Global PPT on Berkovich spaces

I) Complex PPT

1) Local case

Def $\Omega \subseteq \mathbb{C}$, $\phi : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ sub-h if usc

and $\forall\, \overline{B}(z_0, r) \subseteq \Omega$, $\phi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + r e^{i\theta})\, d\theta$

$\Longleftrightarrow$ $\Delta \phi \geq 0$ in the sense of distrib. (local!)

Def $\Omega \subseteq \mathbb{C}^n$, $\phi : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ psh if usc +

- $\forall\, \overline{B}(z_0, r) \subseteq \Omega$, $\xi \in S$, $\phi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + r \xi e^{i\theta})\, d\theta$
- $\forall\, \ell \subseteq \mathbb{C}^n$, $\phi_{|\Omega \cap \ell}$ sub-h.

Equiv $dd^c \phi := \frac{i}{2\pi} \partial \bar{\partial} \phi \geq 0$ in the sense of currents.

examples $*$ $f \in \mathcal{O}(\Omega)$: $|f|$ psh (Cauchy)

$*$ $f \in \mathcal{O}(\Omega)$: $\log |f|$ psh, $\{\phi = -\infty\} = V(f)$

$*$ more gen. $\log (|f_1|^2 + \dots + |f_N|^2)$ psh

$*$ Ω $\mathbb{T}^n$-inv, i.e. $\Omega = \mathrm{Log}^{-1}(U)$ $\quad \mathrm{Log} = (\log|z_1|, \dots, \log|z_n|)$

$\phi = \psi(\log|z_1|, \dots, \log|z_n|)$ psh for ψ cvx.

$*$ $\max(\phi_1, \dots, \phi_n)$ psh.

Def if $\phi \in PSH(\Omega) \cap \mathcal{C}^2(\Omega)$, $MA(\phi) := (dd^c\phi)^n$

$$= \frac{1}{(2\pi)^n} \det\left(\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right) d\lambda$$

more gen. $MA(\phi_1, \dots, \phi_n) = dd^c\phi_1 \wedge \dots \wedge dd^c\phi_n$

multilin, symmetric.

Thm (Bedford-Taylor) MA extends to CPSH.

continuous psh metrics

2) Global setting

(X, L) smooth ~~polarized~~ cpx variety, L semi-ample

Def metric ($\mathcal{C}^0$) on L is the data: $\forall\, s \in H^0(U, L_{|U})$,

of $\|s\|_\phi : U \to \mathbb{R}_{\geq 0}$

- $\forall f \in \mathcal{O}(U)$, $\|fs\|_\phi = |f| \cdot \|s\|_\phi$
- comp. with restr.
- $\|s\|_\phi(x) = 0 \iff s(x) = 0$

in coo. $L_{|U} \simeq U \times \mathbb{C}$

and $\underbrace{\|1\|_\phi}_{\phi} = e^{-\phi}$ so that $\|s\|_\phi = |s| e^{-\phi}$ (Rq $\phi_1 + \phi_2$)

e.g. $L = \mathcal{O}_X \to$ function

Def ϕ psh if the loc. function ϕ is psh.

examples • $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$ $\quad \phi_{FS} = \frac{1}{2}\log(|z_0|^2 + \dots + |z_N|^2)$

i.e. $\|s\|_{\phi_{FS}} = |s| \cdot e^{-\phi} = \dfrac{|s(z)|}{(|z_0|^2 + \dots + |z_N|^2)^{\frac{1}{2}}}$

- $s_0, \dots, s_N \in H^0(X, mL)$ with no com. zeros:

$$\phi = \frac{1}{2m} \log(|s_0|^2 + \dots + |s_N|^2) = m^{-1} \iota^* \phi_{FS} \quad \text{psh}$$

ie $$\|s\|_\phi = \frac{|s(z)|}{(|s_0|^2 + \dots + |s_N|^2)^{\frac{1}{2m}}}$$ Def métrique FS.

Def $\phi \in PSH(X, L)$: $dd^c\phi$ courant de courbure, $dd^c \phi \in c_1(L)$

CPSH = continuous psh

$MA(\phi_1, \dots, \phi_m) := dd^c\phi_1 \wedge \dots \wedge dd^c \phi_m$ MA, ≥ 0 measure of mass $(L_1 \cdot \ldots \cdot L_m)$

+ Properties

Prop (Stokes) $L_0 = L_1 = \mathcal{O}_X$; ϕ_i psh sur L_i :

$$\int_X \phi_0 \, dd^c\phi_1 \wedge \dots \wedge dd^c \phi_m = \int_X \phi_1 \, dd^c \phi_0 \wedge dd^c\phi_2 \wedge \dots \wedge dd^c \phi_m$$

Thm (Demailly cas ample) $\phi \in PSH(X, L)$ lim. décroissante de FS

$\Rightarrow$ $\phi \in$ CPSH lim. uniforme de FS

II) NA case

1) Defs

A Banach ring, $X/_A$ de type fini $+$ L semi-ample sur X ($\overset{\shortparallel}{\mathcal{O}_X}$-mod. inv)

Def ϕ met. $\mathcal{C}^0$ sur L :

