SORBONNE UNIVERSITÉ

HABILITATION À DIRIGER DES RECHERCHES

Discipline : Mathématiques

présentée par

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Finitudes en arithmétique et algébrisation en géométrie rigide

Soutenue le 26 mars 2024 devant le jury composé de :

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Conventions

An *algebraic variety*, or simply a *variety*, over a field k is a separated k-scheme of finite type. A *subvariety* is always assumed to be closed, unless stated otherwise. To facilitate identifying my contributions, statement environments are reserved to results that I obtained (mostly in collaboration). The works presented here are cited as [**1**], while other references are of the form [**And92**].

Works presented

[1] Y. Brunebarbe and M. Maculan. Counting integral points of bounded height on varieties with large fundamental group. *J. reine angew. Math.*, to appear.

[2] A. Javanpeykar, T. Krämer, C. Lehn, and M. Maculan. The monodromy of families of subvarieties on abelian varieties, 2022. arXiv:2210.05166, submitted.

[3] T. Krämer and M. Maculan. Arithmetic finiteness of very irregular varieties, 2023. arXiv:2310.08485, submitted.

[4] M. Maculan. Rigid analytic Stein algebraic groups are affine, 2020. arXiv:2007.04659, submitted.

[5] M. Maculan. The universal vector extension of an abeloid variety. *Épijournal Géom. Algébrique*, to appear.

[6] M. Maculan. Non-densité des points entiers et variations des structures de Hodge. In *Séminaire Bourbaki. Volume 2021/2022. Exposés 1181–1196*, pages 73–119, ex. Paris: Société Mathématique de France (SMF), 2022.

[7] M. Maculan and J. Poineau. Notions of Stein spaces in non-Archimedean geometry. *J. Algebraic Geom.*, 30(2):287–330, 2021.

[8] M. Maculan and J. Poineau. Affine vs. Stein in rigid geometry, 2023. arXiv:2305.08974, submitted.

CHAPTER 1

Scarcity of integral points

One of the mathematical successes of the 20th century was realizing that the arithmetic properties of an algebraic variety were governed by its geometric ones. An outstanding example of the resulting motto 'geometry controls arithmetic' is the Mordell conjecture proved by Faltings: a smooth projective curve of genus $\geqslant 2$ over a number field K admits only finitely many K-rational points. Its higher-dimensional generalization, the *Lang-Vojta conjecture* which is unattainable as of now, predicts the Zariski non-density of K-rational points for smooth projective varieties of general type (an assumption of hyperbolic nature). In this chapter I present a result obtained with Y. Brunebarbe [1] inspired by an earlier paper by Ellenberg, Lawrence and Venkatesh [ELV23]: a variety with a large fundamental group (also a hyperbolicity-related notion) has few integral points.

1. Counting integral points

1.1. Height functions. In a nutshell arithmetic geometry is the study of algebraic varieties X over a non-algebraically closed field K. The questions and the methods highly depend on the properties of the field K: for instance if K is finite, then the set X(K) of the K-rational points of X is also finite and its counting led to the Weil Conjectures proved by Deligne [Del74]. Here I will focus on the case when K is a number field. The set X(K) is in general infinite in this case and one cannot count its points on the nose. To bypass this difficulty one picks a (logarithmic) *height* function

$$h \colon \mathrm{X}(\mathrm{K}) \longrightarrow \mathbb{R}$$

on X. Such a height function h has the property that for each $c \in \mathbb{R}$ the set

$$\{x \in \mathrm{X}(\mathrm{K}) : h(x) \leqslant c\}$$

is finite, so it makes sense to count its elements. As opposed to counting rational points on finite fields, one should not hope for beautiful formulas: the precise value of the function

$$\mathbb{R} \ni c \quad \longmapsto \quad \mathrm{N}_{\mathrm{K}}(\mathrm{X}, h; c) \ := \ |\{x \in \mathrm{X}(\mathrm{K}) : h(x) \leqslant c\}|$$

is not of much interest. Instead the asymptotic of $c \mapsto \mathrm{N}_{\mathrm{K}}(\mathrm{X}, h; c)$ for $c \to \infty$ is a fine invariant of X.

EXAMPLE. Suppose $\mathrm{K} = \mathbb{Q}$ and $\mathrm{X} = \mathbb{P}^n$ for some $n \geqslant 1$. Then any rational point x of $\mathbb{P}^n$ can be written as

$$x = [x_0 : \cdots : x_n] \quad \text{with } x_0, \ldots, x_n \in \mathbb{Z} \text{ coprime}$$

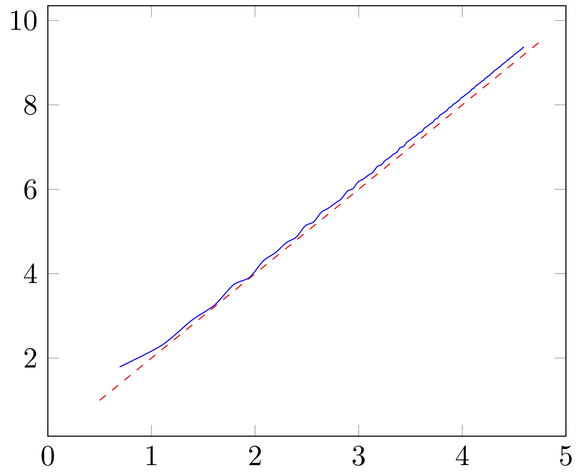

FIGURE 1. Plot of the functions $c \mapsto \log \mathrm{N}_{\mathbb{Q}}(\mathbb{P}^1, h; c)$ in blue and $c \mapsto 2c$ in red.

where coprime means that the ideal generated by $x_0, \ldots, x_n$ is $\mathbb{Z}$. The tuple $(x_0, \ldots, x_n)$ is essentially unique, as the only other possible choice would have been taking its opposite. In particular

$$h(x) = \log \max\{|x_0|, \ldots, |x_n|\}$$

is well-defined and gives rise to the so-called *canonical Weil height* on $\mathbb{P}^n$. A small computation then gives

$$\log \mathrm{N}_{\mathrm{K}}(\mathrm{X}, h; c) \ \sim\ (n+1)c + \mathrm{O}(1).$$

The same formula holds for arbitrary number fields by a theorem of Schanuel [**Sch79**]. Truth to be told, the right-hand side needs to be multiplied by a factor $[\mathrm{K} : \mathbb{Q}]$ due to the convention for heights adopted here. Indeed, when K is arbitrary, in this text I consider

$$h(x) = \frac{1}{[\mathrm{K} \colon \mathbb{Q}]} \sum_{v \in \mathrm{V}_{\mathrm{K}}} \log \max\{|x_0|_v, \ldots, |x_n|_v\}$$

where V_{K} is set of places of K.[1]

Suppose X projective and let L be an ample line bundle on X. Let $n \geqslant 1$ be an integer such that $\mathrm{L}^{\otimes n}$ is very ample. The choice of a basis of the K-vector space $\mathrm{H}^0(\mathrm{X}, \mathrm{L}^{\otimes n})$ yields a closed embedding $i \colon \mathrm{X} \hookrightarrow \mathbb{P}^{d-1}$ where $d = h^0(\mathrm{X}, \mathrm{L}^{\otimes n})$. The function

$$h_{\mathrm{L}}(x) = \tfrac{1}{n} h(i(x))$$

on $\mathrm{X}(\mathrm{K})$ is called 'the' height function relative to L where h is the canonical Weil height on $\mathbb{P}^{d-1}$ considered in the preceding example. Quotes are mandatory as other choices of n and of the basis may perturb h_{L}. However such a perturbation is bounded on $\mathrm{X}(\mathrm{K})$, so the asymptotic behaviour of the function $c \mapsto \mathrm{N}_{\mathrm{K}}(\mathrm{X}, h_{\mathrm{L}}; c)$ is not affected: ultimately it

[1] For a prime p and a p-adic place v the absolute value $|\cdot|_v$ is normalized so that $|p|_v = p^{-[\mathrm{K}_v : \mathbb{Q}_p]}$ where K_v is the v-adic completion of K. In order to have the product formula this forces for a complex place v to set $|\cdot|_v$ to be the square of the usual absolute value—hence technically not an absolute value anymore.

depends only on K, X and L. Because of this when talking about asymptotic behaviours I will abusively write $\mathrm{N}(\mathrm{X}, \mathrm{L}; c)$ instead of $\mathrm{N}(\mathrm{X}, h_{\mathrm{L}}; c)$ and discard the choice of a height function.

Example. For any abelian variety A over K and any ample line bundle L on A,

$$\log \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}; c) \;\sim\; \tfrac{r}{2} \log c + \mathrm{O}(1)$$

where $r = \dim_{\mathbb{Q}} \mathrm{A}(\mathrm{K}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the Mordell-Weil rank [HS00, theorem B.6.3]; see fig. 2.

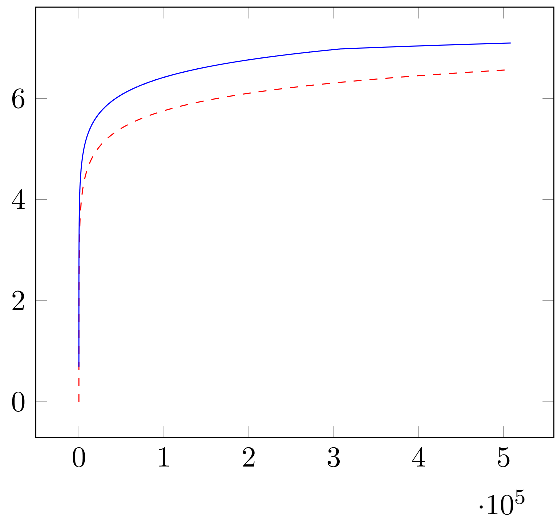

Figure 2. The elliptic curve $\mathrm{E} : y^2 = x^3 + 5x$ has Mordell-Weil rank 1 over $\mathbb{Q}$. In the picture the function $c \mapsto \log \mathrm{N}_{\mathbb{Q}}(\mathrm{E}, h; c)$ is plotted in blue (where h is the Weil height on $\mathbb{P}^2$) and $c \mapsto \frac{1}{2} \log c$ is plotted in red.

Example. Suppose X is a smooth projective curve of genus $g \geqslant 2$. Then Faltings proved that the set X(K) is finite. In other words, the function $c \mapsto \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}; c)$ is eventually constant for each ample line bundle L on X.

1.2. Heuristics. The three examples above are expected to be the only three possible behaviours. Indeed the asymptotic behaviour of $c \mapsto \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}; c)$ for $c \to \infty$ is conjecturally expected to reflect the usual 'trichotomy' of projective varieties into the Fano, Calabi-Yau and general type classes.[2] The birational nature of the above classes forces to refine the counting: the blow-up X at the origin of an abelian g-fold over K satisfies

$$\log \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}; c) \;\sim [\mathrm{K} : \mathbb{Q}] \frac{gc}{d} + o(c)$$

where L is an ample line bundle on X and the restriction of L to the exceptional divisor is isomorphic to $\mathcal{O}(d)$. It is therefore natural to introduce the counting for an open subset $\mathrm{U} \subset \mathrm{X}$,

$$\mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}, h; c) \;:=\; |\{x \in \mathrm{U}(\mathrm{K}) : h(x) \leqslant c\}|$$

[2] The word 'trichotomy' here is improper: according to the Minimal Model Program, any integral variety should be obtained as iterated fibration with generic fibre belonging to one of the preceding three classes.

where L is an ample line bundle on X and h a height function relative to L. As above the asymptotic of $c \mapsto \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}, h; c)$ does not depend on the choice of h so it will be discarded it from the notation. Suppose that X is a non-singular projective variety with canonical bundle ω_{X}.

Fano. When X is Fano, that is, the anti-canonical bundle $\omega_{\mathrm{X}}^{\vee}$ is ample the counting function is expected to have a linear growth. More precisely, assuming the Zariski-density of $\mathrm{X}(\mathrm{K})$, Manin conjectured that there is a non-empty open subset $\mathrm{U} \subset \mathrm{X}$ for which

$$\log \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \omega_{\mathrm{X}}^{\vee}, \mathrm{U}; c) = [\mathrm{K} : \mathbb{Q}]c + (\rho - 1)\log c + \mathrm{O}(1),$$

where ρ is the Picard rank of X. The anti-canonical bundle of $\mathrm{X} = \mathbb{P}^n$ is $\omega_{\mathrm{X}}^{\vee} = \mathcal{O}(n+1)$ hence we recover Schanuel's formula because $h_{\omega_{\mathrm{X}}^{\vee}} = (n+1)h_{\mathcal{O}(1)}$ up to a bounded function. In this form Manin's conjecture is known to be false, as pointed out first by Batyrev and Tschinkel [**BT96**] by considering bundles of cubic surfaces. This will not affect our discussion here, and the reader interested in the modern version of Manin's conjecture may consult [**Pey03**].

Calabi-Yau. When X is a Calabi-Yau variety, that is, the canonical bundle ω_{X} is trivial, there is no precise conjecture to the extent of my knowledge. Vaguely enough, logarithmic growth is expected for the function $c \mapsto \log \mathrm{N}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}; c)$ for a suitable open subset $\mathrm{U} \subset \mathrm{X}$, of course assuming Zariski-density of $\mathrm{X}(\mathrm{K})$. This is the case for abelian varieties as mentioned above. The necessity of restricting to a proper open subset is already seen in the aforementioned case of the blow-up of an abelian variety, or in the case of K3 surfaces where rational curves would destroy the logarithmic growth.

General type. The variety X is said to be of general type if the canonical line bundle ω_{X} is big, for instance if it is ample. The realm of varieties of general type is governed by the Lang-Vojta conjecture, a higher-dimensional generalization of the Mordell conjecture. It predicts that there is a non-empty open subset $\mathrm{U} \subset \mathrm{X}$ for which the set $\mathrm{U}(\mathrm{K}')$ is finite for each finite extension K' of K. The Lang-Vojta conjecture has been proved by Faltings for subvarieties of general type of abelian varieties, but it is otherwise wide open. In terms of counting functions, the Lang-Vojta conjecture can be rephrased by saying that the counting function is eventually constant.

1.3. Integral points. The discussion above admits an 'open' variant, that is, for varieties that are not projective. To guess the correct modifications to perform, it is useful to bare in mind the following example:

Example. Let Σ be a finite subset of places of K containing all the infinite ones and $\mathcal{O}_{\mathrm{K},\Sigma}$ is the ring of Σ-integers. The theorem on the equation of Σ-units states that the set

$$\{x \in \mathcal{O}_{\mathrm{K},\Sigma}^{\times} : 1 - x \in \mathcal{O}_{\mathrm{K},\Sigma}^{\times}\}$$

is finite. The above set can be identified with the set of Σ-integral points of $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, that is points with values in $\mathcal{O}_{\mathrm{K},\Sigma}$. This classical diophantine statement can be rephrased by saying that $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ has finitely many integral points, as opposed as having infinitely many rational points. The geometric explanation for this is that the logarithmic pair

$$(\mathrm{X} = \mathbb{P}^1, \mathrm{D} = [0] + [1] + [\infty])$$

is of log-general type, that is, the logarithmic canonical bundle $\omega_{\mathrm{X}}(\mathrm{D}) = \mathcal{O}(1)$ is big.

In order to speak properly of integral points, fix a proper flat $\mathcal{O}_{\mathrm{K}}$-scheme $\mathcal{X}$ with generic fiber X. For an open subset $\mathrm{U} \subset \mathrm{X}$, a line bundle L on X, a height function h relative to L, and a subring $\mathrm{R} \subset \mathrm{K}$ containing $\mathcal{O}_{\mathrm{K}}$ set

$$\nu(\mathcal{X}, \mathrm{L}, \mathrm{U}, h, \mathrm{R}; c) := \log^+ \#\{x \in \mathcal{U}(\mathrm{R}) : h(x) \leqslant c\},$$

where $\mathcal{U} \subset \mathcal{X}$ is the biggest open subset with generic fiber U, that is, the complement of the Zariski closure of $\mathrm{X} \smallsetminus \mathrm{U}$ in $\mathcal{X}$. Also $\log^+ t = \log \max\{1, t\}$ for a real number t. With this under the belt, given an effective Cartier divisor $\mathrm{D} \subset \mathrm{X}$ with normal crossings, one can easily write down a putative dictionary between the birational type of the logarithmic pair (X, D) and the growth of the counting function

$$\mathbb{R} \ni c \longmapsto \nu(\mathcal{X}, \mathrm{L}, \mathrm{X} \smallsetminus \mathrm{D}, h, \mathcal{O}_{\mathrm{K},\Sigma}; c).$$

For instance when $\mathrm{X} = \mathbb{P}^1$ and $\mathrm{D} = [0] + [\infty]$ the above function is seen to have logarithmic growth, reflecting the fact that the logarithmic canonical bundle $\omega_{\mathrm{X}}(\mathrm{D}) = \mathcal{O}_{\mathrm{X}}$ is trivial, that is, the logarithmic pair (X, D) is log Calabi-Yau.

1.4. Linear growth rate. The main object of interest here will be a much coarser invariant, namely the slope of the counting function,

$$\operatorname{gr.rat}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}) := \limsup_{c \to \infty} \frac{\nu(\mathcal{X}, \mathrm{L}, \mathrm{U}, h, \mathrm{K}; c)}{c},$$

henceforth called the *growth rate of rational points*. As the notation suggests the real number $\operatorname{gr.rat}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U})$ does neither depend on the choice of the model $\mathcal{X}$ nor on that of the height function h. Its variant for integral points is defined as follows. For a finite set of places Σ of K including the Archimedean ones, the real number

$$\limsup_{c \to \infty} \frac{\nu(\mathcal{X}, \mathrm{L}, \mathrm{U}, h, \mathcal{O}_{\mathrm{K},\Sigma}; c)}{c}$$

measures the presence of the Σ-integral points on U. Again this does not depend on $\mathcal{X}$ and h, but *a priori* does depend on the set Σ. Taking the supremum ranging over all such finite sets of places Σ lets one get rid of such a dependence and gives rise to an invariant

$$\operatorname{gr.int}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U})$$

called the *growth rate of integral points* of U with respect to X and L. When $\mathrm{U} = \mathrm{X}$ rational and integral points coincide, thus their growth rate do and in such case the redundant repetition of X is discarded from notation. In terms of counting functions the example section 1.1 become as follows:

Example. Schanuel's formula implies $\operatorname{gr.rat}_{\mathrm{K}}(\mathbb{P}^n, \mathcal{O}(1)) = (n+1)[\mathrm{K} : \mathbb{Q}]$. For a smooth Fano variety X Manin's conjecture predicts

$$\operatorname{gr.rat}_{\mathrm{K}}(\mathrm{X}, \omega_{\mathrm{X}}^{\vee}, \mathrm{U}) = [\mathrm{K} : \mathbb{Q}]$$

for some non-empty open subset $\mathrm{U} \subset \mathrm{X}$.

Example. For an abelian variety A over K and an ample line bundle L on A implies

$$\operatorname{gr.rat}_{\mathrm{K}}(\mathrm{A}, \mathrm{L}) = 0.$$

EXAMPLE. Let X be a smooth projective curve over K, L an ample line on X and U a non-empty open subset in X. If the genus of X is $\geqslant 2$ then Faltings's theorem implies

$$\mathrm{gr.\,rat}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}) = 0.$$

If U is not isomorphic to the affine or projective line, then Siegel's theorem on integral points on affine curves yields

$$\mathrm{gr.\,int}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}) = 0.$$

Note that the growth rate for integral and rational points in general differ: indeed,

$$\mathrm{gr.\,int}_{\mathrm{K}}(\mathbb{P}^1, \mathcal{O}(1), \mathbb{G}_m) = 0 \ < \ 2[\mathrm{K} : \mathbb{Q}] = \mathrm{gr.\,rat}_{\mathrm{K}}(\mathbb{P}^1, \mathcal{O}(1), \mathbb{G}_m).$$

The three examples above show that the growth rate of rational points does not distinguish the Calabi-Yau and the general type classes.

2. Varieties with large fundamental group

Let k be an algebraically closed field of characteristic 0.

2.1. Basic properties. Let X be a normal integral variety over k.

DEFINITION. The variety X *has large étale fundamental group* if for each integral positive-dimensional subvariety $\mathrm{Y} \subset \mathrm{X}$,

$$|\operatorname{Im}(\pi_1^{\text{ét}}(\tilde{\mathrm{Y}}) \to \pi_1^{\text{ét}}(\mathrm{X}))| \ = \ \infty \tag{2.1}$$

where $\tilde{\mathrm{Y}} \to \mathrm{Y}$ is the normalization.[3]

When $k = \mathbb{C}$ this is what Kollár calls having a large algebraic fundamental group [**Kol93**, **Kol95**]; see also [**Cam95**]. It follows immediately from the definitions that:

(1) it suffices to test (2.1) when Y is a curve;

(2) the variety X has large étale fundamental group if and only if, for any non-constant morphism $f\colon \mathrm{Y} \to \mathrm{X}$ with Y integral normal, the image of the induced map $\pi_1^{\text{ét}}(\mathrm{Y}) \to \pi_1^{\text{ét}}(\mathrm{X})$ is infinite [**Kol93**, proposition 2.9.1];

(3) the class of integral normal varieties with large étale fundamental group is closed under products;

(4) given a quasi-finite morphism $f\colon \mathrm{X}' \to \mathrm{X}$ between integral normal varieties, if X has a large étale fundamental group, then X′ has a large étale fundamental group too; the converse holds as soon as f is finite étale.

(5) let $f\colon \mathrm{X}' \to \mathrm{X}$ be a morphism between integral normal varieties with connected normal fibers satisfying the following property: the sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(f^{-1}(x)) \longrightarrow \pi_1^{\text{ét}}(\mathrm{X}') \longrightarrow \pi_1^{\text{ét}}(\mathrm{X}) \longrightarrow 1$$

is exact for all $x \in \mathrm{X}(k)$. If X and all the fibers at k-points of f have large étale fundamental group, then so does X′; this is the case for isotrivial fibrations whose base and fiber have large étale fundamental group.

EXAMPLE. A smooth curve has large étale fundamental group unless it is isomorphic to the affine or the projective line.

[3] Here I dropped the choice of a base-point as property (2.1) does not depend on it.

Example. A g-dimensional abelian variety A over k has large étale fundamental group. Indeed, the étale fundamental group of A is identified with the Tate module:

$$\pi_1^{\text{ét}}(\mathrm{A}, 0) = \mathrm{TA} := \operatorname{proj}\lim_{n \geqslant 1} \mathrm{A}[n] \cong \hat{\mathbb{Z}}^{2g},$$

where $\mathrm{A}[n] \subset \mathrm{A}(k)$ is the n-torsion subgroup and $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$. Let $\mathrm{C} \subset \mathrm{A}$ be a curve and $\mathrm{B} \subset \mathrm{A}$ the subabelian variety generated by A. The inclusion $\mathrm{C} \hookrightarrow \mathrm{B}$ induces a surjective morphism $\operatorname{Jac} \tilde{\mathrm{C}} \twoheadrightarrow \mathrm{B}$ of abelian varieties where $\tilde{\mathrm{C}} \to \mathrm{C}$ is the normalization of C. The image of $\pi_1^{\text{ét}}(\tilde{\mathrm{C}}) \to \pi_1^{\text{ét}}(\mathrm{A}) = \mathrm{TA}$ is TB which is infinite because $\dim \mathrm{B} > 0$. It follows that also semi-abelian varieties, being fibrations over an abelian variety by a torus, have large étale fundamental group.

Smooth projective Fano varieties have trivial étale fundamental group [Deb03], hence the class of varieties having a large fundamental group sits somewhere in between the class of Calabi-Yau varieties and that of general type. Beware that having a large étale fundamental group is not a property stable under birational transformation: the blow-up at a smooth point of a variety having large étale fundamental contains a projective space, thus it does not have large étale fundamental group. Nonetheless taking products of a suitable number of copies of an elliptic curve and of a curve of genus $\geqslant 2$ yields examples of smooth projective varieties X with large étale fundamental group and any possible Kodaira dimension between 0 and $\dim \mathrm{X}$. Kollár more precisely conjectures that a smooth projective variety with large étale fundamental group admits a finite étale cover which is a smooth family of abelian varieties over a projective variety of general type [Kol93, conjecture 1.10]; see [CCE15, theorem 6.5] for a partial result in this direction and [Bru22, theorem 3.16].

