THE MONODROMY OF FAMILIES OF SUBVARIETIES ON ABELIAN VARIETIES

ARIYAN JAVANPEYKAR, THOMAS KRÄMER, CHRISTIAN LEHN, and MARCO MACULAN

Abstract

*Motivated by work of Lawrence, Venkatesh, and Sawin, we show that non-isotrivial families of subvarieties in abelian varieties have big monodromy when twisted by generic rank-*1 *local systems. While Lawrence and Sawin discuss the case of subvarieties of codimension* 1*, our results hold for subvarieties of codimension at least half the dimension of the ambient abelian variety. For the proof, we use a combination of geometric arguments and representation theory to show that the Tannaka groups of intersection complexes on such subvarieties are big.*

Contents

1. Introduction

Recently, Lawrence and Venkatesh [40] have developed a technique to prove nondensity of integral points on varieties that are defined over a number field and support a geometric variation of Hodge structures with big monodromy. They used this

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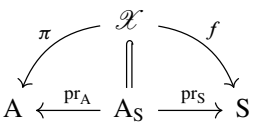
method to give an alternative proof of the Mordell conjecture and to show nondensity for hypersurfaces in projective space of a given (high) degree with good reduction outside a fixed finite set of primes. Later, Lawrence and Sawin [39] applied this strategy to show that (up to translation) any abelian variety over a number field contains only finitely many smooth ample hypersurfaces with given Néron–Severi class and good reduction outside a fixed finite set of primes. The main novelty of their work lies in their way to control monodromy. The arguments of Lawrence and Venkatesh have a topological flavor. For the Mordell conjecture they rely on a judicious choice of Dehn twists; for hypersurfaces in projective space they use the computation of the integral monodromy of the universal family by Beauville [3] (based on the work of Ebeling [17] and Janssen [27]); see also the discussion by Katz in [29]. Instead, the approach by Lawrence and Sawin involves Tannaka groups of perverse sheaves on abelian varieties introduced by Krämer and Weissauer [38]; the relation of these groups to monodromy is reminiscent of the one between the monodromy group of a variation of Hodge structures and its generic Mumford–Tate group (see [2]).

With a view towards new arithmetic applications along these lines (see [35]), we prove a big monodromy theorem for families of subvarieties of higher codimension in abelian varieties. Our results hold for all subvarieties of codimension at least half the dimension of the abelian variety. The geometry in this codimension range is very different from the codimension-1 case in [39], and the results about Tannaka groups that we obtain on the way may be of independent interest.

1.1. Big monodromy

Let S be a smooth irreducible variety over an algebraically closed field k of characteristic 0. Let A be an abelian variety of dimension g over k. Inside the constant abelian scheme $\mathrm{A_S} := \mathrm{A} \times \mathrm{S}$, let $\mathscr{X} \subset \mathrm{A_S}$ be a closed subvariety which is smooth over S with connected fibers of dimension d. The goal of this paper is to understand the monodromy of rank-1 local systems on the smooth proper family $f\colon \mathscr{X} \to \mathrm{S}$ in the following diagram:



Our results apply both in the analytic and in the algebraic setup, using topological local systems with coefficients in $\mathbb{F} = \mathbb{C}$ for $k = \mathbb{C}$ (resp., étale ℓ-adic local systems with coefficients in $\mathbb{F} = \overline{\mathbb{Q}}_\ell$ for a prime ℓ) over an arbitrary algebraically closed field k of characteristic 0. Let $\pi_1(\mathrm{A}, 0)$ be the topological (resp., étale) fundamental group with the discrete (resp., profinite) topology, and denote the group of its continuous characters by

$$\Pi(\mathrm{A},\mathbb{F}) = \mathrm{Hom}\big(\pi_1(\mathrm{A},0),\mathbb{F}^\times\big).$$

In what follows, by a *linear subvariety* we mean a subset $\Pi(\mathrm{B},\mathbb{F}) \subset \Pi(\mathrm{A},\mathbb{F})$ for an abelian quotient variety $\mathrm{A} \twoheadrightarrow \mathrm{B}$ with $\dim \mathrm{B} < \dim \mathrm{A}$. We say that a statement holds for *most* $\chi \in \Pi(\mathrm{A},\mathbb{F})$ if it holds for all χ outside a finite union of torsion translates of linear subvarieties. For $\chi \in \Pi(\mathrm{A},\mathbb{F})$, let L_χ denote the associated rank-1 local system on A. It follows from generic vanishing (see [6], [38], [47]) that for most χ the higher direct images $\mathrm{R}^i f_*\pi^*\mathrm{L}_\chi$ vanish in all degrees $i \neq d$; we consider the local system

$$\mathrm{V}_\chi := \mathrm{R}^d f_*\pi^*\mathrm{L}_\chi$$

of rank $|e|$, where e is the topological Euler characteristic of the fibers of $\mathscr{X} \to \mathrm{S}$. More generally, the study of finite étale covers of the subvariety $\mathscr{X} \subset \mathrm{A}_\mathrm{S}$ induced by finite étale covers of A leads to direct sums

$$\mathrm{V}_{\underline{\chi}} := \mathrm{V}_{\chi_1} \oplus \cdots \oplus \mathrm{V}_{\chi_n},$$

where $\underline{\chi} = (\chi_1,\ldots,\chi_n) \in \Pi(\mathrm{A},\mathbb{F})^n$ is an n-tuple of characters of the fundamental group. Using the natural identification $\Pi(\mathrm{A},\mathbb{F})^n = \Pi(\mathrm{A}^n,\mathbb{F})$, we will also apply the terminology *most* for such n-tuples of characters. Consider for $s \in \mathrm{S}(k)$ the monodromy representation

$$\rho\colon \pi_1(\mathrm{S},s) \longrightarrow \mathrm{GL}(\mathrm{V}_{\underline{\chi},s}) \qquad \text{on the fiber} \qquad \mathrm{V}_{\underline{\chi},s} = \bigoplus_{i=1}^{n} \mathrm{H}^d(\mathscr{X}_s,\mathrm{L}_{\chi_i}).$$

The *algebraic monodromy group* of the local system $\mathrm{V}_{\underline{\chi}}$ is the Zariski closure of the image of ρ. By construction, it is an algebraic subgroup of

$$\mathrm{GL}(\mathrm{V}_{\chi_1,s}) \times \cdots \times \mathrm{GL}(\mathrm{V}_{\chi_n,s}) \subset \mathrm{GL}(\mathrm{V}_{\underline{\chi},s}).$$

This upper bound can sometimes be refined. We say that the subvariety $\mathscr{X} \subset \mathrm{A}_\mathrm{S}$ is *symmetric up to translation* if there exists $a\colon \mathrm{S} \to \mathrm{A}$ such that $\mathscr{X}_t = -\mathscr{X}_t + a(t)$ for all $t \in \mathrm{S}(k)$. In this case, Poincaré duality furnishes a nondegenerate bilinear pairing

$$\theta_{\chi,s}\colon \mathrm{V}_{\chi,s} \otimes \mathrm{V}_{\chi,s} \longrightarrow \mathrm{L}_{\chi,a(s)}$$

for each $\chi \in \Pi(\mathrm{A},\mathbb{F})$, because for the dual of a rank-1 local system and for its inverse image under the translation $\tau_{a(t)}\colon \mathrm{A} \to \mathrm{A}$, $x \mapsto x + a(t)$ we have natural isomorphisms

$$\mathrm{L}_\chi^\vee \simeq [-1]^*\mathrm{L}_\chi, \qquad \tau_{a(t)}^*\mathrm{L}_\chi \simeq \mathrm{L}_\chi \otimes_\mathbb{F} \mathrm{L}_{\chi,a(t)}.$$

The pairing $\theta_{\chi,s}$ is symmetric if d is even, and alternating otherwise. Since the pairing is compatible with the monodromy operation on the fiber, it follows that the algebraic

monodromy group of $V_{\underline{\chi}}$ is contained in an orthogonal (resp., symplectic) group in the two cases. This leads to the following definition.

Definition
We say that $V_{\underline{\chi}}$ has *big monodromy* if its algebraic monodromy group contains the group $G_1 \times \cdots \times G_n$ as a normal subgroup where $G_i \subset \mathrm{GL}(V_{\chi_i,s})$ is defined by

$$G_i := \begin{cases} \mathrm{SL}(V_{\chi_i,s}) & \text{if } \mathscr{X} \text{ is not symmetric up to translation,} \\ \mathrm{SO}(V_{\chi_i,s}, \theta_{\chi_i,s}) & \text{if } \mathscr{X} \text{ is symmetric up to translation and } d \text{ is even,} \\ \mathrm{Sp}(V_{\chi_i,s}, \theta_{\chi_i,s}) & \text{if } \mathscr{X} \text{ is symmetric up to translation and } d \text{ is odd.} \end{cases}$$

Note that the connected component of the algebraic monodromy group of $V_{\underline{\chi}}$ is unaffected by base change along étale morphisms $S' \to S$. To take this into account, we consider the fiber $\mathscr{X}_{\bar{\eta}}$ of $\mathscr{X} \to S$ at a geometric generic point $\bar{\eta}$ of S. There are four obvious cases where the local system $V_{\underline{\chi}}$ does not have big monodromy. We say that $\mathscr{X}_{\bar{\eta}} \subset A_{S,\bar{\eta}}$ is

(1) *constant up to a translation* if it is the translate of a subvariety $Y \subset A$ along a point in $A(\bar{\eta})$. In this case, the algebraic monodromy is finite.

(2) *divisible* if it is stable under translation by a nonzero torsion point x of $A(\bar{\eta})$. In this case, the algebraic monodromy of each V_{χ_i} is itself a group of block matrices which is normalized by the group generated by the point x.

(3) a *symmetric power of a curve* if there is a smooth curve $C \subset A_{S,\bar{\eta}}$ such that the sum morphism $\mathrm{Sym}^d C \to A_{S,\bar{\eta}}$ is a closed embedding with image $\mathscr{X}_{\bar{\eta}}$ and $d \geq 2$. After an étale base change over S, we may assume that C spreads out to a relative curve $\mathscr{C} \subset A_S$ which is smooth and proper over S such that the relative sum morphism $\mathrm{Sym}^d_S \mathscr{C} \to A_S$ is a closed embedding with image $\mathscr{X}$. Then we have an isomorphism compatible with monodromy:

$$H^d(\mathscr{X}_s, L_\chi) \simeq \mathrm{Alt}^d H^1(\mathscr{C}_s, L_\chi).$$

(4) a *product* if there are smooth subvarieties $X_1, X_2 \subset A_{S,\bar{\eta}}$ of dimension greater than 0 such that the sum morphism $X_1 \times X_2 \to A_{S,\bar{\eta}}$ is a closed embedding with image $\mathscr{X}_{\bar{\eta}}$. Again, after an étale base change over S we may assume that X_i spreads out to a subvariety $\mathscr{X}_i \subset A_S$ which is smooth and proper over S such that the relative sum morphism $\mathscr{X}_1 \times \mathscr{X}_2 \to A_S$ is a closed embedding with image $\mathscr{X}$. Then we have the Künneth isomorphism which is compatible with monodromy:

$$H^d(\mathscr{X}_s, L_\chi) \simeq \bigoplus_{i_1+i_2=d} H^{i_1}(\mathscr{X}_{1,s}, L_\chi) \otimes H^{i_2}(\mathscr{X}_{2,s}, L_\chi).$$

If $\mathscr{X}_{\bar{\eta}}$ is nondivisible, then condition (1) holds if and only if the family $\mathscr{X} \to S$ is isotrivial (see Corollary 4.8). To avoid the appearance of the exceptional groups E_6 and E_7 and some low-dimensional half-spin groups, we require a mild assumption on the topological Euler characteristic.

Assumption 1.1
The topological Euler characteristic e of $\mathscr{X}_{\bar{\eta}}$ satisfies

$\lvert e\rvert \neq 27$	if $d \geq 2$ and $\mathscr{X}$ is not symmetric up to a translation,
$\lvert e\rvert \neq 56$	if $d \geq 3$ is odd and $\mathscr{X}$ is symmetric up to translation,
$\lvert e\rvert \neq 2^{2m-1}$	if $d \geq (g-1)/4$, $m \in \{3, \ldots, d\}$ has the same parity as d and $\mathscr{X}$ is symmetric up to translation.

Note that $\lvert e\rvert \geq g$ if $\mathscr{X}_{\bar{\eta}} \subset A_{S,\bar{\eta}}$ has ample normal bundle (see Lemma 2.12). We do not know any example of a smooth subvariety of $A_{S,\bar{\eta}}$ with ample normal bundle and dimension $d < (g-1)/2$ whose Euler characteristic e does not satisfy Assumption 1.1.

MAIN THEOREM (MONODROMY VERSION)
Suppose that $\mathscr{X}_{\bar{\eta}} \subset A_{S,\bar{\eta}}$ has ample normal bundle, dimension $d < (g-1)/2$, and Assumption 1.1 holds. Then the following are equivalent:

(1) *$\mathscr{X}_{\bar{\eta}}$ is nondivisible, not constant up to translation, not a symmetric power of a curve, and not a product.*
(2) *$V_{\underline{\chi}}$ has big monodromy for most torsion n-tuples $\underline{\chi} \in \Pi(A, \mathbb{F})^n$.*

Smooth proper subvarieties of a simple abelian variety have ample normal bundle. Therefore, when A is simple, the preceding theorem is as general as it gets for smooth subvarieties of dimension $d < (g-1)/2$, save the finite list of exceptions in Assumption 1.1. When A is arbitrary, the theorem can be applied in the following concrete cases.

COROLLARY
Suppose that $\mathscr{X}_{\bar{\eta}} \subset A_{S,\bar{\eta}}$ is nondivisible, not constant up to translation, and one of the following holds:

(1) *$\mathscr{X}_{\bar{\eta}}$ is a curve generating $A_{S,\bar{\eta}}$ and $g \geq 4$.*
(2) *$\mathscr{X}_{\bar{\eta}}$ is a surface with ample normal bundle which is neither a symmetric square of a curve nor a product, and $e \neq 27$, $g \geq 6$.*
(3) *$\mathscr{X}_{\bar{\eta}}$ is a complete intersection of ample divisors and $d < (g-1)/2$.*

Then $V_{\underline{\chi}}$ has big monodromy for most n-tuples $\underline{\chi} \in \Pi(A, \mathbb{F})^n$ of torsion characters.

Indeed, a smooth complete intersection of ample divisors is neither a symmetric power of a curve (Corollary 2.10) nor a product (Remark 6.3) and its topological Euler characteristic satisfies $|e| \geq 2^g$ and $|e| \neq 27, 56$ (Corollary 2.17 and Proposition 2.16).

Over $k = \mathbb{C}$, the main theorem in the analytic setup is deduced from the algebraic one by the comparison between classical and étale topology; the hypothesis that the characters are torsion is only used here. For the proof in the algebraic setting, we start as in [39] by relating the algebraic monodromy to the Tannaka group of the rank-1 local systems in question, seen as perverse sheaves on $\mathscr{X}_{\bar{\eta}}$. The idea is similar to the study of monodromy groups via Mumford–Tate groups in the complex case (see [2]). An analogue of the theorem of the fixed part due to Lawrence and Sawin says that the monodromy will be big if we can show that the Tannaka group of the geometric generic fiber is big (see Theorem 4.10); note that the property of the family being symmetric up to translation can be read off from its geometric generic fiber (Corollary 4.8). Thus, we are left with a question about the Tannaka group of the geometric generic fiber of our family. In this setting, we will reset our notation and replace k by an algebraic closure of the function field of S.

1.2. Big Tannaka groups

As before, let A be an abelian variety of dimension g over an algebraically closed field k of characteristic 0. Let $i\colon \mathrm{X} \hookrightarrow \mathrm{A}$ be the inclusion of a smooth connected closed subvariety of dimension d. We define the perverse intersection complex

$$\delta_{\mathrm{X}} := i_* \mathbb{F}_{\mathrm{X}}[d]$$

as the pushforward of the constant sheaf, shifted in cohomological degree $-d$ so that it becomes an object of the abelian category $\mathrm{Perv}(\mathrm{A}, \mathbb{F})$ of perverse sheaves on A as in [4]. As we will recall in Section 3.1, the group law on the abelian variety induces a convolution product on perverse sheaves, and the perverse intersection complex δ_{X} generates a neutral Tannaka category $\langle \delta_{\mathrm{X}} \rangle$ with respect to this convolution. For the rest of this introduction, we fix a character $\chi \in \Pi(\mathrm{A}, \mathbb{F})$ with $\mathrm{H}^i(\mathrm{X}, \mathrm{L}_\chi) = 0$ for all $i \neq d$. Such a character exists by generic vanishing. We then have (see [38, Theorem 13.2]) a fiber functor

$$\omega\colon \quad \langle \delta_{\mathrm{X}} \rangle \longrightarrow \mathrm{Vect}(\mathbb{F}), \qquad \mathrm{P} \longmapsto \mathrm{H}^0(\mathrm{A}, \mathrm{P} \otimes \mathrm{L}_\chi).$$

Applying this fiber functor to $\mathrm{P} = \delta_{\mathrm{X}}$, we recover the vector space

$$\mathrm{V} := \omega(\delta_{\mathrm{X}}) = \mathrm{H}^0(\mathrm{A}, \delta_{\mathrm{X}} \otimes \mathrm{L}_\chi) = \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\chi).$$

The automorphisms of the fiber functor are represented by a reductive algebraic group $\mathrm{G}_{\mathrm{X},\omega} := \mathrm{G}_\omega(\delta_{\mathrm{X}}) \subset \mathrm{GL}(\mathrm{V})$ which we call the *Tannaka group of* X (see also

Definition 3.2). The definitions in Section 1.1 with $S = \mathrm{Spec}(k)$ show that if $X \subset A$ is symmetric up to translation, then V comes with a natural symmetric bilinear form θ which is induced by Poincaré duality. This bilinear form is symmetric or alternating depending on the parity of d, and it is preserved by the action of the group $G_{X,\omega}$ as in [36, Lemma 2.1]. Let $G^{\circ}_{X,\omega} \subset G_{X,\omega}$ be the connected component of the identity, and let

$$G^{*}_{X,\omega} := [G^{\circ}_{X,\omega}, G^{\circ}_{X,\omega}]$$

be its derived group, which is a connected semisimple algebraic group.

Definition
We say that the Tannaka group $G_{X,\omega}$ of X is *big* if the derived group of its connected component of the identity is

$$G^{*}_{X,\omega} = \begin{cases} \mathrm{SL}(V) & \text{if X is not symmetric up to translation,} \\ \mathrm{SO}(V,\theta) & \text{if X is symmetric up to translation and } d \text{ is even,} \\ \mathrm{Sp}(V,\theta) & \text{if X is symmetric up to translation and } d \text{ is odd.} \end{cases}$$

The main theorem from the previous section is obtained by combining the analogue of the theorem of the fixed part by Lawrence and Sawin (Theorem 4.10) with the following result, whose proof will be the main task of this paper. Again we need to exclude a finite list of values of the topological Euler characteristic e of X, for which we refer to Assumption 1.1 with $S = \mathrm{Spec}(k)$ and $\mathscr{X} = X$.

MAIN THEOREM (TANNAKA VERSION)
Suppose that $X \subset A$ *has ample normal bundle, dimension* $d < (g-1)/2$, *and Assumption 1.1 holds. Then the following are equivalent:*
(1) X *is nondivisible, not a symmetric power of a curve, and not a product.*
(2) *The Tannaka group* $G_{X,\omega}$ *is big.*

The preceding statement can be applied in the following special cases.

COROLLARY
Suppose that $X \subset A$ *is nondivisible and one of the following holds:*
(1) X *is a curve generating* A *and* $g \geq 4$.
(2) X *is a surface with ample normal bundle which is neither a product nor the symmetric square of a curve, and* $e \neq 27$, $g \geq 6$.
(3) X *is a complete intersection of ample divisors and* $d < (g-1)/2$.
Then the Tannaka group $G_{X,\omega}$ *is big.*

The Tannaka version of the main theorem also applies when X does not arise from a family as in Section 1.1, so it is stronger than the monodromy version. We also note that over the complex numbers both versions apply in many cases where we have no control on Mumford–Tate groups of the subvarieties. Again, when X is a complete intersection of ample divisors, then automatically $|e| \neq 27, 56$, X is not a symmetric power of a curve nor a product.

1.3. Sum morphisms

Before we describe the proof of the main theorem, let us illustrate the meaning of big Tannaka groups with a simple application. Let $X \subset A$ be a subvariety of dimension d. For any integer $r \geq 1$, the sum morphism induces a morphism

$$\mathrm{Sym}^r X \longrightarrow W_r(X) := X + \cdots + X \subset A$$

onto the r-fold sum of the subvariety inside the abelian variety. If $X \subset A$ has ample normal bundle, or more generally if it is nondegenerate in the sense of Section 2.3 below, then for $r < g/d$ this sum morphism is generically finite onto its image. In general it will not be birational.

Example

Let C be a smooth projective curve of genus $g \geq 2$, seen as a subvariety of its Jacobian variety $A = \mathrm{Pic}^0(C)$ via the Abel–Jacobi embedding for a given basepoint. If C has gonality $> d$, then

$$X := W_d(C) \subset A$$

is a smooth subvariety. For $r > 1$, the map $\mathrm{Sym}^r X \to W_r(X) = W_{rd}(C)$ is not birational. Moreover, the symmetric power $\mathrm{Sym}^r X$ is singular for $d > 1$ but its image $W_r(X) = W_{rd}(C)$ is smooth if C has gonality greater than rd.

In the above example the Tannaka group $G_{X,\omega}$ is not big for $d > 1$ (see Lemma 7.2 below). For subvarieties whose Tannaka group is big, which by our main theorem is true in most cases, we have the following result.

THEOREM

Let $X \subset A$ be a smooth subvariety of dimension d with ample normal bundle, and let r be an integer with $2 \leq r < g/d$. If the Tannaka group $G_{X,\omega}$ is big, then the sum morphism $\mathrm{Sym}^r X \to W_r(X)$ is birational and for $d > 1$ its image $W_r(X)$ is singular.

The key point here is the birationality, which will be shown in Lemma 3.8. Once the birationality is known, the smoothness of $W_r(X)$ implies that the sum morphism

is an isomorphism by Proposition B.4. In particular, $\mathrm{Sym}^r X$ is then also smooth, so $d = 1$ by Proposition B.3. The proof of Lemma 3.8 relates the direct image of the constant sheaf under the sum morphism to the decomposition of wedge or symmetric powers of $V \in \mathrm{Rep}_{\mathbb{F}}(G_{X,\omega})$. In fact Larsen's alternative yields a necessary and sufficient criterion for the Tannaka group to be big, using only the decomposition of the direct image of the constant sheaf under the sum morphism for $r = 2$. But it seems hard to control this direct image in the generality needed for our main theorem, so for the proof of the main theorem we follow a different route that will be described in Sections 1.4, 1.5, and 1.6.

1.4. Simplicity of Tannaka groups

The first step in our proof of the main theorem from the previous section will be to show that under the given assumptions, the algebraic group $G^*_{X,\omega}$ is simple. We refine the methods in [33, Section 6] to obtain the following simplicity criterion (see Theorem 6.1).

