L}}$ is defined in the evident way.

4.1.2. Degree. The *(Arakelov) degree* of a Hermitian line bundle $\bar{\mathcal{L}}$ on $\operatorname{Spec}\mathcal{O}_K$ is

$$\deg \bar{\mathcal{L}} = \log \#(\mathcal{L}/s\mathcal{O}_K) - \sum_{\sigma : K \to \mathbb{C}} \log \|s\|_\sigma,$$

where s is a non-zero element of the $\mathcal{O}_K$-module $\mathcal{L}$; the quantity above does not depend on its choice because of the product formula.

4.1.3. Extension of scalars. Let K' be a finite extension of K and

$$\pi\colon \operatorname{Spec}\mathcal{O}_{K'} \to \operatorname{Spec}\mathcal{O}_K$$

the morphism induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$. Given a Hermitian line bundle $\bar{\mathcal{L}}$ over $\mathcal{O}_K$, the Hermitian line bundle $\pi^*\bar{\mathcal{L}}$ over $\mathcal{O}_{K'}$ has degree

$$\deg \pi^*\bar{\mathcal{L}} = [K' : K] \deg \bar{\mathcal{L}}. \tag{4.1}$$

4.1.4. Height. Let $\bar{\mathcal{L}}$ be a Hermitian line bundle on a proper and flat $\mathcal{O}_K$-scheme $\mathcal{X}$. By the valuative criterion of properness, a K-rational point P of $\mathcal{X}$ extends to an $\mathcal{O}_K$-valued point $\mathcal{P}$ of $\mathcal{X}$. The *height* of the point P with respect to the Hermitian line bundle $\bar{\mathcal{L}}$ is

$$h_{\bar{\mathcal{L}}}(P) = \frac{\deg \mathcal{P}^*\bar{\mathcal{L}}}{[K : \mathbb{Q}]}.$$

Replacing the number field K by a finite extension permits to define the height for any point of $\mathcal{X}$ with values in an algebraic closure $\bar{K}$ of K (equation (4.1) implies that the height is well-defined). A routine variation of the proof of [27, Theorem B.3.2 (d)] yields:

Proposition 4.1. *Let $\mathcal{X}$ and $\mathcal{X}'$ be proper and flat $\mathcal{O}_K$-schemes endowed respectively with Hermitian line bundles $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}'$. Suppose that there exists an isomorphism $f\colon \mathcal{X}_K \to \mathcal{X}'_K$ between the generic fibers of $\mathcal{X}$ and $\mathcal{X}'$, and an isomorphism $\mathcal{L}_{|\mathcal{X}_K} \cong f^*\mathcal{L}'_{|\mathcal{X}'_K}$ of line bundles over $\mathcal{X}_K$. Then*

$$\sup_{P \in \mathcal{X}(\bar{K})} |h_{\bar{\mathcal{L}}}(P) - h_{\bar{\mathcal{L}}'}(f(P))| < +\infty.$$

A line bundle $\mathcal{L}$ on a proper and flat $\mathcal{O}_K$-scheme is *generically ample* if its restriction to the generic fiber $\mathcal{X}_K$ is ample. In this framework the Northcott property can be stated as follows (see for instance [27, Theorem B.2.3]).

Proposition 4.2. *Let $\mathcal{X}$ be a proper and flat $\mathcal{O}_K$-scheme together with a Hermitian line bundle $\bar{\mathcal{L}}$ on $\mathcal{X}$. If $\mathcal{L}$ is generically ample, then, for any real number c,*

$$\#\{P \in \mathcal{X}(K) : h_{\bar{\mathcal{L}}}(P) \leqslant c\} < +\infty.$$

4.1.5. Counting function. Let $\mathcal{X}$ be a proper, flat $\mathcal{O}_K$-scheme, $\bar{\mathcal{L}}$ a generically ample Hermitian line bundle on $\mathcal{X}$ and $\mathcal{U}$ an open subset of $\mathcal{X}$. For a subring R of K containing $\mathcal{O}_K$ and a real number c, Proposition 4.2 permits to define

$$\nu(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, R, c) := \log^+ \#\{P \in \mathcal{U}(R) : h_{\bar{\mathcal{L}}}(P) \leqslant c\},$$

where $\log^+ z := \log \max\{1, z\}$ for $z \in \mathbb{R}$. The *growth rate* of R-points of $\mathcal{U}$ is

$$\operatorname{growth}(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, R) := \limsup_{c \to +\infty} \frac{\nu(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, R, c)}{c}.$$

Clearly, such a function is non-decreasing in the variable R (with respect to inclusion).