2.2. Comparison with the topological fundamental group. Over the complex numbers, the property of having large étale fundamental group can be read off the usual topological fundamental group. For, upon fixing a point $x \in \mathrm{X}(\mathbb{C})$, the natural group homomorphism

$$i_{\mathrm{X}} \colon \pi_1^{\text{top}}(\mathrm{X}(\mathbb{C}), x) \longrightarrow \pi_1^{\text{ét}}(\mathrm{X}, x)$$

identifies the étale fundamental group with the profinite completion of the topological one. Let $\hat{\mathrm{X}}$ be the topological cover of $\mathrm{X}(\mathbb{C})$ corresponding to the normal subgroup $\operatorname{Ker} i_{\mathrm{X}} \subset \pi_1^{\text{top}}(\mathrm{X}(\mathbb{C}), x)$.

When X is proper, the variety X has large étale fundamental group if and only if the complex analytic space $\hat{\mathrm{X}}$ does not contain positive-dimensional compact complex analytic subspaces; see [Kol93, proposition 2.12.3]. This is the case for instance if the complex space $\hat{\mathrm{X}}$ is holomorphically separable, that is, if points in $\hat{\mathrm{X}}$ can be separated by holomorphic functions. In view of this characterization it is useful to be able to determine $\hat{\mathrm{X}}$, but this not so evident. Already understanding when $\hat{\mathrm{X}}$ is a universal cover of $\mathrm{X}(\mathbb{C})$, that is $\operatorname{Ker} i_{\mathrm{X}} = 0$,[4] is quite fruitful. To do this recall that a group Γ is said to be:

- *linear* if it admits a faithful representation $\rho \colon \Gamma \to \mathrm{GL}(\mathrm{V})$ for some vector space V of finite dimension over a field;

[4] This is not always the case as pointed out by Toledo [Tol93].

- *residually finite* if the natural map $\Gamma \to \hat{\Gamma}$ is injective, where $\hat{\Gamma}$ is its profinite completion.

A classical result of Malcev [Mal40] states that a finitely generated linear group is residually finite. Gathering the above considerations, if $\pi_1^{\mathrm{top}}(\mathrm{X}(\mathbb{C}), x)$ is linear, then $\hat{\mathrm{X}}$ is a universal cover of $\mathrm{X}(\mathbb{C})$.

Example. The above discussion can be applied to recover that an abelian variety has large étale fundamental group because it is the quotient of $\mathbb{C}^g$ by a lattice. More interestingly, it can be used to say that X has large étale fundamental group when X is a quotient of a bounded symmetric domain in $\mathbb{C}^n$ by a torsion-free co-compact lattice of its biholomorphism group.

2.3. The role of local systems. Still under the assumption $k = \mathbb{C}$ let $\mathscr{L}$ be a local system of vector spaces over some field on $\mathrm{X}(\mathbb{C})$. Such a local system can be interpreted as a representation of the topological fundamental group by means of the monodromy representation

$$\pi_1^{\mathrm{top}}(\mathrm{X}(\mathbb{C}), x) \longrightarrow \mathrm{GL}(\mathscr{L}_x).$$

Definition. The local system $\mathscr{L}$ is said to be *large* if, given a non-constant morphism $f\colon \mathrm{Y} \to \mathrm{X}$ with Y a normal irreducible complex variety, the local system $f^*\mathscr{L}$ on $\mathrm{Y}(\mathbb{C})$ has infinite monodromy.

Large local systems are called this way because if a complex variety carries one, then it has large étale fundamental group. To see this pick $f\colon \mathrm{Y} \to \mathrm{X}$ as in the definition. Then the image of $\pi_1^{\text{ét}}(\mathrm{Y}, y) \to \pi_1^{\text{ét}}(\mathrm{X}, f(y))$ surjects onto the profinite completion of the image $\Gamma \subset \mathrm{GL}(\mathscr{L}_{f(y)})$ of the monodromy representation of $f^*\mathscr{L}$. The group Γ is finitely generated and by definition linear, thus residually finite by Malcev's result. In particular if Γ is infinite so is its profinite completion.

The main source of large local systems are variations of Hodge structures. More precisely, suppose that $\mathscr{L}$ underlies an admissible graded-polarizable variation of mixed Hodge structures. If the associated period mapping on the universal cover of $\mathrm{X}(\mathbb{C})$ has discrete fibers then the local system $\mathscr{L}$ is large. This furnishes us with plenty of interesting examples:

Example. Let M be the moduli space for a certain class of polarized smooth projective varieties. Suppose that M is a *fine* moduli space, that is, it admits a universal family $\pi\colon \mathfrak{X} \to \mathrm{M}$. Note that in order to do so it may be necessary to endow the varieties in question with extra structure: for instance, in the case of abelian varieties, one may fix a basis for n-torsion subgroup for some $n \geqslant 1$ large enough. By Ehresmann's theorem the sheaf

$$\mathscr{L} \;:=\; \mathrm{R}^d\pi_*\mathbb{Q}_{\mathfrak{X}}$$

of $\mathbb{Q}$-vector spaces on $\mathrm{M}(\mathbb{C})$ is a local system, where d is the dimension of the fibers of $\mathfrak{X} \to \mathrm{M}$. Note that the fiber of $\mathscr{L}$ at some $s \in \mathrm{M}(\mathbb{C})$ is the usual d-th cohomology group

$$\mathscr{L}_s \;=\; \mathrm{H}^d(\mathfrak{X}_s(\mathbb{C}), \mathbb{Q})$$

with rational coefficients. In the jargon of moduli spaces saying that the local system $\mathscr{L}$ is large means exactly that the 'infinitesimal Torelli theorem' is satisfied by the moduli problem in question. By [Bea87] this is the case for a full range of moduli spaces: smooth projective curves, abelian varieties, Calabi-Yau varieties, most complete intersections, ...

Example. One way to think of Shimura varieties, and their mixed variants, is by seeing them as the varieties parameterizing graded polarized integral (mixed) Hodge structures. Somewhat by definition they come equipped with a local system $\mathscr{L}$ whose associated period mapping is the identity, implying that $\mathscr{L}$ is large. Strictly speaking this is only true when the mixed Shimura variety in question is associated with a torsion-free congruence subgroup. The affine line $\mathbb{A}^1$ is a Shimura variety when seen as the quotient

$$\mathfrak{h} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \;/\; \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm \mathrm{id}\},$$

but the étale fundamental group of $\mathbb{A}^1$ is not large; this is because $\mathrm{PSL}_2(\mathbb{Z})$ contains torsion elements. The torsion-freeness hypothesis has to be thought as the Shimura-theoretic analogue of the moduli spaces being fine in the previous example.

3. Subpolynomial growth

3.1. Main statement. Time has come to state the main theorem of this chapter. Let K be a number field and Σ a finite set of places containing all the infinite ones. A geometrically integral normal variety X over K is said to *have large geometric étale fundamental group* if the base-change of X to an algebraic closure of K has large étale fundamental group in the sense of section 2.1. For the sake of brevity I will sometimes omit the adjective 'geometric'. With this notation, the main result obtained with Y. Brunebarbe is the following [1, main theorem]:

Main theorem. *Let* X *be a projective variety over* K, L *an ample line bundle on* X *and* $\mathrm{U} \subset \mathrm{X}$ *an open subset which is geometrically integral and normal. If* U *has large geometric étale fundamental group, then*

$$\operatorname{gr.int}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}) = 0.$$

In other terms, upon fixing a projective flat model $\mathfrak{X}$ of X and a height function h relative to L, for each $\varepsilon > 0$ and each finite subset of places S of K there is $\mathrm{C}_{\mathrm{S},\varepsilon} > 0$ such that

$$\nu(\mathfrak{X}, \mathrm{L}, \mathrm{U}, h, \mathcal{O}_{\mathrm{K},\mathrm{S}}; c) \leqslant \varepsilon c + \mathrm{C}_{\mathrm{S},\varepsilon}, \qquad c \in \mathbb{R}.$$

As mentioned above, when $\mathrm{X} = \mathrm{U}$ and X is non-singular, Kollár conjectures that X is a family of abelian varieties parametrized by a projective variety of general type. Because of this the function $c \mapsto \nu(\mathfrak{X}, \mathrm{L}, \mathrm{U}, h, \mathcal{O}_{\mathrm{K},\mathrm{S}}; c)$ is expected to have logarithmic growth. Needless to say, having null growth rate of integral points is far from having logarithmic growth. However the main theorem above is the first result pointing in this direction. Also note that having null growth rate of integral points is not a property stable under birational transformations, as already seen in the case of the blow-up of an abelian variety. This matches the fact that having a large étale fundamental group is not a birational invariant either.

The main theorem is inspired by a recent paper of Ellenberg, Lawrence and Venkatesh [ELV23]. They show that the growth of integrals points of U vanishes under the following assumption: there is an embedding $\sigma \colon \mathrm{K} \hookrightarrow \mathbb{C}$ for which $\mathrm{U}_\sigma(\mathbb{C})$ admits a geometric variation of Hodge structures whose associated period mapping has discrete fibers. In view of the discussion in section 2.3 the complex variety U_σ has large étale fundamental group, hence U has large geometric étale fundamental group. In this sense the main theorem above generalizes [ELV23]. It also replies positively to the questions raised in *op.cit.* whether the main theorem therein would hold for varieties with large étale fundamental group and mixed Shimura varieties.

3.2. Examples. The main theorem can be applied in the following situations:

(1) Smooth curves (not isomorphic to the affine or projective line) have large étale fundamental group. However the theorems of Siegel [Sie14] and Faltings [Fal83] imply that there are only finitely many integral points.

(2) Abelian varieties and their subvarieties have large étale fundamental group. In the first case the function counting rational points grows logarithmically, as seen in section 1.1. This also happens in the second situation by the particular case of the Lang-Vojta conjecture proved by Faltings [Fal91].

(3) Fine moduli spaces of curves and of abelian varieties have large étale fundamental group. Here the Shafarevich conjecture proved by Faltings [Fal83] implies that these varieties have only finitely many integral points.

(4) Fine moduli spaces of Calabi-Yau varieties and complete intersections (save a finite list of exceptions in low degree) have large étale fundamental group, thus the main theorem applies. This should be put in perspective with the Shafarevich conjecture in chapter 2. Upon fixing numerical invariants, it predicts that over a number field K up to isomorphism there are only finitely many varieties in the above classes having good reduction outside a fixed finite set of places of K. The main theorem above thus gives a first result in direction of the Shafarevich conjecture. Note that by definition these moduli spaces carry a geometric variation of pure Hodge structures, thus [ELV23] applies as well.

(5) Pure or mixed Shimura varieties (associated with a torsion-free congruence subgroup) have large étale fundamental group. When they are not pure, they are the prototypical examples in which the results in [ELV23] cannot be applied, whereas the main theorem here can—this replies to a question raised in *op.cit.* Even in the pure case the main theorem here has a wider range of application than the one in *op.cit.* As mentioned in section 2.3 pure Shimura varieties of exceptional type carry a canonical variation of pure Hodge structures whose underlying local system is large: it is not known whether this variation is of geometric origin [Gro94]. Note that these Shimura varieties are of general type, and so are their subvarieties. Therefore by the Lang-Vojta conjecture they are expected to have only have finitely many integral points.

4. Sketch of the proof

The proof of the main theorem is inspired from [ELV23].

4.1. Arithmetic ingredient. The strategy relies on known uniform bounds on the number of hypersurfaces needed to cover the rational points of a subvariety of the projective space. These bounds were initiated by Heath-Brown [HB02] via the determinant method [BP89], and pursued by many authors including Broberg [Bro04], Salberger

[Sal07], and Chen [Che12a, Che12b]. The crucial remark here is that these bounds become stronger as the degree of the subvariety increases. Building on this intuition, a simple induction on the dimension gives the following statement [1, theorem A]:

THEOREM A. *Let* X *be an integral projective variety of dimension* n *over a number field* K*, and* L *an ample line bundle over* X*. Assume there is a subvariety* $\mathrm{Z} \subset \mathrm{X}$ *and* $d \geqslant 1$ *such that any positive-dimensional integral subvariety* Y *of* X *not contained in* Z *satisfies* $\deg(\mathrm{Y}, \mathrm{L}_{|\mathrm{Y}}) \geqslant d^{\dim \mathrm{Y}}$*. Then,*

$$\mathrm{gr.\,rat}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{X} \smallsetminus \mathrm{Z}) \leqslant \frac{n(n+3)}{2d}[\mathrm{K} : \mathbb{Q}].$$

Recall that the degree of an ample line bundle on a projective variety is its top self intersection number. Of course to make use of the previous results one needs to have a lower bound for all the subvarieties of the one in question. Even when the variety has large étale fundamental there is no whatsoever reason why one should hope for a such a lower bound.

4.2. Geometric ingredient. Getting round of this relies on a previous insight of Brunebarbe [Bru20], who pointed out that one can 'increase hyperbolicity' of a variety with large étale fundamental group by looking at suitable covers; see [1, theorem B].

THEOREM B. *Let* X *be a normal integral projective variety over an algebraically closed field of characteristic* 0*,* L *an ample line bundle over* X*, and* U *a non-empty open subset of* X *whose étale fundamental group is large. Then, given an integer* $d \geqslant 1$*, there is a finite surjective map* $\pi \colon \mathrm{X}' \to \mathrm{X}$ *with* X' *normal integral such that* π *is étale over* U *and, for each positive-dimensional integral subvariety* Y' *of* X' *meeting* $\pi^{-1}(\mathrm{U})$*,*

$$\deg(\mathrm{Y}', \pi^*\mathrm{L}_{|\mathrm{Y}'}) \geqslant d.$$

Let me just sketch how to produce such a cover when $\mathrm{X} = \mathrm{U}$. To do this let me work over the complex numbers. Borrowing the notation from the statement, the crucial point is the finiteness of the following set

$$\left\{ \mathrm{Im}(\pi_1^{\text{ét}}(\tilde{\mathrm{Y}}) \to \pi_1^{\text{ét}}(\mathrm{X})) : \begin{array}{l} \mathrm{Y} \subset \mathrm{X} \text{ integral subvariety with } \dim \mathrm{Y} > 0, \\ \deg(\mathrm{Y}, \mathrm{L}_{|\mathrm{Y}}) < d \text{ and normalization } \tilde{\mathrm{Y}} \to \mathrm{Y} \end{array} \right\} / \text{ conjugation}.$$

This is an immediate consequence of the boundedness of 'weakly complete' families of normal cycles on X in Kollár' terminology, where a normal cycle is the normalization of an integral subvariety of X. The reason to work over $\mathbb{C}$ is that such a finiteness statement relies on purely topological arguments, namely a variation for singular varieties of Ehresmann's theorem due to Goresky-MacPherson. Now let

$$\mathrm{H}_1, \dots, \mathrm{H}_r \subset \pi_1^{\text{ét}}(\mathrm{X})$$

be a set of representatives for the above finite set. Note that each of the H_i is infinite because X has large étale fundamental group. Since $\pi_1^{\text{ét}}(\mathrm{X})$ is profinite, there is a normal subgroup of finite index

$$\mathrm{N} \subset \pi_1^{\text{ét}}(\mathrm{X}) \quad \text{such that} \quad |\,\mathrm{Im}(\mathrm{H}_i \to \pi_1^{\text{ét}}(\mathrm{X})/\mathrm{N})| = \infty \quad \text{for all } i = 1, \dots, r.$$

It is an instructive exercise to prove that the Galois covering $\mathrm{X}' \to \mathrm{X}$ of group $\pi_1^{\text{ét}}(\mathrm{X})/\mathrm{N}$ does the job.

4.3. End of the proof. Theorems A and B are easily combined to give the main theorem. Let $\bar{\mathrm{K}}$ be an algebraic closure of K and for a variety V over K let $\bar{\mathrm{V}}$ denote the variety over $\bar{\mathrm{K}}$ obtained by extending scalars. With the notation of the main theorem, there is no loss of generality in assuming X geometrically integral and normal, as U is so. Fix $\varepsilon > 0$ and pick an integer $d \geqslant 1$ such that

$$\frac{n(n+3)}{2d}[\mathrm{K} : \mathbb{Q}] \leqslant \varepsilon \quad \text{where} \quad n = \dim \mathrm{X}.$$

By Theorem B there is a finite surjective morphism $f : \mathrm{X}' \to \bar{\mathrm{X}}$ over $\bar{\mathrm{K}}$ with X' integral, f étale over $\bar{\mathrm{U}}$ and, for a positive-dimensional integral subvariety Y' of X' not contained in $\mathrm{Z}' := f^{-1}(\bar{\mathrm{Z}})$,

$$\deg(\mathrm{Y}', f^*\mathrm{L}_{|\mathrm{Y}'}) \geqslant d^n.$$

Needless to say, one may assume that U has a K-rational point, otherwise the main theorem is trivial. Such a rational point gives a splitting of the geometric-arithmetic short exact sequence of étale fundamental groups:

$$1 \longrightarrow \pi_1^{\text{ét}}(\bar{\mathrm{X}}) \longrightarrow \pi_1^{\text{ét}}(\mathrm{X}) \longrightarrow \mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K}) \longrightarrow 1.$$

It follows that X' is dominated by a Galois cover of X étale over U defined over K, say $g : \mathrm{X}'' \to \mathrm{X}$. Such a cover clearly satisfies again the needed lower bound on the degrees: for a positive-dimensional integral subvariety Y'' of X'' not contained in $g^{-1}(\mathrm{Z})$ satisfies

$$\deg(\mathrm{Y}'', g^*\mathrm{L}_{|\mathrm{Y}''}) \geqslant d^n \geqslant d^{\dim \mathrm{Y}''}.$$

We are now in perfect position to apply theorem B to the cover X''. Alas there is a little hiccup: not all K-rational points of U lift to K-rational points of $\mathrm{U}'' = g^{-1}(\mathrm{U})$. On the other hand the cover $\mathrm{U}'' \to \mathrm{U}$ is étale, thus an argument in the spirit of the Chevalley-Weil theorem shows that the following inequality holds:

$$\mathrm{gr.int}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}) \leqslant \sup_{t \in \mathrm{H}^1(\mathrm{K}, \mathrm{G})} \mathrm{gr.int}_{\mathrm{K}}(\mathrm{X}''_t, g_t^*\mathrm{L}, g_t^{-1}(\mathrm{U})),$$

where G is the Galois group of $\mathrm{X}'' \to \mathrm{X}$ and, for an isomorphism class $t \in \mathrm{H}^1(\mathrm{K}, \mathrm{G})$ of principal G-bundles over K, the cover $g_t : \mathrm{X}''_t \to \mathrm{X}$ is the twist of $g : \mathrm{X}'' \to \mathrm{X}$ by t. Over $\bar{\mathrm{K}}$ the twist X''_t is isomorphic to X'' thus the needed lower bound of the degree of the subvarieties is still valid. Theorem A can therefore be applied to X''_t to give

$$\mathrm{gr.rat}_{\mathrm{K}}(\mathrm{X}''_t, g_t^*\mathrm{L}, g_t^{-1}(\mathrm{U})) \leqslant \frac{n(n+3)}{2d}[\mathrm{K} : \mathbb{Q}] \leqslant \varepsilon.$$

Combining these two inequalities yields $\mathrm{gr.int}_{\mathrm{K}}(\mathrm{X}, \mathrm{L}, \mathrm{U}) \leqslant \varepsilon$, as the growth rate of integral points is lesser than that of rational points. As $\varepsilon > 0$ is arbitrary, this concludes the proof. □

CHAPTER 2

Finiteness of varieties over number fields

The Lang-Vojta conjecture, when applied to moduli spaces, predicts the finiteness of certain classes of varieties over number fields. The most notorious of such finiteness is the *Shafarevich conjecture* proved by Faltings on the route for the Mordell conjecture: over any number field K, there are only finitely many isomorphism classes of smooth projective curves of a given genus $g \geqslant 2$ with good reduction outside a fixed finite set Σ of places of K. Since then analogous statements for other varieties have also acquired the name of Shafarevich conjecture, not to be confused with the eponymous conjecture on the holomorphic convexity of the universal cover of a complex projective manifold! In this chapter I present the Shafarevich conjecture for smooth projective varieties embedding in their Albanese with ample normal bundle. Such a result is the object of a recent paper in collaboration with T. Krämer [**3**] which is based on a technique of Lawrence, Sawin and Venkatesh [**LV20, LS20**] and a big monodromy result that we earlier proved with C. Lehn and A. Javanpeykar [**2**].

1. The Shafarevich conjecture

1.1. The original conjecture. At the International Congress of Mathematicians in Stockholm in 1962 Shafarevich [**Šaf63**, §4] conjectured that over any number field K, there are only finitely many isomorphism classes of smooth projective curves of a given genus $g \geqslant 2$ with good reduction outside a fixed finite set Σ of places of K. Roughly speaking the latter hypothesis means that the discriminant of an equation for the curve in question is divisible at most by the primes in Σ. More precisely, a smooth projective curve C over K is said to have *good reduction outside* Σ if there is a smooth proper scheme $\mathscr{C}$ over the ring of Σ-integers $\mathcal{O}_{\mathrm{K},\Sigma}$ with geometrically connected fibers of dimension 1 and generic fiber C. The good reduction hypothesis cannot be avoided as shown by quadratic twists of hyperelliptic curves:

Example. Pick a square-free integer $d \in \mathbb{Z}$ and consider the hyperelliptic curve X_d of affine equation

$$\mathrm{X}_d : \quad dy^2 = x(x-1)(x-2)(x-3)(x-4)(x-5).$$

Then X_1 and X_d become isomorphic over $\mathbb{Q}(\sqrt{d})$ but they *are not* isomorphic over $\mathbb{Q}$. On the other hand, given a prime $p \geqslant 7$, the curve X_d has bad reduction at p if and only if p divides d.

The *Shafarevich conjecture* has been proved by Faltings [**Fal83**] in 1984. It is deduced thanks to Torelli's theorem from the analogous statement for abelian varieties: for $g \geqslant 1$ up to isomorphism there are only finitely many abelian varieties of dimension g with good reduction outside Σ. Here an abelian variety A over K *has good reduction outside* Σ

if it extends to an abelian scheme $\mathscr{A}$ over $\mathcal{O}_{K,\Sigma}$. Note that such an extension is unique, as it coincides with the Néron model of A over $\mathcal{O}_{K,\Sigma}$. Let me review briefly the general strategy of Faltings's proof, as it will become useful for the upcoming discussion.