THEOREM A

Suppose that $X \subset A$ *is nondivisible and has ample normal bundle. Then for* $g \geq 3$ *the following are equivalent:*

(1) *The algebraic group* $G^*_{X,\omega}$ *is not simple.*

(2) *There are smooth positive-dimensional subvarieties* $X_1, X_2 \subset A$ *such that the sum morphism induces an isomorphism*

$$X_1 \times X_2 \xrightarrow{\sim} X.$$

A smooth projective curve $C \subset A$ generating A has ample normal bundle, thus the algebraic group $G^*_{C,\omega}$ is simple for $g \geq 3$. When $g = 2$, the simplicity of $G^*_{C,\omega}$ remains open. More generally, Theorem A implies that $G^*_{X,\omega}$ is simple when $X \subset A$ is nondivisible with ample normal bundle and

(1) the image of the Albanese morphism $X \to \mathrm{Alb}(X)$ is nondegenerate in the sense of Section 2.3, or more generally

(2) the natural morphism $\varphi\colon \mathrm{Alb}(X) \to A$ is an isogeny. By Debarre's Barth–Lefschetz theorem for abelian varieties (see [11, Theorem 4.5] or Remark 6.3) this is the case as soon as $d > g/2$ or when X is a complete intersection of ample divisors and $d \geq 2$.

Note that situation (2) above is a particular case of (1).

Our proof of Theorem A uses characteristic cycles on the cotangent bundle T^*A and their link with representation theory (see [33], [34]). The idea is roughly as follows. If the group $G^*_{X,\omega}$ is not simple, then the representation $V = \omega(\delta_X)$ is an external tensor product of representations. This allows us to decompose the characteristic

cycle of the perverse sheaf δ_X as a Pontryagin product. But for smooth subvarieties the characteristic cycle is integral and equal to the conormal bundle to $X \subset A$. Using our assumption that the normal bundle is ample, we can then rule out decompositions as Pontryagin products via computations with Segre classes. For convenience, we recall some relevant background in Section 5, together with computations for the Dynkin types A, B, D to be used later. The integrality of the characteristic cycle also implies that the representation $\omega(\delta_X) \in \mathrm{Rep}_{\mathbb{F}}(G^*_{X,\omega})$ is minuscule in the sense that its weights for a maximal torus form a single Weyl group orbit, due to the following general result (see Corollary 5.15).

FACT

Let $P \in \mathrm{Perv}(A, \mathbb{F})$ *be a perverse sheaf whose characteristic cycle is integral and not stable under any nontrivial translation on the abelian variety. Then* $\omega(P)$ *is a minuscule representation of* $G_\omega(P)$.

There are only few nontrivial minuscule representations V of a simply connected simple algebraic group G, all of which are listed in Table 1. The dimension of $\omega(\delta_X)$ is the absolute value of the topological Euler characteristic of X. Recall that the subvariety $X \subset A$ is symmetric up to a translation if and only if the vector space $\omega(\delta_X)$ carries a nondegenerate bilinear form preserved by the action of $G^*_{X,\omega}$, and this pairing is symmetric if d is even and alternating if d is odd (see [36, Lemma 2.1]). This rules out the occurrence of E_6 for symmetric subvarieties; note that the group E_6 appears as the Tannaka group of the Fano surface in the intermediate Jacobian of a smooth cubic threefold, but $d = (g-1)/2$ here because $d = 2$ and $g = 5$. Similarly, the group E_7 preserves a nondegenerate alternating bilinear form on its 57-dimensional irreducible representation, so subvarieties X with $G^*_{X,\omega} \simeq E_7$ must be odd-dimensional. However, for $d = 1$ this does not happen as we show by a direct geometric argument (see Corollary 3.11), and in higher dimension we do not have any such example. Altogether, to prove that the Tannaka group is big and conclude the proof of the main theorem from

Table 1. Minuscule representations V of simply connected simple groups G.

Dynkin type	G	V	dim V
A_n	SL_{n+1}	rth wedge power	$\binom{n+1}{r}$
B_n	$Spin_{2n+1}$	spin	2^n
C_n	Sp_{2n}	standard	$2n$
D_n	$Spin_{2n}$	standard of SO_{2n}	$2n$
D_n	$Spin_{2n}$	half-spins	2^{n-1}
E_6	E_6	smallest nontrivial or its dual	27
E_7	E_7	smallest nontrivial	56

Section 1.2, we are left with wedge powers and spin representations. The next two sections will characterize the occurrence of the former and rule out the latter.

1.5. Wedge powers

In contrast to the situation for hypersurfaces studied by Lawrence and Sawin in [39], one cannot rule out the occurrence of nontrivial wedge powers for subvarieties of higher codimension by numerical arguments. In fact, wedge powers do appear, but we will use geometric arguments to obtain the following complete classification (see Theorem 7.3).

THEOREM B

Suppose that $\mathrm{X} \subset \mathrm{A}$ *is nondivisible, has ample normal bundle, and* $d < (g-1)/2$. *Then, the following are equivalent:*

(1) *There are integers* r, n *with* $1 < r \leq n/2$ *such that* $\mathrm{G}^*_{\mathrm{X},\omega} \simeq \mathrm{Alt}^r(\mathrm{SL}_n)$ *and* $\omega(\delta_{\mathrm{X}})$ *is the* r*th wedge power of the standard representation.*

(2) *There is a nondegenerate irreducible smooth projective curve* $\mathrm{C} \subset \mathrm{A}$ *such that*
- $\mathrm{X} = \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{A}$ *is the sum of* r *copies of* C, *and*
- *the sum morphism* $\mathrm{Sym}^r \mathrm{C} \to \mathrm{X}$ *is an isomorphism.*

1.6. Spin representations

Recall that for $\mathrm{N} \geq 3$ the group $\mathrm{SO}_{\mathrm{N}}(\mathbb{F})$ admits a double cover by the *spin group* $\mathrm{Spin}_{\mathrm{N}}(\mathbb{F})$, a simply connected algebraic group with a faithful representation $\mathbb{S}_{\mathrm{N}}$, the *spin representation*. We have $\dim \mathbb{S}_{\mathrm{N}} = 2^n$ for $n = \lfloor \mathrm{N}/2 \rfloor$, and if N is odd, then the spin representation is irreducible. If $\mathrm{N} = 2n$ is even, then the spin representation $\mathbb{S}_{\mathrm{N}} \simeq \mathbb{S}^+_{\mathrm{N}} \oplus \mathbb{S}^-_{\mathrm{N}}$ splits as the direct sum of two irreducible representations called the *half-spin representations*. They both have dimension 2^{n-1}. For odd $n = 2m+1$, the half-spin representations are both faithful and dual to each other; for even $n = 2m$, they are both self-dual and their images $\mathrm{Spin}^{\pm}_{4m}(\mathbb{F}) \subset \mathrm{GL}(\mathbb{S}^{\pm}_{4m})$ are called *half-spin groups*. We show that spin or half-spin groups do not occur for smooth nondivisible subvarieties of high enough codimension (see Theorem 8.3).

THEOREM C

Suppose that $\mathrm{X} \subset \mathrm{A}$ *is nondivisible, has ample normal bundle, and* $d < (g-1)/2$. *Then the pair* $(\mathrm{G}_{\mathrm{X},\omega}, \omega(\delta_{\mathrm{X}}))$ *is not isomorphic to any of the above spin or half-spin groups with their spin or half-spin representations unless*

$$\big(\mathrm{G}^*_{\mathrm{X},\omega}, \omega(\delta_{\mathrm{X}})\big) \simeq \big(\mathrm{Spin}^{\pm}_{4m}(\mathbb{F}), \mathbb{S}^{\pm}_{4m}\big) \quad \textit{for some } m \in \{3, \ldots, d\},$$

in which case X *has topological Euler characteristic of absolute value* 2^{2m-1} *and is symmetric up to a translation,* $d - m$ *is even and* $d \geq (g-1)/4$.

The main theorem in Section 1.2 now follows by combining Theorems A, B, C, and from this we also obtain the main theorem in Section 1.1 by the analogue of the theorem of the fixed part given by Theorem 4.10.

1.7. Conventions and notation

We always work over a field k of characteristic 0. A *variety* over k is a separated finite type k-scheme, and a *subvariety* is a closed subvariety unless said otherwise. An *algebraic group* is a finite type group scheme over a field. For a locally free sheaf $\mathscr{E}$ (of finite rank) on a variety X, we denote by $\mathbb{P}(\mathscr{E}) := \operatorname{Proj} \operatorname{Sym}^{\bullet} \mathscr{E}^{\vee}$ the associated projective bundle. If A is an abelian variety over k, we denote by $\operatorname{Lie} \mathrm{A}$ its tangent space at the identity and define $\mathbb{P}_{\mathrm{A}} := \mathbb{P}((\operatorname{Lie} \mathrm{A})^{\vee})$. For a smooth projective connected variety X, we denote by $\mathrm{Pic}^0(\mathrm{X})$ the connected component of the identity in its Picard scheme. This is an abelian variety, and we denote by $\mathrm{Alb}(\mathrm{X})$ its dual abelian variety. Given a locally closed subvariety Y of a variety X over k, let $\mathscr{C}_{\mathrm{Y/X}}$ denote the *conormal sheaf* of Y in X, that is, the $\mathscr{O}_{\mathrm{Y}}$-module I/I^2, where I is the ideal sheaf of the closed immersion $i : \mathrm{Y} \to \mathrm{U}$ for a suitable open subset $\mathrm{U} \subset \mathrm{X}$.

2. Gauss maps, positivity, and nondegeneracy

In this section, we recall from the view of conormal geometry various notions of positivity and nondegeneracy for subvarieties in abelian varieties. We denote by A an abelian variety over an algebraically closed field k of characteristic 0.

2.1. The stabilizer and the abelian variety generated

The *stabilizer* of a subvariety $\mathrm{X} \subset \mathrm{A}$ is the algebraic subgroup $\mathrm{Stab}_{\mathrm{A}}(\mathrm{X}) \subset \mathrm{A}$ whose k-points are

$$\mathrm{Stab}_{\mathrm{A}}(\mathrm{X})(k) = \big\{a \in \mathrm{A}(k) \;\big|\; \mathrm{X} + a = \mathrm{X}\big\}.$$

Write $\mathrm{Stab}(\mathrm{X}) = \mathrm{Stab}_{\mathrm{A}}(\mathrm{X})$ if the ambient abelian variety is clear from the context.

Definition 2.1

We say $\mathrm{X} \subset \mathrm{A}$ is *nondivisible* if it is integral and $\mathrm{Stab}(\mathrm{X})$ is trivial.

If $\mathrm{X} \subset \mathrm{A}$ is a connected subvariety, then the *abelian subvariety generated by* X is defined to be the smallest abelian subvariety $\langle \mathrm{X} \rangle \subset \mathrm{A}$ containing the image of the difference morphism $\mathrm{X} \times \mathrm{X} \to \mathrm{A}$, $(x, x') \mapsto x - x'$. Note that this image $\mathrm{X} - \mathrm{X} \subset \mathrm{A}$ is connected because $\mathrm{X} \times \mathrm{X}$ is so.

2.2. Conormal varieties and Gauss maps

Let us briefly recall the notion of conormal varieties and Gauss maps, which will be crucial later. For abelian varieties, the cotangent bundle Ω^1_{A} is a trivial bundle with

fiber $\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}}) = (\mathrm{Lie}\,\mathrm{A})^\vee$ of rank $g = \dim \mathrm{A}$. Consider the projection

$$p\colon \quad \mathbb{P}(\Omega^1_{\mathrm{A}}) \longrightarrow \mathbb{P}_{\mathrm{A}} = \mathbb{P}\big((\mathrm{Lie}\,\mathrm{A})^\vee\big).$$

If $\mathbb{\Lambda} \subset \mathbb{P}(\Omega^1_{\mathrm{A}})$ is a $(g-1)$-dimensional integral subvariety, then for dimension reasons the morphism

$$\gamma_{\mathbb{\Lambda}} := p_{|\mathbb{\Lambda}}\colon \quad \mathbb{\Lambda} \longrightarrow \mathbb{P}_{\mathrm{A}}$$

is either dominant (and hence generically finite) or not dominant. We say that $\mathbb{\Lambda}$ is *clean* in the first case and *negligible* in the second case. In the clean case, we denote by $\deg \mathbb{\Lambda}$ the generic degree of the generically finite dominant morphism $\gamma_{\mathbb{\Lambda}}$, and in the negligible case, we formally put $\deg \mathbb{\Lambda} = 0$.

We want to apply these definitions to conormal varieties, for which we need some more notation. For any subvariety $\mathrm{X} \subset \mathrm{A}$, its conormal sheaf $\mathscr{C}_{\mathrm{X/A}}$ fits in the exact sequence of coherent sheaves

$$\mathscr{C}_{\mathrm{X/A}} \xrightarrow{\ i\ } \Omega^1_{\mathrm{A}|\mathrm{X}} \longrightarrow \Omega^1_{\mathrm{X}} \longrightarrow 0.$$

If $\mathrm{X} \subset \mathrm{A}$ is regular immersion, then $\mathscr{C}_{\mathrm{X/A}}$ is locally free, and if X is moreover integral, then i is injective. If X is smooth, then all three terms are locally free and the sequence is short exact.

Definition 2.2
For a reduced subvariety $\mathrm{X} \subset \mathrm{A}$ its (projective) *conormal variety* $\mathbb{\Lambda}_{\mathrm{X}} \subset \mathbb{P}(\Omega^1_{\mathrm{A}})$ is the closure of $\mathbb{P}(\mathscr{C}_{\mathrm{X}^{\mathrm{reg}}/\mathrm{A}})$ in $\mathbb{P}(\Omega^1_{\mathrm{A}})$. The *Gauss map* of X is the morphism

$$\gamma_{\mathrm{X}} := \gamma_{\mathbb{\Lambda}_{\mathrm{X}}}\colon \quad \mathbb{\Lambda}_{\mathrm{X}} \longrightarrow \mathbb{P}_{\mathrm{A}}.$$

Let $\mathrm{pr}_{\mathrm{X}}\colon \mathbb{\Lambda}_{\mathrm{X}} \to \mathrm{X}$ be the projection, and let $\mathbb{\Lambda}_{\mathrm{X},x} := \mathrm{pr}_{\mathrm{X}}^{-1}(x)$ for $x \in \mathrm{X}(k)$.

Remark 2.3
As we almost exclusively work with the projective conormal varieties and not with affine ones, we will usually drop the adjective *projective*. We clearly have the following:

(1) The morphism $\gamma_{\mathrm{X}|\mathbb{\Lambda}_{\mathrm{X},x}}\colon \mathbb{\Lambda}_{\mathrm{X},x} \to \mathbb{P}_{\mathrm{A}}$ is injective.
(2) If X is smooth at a point x, then $\mathbb{\Lambda}_{\mathrm{X},x} = \mathbb{P}(\mathscr{C}_{\mathrm{X/A},x})$, where $\mathscr{C}_{\mathrm{X/A},x}$ denotes the fiber at x of the conormal bundle.
(3) If X is smooth, then $\mathbb{\Lambda}_{\mathrm{X}} = \mathbb{P}(\mathscr{C}_{\mathrm{X/A}})$.

The effect of isogenies on conormal varieties is easy to control. For an integer $e \geq 1$ and an integral subvariety $\mathrm{X} \subset \mathrm{A}$, we denote by $[e](\mathrm{X}) \subset \mathrm{A}$ its image under the

isogeny $[e]\colon \mathrm{A} \to \mathrm{A}$. We will always endow this image with the reduced subscheme structure, and we denote by $e_{\mathrm{X}} := [e]_{|\mathrm{X}}\colon \mathrm{X} \to [e](\mathrm{X})$ the finite morphism obtained by restriction of the isogeny to the given subvariety. By abuse of notation, we also denote by $[e] : \mathrm{A} \times \mathbb{P}_{\mathrm{A}} \to \mathrm{A} \times \mathbb{P}_{\mathrm{A}}$ the induced morphism. Then we have the following.

LEMMA 2.4
Let $\mathrm{X} \subset \mathrm{A}$ *be an integral subvariety, and let* $\mathrm{Y} = [e](\mathrm{X}) \subset \mathrm{A}$ *for an integer* $e \geq 1$. *Then we have an identity of cycles*

$$[e]_* \mathbb{\Lambda}_{\mathrm{X}} = \deg(e_{\mathrm{X}}) \cdot \mathbb{\Lambda}_{\mathrm{Y}}.$$

In particular, if the subvariety $\mathrm{X} \subset \mathrm{A}$ *is nondivisible, then* $[e]_* \mathbb{\Lambda}_{\mathrm{X}} = \mathbb{\Lambda}_{\mathrm{Y}}$.

Proof
The first claim follows easily from the fact that by construction the conormal variety to any integral subvariety is integral. The second claim is then clear because the morphism $e_{\mathrm{X}}\colon \mathrm{X} \to \mathrm{Y}$ is birational if X is nondivisible. □

COROLLARY 2.5
Let $\mathrm{X} \subset \mathrm{A}$ *be a smooth integral subvariety, and let* $\mathrm{Y} = [e](\mathrm{X})$ *for an integer* $e \geq 1$. *Then the fibers of* $\mathrm{pr}_{\mathrm{Y}}\colon \mathbb{\Lambda}_{\mathrm{Y}} \to \mathrm{Y}$ *are pure of dimension* $\mathrm{codim}_{\mathrm{A}} \mathrm{Y} - 1$.

Proof
Lemma 2.4 gives a commutative diagram

$$\begin{array}{ccc} \mathbb{\Lambda}_{\mathrm{X}} & \xrightarrow{[e]} & \mathbb{\Lambda}_{\mathrm{Y}} \\ {\scriptstyle \mathrm{pr}_{\mathrm{X}}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathrm{Y}}} \\ \mathrm{X} & \xrightarrow{e_{\mathrm{X}}} & \mathrm{Y} \end{array}$$

where the horizontal arrows are finite morphisms. If X is smooth, then the fibers of the morphism $\mathrm{pr}_{\mathrm{X}} : \mathbb{\Lambda}_{\mathrm{X}} \to \mathrm{X}$ are pure of dimension $\mathrm{codim}_{\mathrm{A}} \mathrm{X} - 1$. □

2.3. Positivity and nondegeneracy of subvarieties
We now discuss various notions of positivity and nondegeneracy for subvarieties of an abelian variety. We say that an integral subvariety $\mathrm{X} \subset \mathrm{A}$ is *degenerate* if there exists a surjective morphism $\pi : \mathrm{A} \to \mathrm{B}$ of abelian varieties with

$$\dim \pi(\mathrm{X}) < \min\{\dim \mathrm{B}, \dim \mathrm{X}\}.$$

Otherwise, we say that X is *nondegenerate*. Any closed point on the abelian variety is a nondegenerate subvariety, and so is the abelian variety itself. Also note that if

the abelian variety A is simple, then any integral subvariety is nondegenerate. We say that a proper integral variety X is *of general type* if there is a proper birational morphism $\nu\colon Y \to X$ from a smooth proper connected variety Y with big canonical bundle. For instance, we have the following:

(1) An integral effective divisor $X \subset A$ is nondegenerate if and only if it is ample. A curve $X \subset A$ is nondegenerate if and only if it generates A. See [11, Section 1, Examples].

(2) For any elliptic curve E and any simple abelian variety B of dimension at least 3, Debarre has constructed in [11, p. 189] a smooth subvariety

$$X \subset A = E \times B$$

of codimension 2 which is nondegenerate but whose normal bundle is not ample. The smooth subvariety is obtained by choosing a general ample divisor $D \subset B$ and intersecting $E \times D$ with a general ample divisor in A.

(3) For $i = 1, 2$, let A_i be an abelian variety, and let $X_i \subset A_i$ be a nondegenerate integral subvariety. By considering the projections onto the factors, one sees that $X_1 \times X_2 \subset A_1 \times A_2$ is of general type but degenerate.

Remark 2.6
Nondegeneracy is invariant under isogenies. Let $f : A \to A'$ be an isogeny of abelian varieties over k. Then an integral subvariety $X \subset A$ is nondegenerate if and only if $f(X) \subset A'$ is.

In what follows, we often consider the sum morphism $\sigma : X \times Y \to A$ for reduced subvarieties $X, Y \subset A$, and we denote by $X + Y \subset A$ its image. For nondegenerate subvarieties, we have the following result by Debarre.

LEMMA 2.7
Let $X, Y \subset A$ *be integral subvarieties.*

(1) *If* X *is nondegenerate, then*

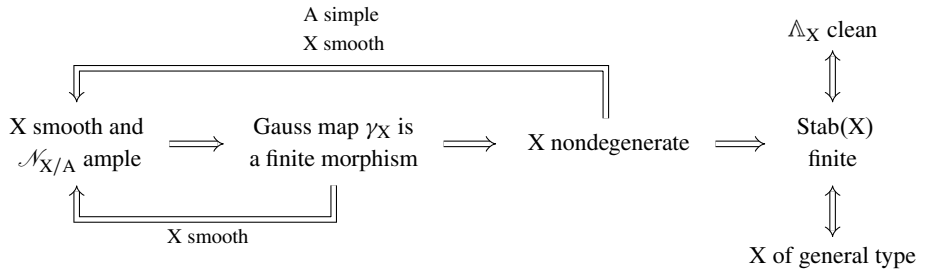
$$\dim(X + Y) = \min\bigl\{\dim(X) + \dim(Y), \dim(A)\bigr\}.$$

(2) *If* X *and* Y *are both nondegenerate, then so is* $X + Y \subset A$.

Proof
See [13, Corollary 8.11]. □

The relations between the various notions of nondegeneracy and positivity that will play a role in this paper are summarized in the following diagram where for a

smooth proper subvariety $X \subsetneq A$ we denote by $\mathcal{N}_{X/A}$ its normal bundle:



More precisely, we have the following.

THEOREM 2.8

Let $X \subset A$ *be an integral subvariety with* $0 < \dim X < \dim A$.

(1) *The following are equivalent:*
   (a) *The conormal cone* $\mathbb{A}_X$ *is clean.*
   (b) *The algebraic group* Stab(X) *is finite.*
   (c) *The variety* X *is of general type.*

(2) *If* X *is nondegenerate, then* Stab(X) *is finite and* $\langle X \rangle = A$.

(3) *If* $\gamma_X \colon \mathbb{A}_X \to \mathbb{P}_A$ *is a finite morphism, then* X *is nondegenerate.*

(4) *Suppose that* X *is smooth. Then the normal bundle* $\mathcal{N}_{X/A}$ *is ample if and only if the Gauss map* $\gamma_X \colon \mathbb{A}_X \to \mathbb{P}_A$ *is a finite morphism.*

(5) *If* A *is a simple abelian variety and* X *is of general type, then* X *is nondegenerate. If* X *is moreover smooth, then* $\mathcal{N}_{X/A}$ *is ample.*

Proof

(1) The equivalence (a) $\Leftrightarrow$ (b) is shown in [52, Theorem 1], while the equivalence (b) $\Leftrightarrow$ (c) follows from Ueno's fibration theorem (see [49, Theorem 3.10]; see also [1, Theorem 3]).