4.1.6. Growth rates. Let X be a proper K-scheme, L an ample line bundle on X and U an open subset of X. Choose a proper and flat $\mathcal{O}_K$-scheme $\mathcal{X}$, a Hermitian line bundle $\bar{\mathcal{L}}$ on $\mathcal{X}$ and an open subset $\mathcal{U}$ of $\mathcal{X}$ whose generic fibers are respectively X, L and U. By Proposition 4.1, the real numbers

$$\begin{aligned} \operatorname{gr.rat}_K(X, L, U) &= \operatorname{growth}(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, K), \\ \operatorname{gr.int}_K(X, L, U) &= \sup_S \operatorname{growth}(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, \mathcal{O}_{K,S}), \end{aligned}$$

where the supremum ranges on the finite set of places S of K containing the Archimedean ones, do not depend on the chosen $\mathcal{X}$, $\bar{\mathcal{L}}$ and $\mathcal{U}$. They are called the *growth rate* respectively of *rational* and *integral points* of U (with respect to X and L). Clearly,

$$\operatorname{gr.int}_K(X, L, U) \leqslant \operatorname{gr.rat}_K(X, L, U).$$

Remark 4.3. Some considerations:

(1) The growth rate of rational and integral points differ in general when U is not proper. For instance, take $X = \mathbb{P}^1_K$, $L = \mathcal{O}(1)$ and $U = \mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}$. Then

$$\begin{aligned} \operatorname{gr.rat}_K(X, L, U) &= \operatorname{gr.rat}_K(X, L, X) = 2[K : \mathbb{Q}], \\ \operatorname{gr.int}_K(X, L, U) &= 0, \end{aligned}$$

because of the theorem on the S-unit equation (see for instance [6, Chapter 5]).

(2) Let K' be a finite extension of K, X' and U' the K'-schemes deduced respectively from X and U by extending scalars to K', and L' the line bundle on X' deduced from L. Then

$$\operatorname{gr.rat}_K(X, L, U) \leqslant \operatorname{gr.rat}_{K'}(X', L', U'),$$

and similarly for the growth rate of integral points. However, as the example above shows, in general the real numbers $\operatorname{gr.rat}_{K'}(X', L', U')$ tend to ∞ as soon as the degree of K' does.

(3) For an integer $n \geqslant 1$,

$$\mathrm{gr.rat}_K(X, L, U) = n\, \mathrm{gr.rat}_K(X, L^{\otimes n}, U)$$

and similarly for the growth rate of integral points. In particular, these growth rates really depend on the line bundle L and not only on its restriction to U.

4.2. Proof of Theorem A.

4.2.1. Statement. Let $N \geqslant 1$ be an integer. For a locally closed subvariety U of $\mathbb{P}^N_K$ and a real number c, with the notation of Section 4.1.5, let

$$\nu_K(U, c) := \nu(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{X} \smallsetminus \mathcal{Z}, K, c),$$

where $\mathcal{X}$ resp. $\mathcal{Z}$ is the Zariski closure of U resp. $\mathcal{X}_K \smallsetminus U$ in $\mathbb{P}^N_{\mathcal{O}_K}$ and $\bar{\mathcal{L}}$ is the Hermitian line bundle on $\mathbb{P}^N_{\mathcal{O}_K}$ obtained by endowing the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N_{\mathcal{O}_K}$ with metrics, for an embedding $\sigma\colon K \to \mathbb{C}$,