$\forall\, \underset{\neq 0}{s} \in H^0(U, L_{|U})$, $U \subseteq X$ Zariski,

$\|s\|_\phi : U^{an} \to \mathbb{R}$ * comp. w. rest

* $\|f s\|_\phi = |f| \cdot \|s\|_\phi \quad \forall f \in \mathcal{O}(U)$

Rq if s sect. an., $s \underset{loc}{=} \underset{\substack{\uparrow \\ an.}}{f} \cdot \underset{alg. triv.}{\tilde{s}} \Rightarrow \|s\|_\phi := |f|\, \|\tilde{s}\|_\phi$ makes sense

examples • $s_0, \dots, s_N \rightsquigarrow \phi_{FS} = \frac{1}{m} \max(\log|s_0|, \dots, \log|s_N|)$

$\underline{FS(X,L)\,;\,DFS(X,L)}$ more gen. + prop. $\phi_{FS} = \frac{1}{m}\max(\log|s_i| + \lambda_i)$ $\lambda_i \in \mathbb{R}$

• $A = k$ NA field, non-triv.

$(\mathcal{X}, \mathcal{L})$ modèle$/_{k^0}$ proj. de (X, mL)

alors $\mathcal{X} = \bigcup_{i \in I} \mathcal{U}_i$, $s_i \in H^0(\mathcal{U}_i, \mathcal{L}_{|\mathcal{U}_i})$ triv.

$\Rightarrow X^{an} = \bigcup_{i \in I} \underbrace{red_{\mathcal{X}}^{-1}(\mathcal{U}_i)}_{\text{cpct } V_i}$ et $\phi_{\mathcal{L}}$ tq $\|s_i\|_{\phi_{\mathcal{L}}} \equiv 1$ sur V_i

(well-def, if $u_i \in \mathcal{O}^*(\mathcal{U}_i)$, $|u_i| \equiv 1$ on V_i)

Notably * $H^0(\mathcal{X}, m\mathcal{L}) \subseteq H^0(X, mL)$
is the unit ball for $\sup \|\cdot\|_\phi$

* ϕ model metric on mL $\Rightarrow$ $m^{-1}\phi$ model on L

$(\phi_1 + \phi_2)$ model metric $\rightsquigarrow$ $\mathbb{Q}$-vector space

~~Prop~~ **Rq** $*$ $(\mathcal{X}, \mathcal{L})$ & $(\mathcal{X}', \mathcal{L}')$ models of mL

with $\mathcal{X}' \xrightarrow{p} \mathcal{X}$ s.t. $p^*\mathcal{L} = \mathcal{L}'$ then $\phi_{\mathcal{L}} = \phi_{\mathcal{L}'}$

$\mathcal{X}' = \bigcup p^{-1}(\mathcal{U}_i)$, $p^* s_i)$

$*$ Converse holds when k $\Big|$ disc. valued / triv or alg. closed

Thm ϕ $\mathcal{C}^0$ metric on L ; the foll are eq:

- ϕ is a pure FS metric (FS with $\lambda_i = 0$)
- ϕ is a model metric with $\mathcal{L}$ semi-ample

Sketch of pf basic idea $\phi_{FS} = \log \max |z_i|$ on $\mathbb{P}^n_k$

$= \phi_{\mathcal{O}(1)}$ on $(\mathbb{P}^n_{k^\circ}, \mathcal{O}(1))$

1) $\Rightarrow$ 2) wlog L glob. gen $\rightsquigarrow \exists\, f : X \to \mathbb{P}^N_k$ $L = f^*\mathcal{O}(1)$

$\phi = \max \log |s_i|$

$\rightsquigarrow \exists\, \mathcal{X}/_{k^\circ}$ proj, $f : \mathcal{X} \to \mathbb{P}^N_{k^\circ}$

$\mathcal{L} := f^*\mathcal{O}(1)$ is s.t. $\phi = \phi_{\mathcal{L}}$

2) $\Rightarrow$ 1) $\mathcal{L}$ semi-ample model : $\mathcal{L}$ glob gen.

$\rightsquigarrow (s_0, \dots, s_N)$ k°-basis of $H^0(\mathcal{X}, \mathcal{L})$ $f : \mathcal{X} \to \mathbb{P}^N_{k^\circ}$

and $\phi_{\mathcal{L}} = \log \max |s_i|$ $\square$

Rq if k triv valued, replace $\mathcal{X}/_{k^\circ}$ $\rightsquigarrow$ $\mathcal{X}/_{\mathbb{A}^1_k}$ model of $X \times \mathbb{G}_m$

$\mathbb{G}_m$-eq.

Def ϕ C^0 metric on L is psh if uniform limit of FS metrics (on A)

Rq: if $k = \mathbb{C}, \mathbb{R}$ usual def by Demailly ($\sum |\cdot|^2$ vs max)

- if k NA, non-triv valued, w.l.o.g. $\Gamma_v(k) \supseteq \mathbb{Z}$

hence $\phi = \max(\log|s_i| + \lambda_i) = \lim \underbrace{\max\left(\log|s_i| + \frac{a_i}{b_i}\right)}_{b_i^{-1} \max(\log|s_i| + a_i)}$

$$= b^{-1} \max \log|t^{a_i} s_i|$$

$\Rightarrow$ uni. limit of pure FS = model metrics.