(2) For the finiteness of the stabilizer, denote by $p \colon A \to B := A/\operatorname{Stab}(X)$ the quotient morphism. This quotient morphism is not surjective, since by construction we have $p^{-1}(p(X)) = X \neq A$. The nondegeneracy of X then forces $p \colon X \to \pi(X)$ to be generically finite, and it follows that Stab(X) is finite as desired. To show that $\langle X \rangle = A$, consider the quotient morphism $q \colon A \to A/\langle X \rangle$. The image $q(X)$ is a point, hence the nondegeneracy of X and the assumption $\dim X > 0$ imply that $\dim A/\langle X \rangle = 0$, which shows that we have $\langle X \rangle = A$.

(3) We prove the contrapositive. If $X \subset A$ is degenerate, then by definition there is a surjective morphism $\pi \colon A \to B$ of abelian varieties such that $\dim Y < \min\{\dim B, \dim X\}$, where $Y := \pi(X)$. We have the following commutative

diagram of $\mathscr{O}_X$-modules with exact rows

$$\begin{array}{ccccccc}
(\pi^*\mathscr{C}_{Y/B})_{|X} & \xrightarrow{j} & (\pi^*\Omega^1_B)_{|X} & \longrightarrow & (\pi^*\Omega^1_Y)_{|X} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle d\pi} & & \downarrow & & \\
\mathscr{C}_{X/A} & \xrightarrow{i} & \Omega^1_{A|X} & \longrightarrow & \Omega^1_X & \longrightarrow & 0
\end{array}$$

where $d\pi$ is the pullback of differential forms along π. Here i is injective over the smooth locus $X^{reg} \subset X$, and likewise j is injective over $\pi^{-1}(Y^{reg})$. Indeed, the short exact sequence

$$0 \longrightarrow (\mathscr{C}_{Y/B})_{|Y^{reg}} \longrightarrow \Omega^1_{B|Y^{reg}} \longrightarrow \Omega^1_{Y^{reg}} \longrightarrow 0$$

of $\mathscr{O}_{Y^{reg}}$-modules is locally split because the $\mathscr{O}_{Y^{reg}}$-module $\Omega^1_{Y^{reg}}$ is locally free; the pullback along π of the above short exact sequence hence stays exact. It follows that ε is also injective over the nonempty open subset

$$U := X^{reg} \cap \pi^{-1}(Y^{reg}).$$

The hypothesis $\dim Y < \dim X$ implies that the morphism $\pi_{|U}\colon U \to Y^{reg}$ is not generically finite. Thus, for y ranging over a dense open subset of Y^{reg}, the fiber $Z := \pi^{-1}(y) \cap U$ is positive-dimensional. Pick $v \in \mathscr{C}_{Y,y}$ nonzero, which exists because $\dim Y < \dim B$. Then

$$0 \neq j(v) \in \bigcap_{x \in Z} \mathscr{C}_{X,x}.$$

Thus, if we denote by $F := \mathrm{pr}_X(\gamma_X^{-1}([j(v)]) \subset X$ the image of $\gamma_X^{-1}([j(v)])$ under the projection $\mathbb{\Lambda}_X \to X$, then the subset Z is contained in F. This shows that the dimension of $\gamma_X^{-1}([j(v)])$ is positive.

(4) Since X is smooth, we have $\mathbb{\Lambda}_X = \mathbb{P}(\mathscr{C}_{X/A})$. The normal bundle $\mathscr{N}_{X/A}$ is globally generated, thus the equivalence is [42, Example 6.1.5].

(5) When A is a simple abelian variety, any integral subvariety is nondegenerate, and the ampleness of the normal bundle of a smooth subvariety X in A follows from [25, Proposition 4.1]. □

2.4. Symmetric powers of curves in abelian varieties

We show here that symmetric powers of a (smooth projective) curve C cannot be embedded as a complete intersection of ample divisors as claimed in Section 1.5. Recall that the curve C has gonality at least $n+1$ if and only if the sum map induces an isomorphism $\mathrm{Sym}^n C \to X := C + \cdots + C \subset \mathrm{Pic}^0(C)$. If so, the normal bundle of X is ample (see [11, Section 1, Examples (2)]). Imposing further positivity properties to the normal bundle is far more restrictive.

PROPOSITION 2.9

Let $\mathrm{C} \subset \mathrm{A}$ *be a smooth irreducible projective curve such that the sum morphism* $\mathrm{Sym}^n \mathrm{C} \to \mathrm{X} := \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{A}$ *is an isomorphism for some* $n \geq 2$*. Then* C *is nonhyperelliptic of genus* $g \geq 3$ *and the following hold:*

(1) *If the normal bundle* $\mathscr{N}_{\mathrm{X/A}} = \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_r$ *is a direct sum of ample vector bundles, then*

$$n \leq \max_{i=1,\ldots,r} \mathrm{rk}\, \mathscr{V}_i + 1.$$

(2) *The normal bundle* $\mathscr{N}_{\mathrm{X/A}}$ *is a direct sum of ample line bundles if and only if* $g = 3$*,* $n = 2$*, and* A *is isomorphic to* $\mathrm{Pic}^0(\mathrm{C})$*.*

Proof

By Lefschetz's principle, we may assume $k = \mathbb{C}$. First of all, the curve C is nonhyperelliptic of genus $g \geq 3$. Otherwise, C would be symmetric when suitably embedded in its Jacobian. In particular, the sum morphism would contract the antidiagonal and thus would not induce an isomorphism $\mathrm{Sym}^n \mathrm{C} \simeq \mathrm{C} + \cdots + \mathrm{C}$.

(1) Arguing by contradiction, suppose that the inequality in the statement does not hold. Then we can apply the Barth–Lefschetz theorem (see [11, Theorem 4.5]) to obtain isomorphisms

$$\mathrm{H}^i(\mathrm{A}) \simeq \mathrm{H}^i(\mathrm{X}), \quad i = 1, 2,$$

of rational cohomology groups. On the other hand, the computation of cohomology of symmetric powers of curves [43, (1.2)] yields the following expressions:

$$\mathrm{H}^1(\mathrm{X}) = \mathrm{H}^1(\mathrm{Sym}^n \mathrm{C}) \simeq \mathrm{H}^1(\mathrm{C}^n)^{\mathfrak{S}_n} = \mathrm{H}^0(\mathrm{C}) \otimes \mathrm{H}^1(\mathrm{C}),$$
$$\mathrm{H}^2(\mathrm{X}) = \mathrm{H}^2(\mathrm{Sym}^n \mathrm{C}) \simeq \mathrm{H}^2(\mathrm{C}^n)^{\mathfrak{S}_n} = \mathrm{Alt}^2 \mathrm{H}^1(\mathrm{C}) \oplus \mathrm{H}^0(\mathrm{C}) \otimes \mathrm{H}^2(\mathrm{C})^{n-1}.$$

Recalling the equality $\mathrm{H}^2(\mathrm{A}) = \mathrm{Alt}^2 \mathrm{H}^1(\mathrm{A})$, we obtain a contradiction.

(2) If C has genus $g = 3$, then the subvariety $\mathrm{C} + \mathrm{C} \subset \mathrm{Pic}^0(\mathrm{C})$ is a theta divisor and hence ample. Conversely, suppose that the normal bundle $\mathscr{N}_{\mathrm{X/A}}$ is a direct sum of ample line bundles. We first claim that then A is isogenous to $\mathrm{Pic}^0(\mathrm{C})$. Indeed, as above we have isomorphisms

$$\mathrm{H}^1(\mathrm{A}) \simeq \mathrm{H}^1(\mathrm{X}) \simeq \mathrm{H}^1(\mathrm{C}).$$

We cannot conclude as in (1) because the Barth–Lefschetz theorem here only says that $\mathrm{H}^2(\mathrm{A}) \to \mathrm{H}^2(\mathrm{X})$ is injective. Instead, write $\mathscr{N}_{\mathrm{X/A}} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{g-2}$ for ample line bundles $\mathscr{L}_i$ on X. By looking at the short exact sequence

$$0 \longrightarrow \mathrm{T}_{\mathrm{X}} \longrightarrow \mathrm{Lie}\,\mathrm{A} \otimes \mathscr{O}_{\mathrm{X}} \longrightarrow \mathscr{N}_{\mathrm{X/A}} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{g-2} \longrightarrow 0,$$

we see that the line bundles $\mathscr{L}_i$ are globally generated and

$$\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_{g-2} \simeq \mathscr{K}_X, \tag{2.1}$$

where $\mathscr{K}_X = \mathrm{Alt}^2 \Omega^1_X$ is the canonical bundle on X. We identify X with $\mathrm{Sym}^2 C$ and write $\pi : C \times C \to X$ for the quotient morphism. Since π ramifies exactly on the diagonal Δ of $C \times C$, we have $\pi^* \mathscr{K}_X = \mathscr{K}_{C\times C}(-\Delta)$. Let us fix a point $p \in C(k)$ and consider the embedding $f : C \to C \times C$, $x \mapsto (x, p)$. Then

$$f^* \pi^* \mathscr{K}_X = \mathscr{K}_C(-p). \tag{2.2}$$

On the other hand, for $i = 1, \ldots, g-2$, the line bundle $\mathscr{M}_i := f^* \pi^* \mathscr{L}_i$ on C is ample and globally generated. Moreover, the curve C being nonhyperelliptic, we necessarily have $\deg \mathscr{M}_i \geq 3$. By combining (2.1) and (2.2) and then by taking degrees, we obtain the inequality

$$2g - 3 = \deg \mathscr{K}_C(-p) = \sum_{i=1}^{g-2} \deg \mathscr{M}_i \geq 3(g-2).$$

This forces $g = 3$. For a suitable Abel–Jacobi embedding $C \hookrightarrow \mathrm{Pic}^0(C)$, there exists an isogeny $\varphi : \mathrm{Pic}^0(C) \to A$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Sym}^2 C & \xrightarrow{\sim} & \Theta & \hookrightarrow & \mathrm{Pic}^0(C) \\ \| & & \downarrow \wr & & \downarrow \varphi \\ \mathrm{Sym}^2 C & \xrightarrow{\sim} & X & \hookrightarrow & A \end{array}$$

Here the leftmost horizontal arrows are induced by the sum and $\Theta \subset \mathrm{Pic}^0(C)$ is a theta divisor. The preimage $\varphi^{-1}(X)$ is smooth, thus its connected components are irreducible. As Θ is one of them, the others are $\Theta + a$ for $a \in \mathrm{Ker}\, \varphi$. Since any two translates of an ample divisor meet, we have $\Theta = \varphi^{-1}(X)$. But the isogeny φ induces an isomorphism $\Theta \simeq X$, thus φ must be injective. □

COROLLARY 2.10

Let $C \subset A$ *be a smooth irreducible projective curve such that the sum morphism* $\mathrm{Sym}^n C \to X := C + \cdots + C \subset A$ *is an isomorphism for some* $n \geq 2$. *Then* C *is nonhyperelliptic of genus* $g \geq 3$ *and the following are equivalent:*

(1) *The subvariety* $X \subset A$ *is a complete intersection of ample divisors.*

(2) *We have* $g = 3$, $n = 2$, *and* A *is isomorphic to* $\mathrm{Pic}^0(C)$.

Proof

For complete intersections of ample divisors, the normal bundle is a direct sum of ample line bundles. Hence, Proposition 2.9(2) applies. □

As an amusing aside, of no use in what follows, note that Proposition 2.9 implies the classical bound for the gonality of a smooth projective curve C of genus g.

COROLLARY 2.11
The gonality of C *is at most* $(g+3)/2$.

Proof
As already mentioned, the curve C has gonality at least $n+1$ if and only if the sum morphism $\mathrm{Sym}^n \mathrm{C} \to \mathrm{X} := \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{Pic}^0(\mathrm{C})$ is an isomorphism, and if this is the case, then X has ample normal bundle in $\mathrm{Pic}^0(\mathrm{C})$. Since the normal bundle has rank $g-n$, Proposition 2.9(1) implies $n \leq g-n+1$, that is, $n+1 \leq (g+3)/2$. □

In particular, Proposition 2.9(1) is sharp in the two extremal cases—that of an indecomposable ample normal bundle and that of a sum of ample line bundles.

2.5. Bounds for the topological Euler characteristic
We now pass to some numerics concerning the topological Euler characteristic of complete intersections. To ease notation below, we define $g := \dim \mathrm{A}$. For a smooth subvariety $\mathrm{X} \subset \mathrm{A}$, let e_{X} denote its topological Euler characteristic. By definition it is the top Chern class of the tangent bundle T_{X} of X. Consider the short exact sequence of vector bundles on X,

$$0 \longrightarrow \mathrm{T}_{\mathrm{X}} \longrightarrow \mathrm{T}_{\mathrm{A}|\mathrm{X}} \longrightarrow \mathscr{N}_{\mathrm{X}/\mathrm{A}} \longrightarrow 0.$$

Since the total Chern class is multiplicative in short exact sequences and the tangent bundle of A is trivial, we have

$$c(\mathrm{T}_{\mathrm{X}}) = c(\mathrm{T}_{\mathrm{A}|\mathrm{X}})c(\mathscr{N}_{\mathrm{X}/\mathrm{A}})^{-1} = c(\mathscr{N}_{\mathrm{X}/\mathrm{A}})^{-1}.$$

First of all, note that we have the following lower bound whenever the normal bundle is ample.

LEMMA 2.12
Let $\mathrm{X} \subsetneq \mathrm{A}$ *be a* d*-dimensional smooth subvariety with ample normal bundle. Then*

$$|e_{\mathrm{X}}| \geq \max\{g, 2^{\min\{d, \lfloor\sqrt{g-1}\rfloor\}}\}.$$

Proof
We may suppose $k = \mathbb{C}$. By definition, the inverse of the total Chern class is the total Segre class. We have $|e_{\mathrm{X}}| = (-1)^d s_d(\mathscr{N}_{\mathrm{X}/\mathrm{A}}) = s_d(\mathscr{N}_{\mathrm{X}/\mathrm{A}}^{\vee})$, where $d = \dim \mathrm{X}$ and s_d is the dth Segre class. Now the normal bundle $\mathscr{N}_{\mathrm{X}/\mathrm{A}}$ is ample and globally generated. Since $\mathrm{H}^1(\mathrm{X}, \mathbb{C}) \neq 0$, we have $|e_{\mathrm{X}}| \geq g$ by [5, Theorem 4]. We also

have $|e_{\mathrm{X}}| \geq 2^{\min\{d,\lfloor\sqrt{g-1}\rfloor\}}$ according to [18, Proposition 2.4] because the cotangent bundle of X is nef.[1] □

The previous lower bound is doubtlessly not sharp. Indeed, for a smooth projective curve X generating A, we have $|e_{\mathrm{X}}| \geq 2g-2$. For surfaces we have the following.

LEMMA 2.13
Let $\mathrm{X} \subset \mathrm{A}$ *be a smooth projective surface generating* A *and with finite stabilizer. Then*

$$e_{\mathrm{X}} \geq 3g - 9.$$

Proof
Write $c_1 = c_1(\mathrm{T}_{\mathrm{X}})$ and $c_2 = c_2(\mathrm{T}_{\mathrm{X}}) = e_{\mathrm{X}}$ and $\chi = \chi(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ as usual. By Theorem 2.8, the surface X is of general type. Thus the Bogomolov–Miyaoka–Yau inequality gives $c_1^2 \leq 3c_2$ which is in turn equivalent to $3\chi \leq c_2$ by Noether's formula. Write $q = h^1(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ and $p = h^2(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ so $\chi = 1 - q + p$. The surface X is minimal, thus we can apply the inequality $p \geq 2q - 4$ (see Beauville's appendix to [10]), which is equivalent to $\chi \geq q - 3$. Combining these inequalities yields $c_2 \geq 3(q-3)$. Since X generates A by hypothesis, we have $q \geq g$ which concludes the proof. □

When the subvariety is a complete intersection of ample divisors, the previous lower bounds can be drastically improved. In order to show this, for integers $n \geq 2$ and $r \in \{1, \ldots, n-1\}$, consider the following subset of partitions of n,

$$\mathrm{P}(n,r) := \left\{ a = (a_1, \ldots, a_r) \in \mathbb{Z}^r \;\middle|\; a_1, \ldots, a_r \geq 1, a_1 + \cdots + a_r = n \right\}.$$

Note that $\mathrm{P}(n,r)$ has cardinality $\binom{n-1}{n-r}$.

LEMMA 2.14
Let X *be a complete intersection of ample divisors* $\mathrm{D}_1, \ldots, \mathrm{D}_r$ *in* A. *If* X *is smooth, then*

$$e_{\mathrm{X}} = (-1)^{\dim \mathrm{X}} \sum_{a \in \mathrm{P}(g,r)} \mathrm{D}_1^{a_1} \cdots \mathrm{D}_r^{a_r}.$$

Proof
Since X is a complete intersection of the divisors $\mathrm{D}_1, \ldots, \mathrm{D}_r$, the normal bundle $\mathscr{N}_{\mathrm{X}/\mathrm{A}}$ is the direct sum of (the restriction to X of) the line bundles $\mathscr{O}(\mathrm{D}_1), \ldots, \mathscr{O}(\mathrm{D}_r)$. In particular,

[1] Beware that in both references the authors adopt the convention dual to the one in [20] for the definition of Segre classes.

$$c(\mathscr{N}_{X/A}) = c\big(\mathscr{O}(D_1)\big)\cdots c\big(\mathscr{O}(D_r)\big) = (1+D_1)\cdots(1+D_r) \in \mathrm{CH}(X).$$

By inverting formally $1+D_i$ we find the following expression

$$c(T_X) = \sum_{n=0}^{g-r}(-1)^n \sum_{\substack{a_1,\dots,a_r\geq 0\\ a_1+\cdots+a_r=n}} D_1^{a_1}\cdots D_r^{a_r} \in \mathrm{CH}(X).$$

Looking at it in the Chow ring of A amounts to multiplying it by $D_1\cdots D_r$. We conclude by then taking the piece of degree g. □

Recall that, for an ample divisor $D \subset A$, the self-intersection D^g is positive and divisible by $g!$, as the ratio $D^g/g!$ is given by $h^0(A, \mathscr{O}(D))$.

LEMMA 2.15
For ample divisors $D_1,\dots,D_g \subset A$*, we have* $D_1\cdots D_g \geq g!$.

Proof
The Khovanskii–Teissier inequality (see [41, Theorem 1.6.1]) states that the lower bound $(D_1\cdots D_g)^g \geq D_1^g\cdots D_g^g$ holds. Since each factor on the right-hand side is a positive multiple of $g!$, this concludes the proof. □

PROPOSITION 2.16
Let $X \subsetneq A$ *be a smooth complete intersection of ample divisors of dimension* $d \geq 1$. *Then* e_X *is even and*

$$|e_X| \geq g!\binom{g-1}{d}.$$

Proof
By assumption, X is the intersection of ample divisors $D_1,\dots,D_r$, where $r = g-d$ is the codimension of X. Lemma 2.14 shows that

$$e_X = (-1)^d \sum_{a\in P(g,r)} D_1^{a_1}\cdots D_r^{a_r}.$$

The divisors $D_1,\dots,D_r$ are ample, thus $D_1^{a_1}\cdots D_r^{a_r} \geq g!$ for each $a \in P(g,r)$. Since the cardinality of $P(g, g-d)$ is $\binom{g-1}{d}$, the inequality in the statement follows. For the parity of e_X, we may assume that $k = \mathbb{C}$ by the Lefschetz principle. It follows that each $[D_i]^{a_i} \in H^{2a_i}(A,\mathbb{Z})$ is divisible by $a_i!$. Since $d \geq 1$, for each $a \in P(g,r)$ we have $a_i \geq 2$ for some i, thus we conclude that e_X is even. □

By Proposition 2.16, the absolute value of the Euler characteristic of a smooth connected complete intersection of ample divisors in A is never equal to 27. We now prove that $|e_X| \neq 56$, except in the case of curves in abelian surfaces and abelian threefolds (in which case there are examples).

COROLLARY 2.17
If $X \subsetneq A$ *is a smooth complete intersection of ample divisors of dimension* $d \geq 1$ *and* $(d, g) \neq (1, 2), (1, 3)$*, then* $|e_X| \neq 56$.

Proof
Proposition 2.16 implies $|e_X| \geq g!\binom{g-1}{d}$ which settles the matter for $g \geq 5$. On the other hand, if X is itself a divisor, that is, $d = g - 1$, then $|e_X| = X^g$ is divisible by $g!$. The two cases left are $(d, g) = (1, 4), (2, 4)$ for which $g!\binom{g-1}{d} = 72$. □

Proposition 2.16 furnishes another proof of Corollary 2.10. Indeed the nth symmetric power of a smooth projective curve of genus $g \geq 2$ has topological Euler characteristic $(-1)^n\binom{2g-2}{n}$; see [43, (4.4)]. Using that the gonality is at most $(g + 3)/2$, we conclude because, for $g \geq 4$ and $n \leq (g + 1)/2$, we have $\binom{2g-2}{n} < g!\binom{g-1}{n}$.

3. Perverse sheaves on abelian varieties

In this section, we collect some general results about perverse sheaves on abelian varieties. We work over a field k with $\mathrm{char}(k) = 0$, but as in [39, Section 3] we do not require this field to be algebraically closed; for the relation with monodromy groups we will later need to work over function fields. For any variety X over k, we denote by

$$\mathrm{Perv}(X, \mathbb{F}) \subset D_c^b(X, \mathbb{F})$$

the abelian category of perverse sheaves with coefficients in $\mathbb{F} = \overline{\mathbb{Q}}_\ell$ for a fixed prime number ℓ. For $k = \mathbb{C}$, we will later also consider perverse sheaves in the analytic sense with coefficients in $\mathbb{F} = \mathbb{C}$, and we will use the above notation also in this case. The results below work both in the ℓ-adic setting over any field k and in the analytic setting with $k = \mathbb{F} = \mathbb{C}$. We let $\pi_1(A, 0)$ be the étale (resp., topological) fundamental group in the two settings, with the profinite (resp., discrete) topology, and write $\Pi(A, \mathbb{F}) = \mathrm{Hom}(\pi_1(A, 0), \mathbb{F}^\times)$ for the group of its continuous characters.