$$\|s\|_\sigma(x) = \frac{|s(x)|_\sigma}{\max_{i=0,\dots,N} |x_i|_\sigma},$$

where s is a local section of $\mathcal{O}(1)$ and $x_0, \dots, x_N$ are the homogeneous coordinates of a point x in $\mathbb{P}^N(\mathbb{C})$. The height function associated with the Hermitian line bundle $\bar{\mathcal{L}}$ is, for a K-rational point x of $\mathbb{P}^N_K$,

$$h(x) := h_{\bar{\mathcal{L}}}(x) = \sum_{v \in \mathrm{V}^0_K} \log \max_{i=0,\dots,N} |x_i|_v + \sum_{\sigma : K \to \mathbb{C}} \log \max_{i=0,\dots,N} |x_i|_\sigma,$$

where V^0_K is the set of finite places of K. In the formula above, for a p-adic place v, the absolute value $|\cdot|_v$ is normalized as $|p|_v = p^{-[K_v:\mathbb{Q}_p]}$ where K_v is the v-adic completion of K.

Theorem 4.4. *Let Z be a subvariety of $\mathbb{P}^N_K$, $\varepsilon > 0$ a real number and $n \geqslant 0$, $D \geqslant 1$ integers. Then there is a real number $C_{n,D} = C_{n,D}(N, K, Z, \varepsilon)$ with the following property: for an integral n-dimensional subvariety X of $\mathbb{P}^N_K$ of degree $\leqslant D$ such that each positive-dimensional integral closed subvariety in X not contained in Z has degree $\geqslant d^{\dim Z}$ for some integer $d \geqslant 1$, and a real number $c \geqslant [K : \mathbb{Q}]\varepsilon$, the following inequality holds:*

$$\nu_K(X \smallsetminus Z, c) \leqslant c[K : \mathbb{Q}](1 + \varepsilon)\frac{n(n+3)}{2d} + C_{n,D}.$$

Note that, for $n = 0$, the above inequality reads $\nu_K(X \smallsetminus Z, c) \leqslant C_{n,D}$. Before going into the proof of the preceding statement, let us see how it permits to prove Theorem A.

Proof of Theorem A. First of all, and rather crucially, notice that the hypotheses and the conclusions are insensitive to taking powers of L. Therefore, up to replacing L with a multiple big enough, one may assume that L is very ample. Via the embedding $X \to \mathbb{P}(\Gamma(X, L)^\vee)$ Theorem 4.4 can be applied to give

$$\mathrm{gr.rat}_K(X, L, X \smallsetminus Z) \leqslant [K : \mathbb{Q}]\frac{n(n+3)}{2d},$$

as desired. □

4.2.2. Proofs. The statement will be deduced by induction from the following:

Theorem 4.5 ([14, Theorem A]). *Let $\varepsilon > 0$ be a real number and $D \geqslant 0$ an integer. Then there are positive integers $A_D = A_D(N, K, \varepsilon)$ and $B_D = B_D(N, K, \varepsilon)$ with the following property: for an integral subvariety X of $\mathbb{P}^N_K$ of degree $D' \leqslant D$ and dimension $n \geqslant 1$, and a real number $c \geqslant [K : \mathbb{Q}]\varepsilon$, the set*

$$\{x \in X^{\mathrm{reg}}(K) : h(x) \leqslant c\}$$

can be covered by no more than

$$A_D \exp\left(\frac{n+1}{D'^{1/n}}(1+\varepsilon)[K : \mathbb{Q}]c\right)$$

hypersurfaces in $\mathbb{P}^N_K$ of degree $\leqslant B_D$ not containing X.