2) The Monge-Ampère measure

k NA, non triv valued

Def $\mathcal{X}$ model + $\mathcal{L}_1, \dots, \mathcal{L}_n$ det. on $\mathcal{X}$; $\mathcal{X}_s$ reduced

$Y_i \subseteq \mathcal{X}_s \to v_{Y_i}$ Shilov pt $\in X^{an}$

$$MA(\phi_{\mathcal{L}_1}, \dots, \phi_{\mathcal{L}_n}) = \sum_{Y_i \subseteq \mathcal{X}_s} (\mathcal{L}_1|_{Y_i} \cdot \dots \cdot \mathcal{L}_n|_{Y_i})\, \delta_{v_{Y_i}}$$

positive measure of mass L^n; well-defined

multilinear, symmetric, measure-valued metric on the $\mathbb{Q}$-vspace of model metrics.

<u>Prop</u> (Stokes) $L_0 = L_1 = \mathcal{O}_X$; ϕ_0 model metrics on $\mathcal{O}_X$

$$\int_{X^{an}} \phi_0 \, MA(\phi_1, \dots, \phi_n) = \int_{X^{an}} \phi_1 \, MA(\phi_0, \phi_2, \dots, \phi_n)$$

<u>Pf</u> $\phi_0 = \phi_{\mathcal{L}_0}$ $\qquad \phi_0(v_{Y_i}) \underset{-\log}{=} \|1\|_{\phi_0}(v_{Y_i}) = a_i$ $\quad\Big|\quad D_1 = \sum b_i Y_i$

$\mathcal{L}_0 = \mathcal{O}_{\mathcal{X}}(D)$ $\qquad\qquad\qquad\qquad \phi_1(v_{Y_i}) = b_i$

$D = \sum a_i Y_i$

$$I_1 = \sum_{Y_i} a_i \big(\underbrace{\mathcal{L}_1|_{Y_i}}_{Y_i \cdot \sum b_j Y_j} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n\big) = \sum_{Y_i, Y_j} a_i b_j \big(\underbrace{Y_i}_{Y_j} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n\big)$$

□

<u>Cor</u> (Chern-Levine-Nirenberg) ϕ_i, ϕ_i' semi-ample model metrics on L_i

$$\left| \int_{X^{an}} (\phi_0 - \phi_0') \, MA(\phi_1, \dots, \phi_n) - \int_{X^{an}} (\phi_0 - \phi_0') \, MA(\phi_1', \dots, \phi_n') \right| \le C \cdot \sum_{i=}^{n} \sup |\phi_i - \phi_i'| .$$

<u>Pf</u> $\Phi = (\phi_2, \dots, \phi_n)$ $\qquad MA(\phi_1, \Phi) = \dots$

<u>Stokes</u> $\int (\phi_0 - \phi_0') \, MA(\phi_1, \Phi) - \int (\phi_0 - \phi_0') \, MA(\phi_1', \Phi) = \int (\phi_0 - \phi_0') \, MA(\phi_1 - \phi_1', \Phi)$

$$= \int (\phi_1 - \phi_1') \, MA(\phi_0 - \phi_0', \Phi) = \int (\phi_1 - \phi_1') \underbrace{MA(\phi_0, \Phi)}_{\ge 0} + \int (\phi_1' - \phi_1) \underbrace{MA(\phi_0', \Phi)}_{\ge 0}$$

mass $(L_0 \cdot L_2 \cdots L_n)$

$|\ \cdot\ | \le (L_0 \cdot L_2 \cdots L_n) \sup |\phi_1 - \phi_1'|$ + symmetry □

Thm $(\phi_1, \dots, \phi_n) \longmapsto MA(\phi_1, \dots, \phi_n)$ extends uniqly to CPSH

continuous / $\|\cdot\|_\infty$ weak topology.

Pf CLN + DFS(X) $\subseteq \mathcal{C}^0(X^{an})$ dense $\square$

IV) Eq. measure

k field, valued, $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ endo. of deg d. i.e. $f^*\mathcal{O}(1) = \mathcal{O}(d)$

ϕ_{FS} std FS metric on $\mathcal{O}(1)$: $\phi_m := \underbrace{\frac{(f^{\circ m})^* \phi_{FS}}{d^m}}_{\text{FS metric on } \mathcal{O}(1)} = d^{-m} \max \log |(f^{\circ m})^* S_i|$

Prop $(\phi_m)_{m \in \mathbb{N}^*}$ cvu dans CPSH$(\mathcal{O}(1))$ vers ϕ_f met. f-eq, tq $\frac{f^*\phi_f}{d} = \phi_f$

Preuve $\phi \in FS(X,L)$; $F(\phi) = \frac{f^*\phi}{d}$: $\|\underbrace{F(\phi) - F(\psi)}_{\frac{(\phi \circ \psi) \circ f}{d}}\| \leq \frac{1}{d} \|\phi - \psi\|$ $\square$

Def $MA(\phi_f) := \mu_f$ eq. measure.