3.1. Convolution on abelian varieties

Let us briefly recall the Tannakian description of perverse sheaves on abelian varieties $X = A$ given in [38]. The sum morphism $\sigma : A \times A \to A$ induces a convolution product

$$* : \mathrm{D}^b_c(\mathrm{A},\mathbb{F}) \times \mathrm{D}^b_c(\mathrm{A},\mathbb{F}) \longrightarrow \mathrm{D}^b_c(\mathrm{A},\mathbb{F}), \qquad \mathrm{K}_1 * \mathrm{K}_2 := \mathrm{R}\sigma_*(\mathrm{K}_1 \boxtimes \mathrm{K}_2)$$

which endows the derived category with the structure of a rigid symmetric monoidal category (see [51]). In [51] this is stated only over algebraically closed fields k, but the proof works in the general case without changes. The subcategory of perverse sheaves is not stable under the convolution product, but it becomes so after passing to a certain quotient category. To explain this, recall that for any $\mathrm{P} \in \mathrm{Perv}(\mathrm{A},\mathbb{F})$ we have

$$\chi(\mathrm{A},\mathrm{P}) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{F}} \mathrm{H}^i(\mathrm{A},\mathrm{P}) \geq 0.$$

Indeed, over $k = \mathbb{C}$ this was observed by Franecki and Kapranov [19, Corollary 1.4]; the case of an arbitrary algebraically closed field k of characteristic 0 can be reduced to the complex case by choosing a model over some algebraically closed subfield of k which embeds into the complex numbers (see Lemma A.1). The additivity of the Euler characteristic in short exact sequences then implies that perverse sheaves of Euler characteristic 0 form a Serre subcategory

$$\mathrm{S}(\mathrm{A},\mathbb{F}) := \left\{ \mathrm{P} \in \mathrm{Perv}(\mathrm{A},\mathbb{F}) \;\middle|\; \chi(\mathrm{A},\mathrm{P}) = 0 \right\} \subset \mathrm{Perv}(\mathrm{A},\mathbb{F})$$

inside the abelian category of perverse sheaves. Let $\mathrm{T}(\mathrm{A},\mathbb{F}) \subset \mathrm{D}^b_c(\mathrm{A},\mathbb{F})$ be the full subcategory of sheaf complexes whose perverse cohomology sheaves are in $\mathrm{S}(\mathrm{A},\mathbb{F})$; its objects will be called *negligible sheaf complexes*.

PROPOSITION 3.1
The quotient category $\overline{\mathrm{D}^b_c}(\mathrm{A},\mathbb{F}) := \mathrm{D}^b_c(\mathrm{A},\mathbb{F})/\mathrm{T}(\mathrm{A},\mathbb{F})$ *is triangulated and inherits from the perverse t-structure on the derived category a t-structure whose heart*

$$\overline{\mathrm{Perv}}(\mathrm{A},\mathbb{F}) \subset \overline{\mathrm{D}^b_c}(\mathrm{A},\mathbb{F})$$

is equivalent to the abelian quotient category $\mathrm{Perv}(\mathrm{A},\mathbb{F})/\mathrm{S}(\mathrm{A},\mathbb{F})$. *It also inherits the structure of a rigid symmetric monoidal category with respect to a convolution product*

$$* : \overline{\mathrm{D}^b_c}(\mathrm{A},\mathbb{F}) \times \overline{\mathrm{D}^b_c}(\mathrm{A},\mathbb{F}) \longrightarrow \overline{\mathrm{D}^b_c}(\mathrm{A},\mathbb{F})$$

induced by the convolution product on the derived category. On the triangulated quotient category, this product is t-exact in both of its arguments. Thus, it restricts to a product

$$* : \overline{\mathrm{Perv}}(\mathrm{A},\mathbb{F}) \times \overline{\mathrm{Perv}}(\mathrm{A},\mathbb{F}) \longrightarrow \overline{\mathrm{Perv}}(\mathrm{A},\mathbb{F}),$$

and $\overline{\mathrm{Perv}}(\mathrm{A},\mathbb{F})$ *is a neutral Tannaka category with respect to this product.*

Proof
Fix an algebraic closure $K \supset k$. The functor $D_c^b(A, \mathbb{F}) \to D_c^b(A_K, \mathbb{F})$ is exact for the perverse t-structure, compatible with the convolution product, and preserves the subcategories of negligible objects. Hence, the result follows from the statement over algebraically closed fields in [31] and [38]; note that by Deligne's internal characterization of neutral Tannaka categories (see [8, Section 6.4]), it suffices to construct a fiber functor on every finitely generated tensor subcategory. □

In what follows, by an *abelian tensor category* we mean a rigid symmetric monoidal abelian $\mathbb{F}$-linear category.

3.2. Tannaka groups of perverse sheaves
Let $\mathscr{C} \subset \overline{\mathrm{Perv}}(A, \mathbb{F})$ be a full abelian tensor subcategory, and let

$$\omega: \quad \mathscr{C} \longrightarrow \mathrm{Vect}(\mathbb{F})$$

be a given fiber functor on this subcategory. The existence of such fiber functors is guaranteed by Proposition 3.1; there is no canonical choice of such a fiber functor, but any two fiber functors on a neutral Tannaka category over an algebraically closed field $\mathbb{F}$ are noncanonically isomorphic (see [15, Theorem 3.2(b)]). Once we have chosen a fiber functor, we get an equivalence of abelian tensor categories between $\mathscr{C}$ and the category $\mathrm{Rep}_{\mathbb{F}}(G_\omega(\mathscr{C}))$ of finite-dimensional algebraic representations of the affine group scheme

$$G_\omega(\mathscr{C}) := \mathrm{Aut}^{\otimes}(\omega)$$

over $\mathbb{F}$ called the *Tannaka group* of $\mathscr{C}$. We are interested in algebraic quotients of this proalgebraic group scheme.

Definition 3.2
For any $P \in \mathscr{C}$, we obtain from the above construction an affine algebraic group

$$G_\omega(P) := \mathrm{Im}\big(G_\omega(\mathscr{C}) \to \mathrm{GL}\big(\omega(P)\big)\big)$$

over $\mathbb{F}$ with a faithful representation on the vector space $\omega(P) \in \mathrm{Vect}(\mathbb{F})$ whose dimension is the Euler characteristic (see [38, proof of Corollary 4.2])

$$\dim_{\mathbb{F}}\big(\omega(P)\big) = \chi(A, P).$$

Let $\iota\colon \langle P\rangle \hookrightarrow \mathscr{C}$ be the smallest abelian tensor subcategory which contains the object P and is stable under subobjects and quotients. Then $G_\omega(P) = G_{\omega\circ\iota}(\langle P\rangle)$ for the fiber functor $\omega \circ \iota\colon \langle P\rangle \to \mathrm{Vect}(\mathbb{F})$ and we have a commutative diagram of abelian tensor categories:

$$\begin{array}{ccc} \langle \mathrm{P} \rangle & \xrightarrow{\sim} & \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{G}_\omega(\mathrm{P})\big) \\ \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow{\sim} & \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{G}_\omega(\mathscr{C})\big) \end{array}$$

For a simple object $\mathrm{P} \in \mathscr{C}$ the faithful representation $\omega(\mathrm{P}) \in \mathrm{Rep}_{\mathbb{F}}(\mathrm{G}_\omega(\mathrm{P}))$ is irreducible and then $\mathrm{G}_\omega(\mathrm{P})$ is reductive by [26, Section 19.1, Proposition (b)]. This is in particular the case when $\mathrm{P} = \delta_{\mathrm{X}}$ is the intersection complex of an integral subvariety $\mathrm{X} \subset \mathrm{A}$, in which case we write

$$\mathrm{G}_{\mathrm{X},\omega} := \mathrm{G}_\omega(\delta_{\mathrm{X}}).$$

In what follows, fix a full abelian tensor subcategory $\mathscr{C} \subset \overline{\mathrm{Perv}}(\mathrm{A}, \mathbb{F})$ and a fiber functor $\omega \colon \mathscr{C} \to \mathrm{Vect}(\mathbb{F})$. When there is no risk of confusion, we also write ω for the restriction of the given fiber functor to any subcategory of $\mathscr{C}$.

3.3. The derived group of the connected component

It is often convenient to pass from arbitrary reductive groups to connected semisimple groups. For a reductive group G, let $\mathrm{G}^\circ \subset \mathrm{G}$ be its connected component of the identity, and note that the derived group

$$\mathrm{G}^* := [\mathrm{G}^\circ, \mathrm{G}^\circ]$$

is a connected semisimple group. For the reductive Tannaka groups from Section 3.2, we will understand the connected components and the center in terms of direct images of perverse sheaves under the morphisms $[d] \colon \mathrm{A} \to \mathrm{A}$, $x \mapsto dx$ for $d \in \mathbb{N}$ and $t_a \colon \mathrm{A} \to \mathrm{A}$, $x \mapsto x + a$ for $a \in \mathrm{A}(k)$. For a perverse sheaf $\mathrm{Q} \in \mathrm{Perv}(\mathrm{A}, \mathbb{F})$ and a point $a \in \mathrm{A}(k)$, we define

$$\mathrm{Q}_a := t_{a*}\mathrm{P}$$

and we say that Q is *nondivisible* if it is simple and satisfies $\mathrm{Q}_a \not\simeq \mathrm{Q}$ for all $a \in \mathrm{A}(k)$ with $a \neq 0$. We denote by

$$\Gamma_{\mathrm{P}} := \big\{ a \in \mathrm{A}(k)_{\mathrm{tors}} \;\big|\; \delta_a \in \langle \mathrm{P} \rangle \big\}$$

the abelian group of torsion points whose associated skyscraper sheaf appears in the Tannaka category $\langle \mathrm{P} \rangle$ generated by a perverse sheaf $\mathrm{P} \in \mathscr{C}$. Note that Γ_{P} is finite. Indeed, every skyscraper sheaf $\delta_a \in \langle \mathrm{P} \rangle$ defines a character of the Tannaka group $\mathrm{G}_\omega(\mathrm{P})$ and algebraic groups have only finitely many torsion characters. In fact, the first part of the following result shows that *all* torsion characters of the Tannaka group are given by skyscraper sheaves in torsion points.

PROPOSITION 3.3
Let k be algebraically closed, and let $P \in \mathscr{C}$ be a simple perverse sheaf.

(1) *The group of connected components of $G := G_\omega(P)$ is given by*

$$G/G^\circ \simeq \mathrm{Hom}(\Gamma_P, \mathbb{G}_m).$$

(2) *Fix an integer $d \geq 1$ with $d \cdot \Gamma_P = \{0\}$. Then for all $Q, Q' \in \langle P \rangle$ we have*

$$\omega(Q)_{|G^\circ} \simeq \omega(Q')_{|G^\circ} \iff [d]_* Q \simeq [d]_* Q',$$
$$\omega(Q)_{|G^\circ} \text{ is irreducible} \iff Q \text{ is nondivisible}.$$

(3) *Let $\det(P) \in \langle P \rangle$ be the unique simple perverse sheaf which corresponds to the top wedge power of $V := \omega(P)$. Then $\det(P)$ is a skyscraper sheaf. If $V_{|G^\circ}$ is irreducible, we have*

$$G^\circ \text{ semisimple} \iff \mathrm{Supp}(\det(P)) \text{ is a torsion point}.$$

Proof
For $k = \mathbb{C}$, parts (1) and (2) are due to Weissauer [53] who also shows that every invertible object in the Tannaka category of perverse sheaves is a skyscraper sheaf (this in particular applies to $\det(P)$); alternatively one could use the Riemann–Hilbert correspondence and the results for holonomic $\mathscr{D}$-modules in [34, Section 3.c]. From $k = \mathbb{C}$ one can pass to an arbitrary algebraically closed field of characteristic 0 because the Tannaka group is invariant under extensions of algebraically closed fields and any perverse sheaf is defined over the algebraic closure of a finitely generated field; see Corollary 4.4 (resp., Lemma A.1). The claim about semisimplicity in (3) follows since by Schur's lemma the center $Z = Z(G^\circ)$ acts on V by scalars and hence $\det(V)$ has finite order if and only if Z is finite. □

Definition 3.4
For perverse sheaves $P \in \mathscr{C}$, we denote the derived group of the connected component of the Tannaka group $G = G_\omega(P)$ by

$$G^*_\omega(P) := [G^\circ, G^\circ].$$

If $P = \delta_X$ is the intersection complex of a subvariety $X \subset A$, we put

$$G^*_{X,\omega} := G^*_\omega(P).$$

Proposition 3.3 allows us to realize this group as the Tannaka group of another perverse sheaf.

COROLLARY 3.5
Suppose that k is algebraically closed. Let $\mathrm{P} \in \mathscr{C}$ be a simple perverse sheaf. Then for any integer $d \geq 1$ with $[d]_*\mathrm{P} \in \mathscr{C}$ and any $a \in \mathrm{A}(k)$ with $\mathrm{P}_a \in \mathscr{C}$ the following properties hold:

(1) $\mathrm{G}^*_\omega(\mathrm{P}_a) \simeq \mathrm{G}^*_\omega(\mathrm{P})$.
(2) $\mathrm{G}^\circ_\omega([d]_*\mathrm{P}) \simeq \mathrm{G}^\circ_\omega(\mathrm{P})$.
(3) $\mathrm{G}_\omega([d]_*\mathrm{P})$ *is connected if and only if* $d \cdot \Gamma_\mathrm{P} = 0$.
(4) *Suppose that* P *is nondivisible with* $[d]_* \det(\mathrm{P}_a) = \delta_0$ *and* $d \cdot \Gamma_{\mathrm{P}_a} = 0$. *If* $[d]_*\mathrm{P}_a$ *belongs to* $\mathscr{C}$, *then*

$$\mathrm{G}_\omega\big([d]_*\mathrm{P}_a\big) \simeq \mathrm{G}^*_\omega(\mathrm{P}).$$

Proof
(1) By [33, Lemma 4.3.2], the inclusions $\langle \mathrm{P} \rangle \subset \langle \mathrm{P} \oplus \delta_a \rangle \supset \langle \mathrm{P}_a \rangle$ induce isomorphisms $\mathrm{G}^*_\omega(\mathrm{P}) \simeq \mathrm{G}^*_\omega(\mathrm{P} \oplus \delta_a) \simeq \mathrm{G}^*_\omega(\mathrm{P}_a)$.

(2) By [53] or [34, Corollary 1.6], the pushforward $[d]_* \colon \langle \mathrm{P} \rangle \to \langle [d]_*\mathrm{P} \rangle$ is a tensor functor which induces an isomorphism between the connected components of the identity of the respective Tannaka groups.

(3) Since $d \cdot \Gamma_\mathrm{P} = \Gamma_{[d]_*\mathrm{P}}$, this follows from Proposition 3.3(1) applied to $[d]_*\mathrm{P}$.

(4) By the previous two steps, the group $\mathrm{G}_\omega([d]_*\mathrm{P}_a)$ is connected. One easily sees that $[d]_*\mathrm{P}_a$ is nondivisible with $\det([d]_*\mathrm{P}_a) = [d]_* \det(\mathrm{P}_a) = \delta_0$ so $\mathrm{G}_\omega([d]_*\mathrm{P}_a)$ is a semisimple group by the last part of Proposition 3.3. It is therefore equal to the derived group of its connected component of the identity, which by (1) and (2) coincides with $\mathrm{G}^*_\omega(\mathrm{P})$. □

Remark 3.6
The isomorphism $\mathrm{G}^\circ_\omega([d]_*\mathrm{P}) \simeq \mathrm{G}^\circ_\omega(\mathrm{P})$ in Corollary 3.5(2) is not canonical, it involves the choice of an isomorphism between the two fiber functors ω and $\omega \circ [d]_*$ on the tensor category $\langle \mathrm{P} \rangle$. But we can choose the isomorphism in a contravariant functorial way with respect to monomorphisms in the full tensor subcategory

$$\mathscr{C} \cap [d]_*^{-1}(\mathscr{C}) := \big\{ \mathrm{Q} \in \mathscr{C} \;\big|\; [d]_*\mathrm{Q} \in \mathscr{C} \big\} \subset \mathscr{C}$$

by fixing an isomorphism between the fiber functors ω and $\omega \circ [d]_*$ on this category.

3.4. Larsen's alternative

Let $\mathrm{X} \subset \mathrm{A}$ be a subvariety such that $\delta_\mathrm{X} \in \mathscr{C}$. We are interested in criteria under which the Tannaka group $\mathrm{G}_{\mathrm{X},\omega}$ is big. Suppose that $\mathrm{X} \subset \mathrm{A}$ is integral, nondegenerate, and $2 \dim \mathrm{X} < \dim \mathrm{A}$, so that by Lemma 2.7 the sum morphism

$$\sigma : X \times X \longrightarrow W := X + X \subset A$$

is generically finite onto its image W, and W is nondegenerate. Let $U \subset W$ be a smooth open dense subset over which σ is finite étale. By adjunction, we have an inclusion $\delta_U \subset \sigma_*(\delta_{X\times X})_{|U}$ as a direct summand. The decomposition theorem in [4] extends this to an inclusion $\delta_W \subset \delta_X * \delta_X = \sigma_*(\delta_{X\times X})$ as a direct summand in the derived category of constructible sheaf complexes. Namely, there is a unique semisimple perverse sheaf $\varepsilon_W \in \mathrm{Perv}(A, \mathbb{F})$ without negligible direct summands, and a unique negligible complex $\nu_X \in D^b_c(A, \mathbb{F})$, such that

$$\delta_X * \delta_X = \delta_W \oplus \varepsilon_X \oplus \nu_X.$$

With this notation, we obtain the following criterion for big Tannaka groups.

LEMMA 3.7

With the notation above, the following are equivalent:

(1) $G_{X,\omega}$ *is big in the sense of Section 1.2.*

(2) ε_X *is either a simple perverse sheaf, or a direct sum of a simple perverse sheaf and a skyscraper sheaf of rank* 1.

Proof

The subvariety $W \subset A$ is nondegenerate and not whole A, so it cannot be the support of a negligible sheaf complex. Now we have $\mathrm{Supp}(\varepsilon_X \oplus \nu_X) = W$ since the sum morphism $\sigma : X \times X \to W$ has generic degree 2, thus $\mathrm{Supp}(\varepsilon_X) = W$. In particular, $V = \omega(\delta_X) \in \mathrm{Rep}_{\mathbb{F}}(G_{X,\omega})$ must have dimension $\dim V > 2$, since otherwise ε_X would be the skyscraper sheaf corresponding to $\det(V)$ by Proposition 3.3(3).

By applying the fiber functor ω, one sees that condition (2) is equivalent to saying that in the decomposition of the tensor square $V \otimes V$ there are only two irreducible direct summands of dimension greater than 1. Since $\dim(V) > 2$, this is equivalent to (1) by Larsen's alternative (see [28, p. 113]) for the subgroup $G_{X,\omega} \subset \mathrm{GL}(V)$. □

3.5. Symmetric powers

If the Tannaka group is big, then similar arguments allow us to control the sum morphism from symmetric powers of the subvariety.

LEMMA 3.8

Let $X \subset A$ *be a nondegenerate subvariety, and let* $r \geq 1$ *be an integer such that* $r \dim X < \dim A$. *If* $G_{X,\omega}$ *is big, then the sum morphism*

$$\tau_r : \mathrm{Sym}^r X \longrightarrow A$$

is birational onto its image $W_r = X + \cdots + X$.

Proof

Consider the following commutative diagram, where q_r denotes the quotient morphism:

$$\begin{array}{ccc} \mathrm{Z}_r = \mathrm{X}^r & \xrightarrow{\sigma_r} & \mathrm{W}_r = \mathrm{X} + \cdots + \mathrm{X} \\ & {\scriptstyle q_r} \searrow \quad \nearrow {\scriptstyle \tau_r} & \\ & \mathrm{Y}_r = \mathrm{Sym}^r \mathrm{X} & \end{array}$$

Since $q_r : \mathrm{Z}_r \to \mathrm{Y}_r$ is a finite branched cover with group $\mathfrak{S}_r$, the decomposition theorem shows that as an $\mathfrak{S}_r$-equivariant perverse sheaf the direct image $q_{r*}(\delta_{\mathrm{Z}_r})$ is a direct sum

$$q_{r*}(\delta_{\mathrm{Z}_r}) \simeq \bigoplus_{\sigma} \sigma \boxtimes \mathrm{P}_\sigma,$$

where σ runs through all irreducible representations of the symmetric group $\mathfrak{S}_r$ and where each P_σ is a semisimple perverse sheaf on Y_r. In this isotypic decomposition the action of the group $\mathfrak{S}_r$ on $\sigma \boxtimes \mathrm{P}_\sigma$ is given by the action on σ. Since the action of the symmetric group on tensor powers of sheaf complexes involves a Koszul sign, the perverse intersection complex on $\mathrm{Y}_r = \mathrm{Sym}^r \mathrm{X}$ is the isotypic piece for the trivial representation $\mathbf{1}$ or the sign representation sgn of $\mathfrak{S}$ depending on the parity of $\dim \mathrm{X}$. We have

$$\delta_{\mathrm{Y}_r} \simeq \mathrm{P}_\varepsilon \quad \text{for } \varepsilon = \begin{cases} \mathrm{sgn} & \text{if } \dim \mathrm{X} \text{ is odd,} \\ \mathbf{1} & \text{if } \dim \mathrm{X} \text{ is even,} \end{cases}$$

as one may check on the open dense subset where q_r is finite étale. So the direct image $\delta_{\mathrm{X},r} := \mathrm{R}\tau_{r*}(\delta_{\mathrm{Y}_r})$ corresponds to the representation

$$\omega(\delta_{\mathrm{X},r}) \simeq \begin{cases} \mathrm{Alt}^r \mathrm{V} & \text{if } \dim \mathrm{X} \text{ is odd,} \\ \mathrm{Sym}^r \mathrm{V} & \text{if } \dim \mathrm{X} \text{ is even,} \end{cases}$$

where $\mathrm{V} := \omega(\delta_{\mathrm{X}})$ is the defining representation of the group $\mathrm{G}_{\mathrm{X},\omega}$. If that group is big, then we are in one of the following cases:

(1) $\mathrm{G}_{\mathrm{X}} = \mathrm{SL}(\mathrm{V})$. Then $\mathrm{Alt}^r \mathrm{V}$ and $\mathrm{Sym}^r \mathrm{V}$ are irreducible representations by Schur–Weyl duality (see [21, Theorem 6.3(4)]).

(2) $\mathrm{G}_{\mathrm{X}} = \mathrm{SO}(\mathrm{V})$. Then we have an embedding $\mathrm{Sym}^{r-2} \mathrm{V} \hookrightarrow \mathrm{Sym}^r \mathrm{V}$ and the quotient $\mathrm{Sym}^r \mathrm{V} / \mathrm{Sym}^{r-2} \mathrm{V}$ is irreducible (see [21, Theorem 19.19]).

(3) $\mathrm{G}_{\mathrm{X}} = \mathrm{Sp}(\mathrm{V})$. Then we have an embedding $\mathrm{Alt}^{r-2} \mathrm{V} \hookrightarrow \mathrm{Alt}^r \mathrm{V}$ and again the quotient $\mathrm{Alt}^r \mathrm{V} / \mathrm{Alt}^{r-2} \mathrm{V}$ is irreducible (see [21, Theorem 17.11]).

In the first case $\delta_{X,r}$ is a simple perverse sheaf modulo negligibles, while in the other two cases we have an embedding $\delta_{X,r-2} \hookrightarrow \delta_{X,r}$ whose cokernel is a simple perverse sheaf (note that $\dim X$ is even in case (2) and odd in case (3)). In all three cases the semisimple perverse sheaf $\delta_{X,r}$ has a unique simple direct summand $\varepsilon_{X,r} \subset \delta_{X,r}$ with full support, that is, with

$$\mathrm{Supp}(\varepsilon_{X,r}) = W_r = X + \cdots + X.$$

The decomposition theorem for the generically finite morphism $\tau_r : Y_r \to W_r$ also shows that

$$\delta_{W_r} \subset \delta_{X,r} = R\tau_{r*}(\delta_{Y_r}),$$

hence $\varepsilon_{X,r} = \delta_{W_r}$. In particular, there exists an open dense subset $U \subset W_r$ such that

$$\big(R\tau_{r*}(\delta_{Y_r})\big)_{|U} \simeq (\delta_{Y_r})_{|U}$$

and by comparing the generic rank on U we obtain $\det(\tau_r) = 1$. □

In fact, the above argument does not require the group $G_{X,\omega}$ to be big, we only need to have sufficient control on the support dimension of the perverse sheaves that enter the relevant wedge or symmetric power. For instance, we have the following result which goes beyond the case of big Tannaka groups.