Proof of Theorem 4.4. The proof goes by induction on n. For $n = 0$, there is nothing to do. Suppose $n \geqslant 1$ and the result true in dimension $< n$. Let A_D and B_D be as in the statement of Theorem 4.5, and R_D as in that of Proposition 3.4. Let X be an integral n-dimensional subvariety of $\mathbb{P}^N_K$ of degree $D' \leqslant D$ with the property that all positive-dimensional integral closed subvarieties Y of X not contained in Z have degree $\geqslant d^{\dim Y}$. Quite trivially, such a property is inherited by integral subvarieties X' of X not contained in Z: any positive-dimensional integral closed subvariety Y of X' not contained in Z has degree $\geqslant d^{\dim Y}$. Now, by Theorem 4.5, there exist r hypersurfaces $H_1, \dots, H_r$ of $\mathbb{P}^N_K$ of degree $\leqslant B_D$ with

$$r \leqslant A_D \exp\left(\frac{n+1}{D^{1/n}}(1+\varepsilon)[K : \mathbb{Q}]c\right) \tag{4.2}$$

not containing X such that the set $\{x \in X^{\mathrm{reg}}(K) : h(x) \leqslant c\}$ is contained in the union of the hypersurfaces $H_1, \dots, H_r$. For $i = 1, \dots, r$, the hyperplane section $X_i := H_i \cap X$ of X is pure of dimension $n - 1$ and has degree $\leqslant DB_D$. Each irreducible component X_i' of X_i has degree $\deg(X_i', \mathcal{O}(1)) \leqslant DB_D$ by Lemma 3.2. Therefore, it is possible to apply the induction hypothesis to such an X_i' and obtain

$$\nu_K(X_i' \smallsetminus Z, c) \leqslant c[K : \mathbb{Q}](1+\varepsilon)\frac{(n-1)(n+2)}{2d} + C_{n-1,DB_D},$$

because $d \leqslant DB_D$. Since the hyperplane section X_i has at most R_{DB_D} irreducible components by Proposition 3.4,

$$\nu_K(X_i \smallsetminus Z, c) \leqslant c[K : \mathbb{Q}](1+\varepsilon)\frac{(n-1)(n+2)}{2d} + C_{n-1,DB_D} + \log R_{DB_D}.$$

Taking into account (4.2), and recalling $D' \geqslant d^n$, the preceding inequality yields

$$\nu_K(X^{\mathrm{reg}} \smallsetminus Z, c) \leqslant c[K : \mathbb{Q}](1+\varepsilon)\frac{n(n+3)}{2d} + C'_{n,D}, \tag{4.3}$$

where $C'_{n,D} := C_{n-1,DB_D} + \log R_{DB_D} + \log A_D$ and the following identity has been noticed:

$$(n+1) + \frac{(n-1)(n+2)}{2} = (n+1) + \sum_{i=1}^{n-1}(i+1) = \sum_{i=1}^{n}(i+1) = \frac{n(n+3)}{2}.$$

Let $Y_1, \dots, Y_s$ be the irreducible components of the singular locus X^{sing} of X. According to Proposition 3.4, the subvariety Y_i has degree $\leqslant R_D$ and $s \leqslant R_D$. The induction hypothesis, applied to an irreducible component Y_i not contained in Z, gives

$$\nu_K(Y_i \smallsetminus Z, c) \leqslant c[K:\mathbb{Q}](1+\varepsilon)\frac{(n-1)(n+2)}{2d} + \widetilde{C}_{n-1,R_D},$$

where $\widetilde{C}_{n-1,R_D} := \max\{C_{0,R_D}, \dots, C_{n-1,R_D}\}$, as Y_i has degree $\leqslant R_D$ and dimension $\leqslant n-1$. In particular,

$$\nu_K(X^{\mathrm{sing}} \smallsetminus Z, c) \leqslant c[K:\mathbb{Q}](1+\varepsilon)\frac{(n-1)(n+2)}{2d} + C''_{n,D}, \tag{4.4}$$

where $C''_{n,D} := \widetilde{C}_{n-1,R_D} + \log R_D$. Combining inequalities (4.3) and (4.4) yields the bound in the statement with $C_{n,D} := \log(\exp(C'_{n,D}) + \exp(C''_{n,D}))$. □

4.3. Growth rates on covers. The last ingredient for the proof of the main theorem is a version of the Chevalley–Weil theorem for growth rates of integral points (Proposition 4.12). Its proof is a variation of the classical one (see [37, 4.2] or Theorem 2.3 in Corvaja's contribution to [15]).