1.2. Faltings's strategy. Let $\bar{K}$ be an algebraic closure of K. For a g-dimensional abelian variety A over K and a prime number ℓ recall that the *ℓ-adic Tate module*

$$\mathrm{T}_\ell \mathrm{A} = \operatorname*{proj\,lim}_{n \geqslant 1} \mathrm{A}[\ell^n](\bar{\mathrm{K}})$$

is a free $\mathbb{Z}_\ell$-module of rank $2g$ and comes equipped with a continuous linear action of the absolute Galois group $\Gamma = \mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K})$. If A has good reduction outside Σ, then for each place $v \notin \Sigma$ the representation $\rho\colon \Gamma \to \mathrm{GL}(\mathrm{T}_\ell \mathrm{A})$ is unramified at v. That is, given an embedding of $\bar{\mathrm{K}}$ into an algebraic closure $\bar{\mathrm{K}}_v$ of the completion K_v of K at v, the composite map

$$\rho_v\colon \quad \Gamma_v := \mathrm{Gal}(\bar{\mathrm{K}}_v/\mathrm{K}_v) \hookrightarrow \Gamma = \mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K}) \xrightarrow{\rho} \mathrm{GL}(\mathrm{T}_\ell \mathrm{A})$$

factors through the quotient $\mathrm{Gal}(\bar{\mathbb{F}}_v/\mathbb{F}_v)$, where $\mathbb{F}_v$ and $\bar{\mathbb{F}}_v$ are the residue fields of K_v and $\bar{\mathrm{K}}_v$ respectively. In this case it is known since Weil that the representation ρ_v is *pure of weight* -1, that is, the Frobenius operator $x \mapsto x^{|\mathbb{F}_v|}$ in $\mathrm{Gal}(\bar{\mathbb{F}}_v/\mathbb{F}_v)$ has characteristic polynomial with integral coefficients and roots of complex absolute value $|\mathbb{F}_v|^{1/2}$. The above procedure sets up a map

$$\begin{array}{ccc} \mathrm{A} & & \left\{ \begin{array}{c} g\text{-dimensional abelian varieties over K} \\ \text{with good reduction outside } \Sigma \end{array} \right\} / \cong \\ \big\downarrow & & \big\downarrow \mathrm{V}_\ell \\ \mathrm{V}_\ell(\mathrm{A}) := \mathrm{T}_\ell \mathrm{A} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell & & \left\{ \begin{array}{c} 2g\text{-dimensional } \ell\text{-adic Galois representations} \\ \text{unramified outside } \Sigma \text{ and pure of weight } -1 \end{array} \right\} / \cong \end{array}$$

and Faltings proceeds by proving that V_ℓ has finite image and fibers. Both statements follow from what is called nowadays *Faltings's isogeny theorem* which represents the hardest part of his proof: given an abelian variety over K the set

$$\left\{ \begin{array}{c} \text{abelian varieties over K} \\ \text{isogenous to A} \end{array} \right\} / \cong$$

is finite. Arguing along the same lines of Tate, Faltings then deduces that the representation $\mathrm{V}_\ell(\mathrm{A})$ is semisimple. A rather elementary argument shows that the set

$$\mathscr{R}_{\mathrm{K},\Sigma,\ell}(d,w) \ := \left\{ \begin{array}{c} \textit{semisimple } d\text{-dimensional } \ell\text{-adic Galois representations} \\ \text{unramified outside } \Sigma \text{ and pure of weight } w \end{array} \right\} / \cong$$

is finite [Del85, théorème 3.1], hence V_ℓ has finite image. On the other hand it was already know that abelian varieties having isomorphic Tate module are isogenous, hence the isogeny theorem lets one conclude. Before moving on, it is useful to notice that the Tate module of A may be thought as the 'homology' of the abelian variety A because of the isomorphism of Galois representations

$$\mathrm{H}^1_{\text{ét}}(\mathrm{A}_{\bar{\mathrm{K}}}, \mathbb{Q}_\ell) \ \cong \ \mathrm{Hom}(\mathrm{T}_\ell \mathrm{A}, \mathbb{Q}_\ell).$$

1.3. Finiteness for other varieties. Since the work of Faltings there has been attempts to generalize the Shafarevich conjecture to other classes of varieties. The Shafarevich conjecture has been proved in the following cases:

(1) K3 surfaces and certain hyper-Kähler manifolds [**And96**], [**She97**], [**Tak20**];
(2) del Pezzo surfaces [**Sch85**];
(3) flag varieties [**JL15**];
(4) complete intersection in the projective space of level $\leqslant 1$ [**JL17**];
(5) fibrations of curves over smooth curves [**Jav15**];
(6) certain Fano threefolds [**JL18**];
(7) smooth cubic threefolds [**JL21**];
(8) Enriques surfaces [**Tak19**].

The proofs of the above results build on the only two techniques that have been available for almost 40 years: the proofs of (1), (4), (5) and (7) rely on the Shafarevich conjecture for abelian varieties, while those of (2), (3) and (6) follow from standard finiteness results in Galois cohomology. A question raises naturally: for which kind of varieties is it reasonable to expect this kind of arithmetic finiteness? As the next example shows, it is quite clear that one cannot hope for it to hold for all varieties:

Example. The set of isomorphism classes of genus 1 curves over K with good reduction outside Σ is infinite. This may look a little odd since the Shafarevich conjecture holds for elliptic curves. The subtlety lies in the fact that elliptic curves come with a rational point by default. Instead for each $n \geqslant 1$ there are genus 1 curves over K with no point of degree $< n$. It would be sufficient to bound the minimal degree of a point on a genus 1 curve to have finiteness *à la* Shafarevich. As it will be useful later, notice that this is equivalent to bound the degree of a polarization on such a curve.

So how does one guess if a given class of varieties ought to undergo the Shafarevich conjecture? The natural point of view is to look at the moduli space M classifying the class of varieties in question. If M is 'hyperbolic', that is, all subvarieties are of log general type, then the Lang-Vojta conjecture predicts that M has only finitely Σ-integral points for each finite set of places Σ of K.[1]

Example. Consider elliptic curves E together with an isomorphism $(\mathbb{Z}/2\mathbb{Z})^2 \cong \mathrm{E}[2]$ of group schemes, where E[2] is the 2-torsion subgroup of E. The moduli space of such

[1] A disclaimer: one should work at the level of fine moduli spaces as in the examples of section 2.3 in chapter 1 or even better at the level of moduli stacks, and for a good reason: the Shafarevich conjecture is really about K-isomorphism classes of varieties, whereas coarse moduli space only classify varieties up to isomorphism over an algebraic closure. This may seem anodyne, but think of the example of elliptic curves:

- the coarse moduli space is the affine line $\mathbb{A}^1$ which is anything but hyperbolic;
- over an algebraically closed field an elliptic curve is the same as a genus 1 curve; however the Shafarevich does not hold for genus 1 curves: this is ultimately due to the pathological nature of the moduli stack of non-polarized genus 1 curves.

To do things properly one should therefore introduce a notion of hyperbolic stack; this appears implicitly in [**JL17**]. Not to overcomplicate this heuristic discussion I will swap these issues under the carpet and stick to the case of fine moduli spaces.

couples is

$$\mathrm{M} = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}.$$

It is a fine as a moduli space: the universal family is the Legendre family of elliptic curves

$$\mathscr{E} \ = \ \{y^2 z = x(x-z)(x-\lambda z)\} \ \subset \ \mathrm{M} \times \mathbb{P}^2$$

together with the isomorphism $\alpha_\lambda \colon (\mathbb{Z}/2\mathbb{Z})^2 \to \mathscr{E}_\lambda[2]$ given by

$$(0,0) \ \mapsto \ [0:1:0], \quad (1,0) \ \mapsto \ [0:0:1], \quad (0,1) \ \mapsto \ [1:0:1], \quad (1,1) \ \mapsto \ [\lambda:0:1].$$

For $\lambda \in \mathrm{K} \smallsetminus \{0,1\}$ saying that the couple $(\mathscr{E}_\lambda, \alpha_\lambda)$ has good reduction outside Σ means that λ is a Σ-integral point of M:

$$\lambda \in \mathrm{M}(\mathcal{O}_{\mathrm{K},\Sigma}) = \{\lambda \in \mathcal{O}_{\mathrm{K},\Sigma}^\times : 1 - \lambda \in \mathcal{O}_{\mathrm{K},\Sigma}^\times\}.$$

As recalled in section 1.3 of chapter 1 the theorem on the Σ-unit equation implies that the set $\mathrm{M}(\mathcal{O}_{\mathrm{K},\Sigma})$ is finite. This agrees with M being of log general type, since the logarithmic canonical bundle of M is the line bundle $\omega_{\mathbb{P}^1} \otimes \mathcal{O}([0] + [1] + [\infty]) \cong \mathcal{O}(1)$ on $\mathbb{P}^1$.

But this only shifts the question: how does one decide if the moduli space M is hyperbolic or not? This is rather well-understood: suppose for simplicity that M is non-singular and is fine as a moduli space, that is, there is a universal family $\pi \colon \mathfrak{X} \to \mathrm{M}$. To avoid issues as in the case genus 1 curves it is also necessary to suppose that the varieties come with a polarization, that is, there is a relatively ample line bundle $\mathscr{L}$ on $\mathfrak{X}$. Fix an embedding $\mathrm{K} \hookrightarrow \mathbb{C}$ and consider the variation of Hodge structures

$$\mathbb{V} \ := \ (\ \mathrm{R}^d \pi_* \mathbb{Q}_{\mathfrak{X}}\ , \quad \mathscr{H}^d_{\mathrm{dR}}(\mathfrak{X}/\mathrm{M})\ , \quad \nabla\ , \quad \mathscr{F}^\bullet\ , \quad \alpha\ )$$

on M where d is the relative dimension of $\mathfrak{X} \to \mathrm{M}$, ∇ is the Gauss-Manin connection, $\mathscr{F}^\bullet$ is the Hodge filtration and α is the C^∞-isomorphism

$$\alpha \colon \quad \mathscr{H}^d_{\mathrm{dR}}(\mathfrak{X}/\mathrm{M}) \ \xrightarrow{\sim} \ \bigoplus_{p+q=d} \mathrm{R}^q \pi_* \Omega^p_{\mathfrak{X}/\mathrm{M}}$$

of complex vector bundles on $\mathrm{M}(\mathbb{C})$ inducing the Hodge decomposition point-wise. Upon fixing a base-point, one can look at the period mapping

$$\Phi \colon \quad \mathrm{M}(\mathbb{C}) \ \longrightarrow \ \mathfrak{H}/\Gamma$$

associated with the primitive part of $\mathbb{V}$, where $\mathfrak{H}$ is some bounded symmetric domain and Γ is some arithmetic subgroup of the biholomorphisms of $\mathfrak{H}$. Note that the hypothesis of M being a fine moduli space implies that the subgroup Γ is torsion-free; see section 2.3 of chapter 1. It follows from work of Zuo [**Zuo00**] and Brunebarbe [**Bru18**] that if Φ is an immersion, then M is hyperbolic. By a theorem of Campana-Paŭn [**CP15**] (formerly a conjecture of Viehweg) the moduli space M is hyperbolic if it parametrizes canonically polarized varieties.

Example. The discussion above shows that smooth hypersurfaces of $\mathbb{P}^n$ of degree $\geqslant 3$ should verify the Shafarevich conjecture. More generally, the Shafarevich conjecture should hold for complete intersections that are of general type or of some specific type for which the period mapping is known to be immersive; see [**JL17**] for details.

2. The Lawrence-Venkatesh method

2.1. Hypersurfaces in the projective space. The situation drastically changed a few years ago when Lawrence and Venkatesh [LV20] came up with a brand new method to prove non-density for the Zariski topology of integral points. Their primary application was to give an alternative proof of the Mordell conjecture: this is possible because for projective curves rational points are all integral and non-density is equivalent to finiteness; see the written account of my Bourbaki seminar [6] for details. As opposed to the proofs of Faltings and Vojta, the arithmetic part of their method did not require anything specific about curves. As an instance of this remark, they proved a weak form of the Shafarevich conjecture for hypersurfaces in a projective space. More precisely let $n, d \geqslant 1$ be integers and consider the scheme $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(d)))$ parametrizing hypersurfaces of degree d in $\mathbb{P}^n$. The complement of the discriminant divisor

$$\mathrm{U}_{n,d} \subset \mathbb{P}(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(d)))$$

then parametrizes smooth hypersurfaces. In particular

$$\mathrm{U}_{n,d}(\mathcal{O}_{\mathrm{K},\Sigma}) = \left\{ \begin{array}{c} \text{hypersurfaces of degree } d \text{ in } \mathbb{P}^n_{\mathrm{K}} \\ \text{with good reduction outside } \Sigma \end{array} \right\}.$$

Here a hypersurface $\mathrm{X} \subset \mathbb{P}^n_{\mathrm{K}}$ is said to *have good reduction outside* Σ if its Zariski closure in $\mathbb{P}^n_{\mathcal{O}_{\mathrm{K},\Sigma}}$ is smooth over $\mathcal{O}_{\mathrm{K},\Sigma}$. With this notation Lawrence and Venkatesh [LV20, proposition 10.2] show that there are $n_0 \geqslant 1$ and a function $d_0 \colon \mathbb{N} \to \mathbb{N}$ such that,

$$\mathrm{U}_{n,d}(\mathbb{Z}[1/\mathrm{N}]) \subset \mathrm{U}_{n,d} \quad \text{is not Zariski-dense for} \quad n \geqslant n_0,\ d \geqslant d_0(n),\ \mathrm{N} \geqslant 1.$$

In other words there is a non-zero homogeneous polynomial $f \in \mathrm{Sym}^{\mathrm{D}}\, \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(d))^\vee$ of some degree $\mathrm{D} \geqslant 1$ vanishing identically on $\mathrm{U}_{n,d}(\mathbb{Z}[1/\mathrm{N}])$. Note that the polynomial f depends on n, d and N. Numerical experiments show that $n_0 \sim 60$ should do the job. Instead given $n \geqslant 1$ it is not clear how to bound the minimal choice for $d_0(n)$. Finally giving an upper bound for D is doubtless out of the scope of the method of Lawrence-Venkatesh. As recalled in the previous section, the Lang-Vojta conjecture implies

$$|\mathrm{U}(\mathbb{Z}[1/\mathrm{N}])/\,\mathrm{PGL}_{n+1}(\mathbb{Z}[1/\mathrm{N}])| < \infty$$

thus non-density may be seen as a preliminary result towards it.

The technique of Lawrence-Venkatesh applies to varieties which support a sufficiently complicated (geometric) variation of Hodge structures, where 'sufficiently complicated' means that the underlying local system must have big monodromy. In the case of the Mordell conjecture this variation is given by a family of Prym varieties over some Hurwitz space of non-abelian covers: big monodromy is then proved by exhibiting carefully some Dehn twists. In the case of hypersurfaces the variation is given by the primitive part of the middle cohomology of the fibers of the universal family $\mathcal{X} \to \mathrm{U}_{n,d}$: a classical computation by Beauville [Bea86] (relying on the study by Ebeling [Ebe84] and Janssen [Jan83] of the topology of singular fibers) shows that the family $\mathcal{X} \to \mathrm{U}_{n,d}$ has big monodromy; see also the discussion by Katz in [Kat04]. In order to have finiteness instead of non-density for hypersurfaces one might consider working by induction and reapply the Lawrence-Venkatesh method to any subvariety. This would require to have big monodromy on any subvariety of $\mathrm{U}_{n,d}(\mathbb{C})/\operatorname{PGL}_{n+1}(\mathbb{C})$. Unfortunately this is certainly not the case for subvarieties mapped to special subvarieties of the corresponding period domain.

2.2. Hypersurfaces of abelian varieties. In 2020 Lawrence-Sawin [LS20] succeeded in adapting the Lawrence-Venkatesh method to prove the Shafarevich conjecture for hypersurfaces in abelian varieties. More precisely let A be a g-dimesional abelian variety over K and suppose that Σ contains the places of bad reduction of A. The abelian variety A then extends uniquely to an abelian scheme $\mathcal{A}$ over $\mathcal{O}_{\mathrm{K},\Sigma}$. Fix an ample class c in the Néron-Severi group A. Assuming $g \neq 3$ Lawrence-Sawin prove that the set

$$\left\{ \begin{array}{c} \text{smooth hypersurfaces in A} \\ \text{algebraically equivalent to } c \\ \text{with good reduction outside } \Sigma \end{array} \right\} / \text{ translation by A(K)}$$

is finite. Here a hypersurface $\mathrm{X} \subset \mathrm{A}$ is said to *have good reduction* if its Zariski closure in the abelian scheme $\mathcal{A}$ is smooth over $\mathcal{O}_{\mathrm{K},\Sigma}$. When $g = 3$ they prove such a statement under the following additional (in all likelihood superfluous) numerical assumption on the top self-intersection c^3 of c. To state it, consider the sequences of integers $(a_i)_{i\geqslant 1}$ et $(d_i)_{i\geqslant 1}$ defined by $a_1 = 1$, $a_2 = 5$ and, for $i \geqslant 1$, by

$$a_{i+2} = 4a_{i+1} - a_i + 1, \qquad\qquad d_i = \frac{1}{6}\binom{a_{i+1} + a_i}{a_i}.$$

Then the above finiteness holds unless $c^3/3!$ is divisible by some d_i for $i \geqslant 2$. Note that the restriction is very mild since the sequence $(d_i)_{i\geqslant 2}$ has double exponential growth: its first terms are

$$d_2 = 8\ 855 \quad \text{and} \quad d_3 = 36\ 030\ 431\ 772\ 522\ 503\ 316.$$

More interestingly, these results can be translated into classical diophantine statements as follows. An ample line bundle L on A extends uniquely to a relatively ample line bundle $\mathcal{L}$ on $\mathcal{A}$. Let

$$\mathrm{U}_{\mathrm{L}} \subset \mathbb{P}(\mathrm{H}^0(\mathcal{A}, \mathcal{L}))$$

be the complement of the discriminant locus. Then the above statement implies the finiteness of the set

$$\mathrm{U}_{\mathrm{L}}(\mathcal{O}_{\mathrm{K},\Sigma}) = \left\{ \begin{array}{c} \text{smooth hypersurfaces in A} \\ \text{linearly equivalent to L} \\ \text{with good reduction outside } \Sigma \end{array} \right\}.$$

As opposed to projective spaces, abelian varieties have an additional degree of freedom: they are not simply connected. Non-trivial unramified covers, which are given in this case by multiplication by an integer, are the essential ingredient permitting one to attain finiteness instead of the mere non-density. The main novelty of their work lies in their way to control monodromy. The arguments of Lawrence and Venkatesh have a topological flavor. Instead, the approach by Lawrence and Sawin involves Tannaka groups of perverse sheaves on abelian varieties introduced by Krämer and Weissauer [KW15c]; the relation of these groups to monodromy is reminiscent of the one between the monodromy group of a variation of Hodge structures and its generic Mumford-Tate group [And92]. I will come back at this later in chapter 3.

3. Finiteness of very irregular varieties

3.1. Very irregular varieties. In collaboration with T. Krämer [3] we adapted the strategy of Lawrence-Venkatesh and Lawrence-Sawin to prove the Shafarevich conjecture for a large class of varieties that we call *very irregular*. To defined them let k be a field of characteristic 0.

Definition. A smooth projective variety X over k is *very irregular* if $\mathrm{X}(k) \neq \emptyset$ and if for any $x \in \mathrm{X}(\mathrm{K})$ the Albanese morphism $\mathrm{alb}_x \colon \mathrm{X} \to \mathrm{Alb}(\mathrm{X})$ is a closed embedding with ample normal bundle, and not an isomorphism.

The above properties do not depend on the choice of the point x. This notion is not standard and the terminology 'very irregular' we introduced is inspired by the name 'irregularity' classically given to the dimension of the Albanese variety of a smooth projective surface. A smooth projective variety X with $\mathrm{X}(k) \neq \emptyset$ is very irregular at least in the following cases:

(1) X is a curve of genus $\geqslant 2$: indeed a theorem of Hartshorne [Har71, proposition 4.1] states that a smooth projective curve in an abelian variety A has ample normal bundle if and only if it generates A.

(2) X is a proper subvariety of an abelian variety A with ample normal bundle and such that the induced morphism $\mathrm{Alb}(\mathrm{X}) \to \mathrm{A}$ is an isogeny. By the Barth-Lefschetz theorem for abelian varieties proved by Debarre, this is the case if X is a complete intersection of ample divisors or if the dimension of X is $> \dim \mathrm{A}/2$.

The canonical bundle of a very irregular variety X is the determinant of the normal bundle of its Albanese embedding and it is therefore ample. It follows that its automorphism group scheme Aut(X) is finite étale. Clearly Aut(X) acts on the Albanese variety and the subgroup of Aut(X) acting on Alb(X) by translations is

$$\mathrm{Stab}(\mathrm{X}) \ := \ \mathrm{Ker}\big(\mathrm{Aut}(\mathrm{X}) \to \mathrm{GL}(\mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}}))\big).$$

3.2. Intrinsic results. Let K be an number field and Σ a finite set of places. Let A be an abelian variety over K with good reduction outside Σ and let $\mathscr{A}$ be the unique abelian scheme over $\mathscr{O}_{\mathrm{K},\Sigma}$ to which A extends. Similarly to the case of hypersurfaces in Lawrence-Sawin, a subvariety $\mathrm{X} \subset \mathrm{A}$ is said to *have good reduction outside* Σ if its Zariski closure in $\mathscr{A}$ is smooth over $\mathscr{O}_{\mathrm{K},\Sigma}$.

Definition. A very irregular variety X over K *has good reduction outside* Σ if its Albanese variety Alb(X) and the subvariety $\mathrm{X} \hookrightarrow \mathrm{Alb}(\mathrm{X})$ do.

In the above definition I implicitly picked a K-point of X to embed it in its Albanese. Note that the good reduction property does not depend on the choice of such a point. In the case of surfaces our main result can be stated as follows (see [3, theorem A]):

Theorem A. *Fix an integer $c \geqslant 1$. Then up to* K*-isomorphism there are only finitely many very irregular surfaces* X *with good reduction outside* Σ *such that*

- $c_2(\mathrm{X}) = c$,
- $h^0(\mathrm{X}, \Omega^1_{\mathrm{X}}) \geqslant 6$,

and such that $c_2(\mathrm{X})/|\mathrm{Stab}(\mathrm{X})| \neq 27$.

A very irregular surface X is minimal hence $\chi := \chi(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \geqslant h^0(\mathrm{X}, \Omega^1_{\mathrm{X}}) - 3$. On the other hand the surface X is of general type thus the Bogomolov-Miyaoka-Yau inequality together with Noether's formula imply $3\chi \leqslant c_2(\mathrm{X})$. Combining this two inequalities one finds $c_2(\mathrm{X}) \geqslant 3(q-3)$. In particular the condition $c_2(\mathrm{X})/|\operatorname{Stab}(\mathrm{X})| \neq 27$ is empty as soon as

$$h^0(\mathrm{X}, \Omega^1_{\mathrm{X}}) \geqslant 13.$$

More generally we show the following result for very irregular varieties of dimension less than half the dimension of the Albanese variety (see [3, theorem B]):

Theorem B. *Let* $\mathrm{P} \in \mathbb{Q}[z]$ *a polynomial of degree* d. *Then up to* K*-isomorphism there are only finitely many smooth projective very irregular varieties* X *with good reduction outside* Σ *such that*

- X *has Hilbert polynomial* P *with respect to the canonical bundle,*
- $g = h^0(\mathrm{X}, \Omega^1_{\mathrm{X}}) \geqslant 2d + 2$,

and such that ($\star$) *and* ($\star\star$) *below hold with* $\mathrm{A} = \operatorname{Alb}(\mathrm{X})$, $e = \chi_{\mathrm{top}}(\mathrm{X})/|\operatorname{Stab}(\mathrm{X})|$.

To see why theorem B implies theorem A note that for a smooth projective surface X the Riemann-Roch theorem and Noether's formula yield

$$\chi(\mathrm{X}, \omega_{\mathrm{X}}^{\otimes n}) = \tfrac{1}{2} n(n-1) c_1(\mathrm{X})^2 + \tfrac{1}{12}(c_1(\mathrm{X})^2 + c_2(\mathrm{X})).$$

If X is of general type, then $c_1(\mathrm{X})^2 \leqslant 3c_2(\mathrm{X})$ by the Bogomolov-Miyaoka-Yau inequality so, once $c_2(\mathrm{X})$ is fixed, there are only finitely many possibilities for the Hilbert polynomial of X with respect to the canonical bundle.

Theorem B applies in particular to all smooth projective curves of genus $\geqslant 4$ with a rational point. However, its proof relies on the Shafarevich conjecture for abelian varieties proved by Faltings in [Fal83], thus theorem B does not furnish us with a new proof of the Shafarevich conjecture for curves. Instead, one can regard theorem B as a generalization of Faltings's theorem to higher dimension.