COROLLARY 3.9
Let $X \subset A$ *be nondegenerate with* $r \dim X < \dim A$*, and consider the representation*

$$V := \begin{cases} \mathrm{Alt}^r\, \omega(\delta_X) & \text{if } 2 \nmid \dim X, \\ \mathrm{Sym}^r\, \omega(\delta_X) & \text{if } 2 \mid \dim X. \end{cases}$$

If $V \in \mathrm{Rep}_{\mathbb{F}}(G^*_{X,\omega})$ *has at most one irreducible direct summand of dimension greater than* 1*, then the sum morphism* $\tau_r : \mathrm{Sym}^r X \to X + \cdots + X$ *is birational.*

Proof
By [53] or [34, Section 3.c], all 1-dimensional representations of the Tannaka group arise from skyscraper sheaves, so for $\dim X > 0$ they cannot contribute to the support $W_r = X + \cdots + X$. Hence we can apply the same argument as in the previous proof. □

COROLLARY 3.10
Let $X \subset A$ *be a smooth irreducible curve generating* A*, and assume* $\dim A \geq 3$*. If the representation* $V = \mathrm{Alt}^2(\omega(\delta_X)) \in \mathrm{Rep}_{\mathbb{F}}(G^*_{X,\omega})$ *is a sum of an irreducible representation and a* 1*-dimensional trivial representation, then*

(1) $\mathrm{X} = p - \mathrm{X}$ *for some point* $p \in \mathrm{A}$,
(2) $\tau\colon \mathrm{Y} = \mathrm{Sym}^2 \mathrm{X} \to \mathrm{W} = \mathrm{X} + \mathrm{X}$ *is finite birational over* $\mathrm{U} = \mathrm{W} \setminus \{p\}$,
(3) $\mathrm{G}^*_{\mathrm{X},\omega} = \mathrm{Sp}(\omega(\delta_{\mathrm{X}}), \theta)$ *for the natural symplectic form* θ *on* $\omega(\delta_{\mathrm{X}})$.

Proof
By assumption, $\mathrm{Alt}^2(\omega(\delta_{\mathrm{X}}))$ contains a 1-dimensional trivial representation, so the representation $\omega(\delta_{\mathrm{X}})$ is isomorphic to its dual. Therefore, $\mathrm{X} = p - \mathrm{X}$ for some point $p \in \mathrm{A}$. Now for dimension reasons $\tau\colon \mathrm{Y} \to \mathrm{W}$ restricts to a finite morphism over the complement $\mathrm{U} = \mathrm{W} \setminus \Sigma$ of a finite set $\Sigma \subset \mathrm{W}$ of points. Note that $\mathrm{Y} = \mathrm{Sym}^2 \mathrm{X}$ is smooth for a smooth curve X, so we have $\delta_{\mathrm{Y}} = \mathbb{F}_{\mathrm{Y}}[2]$. Base change then shows that for any point q we have

$$\mathscr{H}^0\big(\mathrm{R}\tau_*(\delta_{\mathrm{Y}})\big)_q \simeq \mathrm{H}^2\big(\tau^{-1}(q), \mathbb{F}\big) \begin{cases} = 0 & \text{if } q \notin \Sigma, \\ \neq 0 & \text{if } q \in \Sigma. \end{cases}$$

Since $\mathrm{R}\tau_*(\delta_{\mathrm{Y}})$ is a direct sum of a semisimple perverse sheaf P and a negligible sheaf complex and since negligible sheaf complexes cannot have cohomology sheaves which are skyscraper sheaves, it follows that P contains the skyscraper sheaves δ_q in all points $q \in \Sigma$. But by assumption $\mathrm{R}\tau_*(\delta_{\mathrm{Y}})$ contains a unique skyscraper summand; hence, it follows that $\Sigma = \{p\}$ and thus τ is finite over $\mathrm{U} = \mathrm{W} \setminus \{p\}$.

In particular, $\mathrm{R}\tau_*(\delta_{\mathrm{Y}}^-)_{|\mathrm{U}}$ is a perverse sheaf, and $\mathscr{H}^i(\mathrm{R}\tau_*(\delta_{\mathrm{Y}}^-))|_{|\mathrm{U}} = 0$ in all degrees $i \neq -2$ because δ_{Y}^- is a constructible sheaf placed in degree -2. But any semisimple perverse sheaf on a surface with cohomology sheaves only in degrees -2 is the minimal extension of a local system on any open dense subset of the surface. In our case, that local system has rank 1 because δ_{Y}^- has generic rank 1 and $\deg(\tau) = 1$. Local systems of rank 1 are simple; hence, it follows that the minimal extension $\mathrm{R}\tau_*(\delta_{\mathrm{Y}}^-)$ is a simple perverse sheaf.

In conclusion, this shows that $\delta_{\mathrm{X}} * \delta_{\mathrm{X}} = \mathrm{R}\tau_*(\delta_{\mathrm{Y}}) \oplus \mathrm{R}\tau_*(\delta_{\mathrm{Y}}^-)$ is a sum of two simple perverse sheaves and a skyscraper sheaf. It then follows by the same argument as in [37, Theorem 6.1] that $\mathrm{G}^*_{\mathrm{X},\omega} = \mathrm{Sp}(\omega(\delta_{\mathrm{X}}), \theta)$; note that

$$\dim(\omega(\delta_{\mathrm{X}})) = \chi(\delta_{\mathrm{X}}) \geq g > 2$$

since the curve X generates A. □

COROLLARY 3.11
Let $\mathrm{X} \subset \mathrm{A}$ *be a smooth irreducible curve generating* A, *and assume* $\dim \mathrm{A} \geq 3$. *Then the group* $\mathrm{G}^*_{\mathrm{X},\omega}$ *is not isomorphic to* E_7 *acting on* $\omega(\delta_{\mathrm{X}})$ *via its irreducible representation of dimension* 56.

Proof
For the 56-dimensional irreducible representation W of the group E_7, the alternating square $\mathrm{Alt}^2(W)$ is a sum of an irreducible and a 1-dimensional trivial representation. However, Corollary 3.10 says that $\mathrm{Alt}^2(\omega(\delta_X))$ can be a sum of an irreducible and a 1-dimensional trivial representation only if $G^*_{X,\omega} \simeq \mathrm{Sp}_{56}(\mathbb{F})$. □

3.6. Character twists
Recall that $\Pi(A,\mathbb{F}) = \mathrm{Hom}(\pi_1(A,0),\mathbb{F}^\times)$ denotes the group of continuous characters of the étale (resp., topological) fundamental group of the abelian variety. For a character $\chi \in \Pi(A,\mathbb{F})$, let L_χ be the local system of rank 1 with monodromy representation given by the character χ. For a perverse sheaf $P \in \mathrm{Perv}(A,\mathbb{F})$, we call $P_\chi := P \otimes_{\mathbb{F}} L_\chi \in \mathrm{Perv}(A,\mathbb{F})$ the *twist* of the given perverse sheaf by the character. Such twists of perverse sheaves appear in the generic vanishing theorem of [6], [38], and [47]. Let us say that a subset of $\Pi(A,\mathbb{F})$ is a *proper subtorus* if it has the form

$$\Pi(A/B,\mathbb{F}) \subset \Pi(A,\mathbb{F}),$$

where $B \subset A$ is a nonzero abelian subvariety. Then the generic vanishing theorem says that there is a finite union $\mathscr{S}(P) \subset \Pi(A,\mathbb{F})$ of translates of proper subtori such that

$$H^i(A,P_\chi) = 0 \quad \text{for all } i \neq 0 \text{ and all } \chi \in \Pi(A,\mathbb{F}) \smallsetminus \mathscr{S}(P).$$

We will use this in Section 4.3 to write down explicit fiber functors with a natural Galois action. Up to noncanonical isomorphism, the Tannaka group of a perverse sheaf does not change under twists.

LEMMA 3.12
Let $P \in \mathscr{C}$. *Then for any character* $\chi \in \Pi(A,\mathbb{F})$ *with* $P_\chi \in \mathscr{C}$ *we have*

$$G_\omega(P_\chi) \simeq G_\omega(P).$$

Proof
By [38, Proposition 4.1], twisting by χ gives rise to an equivalence of tensor categories

$$\langle P\rangle \xrightarrow{\sim} \langle P_\chi\rangle, \qquad Q \longmapsto Q_\chi$$

in $\overline{\mathrm{Perv}}(A,\mathbb{F})$. This equivalence need not be compatible with the fiber functor ω on the source and target, but since $\mathbb{F}$ is algebraically closed, any two fiber functors are noncanonically isomorphic; hence the same holds for the Tannaka groups. □

4. Galois theory for perverse sheaves

In this section, we discuss the behavior of Tannaka groups of perverse sheaves under extension of the base field and recall the connection between such Tannaka groups and classical monodromy groups in [39, Section 5]. We mostly follow the arguments there but remove the assumption of geometric semisimplicity in the Galois exact sequence by using a result of D'Addezio and Esnault [9].

4.1. Extension of the base field

Let K/k be a field extension, and consider the base change functor

$$(-)_{\mathrm{K}}\colon \operatorname{Perv}(\mathrm{A},\mathbb{F}) \longrightarrow \operatorname{Perv}(\mathrm{A}_{\mathrm{K}},\mathbb{F}), \qquad \mathrm{P} \longmapsto \mathrm{P}_{\mathrm{K}}.$$

Passing to the abelian quotient categories by the subcategories of perverse sheaves of Euler characteristic 0, we have the following.

LEMMA 4.1

The base change functor descends to a faithful exact $\mathbb{F}$-linear tensor functor

$$(-)_{\mathrm{K}}\colon \overline{\operatorname{Perv}}(\mathrm{A},\mathbb{F}) \longrightarrow \overline{\operatorname{Perv}}(\mathrm{A}_{\mathrm{K}},\mathbb{F}).$$

Proof

The functor $(-)_{\mathrm{K}}\colon \operatorname{Perv}(\mathrm{A},\mathbb{F}) \to \operatorname{Perv}(\mathrm{A}_{\mathrm{K}},\mathbb{F})$ is a faithful $\mathbb{F}$-linear exact functor. Let $q_{\mathrm{K}} = q \circ (-)_{\mathrm{K}}$ denote its composite with the quotient functor q as shown below:

$$\begin{array}{ccc} \operatorname{Perv}(\mathrm{A},\mathbb{F}) & \xrightarrow{(-)_{\mathrm{K}}} & \operatorname{Perv}(\mathrm{A}_{\mathrm{K}},\mathbb{F}) \\ & {\scriptstyle q_{\mathrm{K}}} \searrow \quad \swarrow {\scriptstyle q} & \\ & \overline{\operatorname{Perv}}(\mathrm{A}_{\mathrm{K}},\mathbb{F}) & \end{array}$$

Since q_{K} is an exact functor between abelian categories which sends all objects of the Serre subcategory $\mathrm{S}(\mathrm{A},\mathbb{F}) \subset \operatorname{Perv}(\mathrm{A},\mathbb{F})$ to zero, it factors by the universal property of abelian quotient categories (see [22, Corollaire 2, p. 368]) through a unique exact functor

$$(-)_{\mathrm{K}}\colon \overline{\operatorname{Perv}}(\mathrm{A},\mathbb{F}) \longrightarrow \overline{\operatorname{Perv}}(\mathrm{A}_{\mathrm{K}},\mathbb{F}).$$

This functor is clearly $\mathbb{F}$-linear, and it admits the structure of a tensor functor with respect to the natural isomorphisms $(\mathrm{P} * \mathrm{Q})_{\mathrm{K}} \simeq \mathrm{P}_{\mathrm{K}} * \mathrm{Q}_{\mathrm{K}}$ inherited from the derived category. Any exact $\mathbb{F}$-linear tensor functor of rigid abelian tensor categories with $\operatorname{End}(\mathbf{1}) = \mathbb{F}$ is automatically faithful (see [15, Proposition 1.19]), so the claim follows. □

Starting from a given full abelian tensor subcategory $\mathscr{C} \subset \overline{\mathrm{Perv}}(\mathrm{A}, \mathbb{F})$, let us now denote by

$$\mathscr{C}_{\mathrm{K}} = \{\mathrm{Q} \mid \exists \mathrm{P} \in \mathscr{C} \text{ such that Q is a subquotient of } \mathrm{P}_{\mathrm{K}}\} \subset \overline{\mathrm{Perv}}(\mathrm{A}_{\mathrm{K}}, \mathbb{F})$$

the full abelian tensor subcategory generated by the essential image of $\mathscr{C}$ under the functor $(-)_{\mathrm{K}}$ from Lemma 4.1. The category $\mathscr{C}_{\mathrm{K}}$ is again neutral Tannaka as it is a full abelian tensor subcategory of the neutral Tannaka category $\overline{\mathrm{Perv}}(\mathrm{A}_{\mathrm{K}}, \mathbb{F})$. In what follows, we fix a fiber functor

$$\omega \colon \mathscr{C}_{\mathrm{K}} \longrightarrow \mathrm{Vect}(\mathbb{F}).$$

Precomposing with the base extension functor $(-)_{\mathrm{K}}$ we get a fiber functor on $\mathscr{C}$, and we denote by

$$\begin{aligned} \mathrm{G}_{\omega}(\mathscr{C}_{\mathrm{K}}) &= \mathrm{Aut}^{\otimes}(\omega \mid \mathscr{C}_{\mathrm{K}}), \\ \mathrm{G}_{\omega}(\mathscr{C}) &:= \mathrm{Aut}^{\otimes}(\omega \mid \mathscr{C}), \end{aligned}$$

the corresponding Tannaka groups.

COROLLARY 4.2
We have a closed immersion $\mathrm{G}_{\omega}(\mathscr{C}_{\mathrm{K}}) \hookrightarrow \mathrm{G}_{\omega}(\mathscr{C})$.

Proof
The faithful exact $\mathbb{F}$-linear tensor functor $(-)_{\mathrm{K}} \colon \mathscr{C} \to \mathscr{C}_{\mathrm{K}}$ is compatible with our chosen fiber functors by construction; hence, it defines a homomorphism of Tannaka groups. The latter is a closed immersion by [15, Proposition 2.21(b)], since every object of $\mathscr{C}_{\mathrm{K}}$ is isomorphic to a subquotient of P_{K} for some $\mathrm{P} \in \mathscr{C}$. □

4.2. The Galois sequence
Let $k' \subset \mathrm{K}$ be the algebraic closure of k in K. The category of continuous finite-dimensional representations

$$\mathrm{Rep}_{\mathbb{F}}\big(\mathrm{Aut}(k'/k)\big)$$

of the profinite group $\mathrm{Aut}(k'/k)$ over $\mathbb{F}$ is a neutral Tannaka category. If k'/k is Galois, then $\mathrm{Aut}(k'/k) = \mathrm{Gal}(k'/k)$ is a quotient of the absolute Galois group of k. In this case, we can identify objects of the above category with sheaves on $\mathrm{Spec}(k)$ and hence the pushforward under the neutral element $e \colon \mathrm{Spec}(k) \to \mathrm{A}$ gives a fully faithful embedding

$$e_* \colon \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{Gal}(k'/k)\big) \hookrightarrow \overline{\mathrm{Perv}}(\mathrm{A}, \mathbb{F}).$$

We will view Galois representations as a full subcategory of skyscraper sheaves and drop the e_* from the notation. Our chosen fiber functor on $\mathscr{C}$ restricts to a fiber functor

$$\omega\colon \mathscr{C} \cap \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{Gal}(k'/k)\big) \longrightarrow \mathrm{Vect}(\mathbb{F}).$$

We denote its Tannaka group by

$$\mathrm{G}_{\omega,\mathscr{C}}(k'/k) := \mathrm{Aut}^{\otimes}\big(\omega \mid \mathscr{C} \cap \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{Gal}(k'/k)\big)\big).$$

Representations of this group correspond to skyscraper sheaves $\mathrm{P} \in \mathscr{C}$ supported in the origin, whence a morphism $\mathrm{Aut}(k'/k) \to \mathrm{G}_{\omega,\mathscr{C}}(k'/k)$.

THEOREM 4.3
Assume as above that k'/k is Galois. Then we have a short exact sequence of proalgebraic groups

$$1 \longrightarrow \mathrm{G}_{\omega}(\mathscr{C}_{\mathrm{K}}) \longrightarrow \mathrm{G}_{\omega}(\mathscr{C}) \longrightarrow \mathrm{G}_{\omega,\mathscr{C}}(k'/k) \longrightarrow 1.$$

Proof
Corollary 4.2 gives a closed immersion $i\colon \mathrm{G}_{\omega}(\mathscr{C}_{\mathrm{K}}) \to \mathrm{G}_{\omega}(\mathscr{C})$. Moreover, since k'/k is a Galois extension, we have by the above an embedding as a full tensor subcategory

$$\mathscr{C} \cap \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{Gal}(k'/k)\big) \hookrightarrow \mathscr{C},$$

which is stable under subobjects, and this embedding is compatible with the chosen fiber functors on the source and target. By [15, Proposition 2.21(a)], we then have an epimorphism

$$p\colon \mathrm{G}_{\omega}(\mathscr{C}) \twoheadrightarrow \mathrm{G}_{\omega,\mathscr{C}}(k'/k).$$

By construction, $p \circ i$ is trivial. Thus, to complete the proof, by [9, Proposition A.13], it suffices to check that

(1) the functor $(-)_{\mathrm{K}}\colon \mathscr{C} \to \mathscr{C}_{\mathrm{K}}$ is observable (see [9, Appendix A]), and
(2) for every $\mathrm{P} \in \mathscr{C}$ the maximal trivial subobject of P_{K} lies in the essential image of the functor $e_*\colon \mathscr{C} \cap \mathrm{Rep}_{\mathbb{F}}(\mathrm{Gal}(k'/k)) \to \mathscr{C}$.

For part (1) it suffices by [9, Lemma A.4(1)] to show that, for $\mathrm{P} \in \mathscr{C}$, any rank-1 subobject

$$\mathrm{S} \subset \mathrm{P}_{\mathrm{K}}$$

is a direct summand in a semisimple object Q_{K} with $\mathrm{Q} \in \mathscr{C}$. To check this, note that the rank-1 objects in the Tannaka category of perverse sheaves are rank-1 skyscraper

sheaves, and that the sum of all perverse rank-1 skyscraper subsheaves of P_K is semisimple, being a sum of simple objects. To conclude the proof of (1), it suffices to show that this direct sum descends to a perverse subsheaf $Q \subset P$, as it then follows that S is a direct summand of Q_K as desired. To prove that the sum of all rank-1 skyscraper subsheaves descends to k, we first show that the maximal skyscraper subsheaf of P_K descends to a subsheaf of P. Indeed, the Verdier dual of the sum of all perverse skyscraper subsheaves is the maximal perverse skyscraper quotient of the Verdier dual $D(P_K)$ which is $\mathscr{H}^0(D(P_K)) = \mathscr{H}^0(D(P))_K$. Hence, the maximal skyscraper subsheaf descends. Replacing the perverse sheaf P by the maximal skyscraper subsheaf supported at the origin, we are reduced to the case $A = \{0\}$. Then P is given by a representation $V \in \mathrm{Rep}_{\mathbb{F}}(\mathrm{Gal}(\bar{k}/k))$ and the claim reduces to the following two facts:

- A subspace of V is stable under $\mathrm{Gal}(\bar{K}/K)$ if and only if it is so under $\mathrm{Gal}(\bar{k}/k')$ (since $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k')$ is surjective for k' algebraically closed in K).
- The sum of all 1-dimensional subrepresentations of $V_{|\mathrm{Gal}(\bar{k}/k')}$ is stable under $\mathrm{Gal}(\bar{k}/k)$ since the subgroup $\mathrm{Gal}(\bar{k}/k') \subset \mathrm{Gal}(\bar{k}/k)$ is normal.

For (2) we argue similarly. The unit object of the tensor category $\mathscr{C}_K$ is the skyscraper sheaf δ_0 of rank 1 supported in the origin. So the maximal trivial subobject of P_K is the maximal subobject of the form $\delta_0^{\oplus n}$ for some integer $n \geq 0$, and this subobject descends to a subobject $Q \subset P$ as before. □

COROLLARY 4.4

If k is algebraically closed, then for every extension K/k we have a natural isomorphism

$$G_\omega(\mathscr{C}_K) \xrightarrow{\sim} G_\omega(\mathscr{C}).$$

In particular, for every perverse sheaf $P \in \mathscr{C}$, we have $G_\omega(P_K) \simeq G_\omega(P)$.

Proof

If k is algebraically closed, then $k' = k$ and hence $G_\omega(k'/k) \simeq \{1\}$. □

4.3. A splitting of the sequence

We now apply the above when $K = \bar{k}$ is an algebraic closure of k. In the Galois sequence in Theorem 4.3 we have used the fully faithful functor

$$e_*\colon \mathrm{Rep}_{\mathbb{F}}\big(\mathrm{Gal}(\bar{k}/k)\big) \hookrightarrow \overline{\mathrm{Perv}}(A, \mathbb{F})$$

that identifies a Galois representation with the corresponding skyscraper sheaf at the origin. We now describe a splitting of the sequence in Theorem 4.3 for a special

category $\mathscr{C}$ such that the functor $e_*\colon \mathscr{C} \cap \mathrm{Rep}_{\mathbb{F}}(\mathrm{Gal}(\bar{k}/k)) \hookrightarrow \mathscr{C}$ has a left inverse. To do so, let

$$\mathrm{Perv}_0(A, \mathbb{F})$$

be the full subcategory of all $P \in \mathrm{Perv}(A, \mathbb{F})$ for which all simple subquotients Q of $P_{\bar{k}}$ satisfy

$$H^i(A_{\bar{k}}, Q) = 0 \quad \text{for all } i \neq 0.$$

Its image

$$\overline{\mathrm{Perv}}_0(A, \mathbb{F}) \subset \overline{\mathrm{Perv}}(A, \mathbb{F})$$

is a full abelian tensor subcategory which is equivalent to $\mathrm{Perv}_0(A, \mathbb{F})/S_0(A, \mathbb{F})$, where $S_0(A, \mathbb{F}) := S(A, \mathbb{F}) \cap \mathrm{Perv}_0(A, \mathbb{F})$ is the full subcategory of perverse sheaves P with the property that all the subquotients Q of $P_{\bar{k}}$ satisfy $H^\bullet(A_{\bar{k}}, Q) = 0$. We then get a functor

$$\omega\colon \overline{\mathrm{Perv}}_0(A, \mathbb{F}) = \mathrm{Perv}_0(A, \mathbb{F})/S_0(A, \mathbb{F}) \longrightarrow \mathrm{Vect}(\mathbb{F}), \qquad Q \longmapsto H^0(A_{\bar{k}}, Q)$$

which is exact by definition of the source category. Moreover, ω is a tensor functor by the Künneth isomorphism

$$H^\bullet(A_{\bar{k}}, P * Q) \simeq H^\bullet(A_{\bar{k}}, P) \otimes H^\bullet(A_{\bar{k}}, Q),$$

since for $P, Q \in \mathrm{Perv}_0(A, \mathbb{F})$ only the cohomology in degree 0 contributes. For the fiber functor obtained in this way, we can summarize the relation between the Tannaka groups over k and over $\bar{k}$ as follows.