4.3.1. Reminder on twists. Let S be a Noetherian scheme and G a finite étale group scheme over S. A *principal G-bundle* is a finite étale S-scheme P endowed with an action of G such that the morphism $G \times_S P \to P \times_S P$, $(g,p) \mapsto (gp,p)$ is an isomorphism. The set of isomorphism classes of principal G-bundles over S is denoted $\mathrm{H}^1(S,G)$ or simply $\mathrm{H}^1(A,G)$ if the scheme $S = \operatorname{Spec} A$ is affine.

Proposition 4.6. *Let K be a number field, Σ a finite set of places of K containing the Archimedean ones. Then*

$$\#\mathrm{H}^1(\mathcal{O}_{K,\Sigma}, G) < +\infty.$$

Proof. It suffices to show that there are only finitely many G-principal bundles up to *isomorphism of $\mathcal{O}_{K,S}$-schemes*. Indeed, let P be a finite $\mathcal{O}_{K,S}$-scheme and $\bar{K}$ an algebraic closure of K. Then there are only finitely many actions of $G(\bar{K})$ on $P(\bar{K})$ because both sets are finite. Now a principal G-bundle P over $\mathcal{O}_{K,\Sigma}$ is finite étale as an $\mathcal{O}_{K,\Sigma}$-scheme. Thus its generic fiber P_K is the spectrum of a K-algebra A whose dimension as a K-vector space is the rank of G and is the product of (finitely many) finite extensions of K, all of which are unramified outside Σ. The Hermite-Minkowski bound implies that, for any integer $D \geqslant 1$, there are only finitely many isomorphism classes of finite extensions of K of degree $\leqslant D$ unramified outside Σ (see [37, 4.1]). The statement follows. □

Definition 4.7. Let X be an S-scheme with an action of G and P a principal G-bundle over S. The *twist* of X by P, if it exists, is the categorical quotient of $X \times_S P$ by the diagonal action $g(x,p) = (gx, gp)$ of G.

Clearly, isomorphic principal G-bundles give rise to isomorphic twists. With an abuse of notation, for $t \in \mathrm{H}^1(S,G)$, let X_t denote the twist of X by a principal G-bundle in the isomorphism class t. For a scheme X over P, the datum of an equivariant action of G is equivalent to a descent datum. Therefore, the theory of faithfully flat (étale actually) descent shows the following existence result.

Proposition 4.8 ([31, Proposition 4.4.9]). *Let X be an affine S-scheme endowed with an action of G and P a principal G-bundle over S. Then the twist of X by P exists.*

Remark 4.9. The construction of twists is functorial. Namely, let X, Y be S-schemes endowed with an action of G, and P a principal G-bundle over S for which the twists of X and Y by P exist. Then a G-equivariant morphism $f\colon X \to Y$ induces a morphism $f_P\colon X_P \to Y_P$ between twists.

4.3.2. Lifting points. A useful construction consists in twisting schemes to lift points. Let X be an S-scheme and $\pi\colon Y \to X$ a principal G-bundle over X. For an S-valued point x of X, the scheme-theoretic fiber of π at x,

$$P := Y \times_X S,$$

is a principal G-bundle over S. The twist of Y by P exists as an X-scheme: indeed, it is identified with the twist of Y by the G-principal bundle $P \times_S X$; the latter exists because π is a finite morphism (in particular affine) according to the finiteness of G. The G-invariant morphism $\pi\colon Y \to X$ induces a morphism $\pi_P\colon Y_P \to X$.