3.3. Numerical conditions. In order to introduce the numerical conditions that enter the statement of theorem B, let A be an abelian variety of dimension g over a field k and $\mathrm{X} \subset \mathrm{A}$ a smooth subvariety of dimension d. If k is algebraically closed X is said to be *symmetric up to translation* if there is a point $a \in \mathrm{A}(k)$ such that $\mathrm{X} = -\mathrm{X} + a$. If k is an arbitrary field X is said to be symmetric up to translation if its base change to an algebraic closure of k is so. The *stabilizer* of a subvariety $\mathrm{X} \subset \mathrm{A}$ is the algebraic group $\operatorname{Stab}_{\mathrm{A}}(\mathrm{X})$ whose points in a k-scheme S are

$$\operatorname{Stab}_{\mathrm{A}}(\mathrm{X})(\mathrm{S}) \;=\; \{a \in \mathrm{A}(\mathrm{S}) \colon \mathrm{X}_{\mathrm{S}} + a \;=\; \mathrm{X}_{\mathrm{S}}\}.$$

Ueno's fibration theorem implies that X is of general type if and only if its stabilizer is finite. Moreover, when $\mathrm{A} = \operatorname{Alb}(\mathrm{X})$ and X is embedded via the Albanese morphism,

$$\operatorname{Stab}(\mathrm{X}) = \operatorname{Stab}_{\operatorname{Alb}(\mathrm{X})}(\mathrm{X}).$$

To apply the big monodromy criterion below we impose that the topological Euler characteristic $e = \chi_{\mathrm{top}}(\mathrm{X})/|\operatorname{Stab}_{\mathrm{A}}(\mathrm{X})|$ of the quotient $\mathrm{X}/\operatorname{Stab}_{\mathrm{A}}(\mathrm{X})$ satisfies the following conditions:

$$(\star)\qquad \begin{array}{lll} |e| \neq 27 & & \text{if } d \geqslant 2 \text{ and } \mathrm{X} \subset \mathrm{A} \text{ is not symmetric up to a translation,} \\ |e| \neq 56 & & \text{if } d \geqslant 3 \text{ is odd and } \mathrm{X} \subset \mathrm{A} \text{ is symmetric up to translation,} \\ |e| \neq 2^{2m-1} & & \text{if } d \geqslant (g-1)/4,\ m \in \{3, \dots, d\} \text{ has the same parity as } d \\ & & \text{and } \mathrm{X} \subset \mathrm{A} \text{ is symmetric up to translation.} \end{array}$$

For $d < (g-1)/2$ I do not know of any example of subvarieties X with ample normal bundle for which ($\star$) fails. Here the dimension is important as the Fano surface X of a cubic threefold is embedded in its 5-dimensional Albanese variety and has $e = 27$. To deal with the semisimplification of global Galois representations, we also need to assume that

$$(\star\star)\qquad 2\chi(\mathrm{X}\times\mathrm{X}, \Omega^d_{\mathrm{X}\times\mathrm{X}}) \leqslant \chi_{\mathrm{top}}(\mathrm{X}\times\mathrm{X}).$$

Here regardless of the dimension I do not know any example of a subvariety X with ample normal bundle for which ($\star\star$) fails. Indeed, a straightforward computation involving Serre's duality and generic vanishing shows that ($\star\star$) holds if d is odd. When X is a surface, the inequality ($\star\star$) is equivalent to

$$c_2(\mathrm{X}) \leqslant c_1(\mathrm{X})^2 \leqslant 5c_2(\mathrm{X}),$$

and is satisfied because X is of general type with nef cotangent bundle. When X is a hypersurface Lawrence and Sawin show that $\chi(\mathrm{X}, \Omega^i_{\mathrm{X}})$ is the Eulerian number $\mathrm{A}(g, i)$ times the degree of X and ($\star\star$) holds by log concavity of Eulerian numbers.

Let me take advantage of the general setup in this section to introduce two notions that will be useful later:

Definition. If k is algebraically closed X is said to be

- *divisible* if $\mathrm{Stab}_{\mathrm{A}}(\mathrm{X}) \neq 0$;
- a *product* if there are smooth subvarieties $\mathrm{X}_1, \mathrm{X}_2 \subset \mathrm{A}$ with $\dim \mathrm{X}_i > 0$ such that the sum morphism $\mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{A}$ is a closed embedding with image X.

If k is an arbitrary field X is divisible resp. a product if its base change to an algebraic closure of k is so. If the Albanese morphism $\mathrm{X} \to \mathrm{Alb}(\mathrm{X})$ given by some $x \in \mathrm{X}(k)$ is a closed immersion with ample normal bundle, then X is not a product.

3.4. Extrinsic result. The Shafarevich conjecture for abelian varieties proved by Faltings lets one reduce the proof of theorem B to the case where the Albanese is a fixed g-dimensional abelian variety A over a number field K. Let Σ be a finite set of places of K including all places where A has bad reduction. More precisely theorem B follows from a general result about subvarieties of A in the spirit of the work of Lawrence-Sawin (see [3, theorem C]):

Theorem C. *Fix a polynomial* $\mathrm{P} \in \mathbb{Q}[z]$ *of degree* $d < (g-1)/2$ *and an ample line bundle* L *on* A. *Then up to translation by points in* A(K) *there are only finitely many smooth subvarieties* $\mathrm{X} \subset \mathrm{A}$ *over* K *with ample normal bundle and good reduction outside* Σ *that are not a product, that have Hilbert polynomial* P *with respect to* L, *and that satisfy* ($\star$) *and* ($\star\star$) *with* $e = \chi_{\mathrm{top}}(\mathrm{X})/|\mathrm{Stab}_{\mathrm{A}}(\mathrm{X})|$.

A subvariety $\mathrm{X} \subset \mathrm{A}$ is said to be a *geometric complete intersection of ample divisors* if its base change to an algebraic closure of K is a complete intersection of ample divisors. Smooth complete intersections of ample divisors in an abelian variety are never a product and they satisfy ($\star$) with $e = \chi_{\mathrm{top}}(\mathrm{X})$; see remark 6.3, proposition 2.16 and corollary 2.17 in [2]. Hence as a direct consequence of theorem C we obtain:

Corollary. *Fix a polynomial* $\mathrm{P} \in \mathbb{Q}[z]$ *of degree* $d < (g-1)/2$ *and an ample line bundle* L *on* A. *Then up to translation by points in* $\mathrm{A}(\mathrm{K})$ *there exist only finitely many smooth nondivisible geometric complete intersections of ample divisors* $\mathrm{X} \subset \mathrm{A}$ *over* K *with good reduction outside* Σ, *Hilbert polynomial* P *with respect to* L *and satisfying* ($\star\star$).

The proof of theorem C is based on the Lawrence-Venkatesh method [LV20] as elaborated in Lawrence-Sawin [LS20]. As in their work the geometric input is the big monodromy of the families of subvarieties in question. We proved this in collaboration with A. Javanpeykar and C. Lehn [2] and is in my view our main contribution to the story.

3.5. Big monodromy. In this section let me work over $\mathbb{C}$. Let S be a smooth irreducible complex variety and A be a complex abelian variety of dimension g. Inside the constant abelian scheme $\mathrm{A_S} := \mathrm{A} \times \mathrm{S}$, let $\mathfrak{X} \subset \mathrm{A_S}$ be a closed subvariety which is smooth over S with connected fibers of dimension d, seen as a family of subvarieties of A over S:

$$\begin{tikzcd} & \mathfrak{X} \arrow[dl, bend right, "\pi"'] \arrow[d, equal] \arrow[dr, bend left, "f"] & \\ \mathrm{A} & \mathrm{A_S} \arrow[l, "\mathrm{pr_A}"'] \arrow[r, "\mathrm{pr_S}"] & \mathrm{S} \end{tikzcd}$$

It will be useful to consider both the analytic and the algebraic setup, using topological local systems with coefficients in $\Lambda = \mathbb{C}$ resp. étale ℓ-adic local systems with coefficients in $\Lambda = \overline{\mathbb{Q}}_\ell$ for a prime ℓ. Let $\pi_1(\mathrm{A}, 0)$ be the topological resp. étale fundamental group with the discrete resp. profinite topology, and denote the group of its continuous characters by

$$\Pi(\mathrm{A}, \Lambda) = \mathrm{Hom}(\pi_1(\mathrm{A}, 0), \Lambda^\times).$$

In what follows, a *linear subvariety* is a subset $\Pi(\mathrm{B}, \Lambda) \subset \Pi(\mathrm{A}, \Lambda)$ for an abelian quotient variety $\mathrm{A} \twoheadrightarrow \mathrm{B}$ with $\dim \mathrm{B} < \dim \mathrm{A}$. A statement is said to hold for *most* $\chi \in \Pi(\mathrm{A}, \Lambda)$ if it holds for all χ outside a finite union of torsion translates of linear subvarieties. For a character $\chi \in \Pi(\mathrm{A}, \Lambda)$, let L_χ denote the associated rank one local system on A. It follows from generic vanishing [BSS18, KW15c, Sch15] that for most χ the higher direct images $\mathrm{R}^i f_* \pi^* \mathrm{L}_\chi$ vanish in all degrees $i \neq d$. In this case it is interesting to look at the local system

$$\mathrm{V}_\chi \;:=\; \mathrm{R}^d f_* \pi^* \mathrm{L}_\chi$$

of rank $|e|$ where e is the topological Euler characteristic of the fibers of $\mathfrak{X} \to \mathrm{S}$. More generally, for an n-tuple $\underline{\chi} = (\chi_1, \dots, \chi_n) \in \Pi(\mathrm{A})^n$ of characters set

$$\mathrm{V}_{\underline{\chi}} \;:=\; \mathrm{V}_{\chi_1} \oplus \cdots \oplus \mathrm{V}_{\chi_n}.$$

Using the natural identification $\Pi(\mathrm{A},\Lambda)^n = \Pi(\mathrm{A}^n,\Lambda)$ I will also apply the terminology *most* for such n-tuples of characters. Consider for $s \in \mathrm{S}(\mathbb{C})$ the monodromy representation

$$\rho\colon \quad \pi_1(\mathrm{S},s) \longrightarrow \mathrm{GL}(\mathrm{V}_{\underline{\chi},s}) \quad \text{on the fiber} \quad \mathrm{V}_{\underline{\chi},s} = \bigoplus_{i=1}^{n} \mathrm{H}^d(\mathfrak{X}_s, \mathrm{L}_{\chi_i}).$$

The *algebraic monodromy group* of the local system $\mathrm{V}_{\underline{\chi}}$ is the Zariski closure of the image of ρ. By construction it is an algebraic subgroup of

$$\mathrm{GL}(\mathrm{V}_{\chi_1,s}) \times \cdots \times \mathrm{GL}(\mathrm{V}_{\chi_n,s}) \subset \mathrm{GL}(\mathrm{V}_{\underline{\chi},s}).$$

This upper bound can sometimes be refined: say that $\mathfrak{X} \subset \mathrm{A}_\mathrm{S}$ is *symmetric up to translation* if there exists $a\colon \mathrm{S} \to \mathrm{A}$ such that $\mathfrak{X}_t = -\mathfrak{X}_t + a(t)$ for all $t \in \mathrm{S}(\mathbb{C})$. In this case, Poincaré duality furnishes us with a nondegenerate bilinear pairing

$$\theta_{\chi,s}\colon \quad \mathrm{V}_{\chi,s} \otimes \mathrm{V}_{\chi,s} \longrightarrow \mathrm{L}_{\chi,a(s)}$$

for each $\chi \in \Pi(\mathrm{A},\Lambda)$, because for the dual of a rank one local system and for its inverse image under the translation $\tau_{a(t)}\colon \mathrm{A} \to \mathrm{A}, x \mapsto x + a(t)$ we have natural isomorphisms

$$\begin{aligned} \mathrm{L}_\chi^\vee &\cong [-1]^*\mathrm{L}_\chi, \\ \tau_{a(t)}^*\mathrm{L}_\chi &\cong \mathrm{L}_\chi \otimes_\Lambda \mathrm{L}_{\chi,a(t)}. \end{aligned}$$

The pairing $\theta_{\chi,s}$ is symmetric if d is even, and alternating otherwise. Since the pairing is compatible with the monodromy operation on the fiber, it follows that the algebraic monodromy group of $\mathrm{V}_{\underline{\chi}}$ is contained in an orthogonal resp. symplectic group in the two cases. This leads to the following definition:

Definition. The local system $\mathrm{V}_{\underline{\chi}}$ has *big monodromy* if its algebraic monodromy group contains $\mathrm{G}_1 \times \cdots \times \mathrm{G}_n$ as a normal subgroup where $\mathrm{G}_i \subset \mathrm{GL}(\mathrm{V}_{\chi_i,s})$ is defined by

$$\mathrm{G}_i := \begin{cases} \mathrm{SL}(\mathrm{V}_{\chi_i,s}) & \text{if } \mathfrak{X} \text{ is not symmetric up to translation,} \\ \mathrm{SO}(\mathrm{V}_{\chi_i,s}, \theta_{\chi_i,s}) & \text{if } \mathfrak{X} \text{ is symmetric up to translation and } d \text{ is even,} \\ \mathrm{Sp}(\mathrm{V}_{\chi_i,s}, \theta_{\chi_i,s}) & \text{if } \mathfrak{X} \text{ is symmetric up to translation and } d \text{ is odd.} \end{cases}$$

The family $\mathfrak{X} \to \mathrm{S}$ *has big monodromy for most tuples of torsion characters* if $\mathrm{V}_{\underline{\chi}}$ has big monodromy for each $n \geqslant 1$ and most torsion n-tuples $\underline{\chi} \in \Pi(\mathrm{A})^n$.

Note that the connected component of the algebraic monodromy group of $\mathrm{V}_{\underline{\chi}}$ is unaffected by base change along étale morphisms $\mathrm{S}' \to \mathrm{S}$. To take this into account it is convenient to consider the fiber $\mathfrak{X}_{\bar\eta}$ of $\mathfrak{X} \to \mathrm{S}$ at a geometric generic point $\bar\eta$ of S. The local system $\mathrm{V}_{\underline{\chi}}$ does not have big monodromy in each of the following four cases:

(1) if $\mathfrak{X}_{\bar\eta}$ is *constant up to a translation*, *i.e.* it is the translate of a subvariety $\mathrm{Y} \subset \mathrm{A}$ along a point in $\mathrm{A}(\bar\eta)$: in this case the algebraic monodromy is finite. Note that if $\mathfrak{X}_{\bar\eta}$ is nondivisible, then $\mathfrak{X}_{\bar\eta}$ is constant up to translation if and only if the family $\mathfrak{X} \to \mathrm{S}$ is isotrivial.

(2) if $\mathfrak{X}_{\bar\eta}$ is *divisible*, for instance when $\mathfrak{X}_{\bar\eta}$ is stable under translation by a torsion point $0 \neq x \in \mathrm{A}(\bar\eta)$. In this case the algebraic monodromy of each V_{χ_i} is itself a group of block matrices which is normalized by the group generated by the point x.

(3) if $\mathfrak{X}_{\bar{\eta}}$ is a *symmetric power of a curve*, *i.e.* there is a smooth curve $\mathrm{C} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ such that the sum morphism $\mathrm{Sym}^d \mathrm{C} \to \mathrm{A}_{\mathrm{S},\bar{\eta}}$ is a closed embedding with image $\mathfrak{X}_{\bar{\eta}}$ and $d \geqslant 2$. After an étale base change over S, we may assume that C spreads out to a relative curve $\mathscr{C} \subset \mathrm{A}_\mathrm{S}$ which is smooth and proper over S such that the relative sum morphism $\mathrm{Sym}^d_\mathrm{S} \mathscr{C} \to \mathrm{A}_\mathrm{S}$ is a closed embedding with image $\mathfrak{X}$. Then we have an isomorphism compatible with monodromy:

$$\mathrm{H}^d(\mathfrak{X}_s, \mathrm{L}_\chi) \cong \mathrm{Alt}^d \mathrm{H}^1(\mathscr{C}_s, \mathrm{L}_\chi).$$

(4) if $\mathfrak{X}_{\bar{\eta}}$ is a *product*, which by definition means that if there are smooth subvarieties $\mathrm{X}_1, \mathrm{X}_2 \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ with $\dim \mathrm{X}_i > 0$ such that the sum map $\mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{A}_{\mathrm{S},\bar{\eta}}$ is a closed embedding with image $\mathfrak{X}_{\bar{\eta}}$. Again, after an étale base change over S one may assume that X_i spreads out to a subvariety $\mathfrak{X}_i \subset \mathrm{A}_\mathrm{S}$ which is smooth and proper over S such that the relative sum morphism $\mathfrak{X}_1 \times \mathfrak{X}_2 \to \mathrm{A}_\mathrm{S}$ is a closed embedding with image $\mathfrak{X}$. Then the Künneth isomorphism is compatible with monodromy:

$$\mathrm{H}^d(\mathfrak{X}_s, \mathrm{L}_\chi) \cong \bigoplus_{i_1+i_2=d} \mathrm{H}^{i_1}(\mathfrak{X}_{1,s}, \mathrm{L}_\chi) \otimes \mathrm{H}^{i_2}(\mathfrak{X}_{2,s}, \mathrm{L}_\chi).$$

In fact these are the only obstructions to big monodromy:

Big Monodromy Criterion. *Suppose* $\mathrm{X} := \mathfrak{X}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ *has ample normal bundle, dimension* $d < (g-1)/2$*, and satisfies the numerical assumption* ($\star$)*. Then the following conditions are equivalent:*

- X *is nondivisible, not constant up to translation, not a symmetric power of a curve and not a product.*
- $\mathfrak{X} \to \mathrm{S}$ *has big monodromy for most torsion tuples of characters.*

I will discuss the main ingredients of the proof of the above criterion in chapter 3. Before proceeding with its arithmetic applications note that the numerical condition ($\star$) here is needed in order to avoid the appearance of the exceptional groups E_6 and E_7 and some low-dimensional half-spin groups as monodromy groups. Also notice that when A is simple the preceding theorem is as general as it gets for smooth subvarieties of dimension $d < (g-1)/2$ (save the finite list of exceptions in ($\star$)) because smooth proper subvarieties of a simple abelian variety have ample normal bundle.

Now let me put this to work in the framework the Lawrence-Venkatesh method. To do this let K be a number field and Σ a finite set of places of K containing all infinite ones. The abelian variety in theorem C has good reduction outside Σ and hence it extends uniquely to an abelian scheme over $\mathscr{O}_{\mathrm{K},\Sigma}$. For simplicity I reset notation and write A for this abelian scheme. Let S be a smooth separated finite type scheme over $\mathscr{O}_{\mathrm{K},\Sigma}$ with geometrically connected generic fiber, and $\mathfrak{X} \subset \mathrm{A}_\mathrm{S}$ be a closed subscheme which is smooth over S with geometrically connected fibers of dimension d. We say that $\mathfrak{X} \to \mathrm{S}$ has big monodromy for most tuples of torsion characters if for some (equivalently, any) embedding $\sigma\colon \mathrm{K} \hookrightarrow \mathbb{C}$ the family $\mathfrak{X}_\sigma \to \mathrm{S}_\sigma$ does so. By an induction argument Theorem C is reduced to the following criterion for the nondensity of integral points [3, theorem D]:

THEOREM D. *Suppose that*

- $\mathfrak{X} \to \mathrm{S}$ *has big monodromy for most tuples of torsion characters, and*
- *every geometric fiber* X *of* $\mathfrak{X} \to \mathrm{S}$ *satisfies* ($\star\star$).

Then $\mathrm{S}(\mathcal{O}_{\mathrm{K},\Sigma})$ *is not Zariski-dense in* S.

4. Proof of the non-density criterion

In the rest of this introduction I will outline the main ideas in the proof of theorem D assuming for simplicity that $\mathrm{K} = \mathbb{Q}$ so that $\mathcal{O}_{\mathrm{K},\Sigma} = \mathbb{Z}[1/\mathrm{N}]$ for some integer $\mathrm{N} \geqslant 1$. Pick a prime $p \nmid \mathrm{N}$, let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ and let $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}$. This yields an inclusion

$$\Gamma_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

4.1. Auxiliary construction. As often in Diophantine results, the proof starts by constructing auxiliary objects, for example polynomials vanishing at high order with low height. Here the auxiliary object will be a character $\chi_0 \colon \pi_1(\mathrm{A}(\mathbb{C}), 0) \to \mathbb{C}^\times$ of order r with $p \nmid r$ such that

(1) $\mathrm{H}^i(\mathfrak{X}_s(\mathbb{C}), \mathrm{L}_\chi) = 0$ for all $i \neq d$ and all $\chi \in \Gamma\chi_0$, $s \in \mathrm{S}(\mathbb{C})$,
(2) $\mathrm{V}_{\underline{\chi}}$ has big monodromy for the tuple $\underline{\chi}$ of all characters $\chi \in \Gamma\chi_0$.

Note that torsion characters of $\pi_1^{\text{ét}}(\mathrm{A}(\mathbb{C}), 0)$, such as χ_0, can be seen as torsion points of the dual abelian variety of A on which the Galois group Γ acts naturally. There is not much hope of finding such a character: assumption (2) is totally unrealistic. However bypassing it would bury the main ideas under a mass of technicalities which are insignificant at the end of the day. The orbit $\Gamma\chi_0$ corresponds to a direct summand

$$\mathrm{E} \subset [r]_* \mathbb{Q}_{p,\mathrm{A}}$$

where $[r] \colon \mathrm{A} \to \mathrm{A}$ denotes the multiplication by r and $\mathbb{Q}_{p,\mathrm{A}}$ is the constant p-adic étale sheaf. The first step is exactly as in Faltings's proof of the Shafarevich conjecture for abelian varieties. For each $s \in \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ consider the Galois representation

$$\mathrm{V}_s := \mathrm{H}^d_{\text{ét}}(\mathfrak{X}_s \times \bar{\mathbb{Q}}, \mathrm{E})$$

of Γ. By hypothesis (2) it is of dimension $n = |e|.|\Gamma\chi_0|$ where e is the topological Euler characteristic of the fibers of $\mathfrak{X} \to \mathrm{S}$; by [Del74] it is pure of weight d. As a part of the Tate conjecture, the representation V_s is expected to be semisimple. An absolutely fantastic feature of the Lawrence-Venkatesh method is that it lets one circumvent the semisimplicity assumption. To do this consider the semisimplification filtration $\mathrm{V}_s^\bullet$ on V_s. By Faltings's finiteness of pure global Galois representations, that is the finiteness of the set $\mathcal{R}_{\mathrm{K},\Sigma,p}(n, d)$ in section 1.2, the image of the composite map

$$\begin{array}{ccccc} \mathrm{S}(\mathbb{Z}[1/\mathrm{N}]) & \longrightarrow & \left\{ \begin{array}{c} \text{filtered global Galois} \\ \text{representations} \end{array} \right\} / \cong & \longrightarrow & \left\{ \begin{array}{c} \text{graded global Galois} \\ \text{representations} \end{array} \right\} / \cong \\ s & \longmapsto & (\mathrm{V}_s, \mathrm{V}_s^\bullet) & \longmapsto & \mathrm{gr}\, \mathrm{V}_s^\bullet \end{array}$$

is finite. Hence to show that $\mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ is not Zariski dense in S, it suffices to show that the fibers of the map

$$\mathrm{S}(\mathbb{Z}[1/\mathrm{N}]) \ni s \longmapsto \text{isomorphism class of } \operatorname{gr} \mathrm{V}_s^\bullet$$

are not Zariski dense. A natural attempt would be to restrict representations to the local Galois group at a prime $\ell \neq p$ with $\ell \nmid \mathrm{N}$. To see why this is doomed to fail, fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ and consider the inclusion

$$\Gamma_\ell := \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \;\longleftrightarrow\; \Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

For $s \in \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ let $\mathrm{V}_{s,\ell}$ be the p-adic representation of Γ_ℓ obtained from V_s by restriction. By proper base change in étale cohomology, the isomorphism class of $\mathrm{V}_{s,\ell}$ only depends on the image of s in $\mathrm{S}(\mathbb{F}_\ell)$. In particular the set

$$\{\mathrm{V}_{s,\ell} \mid s \in \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])\} \,/ \cong$$

is finite, so there is no hope of proving nondensity by restricting to the subgroup Γ_ℓ for a single $\ell \neq p$. Instead the heart of the Lawrence-Venkatesh method lies in the insight that the restriction of V_s to Γ_p moves enough when $s \in \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ varies, provided that the family $\mathfrak{X} \to \mathrm{S}$ has big monodromy.

4.2. The role of p-adic Hodge theory. To make this precise Lawrence and Venkatesh suggest to apply p-adic Hodge theory: without entering in too much detail, let me simply say that it associates to every finite-dimensional *crystalline* representation V of Γ_p a triple

$$\mathrm{D_{cris}}(\mathrm{V}) := (\mathrm{V_{dR}}, \varphi_\mathrm{V}, h^\bullet \mathrm{V})$$

of a finite-dimensional $\mathbb{Q}_p$-vector space $\mathrm{V_{dR}}$, an endomorphism $\varphi_\mathrm{V} \in \mathrm{End}_{\mathbb{Q}_p}(\mathrm{V_{dR}})$ and a filtration $h^\bullet \mathrm{V_{dR}}$ of $\mathrm{V_{dR}}$ by vector subspaces (not necessarily stable under φ_V). Such triples are called filtered isocristals in the literature, but in this introduction we will informally call them p-adic Hodge structures.[2] Rather than defining the functor $\mathrm{D_{cris}}$ let me only mention what it does to the representations in question. To do so consider the vector bundle $[r]_*\mathcal{O}_\mathrm{A}$ with the connection ∇ induced by the canonical derivation on $\mathcal{O}_\mathrm{A}$. There is a well-defined direct summand

$$\mathcal{E} \subset [r]_*\mathcal{O}_{\mathrm{A}_{\mathbb{Z}_p}}$$

associated to the local system E and stable under ∇. For $s \in \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ the representation V_s is crystalline since $\mathfrak{X}_s$ has good reduction at p. Then Faltings's étale-de Rham comparison theorem [Fal89, theorem 5.6], which should be thought as de Rham theorem over the p-adic numbers, gives

$$\mathrm{D_{cris}}(\mathrm{V}_s) \;=\; (\mathrm{H}^d_{\mathrm{dR}}(\mathfrak{X}_s \times \mathbb{Q}_p, \mathcal{E}),\; \varphi_s,\; \text{Hodge filtration})$$

where φ_s is the endomorphism induced by the Frobenius operator on crystalline cohomology via the de Rham-crystalline comparison theorem. This should give the idea of how to prove that the representations V_s move. In a sense to be made precise below, one identifies the de Rham cohomology groups for s and s' close enough by integrating the Gauss-Manin connection. It then remains to show that the Hodge filtration moves

[2] The name 'filtered isocrystal' is unfortunate as the filtration is not made by subisocrystals but mere vector subspaces. Moreover, we will consider filtrations on filtered isocrystals made by filtered subisocrystals: the name 'bifiltered isocrystals' would be as confusing as imprecise.

enough, i.e. the associated period mapping has large enough image, which will be implied by the hypothesis of big monodromy.