THEOREM 4.5
For $\mathscr{C} = \overline{\mathrm{Perv}}_0(A, \mathbb{F})$ with the fiber functor $\omega := H^0(A_{\bar{k}}, -)$, the above construction induces a splitting of the short exact sequence

$$1 \longrightarrow G_\omega(\mathscr{C}_{\bar{k}}) \longrightarrow G_\omega(\mathscr{C}) \longrightarrow G_{\omega,\mathscr{C}}(\bar{k}/k) \longrightarrow 1.$$

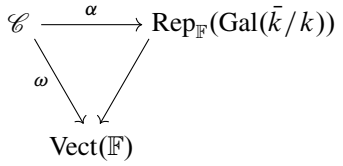
In particular, we have an isomorphism

$$G_\omega(\mathscr{C}) \simeq G_\omega(\mathscr{C}_{\bar{k}}) \rtimes G_{\omega,\mathscr{C}}(\bar{k}/k),$$

and for any $P \in \overline{\mathrm{Perv}}_0(A, \mathbb{F})$, the action of $\mathrm{Gal}(\bar{k}/k)$ on $V = \omega(P)$ factors through the normalizer

$$N\big(G_\omega(P_{\bar{k}})\big) \subset \mathrm{GL}(V).$$

Proof
While the fiber functor $\omega = \mathrm{H}^0(\mathrm{A}_{\bar{k}}, -)$ is only defined on $\mathscr{C}$, it comes with a natural Galois action in the sense that we have a commutative diagram



where α is a left inverse of the functor $e_*\colon \mathrm{Rep}_{\mathbb{F}}(\mathrm{Gal}(\bar{k}/k)) \to \mathscr{C}$. □

4.4. Big monodromy from big Tannaka groups
Now again assume that k is an algebraically closed field of characteristic 0. Consider the constant abelian scheme $\mathrm{A}_\mathrm{S} := \mathrm{A} \times_k \mathrm{S}$, where S is an integral scheme over k. We denote by $\bar{\eta}$ a geometric point over the generic point η of S. Let $\mathscr{X} \subset \mathrm{A}_\mathrm{S}$ be an irreducible closed subscheme which is smooth over S. We want to control the monodromy of the family $\mathscr{X} \to \mathrm{S}$ twisted by a generic rank-1 local system as in [39]. In this context, the following terminology will be useful.

Definition 4.6
We say that $\mathscr{X} \subset \mathrm{A}_\mathrm{S}$ is *constant up to translation in* $\mathrm{A}(\mathrm{S})$ if there is a point $a \in \mathrm{A}(\mathrm{S})$ and a subvariety $\mathrm{Y} \subset \mathrm{A}$ such that $\mathscr{X} = \mathrm{Y}_\mathrm{S} + a$.

In favorable situations, this condition can be read off from the geometric generic fiber of $\mathscr{X} \to \mathrm{S}$ via the following descent result.

LEMMA 4.7
Suppose that S *is a smooth and irreducible variety. Let* $\mathscr{Y}, \mathscr{Z} \subset \mathrm{A}_\mathrm{S}$ *be subvarieties which are flat over* S. *If the subvariety* $\mathscr{Y}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ *has trivial stabilizer, then the following are equivalent:*
(1) $\mathscr{Z} = \mathscr{Y} + a$ *for some* $a \in \mathrm{A}(\mathrm{S})$.
(2) $\mathscr{Z}_{\bar{\eta}} = \mathscr{Y}_{\bar{\eta}} + a$ *for some* $a \in \mathrm{A}(\bar{\eta})$.

Proof
Clearly, the first property implies the second. Conversely, suppose $\mathscr{Z}_{\bar{\eta}} = \mathscr{Y}_{\bar{\eta}} + a$ for some point $a \in \mathrm{A}(\bar{\eta})$. First, we claim that the point a comes from a point $a \in \mathrm{A}(\eta)$. Indeed, let F be the function field of S, and let $y = [\mathscr{Y}_\eta]$ and $z = [\mathscr{Z}_\eta]$ be the F-points of the Hilbert scheme $\mathrm{Hilb}(\mathrm{A})$ defined by the generic fibers of $\mathscr{Y} \to \mathrm{S}$ and $\mathscr{Z} \to \mathrm{S}$, seen as subvarieties of $\mathrm{A}_{\mathrm{S},\eta}$. Now, the abelian variety A acts on the Hilbert scheme

by translation. The transporter

$$\mathrm{T} = \{t \in \mathrm{A}_{\mathrm{S},\eta} \mid z = y + t\}$$

is a subvariety of $\mathrm{A}_{\mathrm{S},\eta}$. Note that $\mathrm{T}(\bar{\eta})$ is nonempty, as it contains the point a. Actually, the point a is the only one of $\mathrm{T}(\bar{\eta})$. For, note that the stabilizer of $\mathscr{Y}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ acts freely and transitively on the base change of T to $\bar{\eta}$. On the other hand, the stabilizer of $\mathscr{Y}_{\bar{\eta}}$ is trivial by assumption, so the transporter $\mathrm{T}(\bar{\eta})$ must be a singleton. The variety T is defined over F and geometrically a singleton, thus $\mathrm{T} = \operatorname{Spec}\mathrm{F}$ which proves the claim.

The point $a \in \mathrm{A}(\eta)$ can be seen as a rational map $a\colon \mathrm{S} \dashrightarrow \mathrm{A}$, which is moreover everywhere defined by the smoothness of S (see [45, Theorem 3.1]). To conclude the proof, note that the generic fibers of $\mathscr{Y} + a$ and $\mathscr{Z}$ coincide; hence $\mathscr{Z} = \mathscr{Y} + a$ by flatness. □

COROLLARY 4.8

If S *is a smooth irreducible variety and if the subvariety* $\mathscr{X}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ *is nondivisible, then the following are equivalent:*

(1) $\mathscr{X} \subset \mathrm{A}_{\mathrm{S}}$ *is constant (resp., symmetric) up to translation in* $\mathrm{A}(\mathrm{S})$.

(2) $\mathscr{X}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ *is constant (resp., symmetric) up to translation in* $\mathrm{A}(\bar{\eta})$.

Moreover, the subvariety $\mathscr{X} \subset \mathrm{A}_{\mathrm{S}}$ *is constant up to translation in* $\mathrm{A}(\mathrm{S})$ *if and only if the family* $\mathscr{X} \to \mathrm{S}$ *is isotrivial.*

Proof

The equivalence of (1) and (2) follows directly from Lemma 4.7. Now suppose that the family $\mathscr{X} \to \mathrm{S}$ is isotrivial. In order to prove that the subvariety $\mathscr{X} \subset \mathrm{A}_{\mathrm{S}}$ is constant up to translation, we may by the equivalence of (1) and (2) replace S by an étale cover and hence assume $\mathscr{X} \simeq \mathrm{Y}_{\mathrm{S}}$ for some $\mathrm{Y} \subset \mathrm{A}$. Fixing $y \in \mathrm{Y}(k)$, we get a section $x\colon \mathrm{S} \to \mathscr{X}$ that gives rise to a commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{X} & \xrightarrow{\mathrm{alb}_x} & \mathrm{Alb}(\mathscr{X}/\mathrm{S}) & \longrightarrow & \mathrm{A}_{\mathrm{S}} & \xrightarrow{z\mapsto z+x} & \mathrm{A}_{\mathrm{S}} \\
\downarrow\wr & & \downarrow\wr & & \| & & \downarrow \\
\mathrm{Y}_{\mathrm{S}} & \xrightarrow{\mathrm{alb}_y} & \mathrm{Alb}(\mathrm{Y}_{\mathrm{S}}/\mathrm{S}) & \longrightarrow & \mathrm{A}_{\mathrm{S}} & \xrightarrow{z\mapsto z+y} & \mathrm{A}_{\mathrm{S}}
\end{array}$$

Here alb_a and alb_y are the relative Albanese morphisms and the composite of the horizontal arrows are the inclusions $\mathscr{X} \subset \mathrm{A}_{\mathrm{S}}$ (resp., $\mathrm{Y}_{\mathrm{S}} \subset \mathrm{A}_{\mathrm{S}}$). Hence, $\mathscr{X} \subset \mathrm{A}_{\mathrm{S}}$ is constant up to translation. □

Example 4.9

The above primitivity is needed. Let $\mathrm{Y} \subset \mathrm{A}$ be a subvariety with finite stabilizer $\mathrm{Stab}(\mathrm{Y}) \neq \{0\}$. Viewing $\mathrm{S} := \mathrm{A}/\mathrm{Stab}(\mathrm{Y})$ as the orbit of the point $[\mathrm{Y}]$ in $\mathrm{Hilb}(\mathrm{A})$

under the translation action of A, we get by restriction of the universal subvariety of $A \times_k \mathrm{Hilb}(A)$ a subvariety $\mathscr{X} \subset A_S$ with fiber $Y + a$ over a point $[a] \in S(k)$. Then the family $\mathscr{X} \to S$ is not constant up to translation in $A(S)$, but it is so up to translation by a section in $A(\bar{\eta})$.

We now assume that the subvariety $\mathscr{X} \subset A_S$ is not constant up to translation in $A(S)$. Then the monodromy of the smooth family $\mathscr{X} \to S$ twisted by a generic rank-1 local system is related to the Tannaka group of the perverse sheaf $\delta_X \in \mathrm{Perv}(A_{S,\bar{\eta}}, \mathbb{F})$ on the geometric generic fiber

$$X := \mathscr{X}_{\bar{\eta}}$$

as follows. For $\chi \in \Pi(A, \mathbb{F})$, let L_χ denote the corresponding rank-1 local system on A. The generic vanishing theorem for perverse sheaves (see [6], [38], [47]) shows that

$$\delta_{X,\chi} := \delta_X \otimes L_\chi \in \overline{\mathrm{Perv}}_0(A_{S,\bar{\eta}}, \mathbb{F})$$

for most $\chi \in \Pi(A, \mathbb{F})$, where *most* means all characters χ outside a finite union of torsion translates of linear subvarieties of $\Pi(A, \mathbb{F})$. From Section 4.3 we get a fiber functor

$$\omega := H^0(A_{S,\bar{\eta}}, -) \colon \langle \delta_{X,\chi} \rangle \longrightarrow \mathrm{Vect}(\mathbb{F}),$$

and we denote by

$$G^*_{X,\chi} := \left[G^\circ_\omega(\delta_{X,\chi}), G^\circ_\omega(\delta_{X,\chi}) \right]$$

the derived group of the connected component of the Tannaka group. Note that by Lemma 3.12 the isomorphism type of this group does not depend on the chosen character; we say that X has a *simple derived connected Tannaka group* if $G^*_{X,\chi}$ is simple for some (hence every) character χ with the above vanishing properties.

To define the monodromy of the family $f \colon \mathscr{X} \to S$ twisted by a rank-1 local system, let $\pi \colon \mathscr{X} \to A$ be the projection to the abelian variety. Using generic vanishing on the geometric generic fiber of $X \subset A_{S,\bar{\eta}}$, one sees that for most χ the higher direct images $R^i f_* \pi^* L_\chi$ vanish in all degrees $i \neq d$, where d denotes the relative dimension of the family $f \colon \mathscr{X} \to S$. For such χ the remaining direct image

$$V_\chi := R^d f_* \pi^* L_\chi$$

is a local system. More generally, we consider for $\underline{\chi} = (\chi_1, \ldots, \chi_n) \in \Pi(A, \mathbb{F})^n$ the direct sum

$$\mathrm{V}_{\underline{\chi}} := \mathrm{V}_{\chi_1} \oplus \cdots \oplus \mathrm{V}_{\chi_n}.$$

Let $\rho\colon \pi_1(\mathrm{S}, \bar{\eta}) \to \mathrm{GL}(\mathrm{V}_{\underline{\chi},\bar{\eta}})$ be the corresponding monodromy representation on the geometric generic fiber. We define the *algebraic monodromy group* of $\mathrm{V}_{\underline{\chi}}$ as the Zariski closure

$$\mathrm{M}(\mathrm{V}_{\underline{\chi}}) := \overline{\mathrm{Im}(\rho)} \subset \mathrm{GL}(\mathrm{V}_{\underline{\chi},\bar{\eta}}).$$

The link between our main theorem from the introduction and the Tannaka groups introduced above is the following result by Lawrence and Sawin, an analogue of the theorem of the fixed part.

THEOREM 4.10

Let S *be a smooth integral variety over* k*, and let* $\mathscr{X} \subset \mathrm{A}_{\mathrm{S}}$ *an integral subvariety such that*

(1) *the family* $f : \mathscr{X} \to \mathrm{S}$ *is smooth of relative dimension* d*, it is not constant up to translation in* $\mathrm{A}(\mathrm{S})$*, and*

(2) *the geometric generic fiber* $\mathrm{X} = \mathscr{X}_{\bar{\eta}} \subset \mathrm{A}_{\mathrm{S},\bar{\eta}}$ *is nondivisible and has a simple derived connected Tannaka group.*

Then for most $\underline{\chi} \in \Pi(\mathrm{A}, \mathbb{F})^n$ *we have*

$$\mathrm{G}^*_{\mathrm{X},\chi_1} \times \cdots \times \mathrm{G}^*_{\mathrm{X},\chi_n} \trianglelefteq \mathrm{M}(\mathrm{V}_{\underline{\chi}}).$$

Proof

In [39, Theorem 5.6] this is stated for hypersurfaces, but the proof works for smooth subvarieties of any codimension. For convenience, we recall the main ideas in our setup. The fiber

$$\mathrm{V}_{\underline{\chi},\bar{\eta}} = \bigoplus_{i=1}^{n} \mathrm{H}^d(\mathrm{X}, \mathrm{L}_{\chi_i})$$

comes with a monodromy action of $\pi_1(\mathrm{S}, \bar{\eta})$ preserving the summands on the right-hand side; the algebraic monodromy is the Zariski closure of the image of $\pi_1(\mathrm{S}, \bar{\eta})$ inside

$$\mathrm{GL}(\mathrm{V}_{\chi_1,\bar{\eta}}) \times \cdots \times \mathrm{GL}(\mathrm{V}_{\chi_n,\bar{\eta}}), \quad \text{where } \mathrm{V}_{\chi_i,\bar{\eta}} = \mathrm{H}^d(\mathrm{X}, \mathrm{L}_{\chi_i}).$$

Since S is smooth, this algebraic monodromy is the Zariski closure of the image of the absolute Galois group of the function field of S. By Theorem 4.5, the Galois action normalizes the subgroups $\mathrm{G}^*_{\mathrm{X},\chi_i} \subset \mathrm{GL}(\mathrm{V}_{\chi_i})$, in fact the algebraic monodromy is a subgroup

$$\mathrm{M}(\mathrm{V}_{\underline{\chi}}) \subset \mathrm{G}_{\mathrm{X}_0,\chi_1} \times \cdots \times \mathrm{G}_{\mathrm{X}_0,\chi_n},$$

where $X_0 := \mathscr{X}_\eta$ denotes the generic fiber and $G_{X_0,\chi_i} := G_\omega(\delta_{X_0,\chi_i})$. We must show that this upper bound on the algebraic monodromy is almost sharp in the sense that for most $\underline{\chi} = (\chi_1, \dots, \chi_n)$, the algebraic monodromy contains the normal subgroup

$$G^*_{X,\chi_1} \times \cdots \times G^*_{X,\chi_n} \trianglelefteq G_{X_0,\chi_1} \times \cdots \times G_{X_0,\chi_n}.$$

In what follows, it will be convenient to identify all factors on the left-hand side with a fixed simple algebraic group. For this, we fix a fiber functor $\xi\colon \langle \delta_X \rangle \to \mathrm{Vect}(\mathbb{F})$ and pick an isomorphism between $H^0(A_{\bar\eta}, -)$ and the fiber functor obtained as the composite

$$\langle \delta_{X_0,\chi_i} \rangle \xrightarrow{\sim} \langle \delta_{X_0} \rangle \xrightarrow{(-)_{\bar\eta}} \langle \delta_X \rangle \xrightarrow{\xi} \mathrm{Vect}(\mathbb{F}),$$

where the isomorphism on the left is the inverse of $P \mapsto P_{\chi_i}$. We get a commutative diagram

$$\begin{array}{ccc} & G^*_{X,\chi_1} \times \cdots \times G^*_{X,\chi_n} & \xrightarrow{\sim} (G^*_X)^n \\ & \big\downarrow & \big\downarrow \\ M(V_{\underline{\chi}}) \hookrightarrow & G_{X_0,\chi_1} \times \cdots \times G_{X_0,\chi_n} & \xrightarrow{\sim} (G_{X_0})^n \end{array}$$

where $G^*_X := [G^\circ_\xi(\delta_X), G^\circ_\xi(\delta_X)] \subset G_{X_0} := G_\xi(\delta_{X_0})$. Note that G_{X_0} is contained in the normalizer

$$N(G^*_X) \subset GL\big(\xi(\delta_{X_0})\big)$$

by Theorem 4.5. Now we use the following general observation (see [39, Lemma 5.4]).

FACT 4.11
Let $G \subset GL(V)$ *be a simple algebraic group, and let* $N(G) \subset GL(V)$ *be its normalizer. Then for every integer* $n \geq 1$ *there exists a finite list of irreducible representations*

$$W_\alpha = W_{\alpha,1} \boxtimes \cdots \boxtimes W_{\alpha,n} \in \mathrm{Rep}_{\mathbb{F}}\big(N(G)^n\big) \quad \big(\alpha \in \{1, \dots, N\}\big)$$

such that for any reductive subgroup $H \subset N(G)^n$ *the following two properties are equivalent:*
(1) $G^n \subset H$.
(2) H *has no invariants on any of the representations* W_α.
In particular, the group G^n *has no invariants on any of the representations* W_α.

We apply the preceding fact to $V = \xi(\delta_{X_0})$, $G = G^*_X$, and $H = M(V_{\underline{\chi}})$. Since $G_{X_0} \subset N(G^*_X)$, each $W_{\alpha,i} \in \mathrm{Rep}_{\mathbb{F}}(N(G))$ defines a representation of the Tannaka group G_{X_0} and hence a perverse sheaf

$$\mathrm{P}_{\alpha,i} \in \langle \delta_{\mathrm{X}_0} \rangle.$$

The representation of $\mathrm{G}_{\mathrm{X}_0,\chi_i}$ obtained by pullback of the representation $\mathrm{W}_{\alpha,i}$ under the isomorphism $\mathrm{G}_{\mathrm{X}_0,\chi_i} \to \mathrm{G}_{\mathrm{X}_0}$ then corresponds to $(\mathrm{P}_{\alpha,i})_{\chi_i} \in \langle \delta_{\mathrm{X}_0,\chi_i} \rangle$. By construction, we have an isomorphism

$$\begin{aligned} \mathrm{W}_\alpha &= \mathrm{W}_{\alpha,1} \boxtimes \cdots \boxtimes \mathrm{W}_{\alpha,n} \\ &\simeq \mathrm{H}^0\big(\mathrm{A}_{\mathrm{S},\bar{\eta}}, (\mathrm{P}_{\alpha,1})_{\chi_1}\big) \boxtimes \cdots \boxtimes \mathrm{H}^0\big(\mathrm{A}_{\mathrm{S},\bar{\eta}}, (\mathrm{P}_{\alpha,n})_{\chi_n}\big) \end{aligned} \tag{4.1}$$

of representations of $\mathrm{N}(\mathrm{G}^*_{\mathrm{X},\chi_1}) \times \cdots \times \mathrm{N}(\mathrm{G}^*_{\mathrm{X},\chi_n})$. To keep track of how the Galois action on the right-hand side depends on the chosen characters, it will be convenient to pass to $\mathrm{A}^n_{\mathrm{S},\bar{\eta}} = \mathrm{A}_{\mathrm{S},\bar{\eta}} \times \cdots \times \mathrm{A}_{\mathrm{S},\bar{\eta}}$ via the Künneth isomorphism. Consider the perverse sheaf

$$\mathrm{K}_0 := e_{1*}\delta_{\mathrm{X}_0} \oplus \cdots \oplus e_{n*}\delta_{\mathrm{X}_0} \in \overline{\mathrm{Perv}}_0(\mathrm{A}^n_{\mathrm{S},\eta}, \mathbb{F}),$$

where $e_i \colon \mathrm{A}_{\mathrm{S},\eta} \hookrightarrow \mathrm{A}^n_{\mathrm{S},\eta}$ is the inclusion of the ith factor. Let K be the base change to $\mathrm{A}^n_{\mathrm{S},\bar{\eta}}$ of the perverse sheaf K_0. Note that we have $\mathrm{G}^*_\zeta(\mathrm{K}) = (\mathrm{G}^*_{\mathrm{X}})^n$ for the fiber functor $\zeta := \xi \boxtimes \cdots \boxtimes \xi \colon \langle \mathrm{K} \rangle \to \mathrm{Vect}(\mathbb{F})$ and

$$\mathrm{Q}_\alpha := e_{1*}\mathrm{P}_{\alpha,1} * \cdots * e_{n*}\mathrm{P}_{\alpha,n} = \mathrm{P}_{\alpha,1} \boxtimes \cdots \boxtimes \mathrm{P}_{\alpha,n} \in \langle \mathrm{K}_0 \rangle.$$

Returning to character twists, consider the local system $\mathrm{L}_{\underline{\chi}} := \mathrm{L}_{\chi_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\chi_n}$ and put

$$\mathrm{K}_{0,\underline{\chi}} := \mathrm{K}_0 \otimes \mathrm{L}_{\underline{\chi}}, \qquad \mathrm{K}_{\underline{\chi}} := \mathrm{K} \otimes \mathrm{L}_{\underline{\chi}}, \qquad \text{and} \qquad \mathrm{Q}_{\alpha,\underline{\chi}} := \mathrm{Q}_\alpha \otimes \mathrm{L}_{\underline{\chi}}.$$

Then we have

$$\mathrm{G}^*_\omega(\mathrm{K}_{\underline{\chi}}) = \mathrm{G}^*_{\mathrm{X},\chi_1} \times \cdots \times \mathrm{G}^*_{\mathrm{X},\chi_n} \qquad \text{and} \qquad \mathrm{Q}_{\alpha,\underline{\chi}} \in \langle \mathrm{K}_{0,\underline{\chi}} \rangle.$$

Combining (4.1) with the Künneth isomorphism, we obtain a Galois equivariant isomorphism

$$\mathrm{W}_\alpha \simeq \mathrm{H}^0(\mathrm{A}^n_{\mathrm{S},\bar{\eta}}, \mathrm{Q}_{\alpha,\underline{\chi}}),$$

where the Galois group acts on the left-hand side via $\mathrm{Gal}(\bar{\eta}/\eta) \to \mathrm{M}(\mathrm{V}_{\underline{\chi}})$ and on the right-hand side by the natural Galois action.