Lemma 4.10. *There is an S-valued point y of Y_P such that $\pi_P(y) = x$.*

Proof. The scheme-theoretic fiber $Y_P \times_X P$ of π_P at x is isomorphic to the twist P_P of P by itself. The diagonal embedding $\Delta\colon P \to P \times_S P$ is G-equivariant and, taking its quotient by G, defines the wanted S-valued point $y\colon S \to P_P$. □

4.3.3. Galois covers on non-algebraically closed fields. Let $f\colon Y \to X$ a finite and surjective morphism between geometrically integral algebraic varieties over a field k of characteristic 0. If k is algebraically closed, the morphism f is Galois if, for each geometric point $\bar{x}$ of X, the group $\mathrm{Aut}(f)$ acts transitively on the geometric fiber $Y_{\bar{x}}$. In general, the morphism f is said to be *geometrically Galois* if, for an algebraic closure $\bar{k}$ of k, the morphism $\bar{f}\colon \bar{Y} \to \bar{X}$ obtained by extending scalars to $\bar{k}$ is Galois. In such a situation $\mathrm{Gal}(\bar{k}/k)$ acts on $\mathrm{Aut}(\bar{f})$ compatibly with its action on $\bar{Y}$. Therefore $\mathrm{Aut}(\bar{f})$ descends to a finite (étale) k-group scheme $\mathrm{Aut}(f)$, called the *geometric Galois group*, acting on Y and such that $f\colon Y \to X$ is the quotient map of Y by $\mathrm{Aut}(f)$.

Example 4.11. For an integer $n \geqslant 1$, the morphism $f\colon \mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^n$ is geometrically Galois; it is Galois if and only if the base field k contains all of the n-th roots of 1. The geometric Galois group $\mathrm{Aut}(f)$ is the finite group k-scheme $\mu_n = \operatorname{Spec} k[t]/(t^n - 1)$.

4.3.4. Statement. Let $f\colon Y \to X$ be a finite and surjective morphism between geometrically integral algebraic varieties over K. Suppose f is geometrically Galois with geometric Galois group $\mathrm{Aut}(f)$. The group $\mathrm{Aut}(f)$ acts by definition on Y. For any $t \in \mathrm{H}^1(K, \mathrm{Aut}(f))$, let Y_t be the twist of Y and $f_t\colon Y_t \to X$ the twist of the $\mathrm{Aut}(f)$-invariant morphism f.

Proposition 4.12. *With the notation above, let U an open subset of X over which f is étale and L an ample line bundle on X. Then*

$$\mathrm{gr.int}_K(X, L, U) \leqslant \sup_{t \in \mathrm{H}^1(K, \mathrm{Aut}(f))} \mathrm{gr.int}_K(Y_t, f_t^* L, f_t^{-1}(U)).$$

Proof. Let $G := \mathrm{Aut}(f)$. Up to taking a power of L – an operation that does not affect the statement – the variety X can be realized as the generic fiber of a scheme $\mathcal{X}$ projective and flat over $\mathcal{O}_K$ on which the line bundle L extends to an ample line bundle $\mathcal{L}$. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in Y and denote again by $f: \mathcal{Y} \to \mathcal{X}$ the induced morphism. Pick continuous metrics $\{\|\cdot\|_{L,\sigma}\}_{\sigma: K \to \mathbb{C}}$ on $\mathcal{L}$ so that $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{L,\sigma})$ is a metrized line bundle. Let $Z := X \smallsetminus U$, $\mathcal{Z}$ its Zariski closure in $\mathcal{X}$, $\mathcal{U} := \mathcal{X} \smallsetminus \mathcal{Z}$ and $\mathcal{V} = f^{-1}(\mathcal{U})$. Let S be a finite set of places of K containing the Archimedean ones. Up to enlarging S, one may assume that the finite group K-scheme G spreads out to a finite étale $\mathcal{O}_{K,S}$-group scheme $\mathcal{G}$ acting on $\mathcal{Y}$ and such that $f: \mathcal{Y} \to \mathcal{X}$ is the categorical quotient of $\mathcal{Y}$ by $\mathcal{G}$. Moreover, one may suppose that the morphism $\mathcal{V} \times_{\mathcal{O}_K} \mathcal{O}_{K,S} \to \mathcal{U} \times_{\mathcal{O}_K} \mathcal{O}_{K,S}$ induced by f is étale. For an isomorphism class of $\mathcal{G}$-principal bundles $t \in \mathrm{H}^1(\mathcal{O}_{K,S}, \mathcal{G})$, let $\mathcal{Y}_{S,t}$ be the twist of $\mathcal{Y}_S$ by t and let $f_{S,t}: \mathcal{Y}_{S,t} \to \mathcal{X} \times_{\mathcal{O}_K} \operatorname{Spec} \mathcal{O}_{K,S}$ be the induced morphism. Consider the relative normalization $f_t: \mathcal{Y}_t \to \mathcal{X}$ of $\mathcal{X}$ in $\mathcal{Y}_{S,t}$ (see [41, Definition 035H]) and set $\mathcal{V}_t := f_t^{-1}(\mathcal{U})$.