Note that the semisimplication filtration $\mathrm{V}_s^\bullet$ on V_s induces a filtration $\mathrm{D}_{\mathrm{cris}}(\mathrm{V}^\bullet)$ on $\mathrm{D}_{\mathrm{cris}}(\mathrm{V}_s)$ which is stable under the crystalline Frobenius. This operation is compatible with taking the associated graded, leading to the following commutative diagram:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{filtered global}\\ \text{Galois representations}\\ \text{crystalline at } p\end{array}\right\} \arrow[r, "\text{graded}"] \arrow[d, "\text{restrict to } \Gamma_p"'] & \left\{\begin{array}{c}\text{graded global}\\ \text{Galois representations}\\ \text{crystalline at } p\end{array}\right\} \arrow[d, "\text{restrict to } \Gamma_p"] \\
\left\{\begin{array}{c}\text{filtered crystalline}\\ \text{local Galois representations}\end{array}\right\} \arrow[r, "\text{graded}"] \arrow[d, "\mathrm{D}_{\mathrm{cris}}"'] & \left\{\begin{array}{c}\text{graded crystalline}\\ \text{local Galois representations}\end{array}\right\} \arrow[d, "\mathrm{D}_{\mathrm{cris}}"] \\
\left\{\begin{array}{c}\text{filtered } p\text{-adic}\\ \text{Hodge structures}\end{array}\right\} \arrow[r, "\text{graded}"] & \left\{\begin{array}{c}\text{graded } p\text{-adic}\\ \text{Hodge structures}\end{array}\right\}
\end{tikzcd}$$

To prove that $\mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ is not Zariski dense, it will be enough to show that the fibers of the map

$$\mathrm{S}(\mathbb{Z}[1/\mathrm{N}]) \ni s \longmapsto \text{isomorphism class of } \operatorname{gr} \mathrm{D}_{\mathrm{cris}}(\mathrm{V}_s^\bullet)$$

are not dense.

4.3. Parallel transport. Consider the d-th relative de Rham cohomology group

$$\mathscr{V} \;:=\; \mathscr{H}^d_{\mathrm{dR}}(\mathfrak{X}_{\mathbb{Z}_p}/\mathrm{S}_{\mathbb{Z}_p}; \pi^*(\mathscr{E}, \nabla))$$

where $\pi\colon \mathfrak{X} \to \mathrm{A}$ is the projection. This vector bundle $\mathscr{V}$ comes equipped with the Gauss-Manin connection that I still denote ∇. Fix a point $o \in \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ and consider the residue disk

$$\Omega = \{s \in \mathrm{S}(\mathbb{Z}_p) \mid s \equiv o \mod p\}.$$

Since $\mathrm{S}(\mathbb{F}_p)$ is finite, it suffices to show that the intersection $\Omega \cap \mathrm{S}(\mathbb{Z}[1/\mathrm{N}])$ is not Zariski dense for any such residue disk. Working on Ω has the advantage that the Gauss-Manin connection can be integrated on Ω: there is a natural isomorphism of K-vector spaces

$$f_s\colon \mathscr{V}_s = \mathrm{H}^d_{\mathrm{dR}}(\mathfrak{X}_s \times \mathbb{Q}_p, \mathscr{E}) \xrightarrow{\sim} \mathscr{V}_o = \mathrm{H}^d_{\mathrm{dR}}(\mathfrak{X}_o \times \mathbb{Q}_p, \mathscr{E}).$$

The filtration on $\mathscr{V}_o$ induced by the Hodge filtration on $\mathscr{V}_s$ via f_s is by definition the image of s via the p-adic period mapping

$$\Phi_p\colon \quad \Omega \longrightarrow \operatorname{Flag}_t(\mathscr{V}_o)$$

where $\operatorname{Flag}_t(\mathscr{V}_o)$ denotes the variety of flags on $\mathscr{V}_o$ whose type t is the type of the flag underlying the Hodge filtration. Define an equivalence relation on Ω by putting

$$s \sim s' \quad \text{if} \quad \operatorname{gr} \mathrm{D}_{\mathrm{cris}}(\mathrm{V}_s^\bullet) \cong \operatorname{gr} \mathrm{D}_{\mathrm{cris}}(\mathrm{V}_{s'}^\bullet).$$

The equivalence classes then give rise to certain constructible subsets $\mathrm{Z} \subset \operatorname{Flag}_t(\mathscr{V}_o)$, and we win as soon as we show that the preimage $\Phi_p^{-1}(\mathrm{Z})$ of each of these subsets is not Zariski dense in Ω. The Ax-Lindemann property of period mappings proved by

Bakker-Tsimerman [**BT19**], or rather its p-adic version given in [**3**, §7] following [**LV20**, lemma 9.3], says that for this it will be enough to show that

- Φ_p has a Zariski dense image in $\mathrm{Flag}_t(\mathscr{V}_o)$, and
- $\dim \mathrm{Z} + \dim \mathrm{S}_{\mathbb{Q}} \leqslant \dim \mathrm{Flag}_t(\mathscr{V}_o)$.

It is time to pay for the careless job done so far. The period mapping Φ_p will *never* have a Zariski dense image in $\mathrm{Flag}_t(\mathscr{V}_o)$, for the simple reason that I forgot all the extra structures that $\mathscr{E}$ comes with. Indeed, over a finite extension K of $\mathbb{Q}_p$ the vector bundle $\mathscr{E}$ splits into a direct sum of line bundles

$$\mathscr{E}_{|\mathrm{A}_\mathrm{K}} = \bigoplus_{\chi\in\Gamma\chi_0} \mathscr{L}_\chi$$

and the connection decomposes accordingly. So over K the image of Φ_p is contained in

$$\mathfrak{H} := \prod_{\chi\in\Gamma\chi_0} \mathrm{Flag}_{t_\chi}(\mathrm{H}^d_{\mathrm{dR}}(\mathfrak{X}_s \times \mathrm{K}, \mathscr{L}_\chi)).$$

Suppose to start the argument above all over again with all the extra structures in place. The Ax-Lindemann property for period mappings implies that it is enough to show that

- Φ_p has a Zariski dense image in $\mathfrak{H}$, and
- $\dim \mathrm{Z} + \dim \mathrm{S}_{\mathbb{Q}} \leqslant \dim \mathfrak{H}$,

for each of the equivalence classes $\mathrm{Z} \subset \mathfrak{H}$ mentioned above. The Zariski-density in $\mathfrak{H}$ of the image of Φ_p is now a consequence of the assumption that the family $\mathfrak{X} \to \mathrm{S}$ has big monodromy with respect to the tuple of consisting of the characters in the orbit $\Gamma\chi_0$. It remains to bound the dimension of the equivalence classes Z mentioned above. A generalization of [**LV20**, proposition 10.6] yields such an upper bound in terms of a combinatorial function and of the dimension of the centralizer of the crystalline Frobenius φ_0. In view of this, to have the needed inequality

$$\dim \mathrm{Z} + \dim \mathrm{S}_{\mathbb{Q}} \leqslant \dim \mathrm{Flag}_t(\mathscr{V}_o)$$

the centralizer of φ_0 must be very small compared to $\dim \mathfrak{H}$. This is achieved by letting r go to infinity.

CHAPTER 3

Sheaf convolution on abelian varieties

In this chapter I introduce the main ingredients entering the proof of the Big Monodromy criterion in section 3.5 of chapter 2. This constitutes the bulk of the paper [**2**] in collaboration with A. Javanpeykar, T. Krämer and C. Lehn. One would expect to see topology at work in such a proof, like in [**Bea86**] or in [**LV20**, §9], but the reader willing to see loops appearing will be utmost deceived. The strategy followed here, inspired from [**LS20**], will be to compare the monodromy group with another group, in a spirit similar to the relation between monodromy and the generic Mumford-Tate groups in a variation of Hodge structures. The 'other group', arguably outlandish, will be the Tannaka group for the convolution product of perverse sheaves on an abelian variety. I will therefore begin with a brief introduction to perverse sheaves.

Let k be a field of characteristic 0 and ℓ a prime number.

1. Perverse sheaves on abelian varieties

1.1. Perverse sheaves. A $\bar{\mathbb{Q}}_\ell$-sheaf F (for the étale topology) on a variety X over k is *constructible* if there is a partition of X on locally closed subsets on which F is locally constant.[1] The category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on X is $\bar{\mathbb{Q}}_\ell$-linear and abelian, hence it makes sense to consider the derived category $\mathrm{D}^b_c(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$ made of bounded complexes of $\bar{\mathbb{Q}}_\ell$-sheaves with constructible cohomology. The category $\mathrm{D}^b_c(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$ is not abelian. However, inspired by the work of Goresky and MacPherson on the intersection homology of a singular topological space, Beilinson, Bernstein, Deligne and Gabber [**BBD82**] discovered an abelian subcategory

$$\mathrm{Perv}(\mathrm{X}, \bar{\mathbb{Q}}_\ell) \subset \mathrm{D}^b_c(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$$

whose objects are called *perverse sheaves*. To define them, say first that $\mathrm{P} \in \mathrm{D}^b_c(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$ is *semiperverse* if its cohomology sheaves $\mathscr{H}^i(\mathrm{P})$ satisfy

$$\dim \operatorname{Supp} \mathscr{H}^i(\mathrm{P}) \leqslant -i \qquad \text{for all } i \in \mathbb{Z}.$$

Then P is called a *perverse sheaf* if P and its Verdier dual are semiperverse.

Example. If X is smooth of dimension d and L is an ℓ-adic local system on X, then $\mathrm{L}[d]$ is a perverse sheaf: $\mathrm{L}[d]$ is semiperverse because it is concentrated in degree $-d$ and so is its Verdier dual $\mathrm{R}\mathscr{H}om(\mathrm{L}[d], \bar{\mathbb{Q}}_\ell(d)[2d]) = \mathrm{L}^\vee(d)[d]$ being of the same shape.

[1] Strictly speaking, one should write lisse instead of locally constant, or locally constant for the pro-étale topology.

The category $\mathrm{Perv}(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$ is abelian and $\bar{\mathbb{Q}}_\ell$-linear. Moreover any object is an iterated extension of finitely many simple objects. For instance, given a d-dimensional smooth subvariety $\mathrm{Y} \subset \mathrm{X}$, the perverse sheaf

$$\delta_{\mathrm{Y}} := i_* \bar{\mathbb{Q}}_\ell[d]$$

is simple, where $i \colon \mathrm{Y} \hookrightarrow \mathrm{X}$ is the closed immersion.

1.2. Convolution of perverse sheaves. Let A be an abelian variety over k and let me apply the above discussion with $\mathrm{X} = \mathrm{A}$. The sum morphism $\sigma \colon \mathrm{A} \times \mathrm{A} \to \mathrm{A}$ induces a convolution product

$$* : \mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \times \mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \longrightarrow \mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell), \quad \mathrm{P}_1 * \mathrm{P}_2 := \mathrm{R}\sigma_* (\mathrm{P}_1 \boxtimes \mathrm{P}_2)$$

Unfortunately the subcategory of perverse sheaves is not stable under the convolution product, as the next examples shows:

Example. Suppose A is an elliptic curve. As recalled above $\mathrm{P} = \bar{\mathbb{Q}}_\ell[1]$ is a perverse sheaf but $\mathrm{P} * \mathrm{P}$ is not: indeed $\mathscr{H}^i(\mathrm{P} * \mathrm{P}) = 0$ for $|i+1| \geqslant 2$ and $\mathscr{H}^i(\mathrm{P} * \mathrm{P})$ is the constant sheaf of A of value $\mathrm{H}^{i+2}(\mathrm{E}, \bar{\mathbb{Q}}_\ell)$ for $i = 0, -1, -2$. In particular,

$$\operatorname{Supp} \mathscr{H}^0(\mathrm{P} * \mathrm{P}) = 1 \ > \ 0.$$

Roughly speaking this is the only issue that may occur. To explain this, recall that for any perverse sheaf P on A,

$$\chi(\mathrm{A}, \mathrm{P}) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\bar{\mathbb{Q}}_\ell} \mathrm{H}^i(\mathrm{A}, \mathrm{P}) \ \geqslant \ 0.$$

Indeed, over $k = \mathbb{C}$ this was observed by Franecki and Kapranov [FK00, corollary 1.4]; the general case can be reduced to the complex case by choosing a model over some algebraically closed subfield of k which embeds into the complex numbers. The additivity of the Euler characteristic in short exact sequences then implies that perverse sheaves of Euler characteristic zero, such as $\mathrm{P} * \mathrm{P}$ in the above example, form a Serre subcategory

$$\mathrm{S}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) := \{\mathrm{P} \in \mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \mid \chi(\mathrm{A}, \mathrm{P}) = 0\} \ \subset \ \mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$$

inside the abelian category of perverse sheaves. Let $\mathrm{T}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \subset \mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ be the full subcategory of sheaf complexes whose perverse cohomology sheaves[2] are in $\mathrm{S}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$; its objects will be called *negligible sheaf complexes*. With this notation the natural functor

$$\overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) := \mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)/\mathrm{S}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \longhookrightarrow \overline{\mathrm{D}^b_c}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) := \mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell)/\mathrm{T}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$$

is faithful. The convolution product descends to the quotient $\overline{\mathrm{D}^b_c}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ and preserves the subcategory $\overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$. With respect to this product

$$* : \overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \times \overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \longrightarrow \overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell),$$

the category $\overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ is a Tannaka category. The internal characterization of Tannaka categories furnishes us with a *fiber functor*

$$\omega \colon \quad \mathscr{C} := \overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \ \longrightarrow \ \mathrm{Vect}(\bar{\mathbb{Q}}_\ell).$$

[2]The derived category of $\mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ turns out to be equivalent to $\mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$. This furnishes us with a perverse cohomological functor ${}^p\mathscr{H}^i \colon \mathrm{D}^b_c(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \to \mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ for each $i \in \mathbb{Z}$. For instance, a complex P is perverse if and only if ${}^p\mathscr{H}^i(\mathrm{P}) = 0$ for all $i \neq 0$.

There is no canonical choice of such a fiber functor, but any two fiber functors are noncanonically isomorphic [DMOS82, theorem 3.2.(b)]. The choice of a fiber functor such as ω induces an equivalence between $\mathscr{C}$ and the category $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\mathrm{G}_\omega(\mathscr{C}))$ of finite-dimensional algebraic representations of the affine group scheme

$$\mathrm{G}_\omega(\mathscr{C}) \ := \ \mathrm{Aut}^{\otimes}(\omega)$$

over $\bar{\mathbb{Q}}_\ell$ called the *Tannaka group* of $\mathscr{C}$. The group scheme $\mathrm{G}_\omega(\mathscr{C})$ is proalgebraic as it is obtained as the limit, over all perverse sheaves P on A, of

$$\mathrm{G}_\omega(\mathrm{P}) \ := \ \mathrm{Im}\big(\mathrm{G}_\omega(\mathscr{C}) \to \mathrm{GL}(\omega(\mathrm{P}))\big) \ \subset \ \mathrm{GL}(\omega(\mathrm{P})).$$

The *dimension* of an object $\mathrm{P} \in \mathscr{C}$ is $\dim_{\bar{\mathbb{Q}}_\ell} \omega(\mathrm{P})$ which does not depend on the chosen fiber functor. For a smooth closed d-dimesional subvariety $i \colon \mathrm{X} \hookrightarrow \mathrm{A}$ the perverse sheaf

$$\delta_{\mathrm{X}} \ := \ i_* \bar{\mathbb{Q}}_\ell[d]$$

has dimension $(-1)^d \chi_{\mathrm{top}}(\mathrm{X})$ where $\chi_{\mathrm{top}}(\mathrm{X})$ is the topological Euler characteristic of X.

Definition. The *Tannaka group* of a smooth subvariety $\mathrm{X} \subset \mathrm{A}$ is the algebraic group

$$\mathrm{G}_{\mathrm{X},\omega} := \mathrm{G}_\omega(\delta_{\mathrm{X}}) \ \subset \ \mathrm{GL}(\omega(\delta_{\mathrm{X}})).$$

The most interesting piece of the Tannaka group is the derived subgroup of its connected component

$$\mathrm{G}^*_{\mathrm{X},\omega} \ := \ [\mathrm{G}^\circ_{\mathrm{X},\omega}, \mathrm{G}^\circ_{\mathrm{X},\omega}].$$

Example. Suppose k algebraically closed and let $x \in \mathrm{A}(k)$. Then,

$$\mathrm{G}_{\{x\},\omega} = \begin{cases} \mu_r & \text{if } x \text{ has finite order } r, \\ \mathbb{G}_m & \text{if } x \text{ has infinite order,} \end{cases}$$

because $\delta_{\{x\}}$ has dimension 1 and $(\delta_{\{x\}})^{*n} = \delta_{\{nx\}}$ for each $n \geqslant 1$.

Example. Let C be a smooth projective curve of genus $g \geqslant 2$ embedded in its Jacobian $\mathrm{A} = \mathrm{Jac}(\mathrm{C})$. Then,

$$\mathrm{G}^*_{\mathrm{X},\omega} = \begin{cases} \mathrm{Sp}_{2g-2} & \text{if C is hyperelliptic,} \\ \mathrm{SL}_{2g-2} & \text{otherwise.} \end{cases}$$

acting on its standard representation; see [KW15b, theorem 6.1].

Example. Suppose $k = \mathbb{C}$ and consider a smooth cubic threefold $\mathrm{X} \subset \mathbb{P}^4_{\mathbb{C}}$. Its intermediate Jacobian A is remarkably an abelian variety, as opposed to a mere complex torus. This is seen identifying A with the Albanese variety of the Fano surface S of lines on X [CG72]. In this case, the topological Euler characteristic of S is $\chi_{\mathrm{top}}(\mathrm{S}) = 27$, the Tannaka group of $\mathrm{S} \subset \mathrm{A}$ is the exceptional group

$$\mathrm{G}^*_{\mathrm{S},\omega} \ \cong \ \mathrm{E}_6$$

and $\omega(\delta_{\mathrm{S}})$ is the 27-dimensional irreducible representation of E_6; see [Krä16].

It is often convenient to have an explicit fiber functor at hand, but it is not clear how to define such a functor on the whole category $\mathscr{C}$. To circumvent this, let me introduce a subcategory $\mathscr{C}_0 \subset \mathscr{C}$. Let $\bar{k}$ be an algebraic closure of k and let $\mathrm{Perv}_0(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ be the full subcategory of $\mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ made of those perverse sheaves P for which all simple subquotients Q of $\mathrm{P}_{\bar{k}}$ satisfy

$$\mathrm{H}^i(\mathrm{A}_{\bar{k}}, \mathrm{Q}) \;=\; 0 \quad \text{for all} \quad i \neq 0.$$

Consider its image

$$\mathscr{C}_0 \;:=\; \overline{\mathrm{Perv}}_0(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \;\subset\; \mathscr{C} \;:=\; \overline{\mathrm{Perv}}(\mathrm{A}, \bar{\mathbb{Q}}_\ell).$$

With this notation, the functor

$$\omega\colon \quad \mathscr{C}_0 \longrightarrow \mathrm{Vect}(\bar{\mathbb{Q}}_\ell), \quad \mathrm{P} \longmapsto \mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{P})$$

is seen to be a fiber functor on $\mathscr{C}_0$. Let $\Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell) = \mathrm{Hom}(\pi_1(\mathrm{A}_{\bar{k}}, 0), \bar{\mathbb{Q}}_\ell^\times)$ be the group of continuous characters of the geometric étale fundamental group of the abelian variety A. For a character $\chi \in \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ let L_χ be the local system of rank 1 with monodromy representation given by χ. For $\mathrm{P} \in \mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ the perverse sheaf

$$\mathrm{P}_\chi \;:=\; \mathrm{P} \otimes_{\bar{\mathbb{Q}}_\ell} \mathrm{L}_\chi \in \mathrm{Perv}(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$$

is called the *twist* of P by χ. With this notation the generic vanishing theorem says that there is a finite union $\mathcal{S}(\mathrm{P}) \subset \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ of translates of linear subvarieties (in the sense of section 3.5 in chapter 2) such that

$$\mathrm{H}^i(\mathrm{A}, \mathrm{P}_\chi) \;=\; 0 \quad \text{for all } i \neq 0 \text{ and all } \chi \in \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell) \smallsetminus \mathcal{S}(\mathrm{P}).$$

Example. Suppose $\mathrm{P} = \delta_\mathrm{X} := \bar{\mathbb{Q}}_\ell[d]$ for a a d-dimensional smooth subvariety $\mathrm{X} \subset \mathrm{A}$. Then $\mathcal{S}(\mathrm{P})$ can be taken to be a finite union of *torsion* translates of linear subvarieties. That is, with the terminology of section 3.5 in chapter 2, the vanishing

$$\mathrm{H}^i(\mathrm{X}, \mathrm{L}_\chi) = 0 \quad \text{for all} \quad i \neq d$$

holds for *most* characters $\chi \in \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$.

For a perverse sheaf P on A the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ acts continuously on the $\bar{\mathbb{Q}}_\ell$-vector space $\mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{P})$ in a natural way. It is absolutely crucial that this action preserves the Tannaka group [2, lemma 4.5]:

Proposition. *Let* P *be a perverse sheaf on* A *such that* $\mathrm{H}^i(\mathrm{A}_{\bar{k}}, \mathrm{Q}) = 0$ *for all simple subquotients* Q *of* $\mathrm{P}_{\bar{k}}$ *and* $i \neq 0$. *Then the natural action of* $\mathrm{Gal}(\bar{k}/k)$ *on* $\mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{P})$ *normalizes the Tannaka group of the perverse sheaf* $\mathrm{P}_{\bar{k}}$ *on* $\mathrm{A}_{\bar{k}}$ *with respect to the fiber functor* $\mathrm{Q} \mapsto \mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{Q})$.

The previous statement is a consequence of a short exact sequence relating the Tannaka group of P and that of $\mathrm{P}_{\bar{k}}$; see [LS20, lemma 3.7] and [2, theorem 4.3] which are both based on [DE21]. This short exact sequence is reminiscent of the geometric-arithmetic short exact sequence of étale fundamental groups

$$1 \longrightarrow \pi_1^{\text{ét}}(\bar{\mathrm{X}}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(\mathrm{X}, \bar{x}) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1,$$

for a geometrically connected variety X, $\bar{\mathrm{X}} = \mathrm{X} \times_k \bar{k}$ and $\bar{x}$ a geometric point of X.

Remark. Convolution of perverse sheaves can be generalized to all commutative algebraic groups, although extra care is needed as the sum morphism is not any longer a proper morphism. Over finite fields this led to ground-breaking equidistribution results such as those in [**Kat12**] for the additive group and [**FFK21**] in general.