Now recall that by the last claim in Fact 4.11, the group $(\mathrm{G}^*_{\mathrm{X}})^n$ has no invariants on W_α. But $(\mathrm{G}^*_{\mathrm{X}})^n = \mathrm{G}^*_\zeta(\mathrm{K})$ is the derived group of the connected component of the Tannaka group of the perverse sheaf K, and $\mathrm{W}_\alpha = \zeta(\mathrm{Q}_\alpha)$ is the representation defined by $\mathrm{Q}_\alpha \in \langle \mathrm{K}_0 \rangle$. Hence, we can apply [39, Lemma 5.2]. The vanishing of invariants of the derived connected Tannaka group on the geometric generic fiber implies that Q_α

has no perverse subquotient coming by pullback from A via $A_{S,\eta} \to A$. By a spreading out argument (see [39, Lemma 5.3]) the last property implies that for most $\underline{\chi}$ the Galois invariants of $H^0(A_{S,\bar{\eta}}, Q_{\alpha,\underline{\chi}})$ vanish. Thus, the algebraic monodromy has no invariants on any of the representations W_α and hence by the equivalence of (1) and (2) above it contains all of $(G_X^*)^n$ as required. □

Remark 4.12
For $n \geq 2$, the above proof gives more precise information on the dependence of n of the locus of characters on which the conclusion of Theorem 4.10 holds. There exists a finite union $\Sigma \subset \Pi(A, \mathbb{F})^2$ of torsion translates of proper linear subvarieties such that the conclusion of the theorem holds for all $n \geq 2$ and all $\underline{\chi} = (\chi_1, \ldots, \chi_n) \in \Pi(A, \mathbb{F})^n$ with

$$(\chi_i, \chi_j) \notin \Sigma \quad \text{for all } i \neq j.$$

This follows from the fact that the list of representations constructed in the proof of Fact 4.11 arises from a finite list of representations of $N(G)^2$ by pullback under the various projections $N(G)^n \to N(G)^2$.

5. From representations to geometry

In this section, we explain the link between representations and characteristic cycles, which will be our main tool to show that under certain assumptions the Tannaka group of a smooth subvariety will be big. We work over an algebraically closed field k with $\mathrm{char}(k) = 0$, and starting from Section 5.3 we assume $k = \mathbb{C}$.

5.1. The ring of clean cycles

Over the complex numbers an important invariant of a perverse sheaf is its characteristic cycle, which is a formal sum of conormal varieties adapted to a suitable Whitney stratification. As we recall in Section 5.3, the convolution product of perverse sheaves is mirrored by a "convolution product" on their characteristic cycles. To define the latter, we need to introduce a convolution product of conormal varieties, which can be done over any algebraically closed field k of characteristic 0 as follows.

Recall that, for an integral subvariety $Z \subset A$, its conormal variety $\mathbb{A}_Z$ is said to be *clean* if its Gauss map $\gamma_Z \colon \mathbb{A}_Z \to \mathbb{P}_A$ is dominant—by Theorem 2.8 this is the case if and only if the variety Z is of general type—and *negligible* otherwise.

Definition 5.1
The group of clean cycles $\mathscr{L}(A)$ is the free abelian group generated by the projective conormal cones $\mathbb{A}_Z$ of integral subvarieties $Z \subset A$, modulo the subgroup generated by the negligible ones. The projection onto the quotient induces an isomorphism

$$\bigoplus_{Z \subset A} \mathbb{Z} \cdot \mathbb{\Lambda}_Z \xrightarrow{\sim} \mathscr{L}(A),$$

where the direct sum ranges over the integral subvarieties $Z \subset A$ of general type.

A *clean cycle* is an element of $\mathscr{L}(A)$ and, by means of the preceding isomorphism, will always be seen as a finite formal sum $\sum_Z m_Z \mathbb{\Lambda}_Z$, $m_Z \in \mathbb{Z}$, indexed by the integral subvarieties $Z \subset A$ of general type.

Recall that in Definition 2.2 we defined the conormal variety $\mathbb{\Lambda}_Z$ for a reduced but not necessarily irreducible subvariety $Z \subset A$. For simplicity, we still write $\mathbb{\Lambda}_Z$ for the conormal variety seen as a cycle on $A \times \mathbb{P}_A$, or merely as a clean cycle. In particular, in the latter case, we have

$$\mathbb{\Lambda}_Z = \sum_{Z' \subset Z} \mathbb{\Lambda}_{Z'},$$

the sum ranging over the irreducible components $Z' \subset Z$ of general type. We will consistently perpetrate this abuse of notation by writing

$$\mathbb{\Lambda}_Z = m_1 \mathbb{\Lambda}_{Z_1} + \cdots + m_n \mathbb{\Lambda}_{Z_n}$$

for a cycle $Z = m_1 Z_1 + \cdots + m_n Z_n$ on A, with $m_i \in \mathbb{Z}$ and $Z_i \subset A$ integral.

Definition 5.2
Let $\mathbb{\Lambda}_{X_1}, \mathbb{\Lambda}_{X_2}$ be clean conormal varieties. Let $U \subset \mathbb{P}_A$ be an open dense subset of the projective cotangent space to the abelian variety such that over this open subset the Gauss maps $\mathbb{\Lambda}_{X_i|U} := \gamma_{X_i}^{-1}(U) \to U$ are finite étale covers. The fiber product of these two finite étale covers embeds into $A \times A \times U \subset A \times A \times \mathbb{P}_A$, and we denote by

$$\mathbb{\Lambda} := \overline{\mathbb{\Lambda}_{X_1|U} \times_U \mathbb{\Lambda}_{X_2|U}} \subset A \times A \times \mathbb{P}_A$$

its Zariski closure. We define the *convolution* of the conormal varieties to be the clean cycle

$$\mathbb{\Lambda}_{X_1} \circ \mathbb{\Lambda}_{X_2} := \sigma_*(\mathbb{\Lambda}) \in \mathscr{L}(A)$$

arising by pushforward under the sum morphism $\sigma\colon A \times A \times \mathbb{P}_A \to A \times \mathbb{P}_A$. We extend this product $\circ$ on conormal varieties bilinearly to a product on the group of clean cycles

$$\circ\colon \mathscr{L}(A) \times \mathscr{L}(A) \longrightarrow \mathscr{L}(A).$$

This endows the group $\mathscr{L}(A)$ with a natural ring structure. The product $\circ$ should not be confused with an intersection of cycles, indeed the intersection product of any two cycles in $\mathscr{L}(A)$ is zero for dimension reasons. For any integer $n \neq 0$, the pushforward

$$[n]_*\colon \mathscr{L}(\mathrm{A}) \longrightarrow \mathscr{L}(\mathrm{A})$$

is a ring homomorphism. For $\mathbb{\Lambda} \in \mathscr{L}$, we denote by $\langle \mathbb{\Lambda} \rangle \subset \mathscr{L}(\mathrm{A})$ the smallest subring of $\mathscr{L}(\mathrm{A})$ which contains $\mathbb{\Lambda}$ and is stable under passing from a clean cycle to its irreducible components.

5.2. A reminder on Segre classes

In the discussion of wedge powers and spin representations to be carried out in Sections 7 to 8, we will need to control the effect that certain tensor constructions on clean cycles have on the dimension of their base. For this we recall in this section some basic facts about Segre classes, or Chern–Mather classes[2] in the terminology of [33, Section 3].

Definition 5.3

The *Segre classes* of a cycle $\mathbb{\Lambda}$ on $\mathrm{A} \times \mathbb{P}_{\mathrm{A}}$ of pure dimension $g-1$ are defined as the cycle classes

$$s_d(\mathbb{\Lambda}) := (\mathrm{pr}_{\mathrm{A}})_*\big([\mathbb{\Lambda}] \cdot [\mathrm{A} \times \mathrm{H}_d]\big) \in \mathrm{CH}_d(\mathrm{A}),$$

where $\mathrm{H}_d \subset \mathbb{P}_{\mathrm{A}}$ is a general linear subspace of dimension $d < g = \dim \mathrm{A}$ and $\mathrm{CH}_d(\mathrm{A})$ denotes the Chow group of dimension d algebraic cycles with $\mathbb{Z}$-coefficients, modulo rational equivalence.

The following observation allows us to control the dimension of the base of a clean cycle in terms of its Segre classes.

Remark 5.4

For any subvariety $\mathrm{Z} \subset \mathrm{A}$ and $d > \dim \mathrm{Z}$, we have $s_d(\mathbb{\Lambda}_{\mathrm{Z}}) = 0$. On the other hand, if Z has a top-dimensional irreducible component of general type, then the Segre classes $s_d(\mathrm{Z})$ are represented by nonzero effective cycles for all $d \in \{0, 1, \ldots, \dim \mathrm{Z}\}$; this follows from the dominance of the Gauss map γ_{Z} and Kleiman's generic transversality theorem (see [33, Lemma 3.1.2(3)]). The top degree Segre class is the fundamental class

$$s_{\dim \mathrm{Z}}(\mathrm{Z}) = [\mathrm{Z}].$$

Clean cycles live on the *projective* cotangent bundle, thus there is no Segre class in degree $g = \dim \mathrm{A}$. The *total Segre class* $s(\mathbb{\Lambda}) := s_0(\mathbb{\Lambda}) + \cdots + s_{g-1}(\mathbb{\Lambda})$ is then seen as an element of quotient

[2] In the case of abelian varieties Segre classes and Chern–Mather classes are the same, since the cotangent bundle to abelian varieties is trivial.

$$\mathrm{CH}_{<g}(\mathrm{A}) := \mathrm{CH}_\bullet(\mathrm{A})/\mathrm{CH}_g(\mathrm{A}).$$

To define a ring structure on this quotient, recall that the group $\mathrm{CH}_\bullet(\mathrm{A})$ comes with a natural ring structure where the product is given by the Pontryagin product

$$[\mathrm{X}] * [\mathrm{Y}] := \sigma_*[\mathrm{X} \times \mathrm{Y}] \quad \text{for the sum morphism } \sigma : \mathrm{X} \times \mathrm{Y} \to \mathrm{X} + \mathrm{Y} \subset \mathrm{A},$$

and that $\mathrm{CH}_g(\mathrm{A}) \subset \mathrm{CH}_\bullet(\mathrm{A})$ is an ideal for the Pontryagin product. Working with the truncated Chow ring has the advantage that the total Segre class is compatible with the convolution product of clean cycles in the following sense.

LEMMA 5.5
Let $\mathbb{\Lambda}_1, \mathbb{\Lambda}_2 \in \mathscr{L}(\mathrm{A})$. *If both Gauss maps* $\gamma_{\mathbb{\Lambda}_i} : \mathrm{Supp}(\mathbb{\Lambda}_i) \to \mathbb{P}_\mathrm{A}$ *are finite morphisms, then*

$$s(\mathbb{\Lambda}_1 \circ \mathbb{\Lambda}_2) = s(\mathbb{\Lambda}_1) * s(\mathbb{\Lambda}_2) \quad \textit{in } \mathrm{CH}_{<g}(\mathrm{A}).$$

Proof
See [33, Lemma 3.3.1]. □

Thus, the convolution product of clean cycles can be controlled via Pontryagin products of Segre classes. For the latter, one can use the following observation.

LEMMA 5.6
Let $\mathrm{X}, \mathrm{Y} \subset \mathrm{A}$ *be proper reduced subvarieties. Suppose that every irreducible component of maximal dimension in* Y *is of general type and that at least one irreducible component of maximal dimension in* X *is nondegenerate. Then the cycle*

$$s(\mathbb{\Lambda}_\mathrm{X}) * s(\mathbb{\Lambda}_\mathrm{Y})$$

is nonzero and effective in all degrees at most $\min\{\dim \mathrm{X} + \dim \mathrm{Y}, \dim \mathrm{A} - 1\}$.

Proof
Since the Pontryagin product is bilinear and the Pontryagin product of two effective cycles is effective or zero, it suffices to show the statement when X and Y are both integral. Let $d = \dim \mathrm{X}$, $e = \dim \mathrm{Y}$, and $g = \dim \mathrm{A}$, and consider the Segre class

$$s_{m-d}(\mathbb{\Lambda}_\mathrm{Y}) \in \mathrm{CH}_{m-d}(\mathrm{A}) \quad \text{for } d \le m \le \min\{d + e, g - 1\}.$$

This class is represented by an effective cycle since $m - d \in \{0, 1, \ldots, e\}$. For any irreducible component $\mathrm{Z}_{m-d} \subset \mathrm{A}$ of an effective cycle representing this class, we have

$$s_d(\mathbb{\Lambda}_X) * s_{m-d}(\mathbb{\Lambda}_Y) = [X] * s_{m-d}(\mathbb{\Lambda}_Y) = [X] * [Z_{m-d}] + \cdots \in \mathrm{CH}_m(A),$$

where $\cdots$ stands for a cycle which is effective or zero. Since by assumption X is nondegenerate, we furthermore know from Lemma 2.7 that the sum morphism

$$\sigma : X \times Z_{m-d} \longrightarrow X + Z_{m-d} \subset A$$

is generically finite onto its image. So $[X] * [Z_{m-d}] = \sigma_*([X \times Z_{m-d}])$ is an effective class in $\mathrm{CH}_m(A)$. Therefore, the Pontryagin product $s(\mathbb{\Lambda}_X) * s(\mathbb{\Lambda}_Y)$ is nonzero and effective in all degrees m with $d \leq m \leq \min\{d+e, g-1\}$. For $0 \leq m < d$ instead the effectivity of the Pontryagin product is trivial because in that range we can look at $s_m(\mathbb{\Lambda}_X) * s_0(\mathbb{\Lambda}_Y) = \deg(\mathbb{\Lambda}_Y) \cdot s_m(\mathbb{\Lambda}_X)$; note that $\deg(\mathbb{\Lambda}_Y) > 0$ because Y is of general type. □

COROLLARY 5.7
Let $X, Y \subset A$ *be reduced subvarieties, possibly reducible. If the Gauss maps* γ_X, γ_Y *are both finite morphisms, then*

$$\dim \pi\big(\mathrm{Supp}(\mathbb{\Lambda}_X \circ \mathbb{\Lambda}_Y)\big) = \min\{\dim X + \dim Y, \dim A - 1\},$$

where $\pi : A \times \mathbb{P}_A \to \mathbb{P}_A$ *is the projection.*

Proof
Combine Lemmas 5.5 and 5.6. □

5.3. Clean characteristic cycles
For the remainder of this section we work over $k = \mathbb{C}$. Recall that to any perverse sheaf $P \in \mathrm{Perv}(A, \mathbb{C})$ one may attach a characteristic cycle (see [16, Definition 4.3.19]), a finite formal sum of conormal varieties

$$\mathrm{CC}(P) = \sum_{Z \subset A} m_Z(P) \cdot \Lambda_Z \quad \text{with } m_Z(P) \in \mathbb{N}.$$

Here the sum runs over all integral subvarieties $Z \subset A$, and only finitely many $m_Z(P)$ are nonzero. These cycles contain a lot of information, for instance, the Dubson–Kashiwara index formula shows that we can read off the topological Euler characteristic as

$$\chi(A, P) = \sum_{Z \subset A} m_Z(P) \cdot \deg(\mathbb{\Lambda}_Z),$$

where $\deg(\mathbb{\Lambda}_Z)$ is the degree of the Gauss map from Section 2.2 (see [19]). Passing from CC(P) to its projectivization and discarding all components which are not clean, we define the *clean characteristic cycle* by

$$\mathrm{cc}(P) := \sum_{\substack{Z \subset A \text{ of} \\ \text{general type}}} m_Z(P) \cdot \mathbb{\Lambda}_Z.$$

It contains all information needed for the Dubson–Kashiwara index formula. This index formula implies that for $P \in \mathrm{Perv}(A, \mathbb{C})$ we have $\mathrm{cc}(P) = 0$ if and only if $P \in S(A, \mathbb{C})$. So the clean characteristic cycle of perverse sheaves is defined on the abelian quotient category $\overline{\mathrm{Perv}}(A, \mathbb{C}) = \mathrm{Perv}(A, \mathbb{C})/S(A, \mathbb{C})$. By additivity in short exact sequences we then obtain a group homomorphism from the Grothendieck group of this abelian quotient category to the group of clean cycles:

$$\mathrm{cc} \colon \mathrm{K}\big(\overline{\mathrm{Perv}}(A, \mathbb{C})\big) \longrightarrow \mathscr{L}(A).$$

The Grothendieck group of an abelian tensor category is not just an abelian group, but also a ring with the product given by the tensor product, which in our case is the convolution product $*$ of perverse sheaves. By [33, Theorem 2.1.1, Example 1.3.2], we have

$$\mathrm{cc}(P_1 * P_2) = \mathrm{cc}(P_1) \circ \mathrm{cc}(P_2) \quad \text{for all } P_1, P_2 \in \overline{\mathrm{Perv}}(A, \mathbb{C}),$$

where $\circ$ is the convolution product of clean cycles introduced previously.

Example 5.8
Passing to characteristic cycles is useful since the convolution of clean cycles is easier to control than the convolution of perverse sheaves. For instance, in Corollary 3.5 we have seen that for any $P \in \mathrm{Perv}(A, \mathbb{C})$ the connected component of its Tannaka group can be realized as the Tannaka group of $[d]_* P$ for any integer $d \geq 1$ with $d \cdot \Gamma_P = \{0\}$. Here

$$\Gamma_P := \big\{ a \in A(\mathbb{C})_{\mathrm{tors}} \;\big|\; \delta_a \in \langle P \rangle \big\} \subset A(\mathbb{C})$$

is a finite abelian group, but usually hard to control. However, we have the inclusion $\Gamma_P \subset \Gamma_{\mathrm{cc}(P)}$ for

$$\Gamma_{\mathrm{cc}(P)} := \big\{ a \in A(\mathbb{C})_{\mathrm{tors}} \;\big|\; \mathbb{\Lambda}_{\{a\}} \in \big\langle \mathrm{cc}(P) \big\rangle \big\},$$

and this latter group depends only on the characteristic cycle of the given perverse sheaf. This will be useful in the proof of part (2) in Theorem 5.11 below.

5.4. Highest weight theory
We want to use characteristic cycles to study the tensor category generated by a given semisimple perverse sheaf P on A. To pass from Tannaka groups to their connected component, we introduce the following notation.

Definition 5.9
Let $m \geq 1$ be the smallest integer $n \geq 1$ with $n \cdot \Gamma_{cc(P)} = \{0\}$, and let

$$P^\circ := [m]_* P.$$

Fix a fiber functor $\xi\colon \langle P^\circ\rangle \to \mathrm{Vect}(\mathbb{C})$, and consider the fiber functor obtained as the composite

$$\omega\colon \ \langle P\rangle \xrightarrow{[m]_*} \langle P^\circ\rangle \xrightarrow{\ \xi\ } \mathrm{Vect}(\mathbb{C}).$$

Its associated Tannaka groups are

$$G := G_\omega(P) \supseteq G^\circ = G_\xi(P^\circ),$$

where the rightmost equality follows from Corollary 3.5.

Recall that G is a reductive group over $\mathbb{C}$ and ω induces an equivalence of abelian tensor categories

$$\omega : \langle P\rangle \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{C}}(G).$$

In what follows, we will assume that the perverse sheaf $P \in \mathrm{Perv}(A, \mathbb{C})$ is nondivisible and $\det(P) = \delta_0$ (the latter can be achieved by replacing P with a translate). Then, by Proposition 3.3 and Corollary 3.5, the connected component G° is a semisimple group. By highest weight theory, its representation ring has the form

$$R(G^\circ) := K\big(\mathrm{Rep}_{\mathbb{C}}(G^\circ)\big) = \mathbb{Z}[\mathsf{X}]^W,$$

where $\mathsf{X} := \mathrm{Hom}(T, \mathbb{G}_m)$ is the character group of a maximal torus $T \subset G$, endowed with the natural action of the Weyl group $W = N_G(T)/Z_G(T)$, and we denote by $\mathbb{Z}[\mathsf{X}]^W \subset \mathbb{Z}[\mathsf{X}]$ the subring of Weyl group invariants. Recall that for semisimple groups the universal cover inherits the structure of an algebraic group and the covering map is an isogeny

$$p\colon \tilde{G} \twoheadrightarrow G^\circ \subset G.$$

Hence, $\tilde{T} := p^{-1}(T)$ is a maximal torus in $\tilde{G}$, and p induces an isomorphism of Weyl groups

$$N_{\tilde{G}}(\tilde{T})/\tilde{T} \xrightarrow{\sim} N_G(T)/T =: W$$

by means of which these groups are identified in what follows. Moreover, p embeds the character group as a subgroup $\mathsf{X} \subset \tilde{\mathsf{X}} := \mathrm{Hom}(\tilde{T}, \mathbb{G}_m)$ of finite index.

Definition 5.10
Let $d \geq 1$ be the smallest integer $e \geq 1$ with $e \cdot \tilde{\mathsf{X}} \subset \mathsf{X}$.

The multiplication by d then gives a morphism $[d]\colon \tilde{\mathsf{X}} \to \mathsf{X}$, and we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{R}(\mathrm{G}^\circ) & \hookrightarrow & \mathrm{R}(\tilde{\mathrm{G}}) & \longrightarrow & \mathrm{R}(\mathrm{G}^\circ) \\ \| & & \| & & \| \\ \mathbb{Z}[\mathsf{X}]^{\mathrm{W}} & \hookrightarrow & \mathbb{Z}[\tilde{\mathsf{X}}]^{\mathrm{W}} & \xrightarrow{[d]_*} & \mathbb{Z}[\mathsf{X}]^{\mathrm{W}} \end{array}$$

where the top row is the dth Adams operation $\Psi_d : \mathrm{R}(\mathrm{G}^\circ) \to \mathrm{R}(\mathrm{G}^\circ)$. Even though the universal cover $\tilde{\mathrm{G}}$ might not be realized as the Tannaka group of a perverse sheaf, we can relate its representations to clean cycles as follows (here we say that a statement holds for a *very general* point of $\mathbb{P}_{\mathrm{A}}(\mathbb{C})$ if it holds for all points outside a countable union of proper subvarieties).

THEOREM 5.11
Let m, $d \geq 1$ be as above.
(1) *The following diagram of ring homomorphisms is commutative:*

$$\begin{array}{ccccccc} \mathrm{R}(\mathrm{G}) & \xrightarrow{(-)_{|\mathrm{G}^\circ}} & \mathrm{R}(\mathrm{G}^\circ) & \hookrightarrow & \mathrm{R}(\tilde{\mathrm{G}}) & \longrightarrow & \mathrm{R}(\mathrm{G}^\circ) \\ {\scriptstyle\omega^{-1}}\downarrow & & {\scriptstyle\xi^{-1}}\downarrow & & & & \downarrow{\scriptstyle\xi^{-1}} \\ \mathrm{K}(\langle \mathrm{P} \rangle) & \xrightarrow{[m]_*} & \mathrm{K}(\langle \mathrm{P}^\circ \rangle) & & & & \mathrm{K}(\langle \mathrm{P}^\circ \rangle) \\ {\scriptstyle\mathrm{cc}}\downarrow & & {\scriptstyle\mathrm{cc}}\downarrow & & & & \downarrow{\scriptstyle\mathrm{cc}} \\ \mathscr{L}(\mathrm{A}) & \xrightarrow{[m]_*} & \mathscr{L}(\mathrm{A}) & & \xrightarrow{\qquad [d]_* \qquad} & & \mathscr{L}(\mathrm{A}) \end{array}$$

(2) *For any very general $v \in \mathbb{P}_{\mathrm{A}}(\mathbb{C})$, there is a morphism $\varphi_v : \mathsf{X} \to \mathrm{A}(\mathbb{C})$ of groups such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{R}(\mathrm{G}^\circ) & = & \mathbb{Z}[\mathsf{X}]^{\mathrm{W}} & \hookrightarrow & \mathbb{Z}[\mathsf{X}] \\ {\scriptstyle\mathrm{cc}\circ\xi^{-1}}\downarrow & & & & \downarrow{\scriptstyle\varphi_v} \\ \mathscr{L}(\mathrm{A}) & & \xrightarrow{\qquad \mathbb{A} \mapsto \mathbb{A}_v \qquad} & & \mathbb{Z}[\mathrm{A}(\mathbb{C})] = \mathrm{Z}_0(\mathrm{A}) \end{array}$$

where $\mathbb{A}_v$ is the fiber of the Gauss map $\mathbb{A} \to \mathbb{P}_{\mathrm{A}}$ seen as a 0-cycle on A.