Claim 4.13. *With the notation above,*

$$\mathrm{growth}(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, \mathcal{O}_{K,S}) \leqslant \max_{t \in \mathrm{H}^1(\mathcal{O}_{K,S}, \mathcal{G})} \mathrm{growth}(\mathcal{Y}_t, f_t^* \bar{\mathcal{L}}, \mathcal{V}_t, \mathcal{O}_{K,S}). \tag{4.5}$$

Proof of the Claim. According to Proposition 4.6, the set $\mathrm{H}^1(\mathcal{O}_{K,S}, \mathcal{G})$ is finite. Therefore, the $\mathcal{O}_K$-scheme

$$\widetilde{\mathcal{Y}} := \bigsqcup_{t \in \mathrm{H}^1(\mathcal{O}_{K,S}, \mathcal{G})} \mathcal{Y}_t$$

is proper. Let $\tilde{f}: \widetilde{\mathcal{Y}} \to \mathcal{X}$ the morphism induced by the twists of f. Given an $\mathcal{O}_{K,S}$-point x of $\mathcal{U}$, there is an $\mathcal{O}_{K,S}$-point $\tilde{x}$ of $\widetilde{U}$ mapping to x. Indeed, the scheme-theoretic fiber

$$\mathcal{P} := \mathcal{Y} \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}_{K,S}$$

of f at x is a principal G-bundle over $\mathcal{O}_{K,S}$. Letting t denote the isomorphism class of $\mathcal{P}$, Lemma 4.10 states that there is an $\mathcal{O}_{K,S}$-point y of $\mathcal{V}_t$ such that $f_t(y) = x$. By definition,

$$h_{f_t^* \bar{\mathcal{L}}}(y) = h_{\bar{\mathcal{L}}}(x).$$

In particular,

$$\mathrm{growth}(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, \mathcal{O}_{K,S}) \leqslant \mathrm{growth}(\widetilde{\mathcal{Y}}, \tilde{f}^* \bar{\mathcal{L}}, \tilde{f}^{-1}(\mathcal{U}), \mathcal{O}_{K,S}),$$

whence the claim. □

One concludes by bounding the right-hand side of (4.5) by

$$\sup_{t \in \mathrm{H}^1(K,G)} \mathrm{gr.int}_K(Y_t, f_t^* L, V_t),$$

where $V_t = f_t^{-1}(U)$, so that the right-hand side of the so-obtained inequality

$$\mathrm{growth}(\mathcal{X}, \bar{\mathcal{L}}, \mathcal{U}, \mathcal{O}_{K,S}) \leqslant \sup_{t \in \mathrm{H}^1(K,G)} \mathrm{gr.int}_K(Y_t, f_t^* L, V_t),$$

is independent of S. As S is arbitrary, taking the supremum over all finite sets of places of K (containing the Archimedean ones) finishes the proof. □

5. Proof of the main theorem

5.1. Setup. Let $\bar{K}$ be an algebraic closure of K. For a variety V over K, let $\bar{V}$ denote the variety over $\bar{K}$ obtained by extending scalars. We will need the following descent argument:

Lemma 5.1. *Let X be a normal geometrically integral variety and let $f : X' \to \bar{X}$ be a finite surjective morphism which is Galois. If $X(K) \neq \emptyset$, then there exists a geometric Galois finite surjective morphism $g : X'' \to X$ of varieties over K with X'' geometrically integral and normal such that g is étale over U and the morphism $\bar{g} : \bar{X}'' \to \bar{X}$ factors through the morphism $f : X' \to \bar{X}$.*

Proof. Argue as in the proof of the implication (iii) ⇒ (iv) in [33, Lemma 3.5.57]. First of all, the morphism $X' \to \bar{X}$ may be assumed to be Galois, for it suffices to replace the function field $\bar{K}(X')$ of X' by its Galois closure F over the function field $\bar{K}(\bar{X})$ of $\bar{X}$, and X' by its normalization in F. Let us show that X' comes by extension of scalars from some cover $X'' \to X$ defined over K with X'' geometrically integral and normal. For, pick a K-rational point of U (which exists by assumption) and a $\bar{K}$-point x' in $f^{-1}(x)$. Let G be the stabilizer of x' in the Galois group $\mathrm{Gal}(\bar{K}(X')/K(X))$. The sought-for X'' is obtained as the normalization of X in the finite extension $\bar{K}(X')^G$ of $K(X)$. See [33] for details. □

Let us move to the proof of the main theorem and borrow notation from its statement. There is no loss of generality in assuming X geometrically integral and normal, as U is so. Also, one may assume that U has a K-rational point, for the main theorem is trivial otherwise. Let $\varepsilon > 0$ be a real number and pick an integer $d \geqslant 1$ such that

$$\frac{n(n+3)}{2d}[K : \mathbb{Q}] \leqslant \varepsilon,$$

where $n = \dim X$.

5.2. Geometric input. Note that Theorem B gives a morphism $f : X' \to \bar{X}$ of varieties over $\bar{K}$ which is finite surjective Galois and étale over $\bar{U}$ with X' integral and with the property that, for a positive-dimensional integral subvariety Y' of X' not contained in $Z' := f^{-1}(\bar{Z})$,

$$\deg(Y', f^* L_{|Y'}) \geqslant d^n.$$

Let $g : X'' \to X$ as in Lemma 5.1. Then a positive-dimensional integral subvariety Y'' of X'' not contained in $g^{-1}(Z)$ satisfies

$$\deg(Y'', g^* L_{|Y''}) \geqslant d^n \geqslant d^{\dim Y''}.$$

Note that Y'' is not necessarily geometrically irreducible, but no irreducible component of $\bar{Y}''$ is contained in $g^{-1}(\bar{Z})$.

5.3. Arithmetic input. Let $G = \mathrm{Aut}(g)$ be the geometric Galois group of the cover g. For an isomorphism class t of principal G-bundles over K, let X''_t be the twist of the variety X'' by t and $g_t : X''_t \to X$ that of the morphism g. According to Proposition 4.12,

$$\mathrm{gr.int}_K(X, L, U) \leqslant \sup_{t \in \mathrm{H}^1(K,G)} \mathrm{gr.int}_K(X''_t, g_t^* L, g_t^{-1}(U)).$$

For an irreducible subvariety Y'' of X''_t not contained in $g_t^{-1}(Z)$,

$$\deg(Y'', g_t^* L_{|Y''}) \geqslant d^{\dim Y''},$$

because, after extending scalars to $\bar{K}$, the cover g_t is isomorphic to g. Theorem A can therefore be applied to the integral projective variety X''_t, the subvariety $g_t^{-1}(Z)$, and the ample line bundle $g_t^* L$, to give

$$\mathrm{gr.rat}_K(X''_t, g_t^* L, g_t^{-1}(U)) \leqslant \frac{n(n+3)}{2d}[K:\mathbb{Q}] \leqslant \varepsilon.$$

Combining these two inequalities yields $\mathrm{gr.int}_K(X, L, U) \leqslant \varepsilon$, as the growth rate of integral points is lesser than that of rational points. As $\varepsilon > 0$ is arbitrary, this concludes the proof. □

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