1.3. Big monodromy from big Tannaka groups. In this section I explain how to reduce the proof of the Big Monodromy criterion in section 3.5 of chapter 2 to computing some Tannaka group. With the notation introduced therein I am only going to treat the case $\Lambda = \bar{\mathbb{Q}}_\ell$: since characters in the statement of the Big Monodromy criterion are all torsion, the case $\Lambda = \mathbb{C}$ is deduced from the case $\Lambda = \bar{\mathbb{Q}}_\ell$ by the comparison of the étale topology with the usual complex topology. So let A be a complex abelian variety and S a connected smooth complex variety. Inside the constant abelian scheme $\mathrm{A_S} := \mathrm{A} \times_k \mathrm{S}$ let $\mathfrak{X} \subset \mathrm{A_S}$ be an irreducible closed subvariety which is smooth over S:

$$\begin{tikzcd}
& \mathfrak{X} \arrow[dl, bend right, "\pi"'] \arrow[dr, bend left, "f"] \arrow[d, equal] & \\
\mathrm{A} & \mathrm{A_S} \arrow[l, "\mathrm{pr_A}"'] \arrow[r, "\mathrm{pr_S}"] & \mathrm{S}
\end{tikzcd}$$

Let $\eta \in \mathrm{S}$ be the generic point and $\bar{k}$ an algebraic closure of the function field $k = \kappa(\eta)$ of S. Write

$$\bar{\eta} \colon \operatorname{Spec} \bar{k} \longrightarrow \mathrm{S}$$

for the so-defined geometric generic point. Recall that the fiber $\mathfrak{X}_{\bar{\eta}}$ of $\mathfrak{X}$ over $\bar{\eta}$ is constant up to translation if there is a subvariety $\mathrm{Y} \subset \mathrm{A}$ and a point $a \in \mathrm{A}(\bar{\eta})$ such that $\mathfrak{X}_{\bar{\eta}} = \mathrm{Y} + a$. When $\mathfrak{X}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ is nondivisible, $\mathfrak{X}_{\bar{\eta}}$ being constant up to translation is equivalent to the family $\mathfrak{X} \to \mathrm{S}$ being isotrivial.

By the generic vanishing theorem [**BSS18**, **KW15c**, **Sch15**] for most characters $\chi \in \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ the higher direct images $\mathrm{R}^i f_* \pi^* \mathrm{L}_\chi$ vanish in all degrees $i \neq d$, where d denotes the relative dimension of the family $f \colon \mathfrak{X} \to \mathrm{S}$. For such χ consider the local system

$$\mathrm{V}_\chi \ := \ \mathrm{R}^d f_* \pi^* \mathrm{L}_\chi.$$

More generally for an n-tuple of characters $\underline{\chi} = (\chi_1, \dots, \chi_n)$ write $\mathrm{V}_{\underline{\chi}} := \mathrm{V}_{\chi_1} \oplus \cdots \oplus \mathrm{V}_{\chi_n}$ and let $\rho \colon \pi_1(\mathrm{S}, \bar{\eta}) \to \mathrm{GL}(\mathrm{V}_{\underline{\chi},\bar{\eta}})$ be the corresponding monodromy representation on the geometric generic fiber. The *algebraic monodromy group* of $\mathrm{V}_{\underline{\chi}}$ is defined as the Zariski closure

$$\mathrm{M}(\mathrm{V}_{\underline{\chi}}) \ := \ \overline{\mathrm{Im}(\rho)} \ \subset \ \mathrm{GL}(\mathrm{V}_{\underline{\chi},\bar{\eta}}).$$

The key remark is to relate this to the Tannaka group of the geometric generic fiber

$$\mathrm{X} \ := \ \mathfrak{X}_{\bar{\eta}}$$

seen as a smooth subvariety of the abelian variety $\mathrm{A}_{\bar{k}}$. Let $\delta_\mathrm{X} = i_* \bar{\mathbb{Q}}_\ell[d]$ be the constant sheaf on X shifted in degree $-d$ where $i \colon \mathrm{X} \hookrightarrow \mathrm{A}_{\bar{k}}$ is the inclusion. With this notation, given χ such that $\mathrm{R}^i f_* \pi^* \mathrm{L}_\chi = 0$ for $i \neq d$,

$$\mathrm{V}_{\chi,\bar{\eta}} \ = \ \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\chi) \ = \ \mathrm{H}^0(\mathrm{A}_{\bar{k}}, \delta_{\mathrm{X},\chi}).$$

Moreover the composite map $\mathrm{Gal}(\bar{k}/k) \twoheadrightarrow \pi_1(\mathrm{S}, \bar{\eta}) \to \mathrm{GL}(\mathrm{V}_{\chi,\bar{\eta}})$ simply defines the natural linear action of $\mathrm{Gal}(\bar{k}/k)$ on $\mathrm{V}_{\chi,\bar{\eta}} = \mathrm{H}^0(\mathrm{A}_{\bar{k}}, \delta_{\mathrm{X},\chi})$. In particular, the discussion in section 1.2 then shows that the algebraic monodromy group $\mathrm{M}(\mathrm{V}_\chi)$ normalizes the Tannaka group

$$\mathrm{G}_{\mathrm{X},\chi} \subset \mathrm{GL}(\mathrm{V}_{\chi,\bar{\eta}}).$$

Definition. Say that X has a *simple derived connected Tannaka group* if $\mathrm{G}^*_{\mathrm{X},\chi}$ is simple for some (hence every) character χ with the above vanishing properties.

The main statement is the following relation between the algebraic monodromy group and the above Tannaka groups; see [**LS20**, theorem 5.6] and [**2**, theorem 4.10]. It is reminiscent of the relation between the algebraic monodromy group and the generic Mumford-Tate group for a variation of Hodge structures:

Theorem A. *Let* S *be a smooth integral variety over* k*, and let* $\mathfrak{X} \subset \mathrm{A}_\mathrm{S}$ *an integral subvariety such that*

(1) the family $f\colon \mathfrak{X} \to \mathrm{S}$ *is smooth of relative dimension* d*, it is not constant up to translation in* $\mathrm{A}(\mathrm{S})$*, and*

(2) the geometric generic fiber $\mathrm{X} = \mathfrak{X}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ *is nondivisible and has a simple derived connected Tannaka group.*

Then, for most $\underline{\chi} \in \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell)^n$*,*

$$\mathrm{G}^*_{\mathrm{X},\chi_1} \times \cdots \times \mathrm{G}^*_{\mathrm{X},\chi_n} \ \trianglelefteq\ \mathrm{M}(\mathrm{V}_{\underline{\chi}}).$$

Theorem A therefore reduces the proof of the Big Monodromy criterion in section 3.5 to computing the Tannaka group of X. This will be the main task of the upcoming section.

2. Computation of Tannaka groups

Let k be an algebraically closed field of characteristic 0, A an abelian variety of dimension g over k and $\mathrm{X} \subset \mathrm{A}$ a d-dimensional smooth connected closed subvariety.

2.1. Big Tannaka groups. Consider the perverse sheaf

$$\delta_\mathrm{X} \ :=\ i_* \bar{\mathbb{Q}}_\ell[d]$$

and fix a character $\chi \in \Pi(\mathrm{A}, \bar{\mathbb{Q}}_\ell)$ with $\mathrm{H}^i(\mathrm{X}, \mathrm{L}_\chi) = 0$ for all $i \neq d$. As discussed above such a character furnishes us with a fiber functor

$$\omega\colon \quad \langle \delta_\mathrm{X} \rangle \longrightarrow \mathrm{Vect}(\bar{\mathbb{Q}}_\ell), \quad \mathrm{P} \longmapsto \mathrm{H}^0(\mathrm{A}, \mathrm{P} \otimes \mathrm{L}_\chi),$$

where $\langle \delta_\mathrm{X} \rangle \subset \mathscr{C}$ is the abelian tensor subcategory generated by δ_X. In particular for the perverse sheaf $\mathrm{P} = \delta_\mathrm{X}$ the value of such fiber functor is

$$\mathrm{V} := \omega(\delta_\mathrm{X}) = \mathrm{H}^0(\mathrm{A}, \delta_\mathrm{X} \otimes \mathrm{L}_\chi) = \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\chi).$$

When $\mathrm{X} \subset \mathrm{A}$ is symmetric up to translation in the sense of section 3.3 of chapter 2, then V comes with a natural symmetric bilinear form θ which is induced by Poincaré duality. This bilinear form is symmetric or alternating depending on the parity of d, and it is preserved by the action of the group $\mathrm{G}_{\mathrm{X},\chi}$ as in [**KW15a**, lemma 2.1]. Let $\mathrm{G}^\circ_{\mathrm{X},\chi} \subset \mathrm{G}_{\mathrm{X},\chi}$ be the connected component of the identity and

$$\mathrm{G}^*_{\mathrm{X},\omega} := [\mathrm{G}^\circ_{\mathrm{X},\chi}, \mathrm{G}^\circ_{\mathrm{X},\chi}]$$

its derived group, which is a connected semisimple algebraic group.

Definition. Say that the Tannaka group $\mathrm{G}_{\mathrm{X},\chi}$ of X is *big* if the derived group of its connected component of the identity is

$$\mathrm{G}^*_{\mathrm{X},\chi} = \begin{cases} \mathrm{SL}(\mathrm{V}) & \text{if X is not symmetric up to translation,} \\ \mathrm{SO}(\mathrm{V},\theta) & \text{if X is symmetric up to translation and } d \text{ is even,} \\ \mathrm{Sp}(\mathrm{V},\theta) & \text{if X is symmetric up to translation and } d \text{ is odd.} \end{cases}$$

The main theorem of this section is the following computation of Tannaka groups.

Main theorem. *Suppose* $\mathrm{X} \subset \mathrm{A}$ *has ample normal bundle, dimension* $d < (g-1)/2$, *and* $(\star)$ *in section 3.3 of chapter 2 holds. Then the following are equivalent:*

(1) X *is nondivisible, not a symmetric power of a curve and not a product;*

(2) The Tannaka group $\mathrm{G}_{\mathrm{X},\chi}$ *is big.*

Thanks to the discussion in section 1.3 the main theorem above implies the Big Monodromy criterion in section 3.5 of chapter 2. As we will see in a moment the numerical condition $(\star)$ prevents the occurence as Tannaka group of the exceptional groups E_6 and E_7, and of some low-dimensional spin representations. Similarly to the Big Monodromy criterion, the preceding statement is substantially sharp for simple abelian varieties and can be applied in the following special cases:

Corollary. *Suppose* $\mathrm{X} \subset \mathrm{A}$ *is nondivisible and one of the following holds:*

(1) X *is a curve generating* A *and* $g \geqslant 4$;

(2) X *is a surface with ample normal bundle which is neither a product nor the symmetric square of a curve, and* $e \neq 27$, $g \geqslant 6$;

(3) X *is a complete intersection of ample divisors and* $d < (g-1)/2$.

Then the Tannaka group $\mathrm{G}_{\mathrm{X},\chi}$ *is big.*

When X is a complete intersection of ample divisors, then $|e| \neq 27, 56$ holds automatically, and X is not a symmetric power of a curve nor a product. Before describing the ingredients of the proof of the main theorem, let me mention the following observation to illustrate the information captured by Tannaka groups:

Fact. *Suppose that* $\mathrm{X} \subset \mathrm{A}$ *has ample normal bundle and* $d < g/2$. *If* $\mathrm{G}_{\mathrm{X},\chi}$ *is big, then the sum morphism* $\mathrm{Sym}^2(\mathrm{X}) \to \mathrm{X} + \mathrm{X}$ *is birational.*

This follows from the observation that the direct image of the constant sheaf under the sum morphism is related to the decomposition of $\mathrm{V} \otimes \mathrm{V} \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\mathrm{G}_{\mathrm{X},\chi})$. In fact, Larsen's alternative yields a necessary and sufficient criterion for the Tannaka group to be big, using only the decomposition of the direct image of the constant sheaf under the sum morphism; see [2, corollary 3.8]. However, it seems hard to control this direct image in the generality needed for the main theorem, so the proof of the main theorem follows a different route that will be described in the next section.

2.2. Sketch of the proof. The first step in the proof of the main theorem from the previous section is to show that under the given assumptions, the algebraic group $\mathrm{G}^*_{\mathrm{X},\chi}$ is simple [2, theorem A]:

THEOREM B. *Suppose* $\mathrm{X} \subset \mathrm{A}$ *has ample normal bundle and is nondivisible. Then for* $g \geqslant 3$ *the following are equivalent:*

(1) The algebraic group $\mathrm{G}^*_{\mathrm{X},\chi}$ *is not simple;*
(2) There are smooth positive-dimensional subvarieties $\mathrm{X}_1, \mathrm{X}_2 \subset \mathrm{A}$ *such that the sum morphism induces an isomorphism*

$$\mathrm{X}_1 \times \mathrm{X}_2 \xrightarrow{\sim} \mathrm{X}.$$

A smooth projective curve $\mathrm{C} \subset \mathrm{A}$ generating A has ample normal bundle, thus the algebraic group $\mathrm{G}^*_{\mathrm{C},\chi}$ is simple for $g \geqslant 3$. When $g = 2$ the simplicity of $\mathrm{G}^*_{\mathrm{C},\chi}$ remains open. More generally theorem B implies that $\mathrm{G}^*_{\mathrm{X},\chi}$ is simple when $\mathrm{X} \subset \mathrm{A}$ is nondivisible with ample normal bundle and the natural morphism $\mathrm{Alb}(\mathrm{X}) \to \mathrm{A}$ is an isogeny. By Debarre's Barth-Lefschetz theorem for abelian varieties [**Deb95**, theorem 4.5] this is the case as soon as $d > g/2$ or when X is a complete intersection of ample divisors and $d \geqslant 2$.

The proof of theorem B relies deeply on results of Krämer linking characteristic cycles on the cotangent bundle of A and representation theory [**Krä22**, **Krä21**]. Over $k = \mathbb{C}$, a look at characteristic cycles of the corresponding complex $\mathscr{D}_{\mathrm{A}}$-modules shows that the representation $\omega(\delta_{\mathrm{X}}) \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\mathrm{G}^*_{\mathrm{X},\omega})$ is minuscule in the sense that its weights form a single orbit under the Weyl group of $\mathrm{G}^*_{\mathrm{X},\omega}$. There are only few nontrivial minuscule representations V of a simply connected simple algebraic group G, all of which are listed below:

Dynkin type	G	V	dim V
A_n	SL_{n+1}	r-th wedge power	$\binom{n+1}{r}$
B_n	Spin_{2n+1}	spin	2^n
C_n	Sp_{2n}	standard	$2n$
D_n	Spin_{2n}	standard of SO_{2n}	$2n$
D_n	Spin_{2n}	half-spins	2^{n-1}
E_6	E_6	smallest nontrivial or its dual	27
E_7	E_7	smallest nontrivial	56

The dimension of $\omega(\delta_{\mathrm{X}})$ is the absolute value of the topological Euler characteristic of X. Recall that if $\mathrm{X} \subset \mathrm{A}$ is symmetric up to a translation, then $\omega(\delta_{\mathrm{X}})$ carries a nondegenerate bilinear form preserved by the action of $\mathrm{G}^*_{\mathrm{X},\chi}$. Moreover, this pairing is symmetric if d is even and alternating if d is odd; see [**KW15a**, lemma 2.1]. This rules out the occurence of E_6 for symmetric subvarieties; note that the group E_6 appears as the Tannaka group of the Fano surface in the intermediate Jacobian of a smooth cubic threefold, but $d = (g-1)/2$ here because $d = 2$ and $g = 5$. Similarly, the group E_7 preserves a nondegenerate alternating bilinear form on its 57-dimensional irreducible representation, so subvarieties X with $\mathrm{G}^*_{\mathrm{X},\chi} \cong \mathrm{E}_7$ must be odd-dimensional. However, a direct geometric argument shows that this does not happen for $d = 1$. In higher dimension I do not know any such example and it seems plausible that there should not be any.

Altogether, to prove that the Tannaka group is big and conclude the proof of the main theorem from section 2.1, it suffices to exclude wedge powers and spin representations. Concerning wedge powers, in contrast to the situation for hypersurfaces studied by Lawrence and Sawin in [LS20], one cannot rule them by numerical arguments for subvarieties of higher codimension. In fact, wedge powers do appear:

EXAMPLE. Let $\mathrm{C} \subset \mathrm{A}$ be a smooth projective curve of genus n and $r \geqslant 2$ an integer. Suppose that the sum morphism $\mathrm{Sym}^r \mathrm{C} \to \mathrm{X} := \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{A}$ is an isomorphism. Then,

$$\mathrm{G}^*_{\mathrm{X},\chi} \cong \mathrm{Alt}^r \mathrm{SL}_n := \mathrm{Im}(\mathrm{SL}_n \to \mathrm{SL}(\mathrm{Alt}^r \bar{\mathbb{Q}}_\ell^n)) \quad \text{and} \quad \omega(\delta_\mathrm{X}) = \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\chi) \cong \mathrm{Alt}^r \bar{\mathbb{Q}}_\ell^n.$$

However we show that this is the only possible case [2, theorem B]:

THEOREM C. *Suppose* $\mathrm{X} \subset \mathrm{A}$ *has ample normal bundle and is nondivisible. Then for* $d < (g-1)/2$ *the following are equivalent:*

(1) There are integers r *and* n *with* $1 < r \leqslant n/2$ *such that* $\mathrm{G}^*_{\mathrm{X},\chi} \cong \mathrm{Alt}^r(\mathrm{SL}_n)$ *and* $\omega(\delta_\mathrm{X})$ *is the* r*-th wedge power of the standard representation.*

(2) There is a nondegenerate irreducible smooth projective curve $\mathrm{C} \subset \mathrm{A}$ *such that*
- $\mathrm{X} = \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{A}$ *is the sum of* r *copies of* C*, and*
- *the sum morphism* $\mathrm{Sym}^r \mathrm{C} \to \mathrm{X}$ *is an isomorphism.*

Concerning spin representations, recall that for $\mathrm{N} \geqslant 3$ the group $\mathrm{SO}_\mathrm{N}(\bar{\mathbb{Q}}_\ell)$ admits a double cover

$$\mathrm{Spin}_\mathrm{N}(\bar{\mathbb{Q}}_\ell) \longrightarrow \mathrm{SO}_\mathrm{N}(\bar{\mathbb{Q}}_\ell)$$

by the spin group. The spin group is a simply connected algebraic group and admits a faithful representation $\mathbb{S}_\mathrm{N} \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\mathrm{Spin}_\mathrm{N}(\bar{\mathbb{Q}}_\ell))$, the *spin representation* of dimension $\dim \mathbb{S}_\mathrm{N} = 2^n$ where $n = \lfloor \mathrm{N}/2 \rfloor$. The behavior of this representation depends on the Dynkin type [FH91, §20]:

B_n: If $\mathrm{N} = 2n+1$ is odd, then the spin representation $\mathbb{S}_\mathrm{N}$ is irreducible.

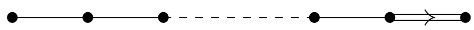

FIGURE 1. Dynkin diagram of type B_n.

D_n: If $\mathrm{N} = 2n$ is even, then $\mathbb{S}_\mathrm{N} \cong \mathbb{S}_\mathrm{N}^+ \oplus \mathbb{S}_\mathrm{N}^-$ splits as the direct sum of two irreducible representations called the *half-spin representations*. They both have dimension $\dim \mathbb{S}_\mathrm{N}^+ = \dim \mathbb{S}_\mathrm{N}^- = 2^{n-1}$ but are not isomorphic to each other, they are only related by an outer automorphism of the spin group. The dual of the half-spin representations and the center of the spin group are given by the following table:

	dual of $\mathbb{S}_\mathrm{N}^+$	center of $\mathrm{Spin}_{2n}(\bar{\mathbb{Q}}_\ell)$
n even	$\mathbb{S}_\mathrm{N}^+$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
n odd	$\mathbb{S}_\mathrm{N}^-$	$\mathbb{Z}/4\mathbb{Z}$

For $n = 2m+1$ odd, the half-spin representations are faithful. For $n = 2m$ even, the half-spin representation $\mathbb{S}_\pm$ is self-dual and the natural pairing is symmetric if m is even and alternating if m is odd. The images of $\mathrm{Spin}_{4m}(\bar{\mathbb{Q}}_\ell)$ via the half-spin representations

$$\mathrm{Spin}^{\pm}_{4m}(\bar{\mathbb{Q}}_\ell) \;\subset\; \mathrm{GL}(\mathbb{S}^{\pm}_{4m})$$

are called the *half-spin groups*. They are isomorphic to each other and fit in the following diagram of isogenies given by dividing out the subgroups of $\mathrm{Z}(\mathrm{Spin}_{4m}(\bar{\mathbb{Q}}_\ell)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$\begin{tikzcd}
& \mathrm{Spin}_{4m}(\bar{\mathbb{Q}}_\ell) \arrow[dl, bend right] \arrow[d] \arrow[dr, bend left] & \\
\mathrm{Spin}^{-}_{4m}(\bar{\mathbb{Q}}_\ell) \arrow[dr, bend right] & \mathrm{SO}_{4m}(\bar{\mathbb{Q}}_\ell) \arrow[d] & \mathrm{Spin}^{+}_{4m}(\bar{\mathbb{Q}}_\ell) \arrow[dl, bend left] \\
& \mathrm{SO}_{4m}(\bar{\mathbb{Q}}_\ell)/\pm 1 &
\end{tikzcd}$$

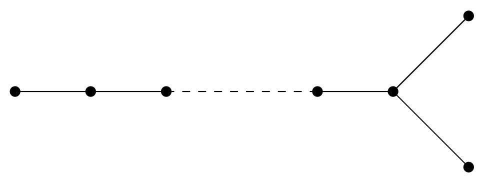

FIGURE 2. Dynkin diagram of type D_n.

We show that spin or half-spin groups do not occur for smooth nondivisible subvarieties of high enough codimension [2, theorem C]:

THEOREM D. *Suppose that* $\mathrm{X} \subset \mathrm{A}$ *has ample normal bundle, is nondivisible and has dimension* $d < (g-1)/2$. *Then the pair* $(\mathrm{G}_{\mathrm{X},\chi}, \omega(\delta_\mathrm{X}))$ *is not isomorphic to any of the above spin or half-spin groups with their spin or half-spin representations unless*

$$(\mathrm{G}^*_{\mathrm{X},\omega}, \omega(\delta_\mathrm{X})) \cong (\mathrm{Spin}^{\pm}_{4m}(\bar{\mathbb{Q}}_\ell), \mathbb{S}^{\pm}_{4m}) \quad \textit{for some } m \in \{3, \dots, d\},$$

in which case X *has topological Euler characteristic of absolute value* $|e| = 2^{2m-1}$ *and is symmetric up to a translation,* $d - m$ *is even and* $d \geqslant (g-1)/4$.

The main theorem in section 2.1 now follows by combining theorems B, C, D, and from this we also obtain the Big Monodromy criterion section 3.5 of chapter 2 by the analog of the theorem of the fixed part given by theorem A.

CHAPTER 4

Stein spaces in rigid geometry

A complex manifold is Stein if it can be embedded holomorphically in some $\mathbb{C}^n$ as a closed subspace. It is the complex-analytic analogue of the notion of an affine variety in algebraic geometry. It would be tempting to say that if a complex algebraic variety is Stein, then it is affine; that is, one could choose an embedding as above to be given by polynomial functions. This is false, as a classical example of Serre shows. In this chapter I broach the analogous question over $\mathbb{Q}_p$ and more generally over any non-Archimedean field. This will start by defining what a Stein space in the non-Archimedean context following the paper [7] in collaboration with J. Poineau. Then I will discuss the non-Archimedean analogue of Serre's example as in [4, 5]. Finally I will present some results obtained with J. Poineau [8] showing that the affine and Stein notions are way closer in the non-Archimedean framework than in the complex one.

1. Affine versus Stein in complex geometry

1.1. Complex Stein spaces. Stein spaces have their origin in the Mittlag-Leffler theorem: if one prescribes the principal parts of a meromorphic function on a domain of the complex plane, then there exists a meromorphic function defined on that domain having exactly those principal parts. It was clear to Cousin that already in dimension 2 this does not hold for an arbitrary domain $\mathrm{X} \subset \mathbb{C}^2$: for instance it fails for the complement H of

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leqslant \tfrac{1}{2}, |z_2| \geqslant \tfrac{1}{2}\} \ \subset \ \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|, |z_2| < 1\}$$

in the unit bidisk. In the language of sheaves developed by Leray and implemented by

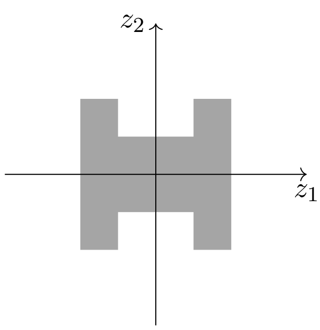

FIGURE 1. The domain H.

Cartan and Serre, the problem of Cousin is reformulated as follows: let $\mathcal{O}_\mathrm{X}$ resp. $\mathcal{M}_\mathrm{X}$ be the sheaf of holomorphic resp. meromorphic functions on X. In this framework, a prescription of principal parts of a meromorphic function on X can be seen as a global section c of the quotient sheaf $\mathscr{M}_X/\mathscr{O}_X$. In turn, finding a meromorphic function on X with such principal parts corresponds to the existence of a global section f of $\mathscr{M}_X$ mapping to c via the linear map $H^0(X, \mathscr{M}) \to H^0(X, \mathscr{M}_X/\mathscr{O}_X)$. Now the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{M}_X \longrightarrow \mathscr{M}_X/\mathscr{O}_X \longrightarrow 0$$

yields a long exact sequence of cohomology groups:

$$\begin{tikzcd}
H^0(X, \mathscr{O}_X) \arrow[r] & H^0(X, \mathscr{M}_X) \arrow[r] & H^0(X, \mathscr{M}_X/\mathscr{O}_X) \arrow[dll, "\delta"] \\
H^1(X, \mathscr{O}_X) \arrow[r] & H^1(X, \mathscr{M}_X) \arrow[r] & H^1(X, \mathscr{M}_X/\mathscr{O}_X) \arrow[r] & \cdots
\end{tikzcd}$$

and the existence of the above f is equivalent to the vanishing $\delta(c) = 0$. In particular the Cousin problem for X can be solved for all $c \in H^0(X, \mathscr{M}_X/\mathscr{O}_X)$ if $H^1(X, \mathscr{O}_X) = 0$. In this special case of a domain of $\mathbb{C}^2$ the vanishing $H^1(X, \mathscr{O}_X) = 0$ implies, for each coherent sheaf $\mathscr{F}$ on X,

$$H^q(X, \mathscr{F}) = 0 \quad \text{for all} \quad q \geqslant 1.$$

The celebrated theorem B of Cartan states that this vanishing result holds more generally for Stein spaces and it characterizes them. Recall that a complex space X is said to be *Stein* if it is

- *holomorphically separable*, that is, for distinct points $x, y \in X$ there is a holomorphic function f on X such that $f(x) \neq f(y)$;
- *holomorphically convex*, that is, for each compact subset $K \subset X$, the holomorphic convex hull

$$\hat{K}_X = \{x \in X : |f(x)| \leqslant \sup_K |f| \text{ for all } f \in H^0(X, \mathscr{O}_X)\}$$

  is compact.

Examples of Stein spaces are $\mathbb{C}^n$, disks, products of Stein spaces and closed subspaces of Stein spaces. Conversely Narasimhan's embedding theorem states that any Stein space (under a mild finiteness condition) admits a closed embedding in $\mathbb{C}^n$ for some $n \geqslant 1$.

Example. An example of space which is not Stein is $X = \mathbb{C}^2 \setminus \{0\}$. This can be seen by showing that the cohomology group $H^1(X, \mathscr{O}_X)$ does not vanish. Alternatively, since any holomorphic function on X extends to $\mathbb{C}^2$, the holomorphic convex hull of the compact subset

$$K = \{(x, y) \in \mathbb{C}^2 : \max\{|x|, |y|\} = 1\} \ \subset \ X = \mathbb{C}^2 \setminus \{0\}$$

is the closed bi-disk deprived of the origin, hence not compact.

1.2. Comparison with affine varieties. These results surely inspired Serre while developing the theory of coherent sheaves for the Zariski topology over an algebraic variety [Ser55]. The notion analogous to Stein complex space is that of an *affine* algebraic variety, that is, a variety admitting a closed embedding in some affine space $\mathbb{A}^n$. Serre's version of Cartan's theorem B then says that X is affine if and only if for each coherent sheaf $\mathcal{F}$ on X,

$$\mathrm{H}^q(\mathrm{X}, \mathcal{F}) = 0 \quad \text{for all} \quad q \geqslant 1.$$

When $k = \mathbb{C}$ it is natural to compare the two notions. In this case for a complex algebraic variety X let X^{an} denote the underlying complex space. When X is proper (that is $\mathrm{X}(\mathbb{C})$ is compact) Serre's GAGA theorem says that for each coherent sheaf $\mathcal{F}$ and each $q \geqslant 0$ the natural map

$$\mathrm{H}^q(\mathrm{X}, \mathcal{F}) \longrightarrow \mathrm{H}^q(\mathrm{X}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \tag{1.1}$$

is an isomorphism. When X is not proper, this is no longer true: the map (1.1) is in general not injective nor surjective. The lack of surjectivity is not surprising: the vector space $\mathrm{H}^q(\mathrm{X}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ comes equipped with a topology for which it is complete, thus its dimension as a complex space is never countable infinite; on the other hand the dimension of $\mathrm{H}^q(\mathrm{X}, \mathcal{F})$ is always countable. As a result, for $\mathrm{X} = \mathbb{A}^2 \smallsetminus \{0\}$ and $\mathcal{F} = \mathcal{O}_{\mathrm{X}}$ the map (1.1) is not surjective.

Producing an example where (1.1) is not injective is more subtle. In view of the above cohomological characterizations of affine varieties and Stein spaces, it would be enough to find a non-affine variety X which is Stein, that is, the underlying complex space X^{an} is Stein. This is a classical example by Serre: consider a complex elliptic curve or, more generally, a complex abelian variety A. A *rank* 1 *connection* on A is then a couple

$$(\mathcal{L}, \nabla) \quad \text{with} \quad \mathcal{L} \text{ a line bundle on A}, \quad \nabla \colon \mathcal{L} \to \Omega^1_{\mathrm{A}} \otimes \mathcal{L} \text{ a connection.}$$

Isomorphism classes of rank 1 connections on A are parametrized by the so-called *universal vector extension* $\mathrm{A}^\natural$ of A. Given rank 1 connections $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ on A the formula

$$s \otimes s' \longmapsto \nabla(s) \otimes s' + s \otimes \nabla'(s')$$

defines a connection on $\mathcal{L} \otimes \mathcal{L}'$, thus leading to a group law on $\mathrm{A}^\natural$. The line bundle underlying a rank 1 connection is algebraically trivial, hence the forgetful map $(\mathcal{L}, \nabla) \mapsto \mathcal{L}$ takes values in the dual abelian variety $\check{\mathrm{A}}$. According to a theorem of Atiyah, any algebraically trivial line bundle on A admits a connection, thus the forgetful map is surjective. These considerations lead to the following short exact sequence of algebraic groups:

$$0 \longrightarrow \mathbb{V}(\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}})) \longrightarrow \mathrm{A}^\natural \longrightarrow \check{\mathrm{A}} \longrightarrow 0.$$

Here $\mathbb{V}(\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}}))$ stands for the complex vector space $\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}})$ seen as an algebraic group and the morphism $\mathbb{V}(\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}})) \to \mathrm{A}^\natural$ associates to a global differential form ω on A the rank 1 connection $(\mathcal{O}_{\mathrm{A}}, \mathrm{d} + \omega)$ where $\mathrm{d} \colon \mathcal{O}_{\mathrm{A}} \to \Omega^1_{\mathrm{A}}$ is the canonical derivation. Connections on abelian varieties are all integrable, thus for a rank 1 connection $(\mathcal{L}, \nabla)$ one can consider its monodromy representation

$$\rho_{\mathcal{L},\nabla} \colon \quad \pi_1(\mathrm{A}, 0) \longrightarrow \mathrm{GL}(\mathcal{L}_0) = \mathbb{C}^\times$$

where $\pi_1(\mathrm{A}, 0)$ is the topological fundamental group of A. This sets up a biholomorphism

$$\mathrm{A}^{\natural,\mathrm{an}} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(\mathrm{A}, 0), \mathbb{C}^\times), \qquad (\mathcal{L}, \nabla) \longmapsto \rho_{\mathcal{L},\nabla}$$

usually called *Riemann-Hilbert correspondence*. The complex manifold

$$\mathrm{Hom}(\pi_1(\mathrm{A}, 0), \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{2g} \quad \text{where} \quad g = \dim \mathrm{A}$$

is Stein, hence so is $A^{\natural,\mathrm{an}}$. On the other hand the algebraic variety $A^\natural$ is *not* affine: otherwise $\check{A}$ would be affine, since the quotient of an affine algebraic group by a normal subgroup is so. Actually more is true: any algebraic function on $A^\natural$ is constant. This extra information lets one give an example where (1.1) is not injective.

Example. Let $X = A^\natural$ and let $Y \subset X$ be a closed affine subvariety, for instance any fiber of the forgetful map $A^\natural \to \check{A}$. If $\mathscr{F} \subset \mathcal{O}_X$ is the ideal sheaf defining Y, then the short exact sequence $0 \to \mathscr{F} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ yields a long exact sequence of cohomology groups

$$\begin{array}{l} H^0(X,\mathscr{F}) = 0 \longrightarrow H^0(X,\mathcal{O}_X) = \mathbb{C} \longrightarrow H^0(Y,\mathcal{O}_Y) \\ \hookrightarrow H^1(X,\mathscr{F}) \longrightarrow H^1(X,\mathcal{O}_X) = 0 \longrightarrow H^1(Y,\mathcal{O}_Y) = 0 \end{array}$$

where the vanishing of $H^1(X,\mathcal{O}_X)$ can be found in [Col98, Lau96]. In particular, if $\dim Y > 0$, then the cohomology group $H^1(X,\mathscr{F})$ does not vanish and the map (1.1) is not injective.

Following Simpson [Sim94a, Sim94b] the Riemann-Hilbert correspondence can be seen more generally as an isomorphism between the de Rham and the Betti moduli space. To fix ideas let C be a complex smooth projective curve of genus $g \geqslant 1$. For $r \geqslant 1$ consider the (coarse) moduli space $M_{\mathrm{dR},r}$ of rank r vector bundles on C endowed with a (necessarily integrable) connection. The Riemann-Hilbert correspondence here is incarnated by a biholomorphism

$$M^{\mathrm{an}}_{\mathrm{dR},r} \xrightarrow{\sim} M^{\mathrm{an}}_{\mathrm{B},r}.$$

Here $M_{\mathrm{B},r}$ is the GIT quotient of the affine variety $\mathrm{Hom}(\pi_1(C,x),\mathrm{GL}_r)$ parametrizing group morphisms $\pi_1(C,x) \to \mathrm{GL}_r(\mathbb{C})$ under the action of GL_r by conjugation. Again, it is not hard to see that the variety $M_{\mathrm{dR},r}$ is not quasi-affine, while GIT ensures that $M_{\mathrm{B},r}$ is. In particular $M_{\mathrm{dR},r}$ is a Stein non-affine variety.

1.3. Relation with Hilbert's fourteenth problem. As pointed out by Neeman [Nee88] this construction can be adapted to give counter-examples to Hilbert's fourteenth problem as extended by Zariski. Recall that Hilbert's fourteenth problem asks if, given a linear algebraic group G acting on an affine variety X, the algebra $\Gamma(X,\mathcal{O}_X)^G$ of G-invariant regular functions is finitely generated. This is true if G is reductive (as shown by Hilbert) but it is false for $G = \mathbb{A}^{13}$ acting on $X = \mathbb{A}^{32}$ as famously shown by Nagata; let me mention the beautiful alternative counter-example for $G = \mathbb{A}^3$ acting on $X = \mathbb{A}^{18}$ by Totaro [Tot08]. In a rather different direction Zariski [Zar50] proved that for a normal surface X over a field k of characteristic 0 the algebra of regular functions $\Gamma(X,\mathcal{O}_X)$ is finitely generated. Rees showed that this is false in dimension $\geqslant 3$. Following Neeman, a counter-example can be produced as follows. Let $A^\natural$ be the universal vector extension of a complex abelian variety A. Let $\mathscr{L}$ be an ample line bundle on $A^\natural$ and

$$X = \mathbb{V}(\mathscr{L}^\vee) \smallsetminus e(A^\natural)$$

the total space of $\mathscr{L}^\vee$ deprived of its zero section $e\colon A^\natural \to \mathbb{V}(\mathscr{L}^\vee)$. By ampleness of $\mathscr{L}$ the variety X is seen to be quasi-affine, that is, admitting an open immersion into an affine variety. The variety X is Stein, as any principal $\mathbb{C}^\times$-bundle over a Stein space. Neeman shows that a Stein quasi-affine variety Y is affine if and only if $\Gamma(\mathrm{Y}, \mathcal{O}_\mathrm{Y})$ is finitely generated. This is clearly not the case for X: if it were affine, then $\mathrm{A}^\natural$ would be affine too.

2. Stein spaces in rigid geometry

2.1. Rigid geometry. The first trace of Tate's rigid analysis can be found in a letter of Serre of July 31, 1959 [Cor15]:

> Il paraît que vous faites des choses rupinantes avec les courbes elliptiques sur les p-adiques (j non entier), m'a raconté Lang; vous savez faire marcher ce que nos pères appelaient les "fonctions loxodromiques" sur les p-adiques.

Given a complex elliptic curve E Serre is alluding to the partial uniformization $\mathbb{C}^\times \to \mathrm{E}$ obtained by writing E as a quotient $\mathbb{C}^\times/q^\mathbb{Z}$ for some $|q| > 1$. Tate noticed that this is given by explicit power series that, after opportune manipulation, converge p-adically and give a bijection

$$\mathrm{E}(\mathbb{C}_p) \cong \mathbb{C}_p^\times/q^\mathbb{Z} \quad \text{with} \quad |q| > 1$$

where E is any elliptic curve over $\mathbb{C}_p$ whose j-invariant is not integral. The driving force behind his foundation of rigid analysis was endowing both sides of the above equation with a natural structure of *analytic space* over $\mathbb{C}_p$ so that the above bijection would become an analytic isomorphism. As nowadays there are several approaches to rigid analytic geometry: here I will take the point of view of Berkovich [Ber90, Ber93], as it is closer to the complex intuition.

Given a non-Archimedean field K, understood to be complete and non-trivially valued, analytic spaces over K are build by gluing local models, like balls for real manifolds, affine varieties for algebraic varieties, ... Similarly to algebraic geometry here local models are given by some kind of K-algebras. More precisely, a Banach K-algebra is said to be *affinoid* if it can be written as a quotient of a Tate algebra,

$$\mathrm{K}\{z_1/r_1, \dots, z_n/r_n\} = \left\{ f = \sum_{\alpha \in \mathbb{N}^n} |c_\alpha| z^\alpha : c_\alpha r^\alpha \to 0 \text{ as } |\alpha| \to \infty \right\},$$

where I used the notation $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and real numbers $r_1, \dots, r_n > 0$. Such a Tate algebra is endowed with the norm

$$\|f\|_r = \max_{\alpha \in \mathbb{N}^n} |c_\alpha| r^\alpha$$

and a quotient $\mathrm{K}\{t_1/r_1, \dots, t_n/r_n\} \to \mathrm{A}$ is endowed with the quotient norm $\| \cdot \|_\mathrm{A}$. The role of the spectrum of a ring in algebraic geometry is played by the Banach spectrum of an affinoid algebra A: it is a locally ringed space $\mathrm{X} = (|\mathrm{X}|, \mathcal{O}_\mathrm{X})$ whose underlying topological space is the set

$$|\mathrm{X}| = \mathscr{M}(\mathrm{A}) = \left\{ x \colon \mathrm{A} \to \mathbb{R}_+ \;\middle|\; \begin{array}{c} x \text{ is a multiplicative seminorm with} \\ |f(x)| := x(f) \leqslant \|f\|_\mathrm{A} \text{ for all } f \in \mathrm{A} \end{array} \right\}$$

endowed with the coarsest topology for which the map $\mathscr{M}(\mathrm{A}) \to \mathbb{R}_+$, $x \mapsto |f(x)|$ is continuous for any $f \in \mathrm{A}$. The topological space $\mathscr{M}(\mathrm{A})$ is compact Hausdorff, which is rather surprising since the field K is not locally compact in general. Rather than giving the precise definition of the sheaf of functions and of the gluing procedure, let me describe some examples that it will be useful to bare in mind.

Example. In Berkovich's framework, analytic spaces have a natural boundary (which may be empty) similarly to real manifolds with boundaries. Given an affinoid algebra $(\mathrm{A}, \|\cdot\|_{\mathrm{A}})$, the quotient

$$\tilde{\mathrm{A}} \;=\; \{f \in \mathrm{A} : \|f\|_{\mathrm{A}} \leqslant 1\} \;/\; \{f \in \mathrm{A} : \|f\|_{\mathrm{A}} < 1\}$$

of the subring of power-bounded elements by its ideal of topological nilpotent elements is an algebra of finite type over the residue field k of K. There is a reduction map

$$\mathrm{red}\colon \quad \mathscr{M}(\mathrm{A}) \longrightarrow \mathrm{Spec}(\tilde{\mathrm{A}})$$

which is anti-continuous, that is, the preimage of an open subset is closed. In particular, the subset

$$\mathrm{Int}\, \mathscr{M}(\mathrm{A}) = \mathrm{red}^{-1}(\{x \in \mathrm{X} : x \text{ closed}\}) \;\subset\; \mathscr{M}(\mathrm{A})$$

is open and its complement $\partial\mathscr{M}(\mathrm{A})$ is the *boundary* of $\mathscr{M}(\mathrm{A})$. In the case of the Tate algebra $\mathrm{A} = \mathrm{K}\{z\}$ one has $\tilde{\mathrm{A}} = k[t]$. In the affine line $\mathbb{A}^1 = \mathrm{Spec}\,\tilde{\mathrm{A}}$ over k all points are closed save the generic one η. It is not hard to see that the preimage $\mathrm{red}^{-1}(\eta)$ is made of one single point in $\mathscr{M}(\mathrm{A})$ given by the so-called Gauss norm

$$\mathrm{K}\{z\} \longrightarrow \mathbb{R}_+, \qquad \sum_{n=0}^{\infty} c_n z^n \longmapsto \max_{n \in \mathbb{N}} |c_n|.$$

Example. Let X be an algebraic variety over K. Similarly to the complex case one can functorially associate to X a K-analytic space X^{an}. Its underlying set is

$$|\mathrm{X}^{\mathrm{an}}| = \{(x, |\cdot|) : x \in \mathrm{X}, |\cdot|\colon \kappa(x) \to \mathbb{R}_+ \text{ absolute value extending the one on K}\},$$

where $\kappa(x)$ stands for the residue field at a point $x \in \mathrm{X}$. Note that here it is vital to see X as a scheme, so that the points $x \in \mathrm{X}$ are not necessarily closed. In fact, unlike the complex picture, here the forgetful map $\pi\colon |\mathrm{X}^{\mathrm{an}}| \to |\mathrm{X}|$, $(x, |\cdot|) \mapsto x$ is surjective: this stems from to the fact that any non-Archimedean field can be embedded in a strictly larger complete field. The set $|\mathrm{X}^{\mathrm{an}}|$ is endowed with the coarsest topology for which the forgetful map π is continuous and, for each open subset $\mathrm{U} \subset \mathrm{X}$ and any $f \in \Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$, the map

$$\pi^{-1}(\mathrm{U}) \longrightarrow \mathbb{R}_+, \quad (x, |\cdot|) \longmapsto |f(x)|$$

is continuous. The topological space $|\mathrm{X}^{\mathrm{an}}|$ is locally compact, thus it makes sense to talk about uniform convergence. When X is reduced, this lets one describe analytic functions on an open subset $\mathrm{U} \subset |\mathrm{X}^{\mathrm{an}}|$ as locally uniform limits of rational functions without poles. When X is smooth, the topological space $|\mathrm{X}^{\mathrm{an}}|$ is locally contractible [Ber99], which is again a quite remarkable property considering the totally disconnected nature of K. Even more remarkably, if E is an elliptic curve with integral j-invariant, then E^{an} is contractible. Instead, when the j-invariant is not integral (and K is algebraically closed, to fix ideas) then E^{an} is not simply connected and its topological universal cover is $\mathbb{G}_m^{\mathrm{an}}$: this explains why Tate's uniformization worked only in this case.

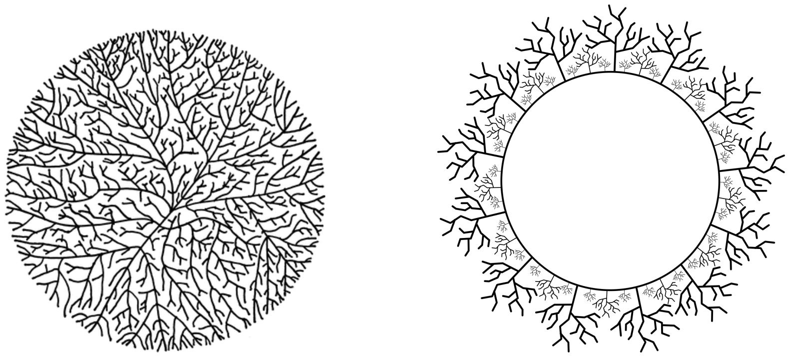

FIGURE 2. On the left an elliptic curve with integral j-invariant and on the right one with non-integral j-invariant. Drawings by Mattias Jonsson.

EXAMPLE. Let R be the ring of integers of K and $\mathfrak{X}$ a flat separated R-scheme of finite type with generic fiber X. Then one associates to $\mathfrak{X}$ a compact subspace $\mathfrak{X}_\eta \subset \mathrm{X}^{\mathrm{an}}$ called the *Raynaud generic fiber* of X. A point $(x, |\cdot|)$ in X^{an} lies in $\mathfrak{X}_\eta$ if and only if the morphism $\operatorname{Spec}\kappa(x) \to \mathrm{X}$ extends to a morphism of R-schemes $\operatorname{Spec}\kappa(x)^\circ \to \mathfrak{X}$ where $\kappa(x)$ is the residue field at x and $\kappa(x)^\circ \subset \kappa(x)$ its ring of integers with respect to the absolute value $|\cdot|$. When $\mathfrak{X} = \mathbb{A}^1_{\mathrm{R}}$ then

$$\mathfrak{X}_\eta = \{x \in \mathbb{A}^{1,\mathrm{an}}_{\mathrm{K}} : |z(x)| \leqslant 1\} \ \subset \ \mathbb{A}^{1,\mathrm{an}}_{\mathrm{K}}$$

where z is the coordinate function on $\mathbb{A}^1_{\mathrm{K}}$.

2.2. What is a Stein space? In the context of rigid analytic geometry, the analogue of Serre's cohomological characterization of affine varieties is called Tate's acyclicity theorem. Let X be an affinoid space over K, that is, the Banach spectrum of an affinoid algebra. Then,

$$\mathrm{H}^q(\mathrm{X}, \mathscr{F}) = 0 \quad \text{for any coherent sheaf } \mathscr{F} \text{ on X and all } q \geqslant 1. \tag{2.1}$$

Unlike the theorem of Serre such a cohomological vanishing does not characterize affinoid spaces: this is the beginning of the story. Of course, there are plenty of non-compact spaces (hence non-affinoid) that satisfy (2.1): for instance, the analytic space attached to any affine variety does. This led Kiehl to translate the complex notion of Stein exhaustion by Stein compact blocks into the rigid analytic jargon: he calls an analytic space X *quasi-Stein* if it can be covered[1] by affinoid domains

$$\mathrm{X}_0 \ \subset \ \mathrm{X}_1 \ \subset \ \cdots \ \subset \ \mathrm{X}_n \ \subset \ \mathrm{X}_{n+1} \ \subset \ \cdots$$

such that the restriction map $\Gamma(\mathrm{X}_{n+1}, \mathscr{O}_{\mathrm{X}}) \to \Gamma(\mathrm{X}_n, \mathscr{O}_{\mathrm{X}})$ has a dense image for any $n \geqslant 0$. For instance compact quasi-Stein spaces are exactly affinoid spaces. Kiehl goes on proving that quasi-Stein spaces satisfy the cohomological vanishing (2.1). But the reason why he hesitated in calling them Stein is that he could not prove that (2.1) characterizes them. In fact, it does not: Liu later exhibited compact analytic spaces satisfying (2.1)

[1] Technically one should ask this to be an admissible cover in Tate's terminology, or a G-cover in Berkovich's one.

without being affinoid (hence not quasi-Stein). First, Liu exhibited a non-affinoid compact analytic space whose normalization is affinoid [**Liu88**]; this construction has been recently reinterpreted by Temkin [**Tem22**]. Second, Liu showed in [**Liu90**] (taking into account some amendment in [**7**]) that a compact analytic space X satisfies (2.1) if and only if

- it is *holomorphically separable*, that is, given distinct points $x, y \in \mathrm{X}$ there is an analytic function $f \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ such that $|f(x)| \neq |f(y)|$;
- $\mathrm{H}^q(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for all $q \geqslant 1$.

As a consequence he produced a non-affinoid compact domain of the unit closed bidisk satisfying (2.1).This leads naturally to the following definition:

Definition. Let X be an analytic space countable at infinity. If X is compact, then it is said to be *Stein* if it satisfies (2.1). In general, it is said to be *Stein* if it can be covered by compact Stein domains

$$\mathrm{X}_0 \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_n \subset \mathrm{X}_{n+1} \subset \cdots$$

such that the restriction map $\Gamma(\mathrm{X}_{n+1}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{X}_n, \mathcal{O}_{\mathrm{X}})$ has a dense image for any $n \geqslant 0$.

The main advantage of adopting Berkovich's language is that the underlying topological subspace is locally compact. This lets one translate *verbatim* in rigid analysis the concept of a *holomorphically convex* analytic space. With J. Poineau we took profit of this to prove the following characterization of Stein spaces [**7**, theorems 1.11, 1.12]:

Theorem A. *Let* X *be a separated* K*-analytic space which is countable at infinity. Then the following are equivalent:*

(1) X *is Stein;*

(2) the cohomological vanishing (2.1) *holds for the base-change of* X *to any complete extension of* K*;*

(3) X *is holomorphically separable, holomorphically convex and* $\mathrm{H}^q(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ *for all* $q \geqslant 1$*.*

Moreover, when X *is without boundary, the preceding conditions are equivalent to:*

(4) X *is holomorphically separable and holomorphically convex;*

(5) X *is quasi-Stein in the sense of Kiehl.*

In (2) I ignore whether it suffices to test (2.1) only on the base field K. The reason why Kiehl could not access to such a characterization is undoubtedly the lack of local (sequential) compactness of rigid spaces *à la* Tate which he was using. The above theorem can be applied to prove that given a finite surjective morphism $\mathrm{X} \to \mathrm{Y}$ between analytic space without boundary X is (quasi-)Stein if and only if Y is: this is not at all evident from the definition of quasi-Stein because of Liu's first counter-example. Later on, we realized in [**8**, corollary 4.3] that the equivalence with (4) and (5) holds under a very mild hypothesis on the boundary (instead of being empty): it suffices that X admits a morphism $\mathrm{X} \to \mathrm{S}$ without relative boundary where S is quasi-Stein. This shows that in Liu's counter-examples the boundary is of a rather bizarre nature compared to that of affinoid spaces. On other direction, Lütkebohmert proved that under the finiteness condition

$$\sup_{x \in \mathrm{X}} \dim_{\mathscr{H}(x)} \Omega^1_{\mathrm{X},x} \otimes_{\mathscr{O}_{\mathrm{X},x}} \mathscr{H}(x) < +\infty \tag{2.2}$$

where $\mathscr{H}(x)$ is the completed residue field at a point $x \in \mathrm{X}$, an analytic space X without boundary is Stein if and only if it admits a closed embedding in some analytic affine space $\mathbb{A}^{n,\mathrm{an}}$. Note that the condition (2.2) is exactly the one appearing in Narasimhan's embedding theorem for complex Stein spaces.

3. Stein vs. affine in rigid geometry

The main question in this section is: is there a non-affine variety on a non-Archimedean field whose associated analytic space is Stein?

3.1. The case of algebraic groups. To get started with, an algebraic variety X over a non-Archimedean field K is said to be *Stein* if the associated analytic space X^{an} is so. In this case X^{an} obviously satisfies the condition (2.2) so being Stein is the same as admitting an analytic closed embedding $\mathrm{X}^{\mathrm{an}} \hookrightarrow \mathbb{A}^{n,\mathrm{an}}$. In view of Serre's example in the complex case explained in section 1.2 it is natural to start by looking at the universal vector extension of an abelian variety. Unlike the complex case, here the affine and the Stein notions are equivalent more generally for algebraic groups [4, main theorem]:

Theorem B. *An algebraic group over* K *is Stein if and only if it is affine.*

Of course to prove such a statement one has to understand analytic functions on algebraic groups. For an algebraic group G, the K-algebra $\Gamma(\mathrm{G}, \mathscr{O}_\mathrm{G})$ is of finite type. Let

$$\pi_\mathrm{G} \colon \quad \mathrm{G} \longrightarrow \mathrm{G}_{\mathrm{aff}} := \operatorname{Spec} \Gamma(\mathrm{G}, \mathscr{O}_\mathrm{G})$$

be the canonical morphism. Theorem B is a straightforward consequence of the following result [4, theorem A]:

Theorem C. *All analytic functions on* G *come from* $\mathrm{G}_{\mathrm{aff}}$ *by precomposing with* π_G.

Employing Brion's nomenclature an algebraic group G over K is said to be *anti-affine* if $\Gamma(\mathrm{G}, \mathscr{O}_\mathrm{G}) = \mathrm{K}$. For instance, as recalled above, when K is of characteristic 0, the universal vector extension of a non-zero abelian variety is anti-affine. In the anti-affine case, the above theorem states that anti-affine algebraic groups over K do not admit non-constant analytic functions. This is of course in contrast with the complex situation. One of Brion's motivation in studying anti-affine algebraic groups was understanding the example of Neeman given in section 1.3. Let G be an anti-affine extension of an abelian variety A and H an ample line bundle on A. Let P denote the total space of $p^*\mathrm{H}^\vee$ deprived of its zero section where $p \colon \mathrm{G} \to \mathrm{A}$ is the projection. By ampleness the variety P is quasi-affine and Brion shows that the ring $\Gamma(\mathrm{P}, \mathscr{O}_\mathrm{P})$ is not Noetherian [**Bri09**, theorem 3.9]. The equivalence between affine and Stein holds even in this more intricate example [4, theorem C]:

Corollary. *Let* G *be an algebraic group,* L *a line bundle on* G *and* P *the total space of* L *deprived of its zero section. Then the following are equivalent:*

$$\mathrm{G} \textit{ is affine} \iff \mathrm{G} \textit{ is Stein} \iff \mathrm{P} \textit{ is affine} \iff \mathrm{P} \textit{ is Stein.}$$

Let me try to explain the content of theorem C by sketching its proof when G is the universal vector extension of an abelian variety A over K and the characteristic of K is 0. Actually to simplify notation G will be the universal vector extension of the dual abelian variety $\check{\mathrm{A}}$ so that G is a vector extension of A. Also the superscript 'an' will be dropped. The proof has a rather different flavour depending on the reduction behaviour of the abelian variety A.

Totally degenerate reduction. The case of totally degenerate reduction is perhaps the more intuitive, since it is quite close to the complex picture. When A is a complex abelian variety, the biholomorphism

$$\mathrm{G} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(\mathrm{A}(\mathbb{C}), 0), \mathbb{C}^\times)$$

given by the Riemann-Hilbert correspondence admits an explicit description. To explain it let $\exp\colon \mathrm{V} := \mathrm{Lie}\,\mathrm{A} \to \mathrm{A}(\mathbb{C})$ be the exponential map and identify $\Lambda := \mathrm{Ker}\exp$ with the fundamental group of $\mathrm{A}(\mathbb{C})$. With this notation Hodge theory identifies $\mathrm{H}^0(\check{\mathrm{A}}, \Omega^1)$ with the conjugated complex vector space $\bar{\mathrm{V}}$. Let $\theta_\Lambda\colon \Lambda \to \bar{\mathrm{V}}$ be the inclusion. Then

$$\mathrm{G}(\mathbb{C}) \;=\; (\mathrm{V} \times \bar{\mathrm{V}})/\Lambda.$$

This can be mimicked over the non-Archimedean field K when A has totally degerate reduction. In this case, passing to finite extension of K, the universal cover of A is a K-torus T and the topological fundamental group is identified with a free abelian group $\Lambda \subseteq \mathrm{T}(\mathrm{K})$ of rank $\dim \mathrm{T}$. Consider the torus $\check{\mathrm{T}}$ with group of characters Λ and $\omega_{\check{\mathrm{T}}}$ the dual of its Lie algebra. Then the universal vector extension G can be identified with the quotient $(\mathrm{T} \times \mathbb{V}(\omega_{\check{\mathrm{T}}}))/\Lambda$ thanks to the results in [5]. Here the action is given by

$$\chi.(t, v) = (\chi t, v + \theta_\Lambda(\chi))$$

where $\theta_\Lambda(\chi) = \chi^* \frac{\mathrm{d}z}{z}$ and $\chi \in \Lambda$ is seen as a character $\chi\colon \check{\mathrm{T}} \to \mathbb{G}_m$. Now an analytic function on G is an analytic function on $\mathrm{T} \times \mathbb{V}(\omega_{\check{\mathrm{T}}})$ invariant under the action of Λ. Since the image of $\theta_\Lambda\colon \Lambda \to \omega_{\check{\mathrm{T}}}$ spans the K-vector space $\omega_{\check{\mathrm{T}}}$ and accumulates to 0, such an invariant function is necessarily constant.

Good reduction. Suppose that A has good reduction, that is, it is the generic fiber of an abelian scheme $\mathscr{A}$. In this case the topology offers no information as the topological space underlying the universal vector extension G is contractible. When K has 0 residue characteristic Coleman proved that all algebraic functions on each successive thickening of the universal extension of $\mathscr{A}$ are constant. Passing to the limit then lets one conclude. When K is a valued extension of $\mathbb{Q}_p$ the situation is more interesting. There is no loss of generality in supposing K algebraically closed. The idea is to replace the topological universal cover by the 'perfectoid' one

$$\tilde{\mathscr{A}} = \varprojlim_{\times p} \mathscr{A},$$

where the transition maps are the multiplication by p. Then the universal vector extension of the dual abelian scheme $\check{\mathscr{A}}$ can be identified with the quotient $(\tilde{\mathscr{A}} \times \mathbb{V}(\omega_{\check{\mathscr{A}}}))/\mathrm{T}_p\mathscr{A}$ where $\omega_{\check{\mathscr{A}}}$ is the dual of the Lie algebra of $\check{\mathscr{A}}$ and

$$\mathrm{T}_p\mathscr{A} = \varprojlim_{n \geqslant 1} \mathscr{A}[p^n]$$

is the Tate module, where the torsion subgroups are seen as finite flat group schemes. The action of $\mathrm{T}_p\mathscr{A}$ is given by some morphism of group schemes $\theta_{\mathrm{T}_p\mathscr{A}}\colon \mathrm{T}_p\mathscr{A} \to \mathbb{V}(\omega_{\check{\mathrm{A}}})$. Following an insight of Coleman and Faltings, the K-linear map

$$\theta_{\mathrm{T}_p\mathscr{A}}\colon \mathrm{T}_p\mathscr{A}(\mathrm{R}) \otimes_{\mathbb{Z}_p} \mathrm{K} \longrightarrow \omega_{\check{\mathscr{A}}} \otimes_{\mathrm{R}} \mathrm{K} = \omega_{\check{\mathrm{A}}}$$

is surjective and leads to the Hodge-Tate decomposition of $\mathrm{H}^1_{\text{ét}}(\mathrm{A}, \mathbb{Q}_p)$. Moreover the analytic functions on $\mathrm{A}^\natural$ are those on $\mathbb{V}(\omega_{\check{\mathrm{A}}})$ invariant under the translation by $\mathrm{T}_p\mathscr{A}(\mathrm{R})$. Since the image of $\theta_{\mathrm{T}_p\mathscr{A}}$ spans the K-vector space $\omega_{\check{\mathrm{A}}}$ and accumulates to 0, such functions are necessarily constant.

Intermediate reduction. Finally, when A has intermediate reduction, the proof mixes the two techniques. The main ingredient is the study of the universal cover of G carried out in [5]. To explain this recall that the universal cover of A is an extension

$$0 \longrightarrow \mathrm{T} \longrightarrow \mathrm{E} \longrightarrow \mathrm{B} \longrightarrow 0$$

where T is a K-torus and B an abelian variety with good reduction. The topological fundamental group of A is identified with a free abelian group $\Lambda \subseteq \mathrm{E}(\mathrm{K})$ of rank $\dim \mathrm{T}$. The universal cover of the dual abeloid variety is a similar extension

$$0 \longrightarrow \check{\mathrm{T}} \longrightarrow \check{\mathrm{E}} \longrightarrow \check{\mathrm{B}} \longrightarrow 0$$

where $\check{\mathrm{B}}$ is the dual of B and $\check{\mathrm{T}}$ is the K-torus with group of characters Λ. Then the universal cover $\tilde{\mathrm{G}}$ of G is the push-out of $\mathrm{B}^\natural \times_{\mathrm{B}} \mathrm{E}$ along the K-linear map $\omega_{\check{\mathrm{B}}} \to \omega_{\check{\mathrm{E}}}$ induced by the projection $\check{\mathrm{E}} \to \check{\mathrm{B}}$. Here $\omega_{\check{\mathrm{B}}}$ is the dual of the Lie algebra of $\check{\mathrm{B}}$, and similarly for $\omega_{\check{\mathrm{E}}}$. In other words, the following diagram is exact and commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{V}(\omega_{\check{\mathrm{B}}}) \arrow[r] \arrow[d] & \mathrm{B}^\natural \times_{\mathrm{B}} \mathrm{E} \arrow[r] \arrow[d] & \mathrm{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{V}(\omega_{\check{\mathrm{E}}}) \arrow[r] & \tilde{\mathrm{G}} \arrow[r] & \mathrm{E} \arrow[r] & 0
\end{tikzcd}$$

The topological fundamental group of G is identified with a subgroup of $\tilde{\mathrm{G}}(\mathrm{K})$ mapping isomorphically to Λ via the projection $\tilde{\mathrm{G}} \to \mathrm{E}$. Once again analytic functions on G correspond to analytic functions on the universal cover $\tilde{\mathrm{G}}$ invariant under the fundamental group and a careful analysis lets one show that the latter are all constant.

3.2. General results. The case of algebraic groups pushed me to look further. Together with J. Poineau we obtained the following criterion in [8, theorem A]:

THEOREM D. *A Stein variety* X *is quasi-affine and, for any closed subscheme* Y *of dimension* $\leqslant 1$, *the restriction map* $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}})$ *is surjective. Furthermore, if the ring* $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ *is Noetherian, then* X *is affine.*

Recall that a variety X is quasi-affine if it admits an open embedding into an affine variety. Since the K-algebra of regular functions on an algebraic group is finitely generated, theorem D implies theorem B directly. More interestingly, a result of Goodman-Landman says that an algebraic variety X is affine if it is quasi-affine and for each subvariety $\mathrm{Y} \subset \mathrm{X}$ the restriction map is $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}})$ is surjective. Therefore theorem D in the case of surfaces yields the following:

Corollary. *An algebraic surface is Stein if and only if it is affine.*

Example. Suppose K of characteristic 0 and let C be a smooth projective curve of genus $\geqslant 1$ over K. For $r \geqslant 1$ let $\mathrm{M}_{\mathrm{dR},r}$ be the moduli space of vector bundles with a connection on C. As recalled in section 1.2 the variety $\mathrm{M}_{\mathrm{dR},r}$ is not quasi-affine. Therefore theorem D implies that $\mathrm{M}_{\mathrm{dR},r}$ is not Stein, whereas it was the case over $\mathbb{C}$.

As for theorem B, proving theorem D requires to understand analytic functions on algebraic varieties. The crucial point is the following approximation theorem [8, theorem B]:

Theorem E. *Let* F *be a semi-reflexive coherent sheaf on a variety* X. *Then* $\Gamma(\mathrm{X},\mathrm{F})$ *is dense in* $\Gamma(\mathrm{X}^{\mathrm{an}},\mathrm{F}^{\mathrm{an}})$.

When X is reduced the topology on $\Gamma(\mathrm{X}^{\mathrm{an}},\mathrm{F}^{\mathrm{an}})$ is that of uniform convergence; when X is not reduced, as in complex analysis, the topology is finer and I will not define it here. Also recall that an $\mathcal{O}_{\mathrm{X}}$-module F is *semi-reflexive* if the natural homomorphism $\mathrm{F} \to \mathrm{F}^{\vee\vee}$ is injective. Therefore theorem E applies in particular to vector bundles and coherent sheaves of ideals, and I ignore if its conclusion holds for any coherent sheaf on X. Theorem E is quite easy to prove when X admits a proper morphism onto an affine variety: this is a rather strong condition, and certainly not the 'generic' case. Finally, note that theorem E implies theorem C at once.

Theorem E is the breaking point of the analogy between complex and rigid geometry: roughly speaking, there are fewer rigid analytic functions with essential singularities than holomorphic ones. For instance, there are plenty of non-constant holomorphic functions on the universal vector extension of a complex abelian, whereas in the non-Archimedean case analytic functions are all constant (as the algebraic ones). Mirroring this discrepancy, the proof of theorem E has to rely on some technique unavailable in the complex framework: with no big surprise, it is the involvement of models over the ring of integers R of K. To state the result let $\varpi \in \mathrm{R} \smallsetminus \{0\}$ be a topological nilpotent element and for an R-scheme S let S_n be the closed subscheme $\mathrm{S} \times_{\mathrm{R}} (\mathrm{R}/\varpi^n\mathrm{R})$ of S. The density result lying at the core of the method is the following [8, theorem C]:

Theorem F. *Let* $\mathcal{D}$ *be an effective divisor in a proper and flat* R*-scheme* $\mathcal{X}$. *If* $\mathcal{D}$ *is flat over* R, *then for any semi-reflexive coherent* $\mathcal{O}_{\mathrm{X}}$*-module* $\mathcal{F}$ *the natural map*

$$\Gamma(\mathcal{X} \smallsetminus \mathcal{D}, \mathcal{F}) \longrightarrow \operatorname*{proj\,lim}_{n\in\mathbb{N}} \Gamma(\mathcal{X}_n \smallsetminus \mathcal{D}_n, \mathcal{F}),$$

is injective, and has a dense image.

Recall that the topology on $\operatorname{proj\,lim}_{n\in\mathbb{N}} \Gamma(\mathrm{U}_n,\mathrm{F})$ is defined to be the prodiscrete one. Let us relate this to theorem E. Let U be the generic fiber of $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{D}$ and consider the Raynaud generic fiber $\mathcal{U}_\eta \subset \mathrm{U}^{\mathrm{an}}$. With this notation theorem F states that any analytic section of $\mathcal{F}^{\mathrm{an}}$ over the compact subset $\mathcal{U}_\eta$ can be globally approximated by algebraic sections in $\Gamma(\mathrm{U},\mathcal{F})$.

Example. In general theorem F fails to be true when the flatness of $\mathcal{D}$ is discarded. Suppose for simplicity R discretely valued and let k be the residue field of R. Consider the blow-up $\pi\colon \mathfrak{X} \to \mathbb{P}^1_{\mathrm{R}}$ at a k-rational point of the special fiber. Let $\mathfrak{D}$ be the exceptional divisor. The completion of the open subset $\mathfrak{U} := \mathfrak{X} \smallsetminus \mathfrak{D}$ along its special fiber is isomorphic to the completion of the affine line $\mathbb{A}^1_{\mathrm{R}}$ along its special fiber. Thus

$$\operatorname*{proj\,lim}_{n\in\mathbb{N}} \Gamma(\mathfrak{U}_n, \mathcal{O}_{\mathfrak{X}}) \;\cong\; \mathrm{R}\{z\} \;:=\; \operatorname*{proj\,lim}_{n\in\mathbb{N}} \mathrm{R}/\varpi^n\mathrm{R}[z],$$

where z is the coordinate function on the affine line $\mathbb{A}^1_{\mathrm{R}}$. However the generic fiber of $\mathfrak{U}$ is the projective line $\mathbb{P}^1_{\mathrm{K}}$ over K so by flat base change $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \hookrightarrow \Gamma(\mathbb{P}^1_{\mathrm{K}}, \mathcal{O}_{\mathbb{P}^1}) = \mathrm{K}$. In particular, the image of

$$\mathrm{R} = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \;\longrightarrow\; \operatorname*{proj\,lim}_{n\in\mathbb{N}} \Gamma(\mathfrak{U}_n, \mathcal{O}_{\mathfrak{X}}) \cong \mathrm{R}\{z\},$$

is not dense.

For theorem F to be of any use, given a variety U over K, one has to be able to produce a model $\mathfrak{U}$ of U which is the complement of an effective Cartier divisor $\mathfrak{D}$ flat over R in a proper flat R-scheme $\mathfrak{X}$. We obtain the following precise statement [8, theorem D]:

Theorem G. *Let* D *be an effective Cartier divisor in a proper variety* X*. Then, there is an effective Cartier divisor* $\mathfrak{D}$ *in a proper flat* R*-scheme* $\mathfrak{X}$ *and an isomorphism of* K*-schemes* $\mathrm{X} \cong \mathfrak{X} \times_{\mathrm{R}} \mathrm{K}$ *inducing an isomorphism* $\mathrm{D} \cong \mathfrak{D} \times_{\mathrm{R}} \mathrm{K}$.

In order to construct such an $\mathfrak{X}$ and $\mathfrak{D}$, the first reflex is to take any proper flat model $\mathfrak{X}$ of X and the Zariski closure $\mathfrak{D}$ of D in $\mathfrak{X}$. The closed subscheme $\mathfrak{D}$ is without doubt flat over R but may fail to be a Cartier divisor. To repair such an issue, one considers the blow-up $\mathfrak{X}'$ of $\mathfrak{X}$ in $\mathfrak{D}$. The inverse image $\mathfrak{D}'$ of $\mathfrak{D}$ in $\mathfrak{X}'$ is thus Cartier, but will not be flat over R anymore in general. Looking at the Zariski closure in $\mathfrak{X}'$ of the generic fiber of $\mathfrak{D}'$ brings us back to square one. Not to be caught in a vicious circle, one has to choose carefully the blow-up of $\mathfrak{X}$ to perform. Namely, if $\mathcal{J}$ is the sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$ defining the closed subscheme $\mathfrak{D}$, then we consider the blow-up $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ of $\mathfrak{X}$ in the ideal $\mathcal{J} + \varpi^n\mathcal{O}_{\mathrm{X}}$, for some big enough $n \geqslant 1$. The strict transform $\mathfrak{D}'$ of $\mathfrak{D}$ in $\mathfrak{X}'$ is shown to be the effective Cartier divisor $\pi^*\mathfrak{D} - \mathcal{E}$, where $\mathcal{E}$ is the exceptional divisor and the minus sign stands for the difference of effective Cartier divisors.

With theorem G under the belt, the proof of theorem E is achieved by arbitrarily enlarging the compact subset obtained as Raynaud's generic fiber of the model—a routine operation.

3.3. Further questions. The results above leave at least three questions without an answer. The first one was the one I began with:

Question. *Is there a non-affine algebraic variety* X *over* K *such that* X^{an} *is Stein?*

Thanks to the results in this chapter, now we know that if such a variety X exists, then we may assume that it satisfies the following conditions:

(1) It is integral normal quasi-affine and of dimension $\geqslant 3$.
(2) If Y is a normal affine variety containing X as an open subset, then Y has to be singular: the complement $\mathrm{Y} \smallsetminus \mathrm{X}$ is pure of codimension 1 by Hartogs' phenomenon. If Y were to be smooth, then $\mathrm{Y} \smallsetminus \mathrm{X}$ would be an effective Cartier divisor and X affine.

(3) Each closed algebraic subvariety X' of X of dimension $\leqslant 2$ is affine.
(4) The ring $\Gamma(X, \mathcal{O}_X)$ is not Noetherian.
(5) The restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(C, \mathcal{O}_C)$ is surjective for any subvariety $C \subset X$ of dimension $\leqslant 1$.

It is worth noting that Neeman's threefold does not give such an example: it satisfies (1)–(4) but not (5). A weaker question to which I have no answer to offer is:

Question. *Is there a variety* X *over* K *and a coherent sheaf* $\mathcal{F}$ *on* X *such that the natural map* $H^1(X, \mathcal{F}) \to H^1(X^{an}, \mathcal{F}^{an})$ *is not injective?*

Leaving behind the non-Archimedean world, the first question has its counterpart in the complex framework:

Question. *Is there a non-affine complex algebraic variety* X *such that* X^{an} *is Stein and* $\Gamma(X, \mathcal{O}_X) \subset \Gamma(X^{an}, \mathcal{O}_{X^{an}})$ *is dense?*

A sought-for example can be assumed to undergo restrictions (1)–(5). Again, Neeman's example X is not of this kind because $\Gamma(X, \mathcal{O}_X) \subset \Gamma(X^{an}, \mathcal{O}_{X^{an}})$ is not dense, despite containing plenty of non-constant functions.

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