Proof
(1) See [33, Theorem 2.2.3]. For part (2), let $\Gamma \leq \mathrm{A}(\mathbb{C})$ be the subgroup generated by the points in the fiber of the Gauss map $\gamma\colon \mathrm{cc}(\mathrm{P}^\circ) \to \mathbb{P}_{\mathrm{A}}$ over a very general point v.

Then [33, Theorem 2.2.3] gives $\varphi_v : \mathsf{X} \to \Gamma/\Gamma_{\mathrm{tors}}$ with the required properties. It then only remains to note that the group Γ is free since our assumption $m \cdot \Gamma_{\mathrm{cc(P)}} = \{0\}$ implies that the subring $\langle \mathrm{cc}(\mathrm{P}^\circ)\rangle \subset \mathscr{L}(\mathrm{A})$ does not contain any conormal variety to a torsion point in $\mathrm{A}(\mathbb{C})$. □

Definition 5.12
For $\beta \in \tilde{\mathsf{X}}$, let $[\beta] \in \mathbb{Z}[\tilde{\mathsf{X}}]$ denote the corresponding basis vector in the group algebra. Note that the multiplication of basis vectors in the group algebra is defined by $[\alpha] \cdot [\beta] = [\alpha + \beta]$, and $[\alpha + \beta] \neq [\alpha] + [\beta]$. The subring $\mathbb{Z}[\tilde{\mathsf{X}}]^{\mathrm{W}} \subset \mathbb{Z}[\tilde{\mathsf{X}}]$ of Weyl group invariants has as its underlying additive group the free abelian group with $\mathbb{Z}$-basis consisting of the vectors

$$[\mathrm{W}.\alpha] := \sum_{\beta \in \mathrm{W}.\alpha} [\beta] \in \mathbb{Z}[\tilde{\mathsf{X}}]^{\mathrm{W}},$$

where α runs through the dominant integral weights in $\tilde{\mathsf{X}}$ and $\mathrm{W}.\alpha \subset \tilde{\mathsf{X}}$ denotes its orbit under the Weyl group. Multiplying by the integer d from above, we obtain an element $[\mathrm{W}.d\alpha] \in \mathbb{Z}[\mathsf{X}]^{\mathrm{W}} = \mathrm{R}(\mathrm{G}^\circ)$. Applying the inverse of $\xi : \mathrm{K}(\langle \mathrm{P}^\circ\rangle) \xrightarrow{\sim} \mathrm{R}(\mathrm{G}^\circ)$ to this element of the representation ring and taking its characteristic cycle, we obtain a clean cycle

$$\mathrm{cc}(\mathrm{P}, \alpha) := \mathrm{cc}\big(\xi^{-1}[\mathrm{W}.d\alpha]\big) \in \mathscr{L}(\mathrm{A}).$$

Note that for any integer $n \neq 0$ we have $\mathrm{cc}(\mathrm{P}, n\alpha) = [n]_* \, \mathrm{cc}(\mathrm{P}, \alpha)$.

Remark 5.13
By design $\mathrm{cc}(\mathrm{P}, \alpha)$ lies in the subring $\langle \mathrm{cc}(\mathrm{P}^\circ)\rangle \subset \mathscr{L}(\mathrm{A})$. In particular, if the cycle $\mathrm{cc}(\mathrm{P})$ is defined over a given algebraically closed subfield of $\mathbb{C}$, then so is $\mathrm{cc}(\mathrm{P}, \alpha)$.

LEMMA 5.14
For any $\alpha \in \tilde{\mathsf{X}}$, the cycle $\mathrm{cc}(\mathrm{P}, \alpha) \in \mathscr{L}(\mathrm{A})$ is effective. Moreover, with m and d as in Definition 5.9 and Definition 5.10, we have

$$[d\,m]_* \, \mathrm{cc}(\mathrm{Q}) = \sum_{\alpha} m_\alpha(\mathrm{V}) \cdot \mathrm{cc}(\mathrm{P}, \alpha)$$

for any $\mathrm{Q} \in \langle \mathrm{P}\rangle$ and $\mathrm{V} = \omega(\mathrm{Q}) \in \mathrm{Rep}(\mathrm{G})$ with weight multiplicities $m_\alpha(\mathrm{V})$.

Proof
The morphism $\varphi_v : \mathbb{Z}[\mathsf{X}] \to \mathbb{Z}[\mathrm{A}(\mathbb{C})]$ of groups is induced by $\varphi_v : \mathsf{X} \to \mathrm{A}(\mathbb{C})$ in Theorem 5.11(2). Hence, it sends the submonoid $\mathbb{N}[\mathsf{X}] \subset \mathbb{Z}[\mathsf{X}]$ into the submonoid $\mathbb{N}[\mathrm{A}(\mathbb{C})]$

of effective 0-cycles. This construction works for *very* general $v \in \mathbb{P}_A(\mathbb{C})$ only. However, if a clean cycle is known to have effective fibers over a very general cotangent vector $v \in \mathbb{P}_A(\mathbb{C})$, then the cycle is effective. Hence, the claim about effectivity follows. The formula for multiplicities holds by construction. □

By Lemma 5.14, the weight multiplicities $m_\alpha(V)$ give us information about the multiplicities in characteristic cycles and vice versa. For example, we say that a representation of a connected reductive group is *minuscule* if it is an irreducible nontrivial representation whose weights for a maximal torus form a single orbit under the Weyl group. A representation of an arbitrary reductive group is called minuscule if its restriction to the connected component of the identity is so. This is a very restrictive condition: For the simple Dynkin types the table in Section 1.4 shows that the only minuscule representations other than standard representations of classical groups are the wedge powers of the standard representation in type A, spin and half-spin representations in types B and D, and the representations of dimension 27 and 56 of the exceptional groups of type E_6 and E_7. The previous lemma shows that for any perverse sheaf whose characteristic cycle is integral, the corresponding representation must be minuscule (see [34, Corollary 1.10]).

COROLLARY 5.15
Let $Y \subset A$ *be a nondivisible subvariety, and let* P *be a simple perverse sheaf on* A *with clean characteristic cycle* $\mathrm{cc}(P) = \mathbb{A}_Y$. *Then* $\omega(P)$ *is a minuscule representation of the group* $G = G_\omega(P)$.

Proof
Since Y is nondivisible, the cycle $[dm]_* \mathrm{cc}(P)$ is integral. In the above identity for the weight multiplicities of the representation $V = \omega(\delta_Y)$ then $m_\alpha(V) \neq 0$ for at most one dominant weight α, and this weight must enter with multiplicity 1. □

We now give a geometric description of the cycles $\mathrm{cc}(P, \alpha) \in \mathscr{L}(A)$ in cases when $G = G_\omega(P)$ is a classical group. The idea is to express arbitrary weights in terms of the weights in the standard representation, similar to the argument in [32]. We do this for each of the classical Dynkin types A, B, D separately; the computation for the Dynkin type C is similar, but we omit it since in type C there are no minuscule representations other than the standard representation.

5.5. Weyl orbits for type A
For $\tilde{G} = SL_n$, let us denote by $\varepsilon_1, \ldots, \varepsilon_n \in \tilde{X}$ the weights of the standard representation. The triviality of the determinant implies that $\varepsilon_1 + \cdots + \varepsilon_n = 0$, and the

wedge powers of the standard representation have as highest weights the fundamental weights

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 0 < i < n.$$

The dominant integral weights are the $\mathbb{N}$-linear combinations of the fundamental weights, that is, the weights

$$\alpha = \sum_{\nu=1}^{n-1} \alpha_\nu \varepsilon_\nu \quad \text{with integers } \alpha_1 \geq \cdots \geq \alpha_{n-1} \geq 0.$$

More generally, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, let $\ell = \max\{\nu \mid \alpha_\nu \neq 0\}$ be the length of the n-tuple α, and let $\mathrm{I}(\ell, n)$ be the set of injective maps

$$\iota: \quad \{1, \ldots, \ell\} \hookrightarrow \{1, \ldots, n\}.$$

Then α has the Weyl group orbit

$$\mathrm{W}.\alpha = \left\{ \alpha_1 \varepsilon_{\iota(1)} + \cdots + \alpha_\ell \varepsilon_{\iota(\ell)} \in \tilde{\mathsf{X}} \;\middle|\; \iota \in \mathrm{I}(\ell, n) \right\}.$$

Each weight in $\mathrm{W}.\alpha$ is obtained for precisely $\mathrm{N}(\alpha)$ distinct choices of ι where

$$\mathrm{N}(\alpha) := \prod_{i \in \mathbb{Z}} \ell_i! \quad \text{with } \ell_i := \#\{\nu \mid \alpha_\nu = i\},$$

with the convention $0! = 1$ so that the above product is finite. Hence, we have

$$[\mathrm{W}.\alpha] = \frac{1}{\mathrm{N}(\alpha)} \cdot \sum_{\iota \in \mathrm{I}(\ell, n)} [\alpha_1 \varepsilon_{\iota(1)} + \cdots + \alpha_\ell \varepsilon_{\iota(\ell)}] \in \mathbb{Z}[\mathsf{X}].$$

To describe the clean cycle $\mathrm{cc}(\mathrm{P}, \alpha)$ in these terms, we need some more notation.

Definition 5.16
Let $\mathbb{A} \in \mathscr{L}(\mathrm{A})$ be a reduced clean cycle. Let $\mathrm{U} \subset \mathbb{P}_\mathrm{A}$ be any open dense subset over which the Gauss map $\gamma_\mathbb{A}: \mathbb{A} \to \mathbb{P}_\mathrm{A}$ restricts to a finite flat morphism, and denote its preimage by $\mathbb{A}_{|\mathrm{U}} := \gamma_\mathbb{A}^{-1}(\mathrm{U})$. For an integer $\ell \geq 1$, consider the fiber product

$$\mathbb{A}_{|\mathrm{U}}^{\times \ell} := \mathbb{A}_{|\mathrm{U}} \times_\mathrm{U} \times \cdots \times_\mathrm{U} \mathbb{A}_{|\mathrm{U}} \subset \mathrm{A}^\ell \times \mathrm{U}$$

and inside it the big diagonal

$$\Delta_\ell := \left\{ (p_1, \ldots, p_\ell, v) \;\middle|\; p_i = p_j \text{ for some } (i, j) \text{ with } i \neq j \right\} \subset \mathbb{A}_{|\mathrm{U}}^{\times \ell}.$$

Since the fiber product of finite flat morphisms is again a finite flat morphism, every irreducible component of the fiber product $\mathbb{A}_{|\mathrm{U}}^{\times \ell}$ is a finite flat cover of U. So the Zariski closure

$$\Lambda^{[\ell]} := \overline{\Lambda^{\times \ell}_{|\mathrm{U}} \smallsetminus \Delta_\ell} \subset \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A}$$

does not depend on the specific choice of the open dense subset $\mathrm{U} \subset \mathbb{P}_\mathrm{A}$ over which $\gamma_\mathbb{\Lambda}$ is finite and flat. For any $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$, we then define a clean cycle as the pushforward

$$\mathbb{\Lambda}^\alpha := \frac{1}{\mathrm{N}(\alpha)} \cdot \sigma_{\alpha *}(\mathbb{\Lambda}^{[\ell]}) \in \mathscr{Z}(\mathrm{A})$$

for the "sum" morphism

$$\sigma_\alpha \colon \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A} \longrightarrow \mathrm{A} \times \mathbb{P}_\mathrm{A}, \qquad (p_1, \ldots, p_\ell, v) \longmapsto (\alpha_1 p_1 + \cdots + \alpha_\ell p_\ell, v).$$

Remark 5.17
The group $\mathfrak{S}_\ell$ acts on $\mathrm{A}^\ell \times \mathbb{P}_\mathrm{A}$ by permutations of the abelian variety factors, and this action restricts to an action on $\mathbb{\Lambda}^{[\ell]} \subset \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A}$. The morphism σ_α factors through the quotient by the subgroup

$$\mathfrak{S}_\alpha := \prod_{i \in \mathbb{Z}} \mathfrak{S}_{\ell_i} \subset \mathfrak{S}_\ell, \quad \text{where } \ell_i := \#\{\nu \mid \alpha_\nu = i\},$$

as shown in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A} & \xrightarrow{\sigma_\alpha} & \mathrm{A} \times \mathbb{P}_\mathrm{A} \\ & {\scriptstyle q} \searrow \quad \nearrow {\scriptstyle \exists ! \bar{\sigma}_\alpha} & \\ & (\mathrm{A}^\ell \times \mathbb{P}_\mathrm{A})/\mathfrak{S}_\alpha & \end{array}$$

Here, the quotient morphism q is finite of degree $\mathrm{N}(\alpha)$. The restriction of q to the subvariety $\mathbb{\Lambda}^{[\ell]} \subset \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A}$ is still finite of the same degree over its image, as one may see by looking at a general fiber of the Gauss map. Hence, Definition 5.16 amounts to

$$\mathbb{\Lambda}^\alpha = \tilde{\sigma}_{\alpha *}(\tilde{\mathbb{\Lambda}}^\alpha), \quad \text{where } \tilde{\mathbb{\Lambda}}^\alpha := \mathbb{\Lambda}^{[\ell]}/\mathfrak{S}_\alpha \subset (\mathrm{A}^\ell \times \mathbb{P}_\mathrm{A})/\mathfrak{S}_\alpha.$$

In particular, it follows that if $\mathbb{\Lambda}^\alpha$ is reduced (resp., integral), then so is $\tilde{\mathbb{\Lambda}}^\alpha$.

We now obtain from Theorem 5.11 the following description of the clean cycles corresponding to the dominant weights.

LEMMA 5.18
Suppose that $\mathrm{G} = \mathrm{G}_\omega(\mathrm{P})$ *is semisimple with universal cover* SL_n*, and let* $\alpha \in \tilde{\mathrm{X}}$ *be a dominant weight. If the clean cycle* $\mathrm{cc}(\mathrm{P}, \alpha)$ *is reduced (resp., integral), then the cycle* $\mathbb{\Lambda} := \mathrm{cc}(\mathrm{P}, \varepsilon_1)$ *is reduced (resp., integral), and*

$$\mathrm{cc}(\mathrm{P},\alpha) = \mathbb{\Lambda}^{\alpha},$$

where on the right-hand side we identify $\alpha = \sum_{i=1}^{n} \alpha_i \varepsilon_i$ *with* $(\alpha_1, \ldots, \alpha_n)$.

Proof

Let $v \in \mathbb{P}_{\mathrm{A}}(\mathbb{C})$ be a very general point, and let $p_1, \ldots, p_n \in \mathrm{A}(\mathbb{C})$ be the points in the fiber of the Gauss map $\mathbb{\Lambda} \to \mathbb{P}_{\mathrm{A}}$ over v, counted with multiplicities so that

$$\mathbb{\Lambda}_v = [p_1] + \cdots + [p_n]$$

inside $\mathbb{Z}[\mathrm{A}(\mathbb{C})]$. By Theorem 5.11, the points in the fiber are precisely the images of the weights in the Weyl group orbit $\mathrm{W} \cdot d\varepsilon_1 = \{d\varepsilon_1, \ldots, d\varepsilon_n\} \subset \mathsf{X}$ under the homomorphism $\varphi_v \colon \mathsf{X} \to \mathrm{A}(\mathbb{C})$. Up to relabeling indices, we may assume $\varphi_v(d\varepsilon_i) = p_i$ for all i. Writing the weight as $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $\ell < n$, we have

$$\begin{aligned}
\mathrm{cc}(\mathrm{P},\alpha)_v &= \mathrm{cc}\big(\xi^{-1}[\mathrm{W}.d\alpha]\big)_v = \varphi_v\big([\mathrm{W}.d\alpha]\big) \\
&= \frac{1}{\mathrm{N}(\alpha)} \cdot \sum_{\iota \in \mathrm{I}(\ell,n)} \big[\varphi_v(\alpha_1 d\varepsilon_{\iota(1)} + \cdots + \alpha_\ell d\varepsilon_{\iota(\ell)})\big] \\
&= \frac{1}{\mathrm{N}(\alpha)} \cdot \sum_{\iota \in \mathrm{I}(\ell,n)} [\alpha_1 p_{\iota(1)} + \cdots + \alpha_\ell p_{\iota(\ell)}].
\end{aligned}$$

For very general v this 0-cycle contains no multiple points, since we assumed $\mathrm{cc}(\mathrm{P},\alpha)$ to be reduced. Therefore, each point

$$p = \alpha_1 p_{\iota(1)} + \cdots + \alpha_\ell p_{\iota(\ell)}$$

enters in the above expression for $\mathrm{cc}(\mathrm{P},\alpha)_v$ only for $\mathrm{N}(\alpha)$ different choices of ι. But p does not change if we replace the map $\iota \colon \{1, \ldots, \ell\} \hookrightarrow \{1, \ldots, n\}$ by $\iota \circ \tau$ for any permutation

$$\tau \in \mathfrak{S}_\alpha = \prod_{i \in \mathbb{Z}} \mathfrak{S}_{\ell_i} \subset \mathfrak{S}_\ell.$$

Since $\mathrm{N}(\alpha) = |\mathfrak{S}_\alpha|$, it follows from the above that for all $\iota, \iota^* \in \mathrm{I}(\ell, n)$ we have the equivalence

$$\sum_{\nu=1}^{\ell} \alpha_\nu p_{\iota(\nu)} = \sum_{\nu=1}^{\ell} \alpha_\nu p_{\iota^*(\nu)} \quad \Longleftrightarrow \quad \exists \tau \in \prod_{i \in \mathbb{Z}} \mathfrak{S}_{\ell_i} : \iota^* = \iota \circ \tau.$$

Since $\ell < n$, this forces $p_1, \ldots, p_n \in \mathrm{A}(\mathbb{C})$ to be pairwise distinct, so the 0-cycle $\mathbb{\Lambda}_v$ is reduced and $\mathbb{\Lambda}$ must be reduced as well. Now let $\mathrm{U} \subset \mathbb{P}_{\mathrm{A}}$ be an open neighborhood

of v such that the Gauss map $\mathbb{\Lambda}_{|\mathrm{U}} \to \mathrm{U}$ is finite and flat. For the complement of the big diagonal, we then get a bijection

$$\mathrm{I}(\ell, n) \longrightarrow (\mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell} \smallsetminus \Delta_\ell)_v, \qquad \iota \longmapsto (p_{\iota(1)}, \dots, p_{\iota(\ell)}, v)$$

and therefore

$$\mathrm{cc}(\mathrm{P}, \alpha)_v = \frac{1}{\mathrm{N}(\alpha)} \cdot \sum_\iota [\alpha_1 p_{\iota(1)} + \cdots + \alpha_\ell p_{\iota(\ell)}] = \frac{1}{\mathrm{N}(\alpha)} \cdot \big(\sigma_{\alpha *}(\mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell} \smallsetminus \Delta_\ell)\big)_v$$

which is by definition the fiber of the cycle $\mathbb{\Lambda}^\alpha$ over v. Since this holds for very general v, it follows that $\mathrm{cc}(\mathrm{P}, \alpha) = \mathbb{\Lambda}^\alpha$ as claimed.

If we moreover assume that the cycle $\mathrm{cc}(\mathrm{P}, \alpha)$ is integral, then it follows from Remark 5.17 that the cycle

$$\tilde{\mathbb{\Lambda}}^\alpha = \mathbb{\Lambda}^{[\ell]} / \mathfrak{S}_\alpha$$

is integral as well. We claim that in this case, the cycle $\mathbb{\Lambda}$ must be integral. Indeed, suppose for a contradiction that $\mathbb{\Lambda} = \mathbb{\Lambda}_1 + \mathbb{\Lambda}_2$ with effective cycles $\mathbb{\Lambda}_1, \mathbb{\Lambda}_2 \in \mathscr{L}(\mathrm{A})$. We know that the fiber of the Gauss map $\gamma_{\mathbb{\Lambda}} \colon \mathbb{\Lambda} \to \mathbb{P}_\mathrm{A}$ over a general point $v \in \mathbb{P}_\mathrm{A}(\mathbb{C})$ is reduced, hence every point in this fiber lies either on $\mathbb{\Lambda}_1$ or on $\mathbb{\Lambda}_2$ but not on both. Thus, each point on the big diagonal comes from a point on the big diagonal of one of the two summands. For the complement of the big diagonal, we therefore obtain that

$$\mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell} \smallsetminus \Delta_\ell \supset \sum_{\ell_1 + \ell_2 = \ell} (\mathbb{\Lambda}_{1|\mathrm{U}}^{\times \ell_1} \smallsetminus \Delta_{1,\ell}) \times_\mathrm{U} (\mathbb{\Lambda}_{2|\mathrm{U}}^{\times \ell_2} \smallsetminus \Delta_{2,\ell}),$$

where we denote the big diagonals in the summands by $\Delta_{i,\ell} \subset \mathbb{\Lambda}_{i|\mathrm{U}}^{\times \ell_i}$. Taking Zariski closure gives

$$\mathbb{\Lambda}^{[\ell]} \supset \sum_{\ell_1 + \ell_2 = \ell} \mathbb{\Lambda}^{[\ell_1, \ell_2]},$$

where

$$\mathbb{\Lambda}^{[\ell_1, \ell_2]} := \overline{(\mathbb{\Lambda}_{1|\mathrm{U}}^{\times \ell_1} \smallsetminus \Delta_{1,\ell}) \times_\mathrm{U} (\mathbb{\Lambda}_{2|\mathrm{U}}^{\times \ell_2} \smallsetminus \Delta_{2,\ell})}.$$

From a look at the degree of the respective Gauss maps we have $\mathbb{\Lambda}^{[\ell_1, \ell_2]} \neq 0$ if and only if $0 \leq \ell_1 \leq \deg(\mathbb{\Lambda}_1)$ and $0 \leq \ell_2 \leq \deg(\mathbb{\Lambda}_2)$. For $\ell_1 + \ell_2 = \ell$, these four inequalities are equivalent to

$$\max\big\{0, \ell - \deg(\mathbb{\Lambda}_2)\big\} \leq \ell_1 \leq \min\big\{\deg(\mathbb{\Lambda}_1), \ell\big\}.$$

This set of inequalities has at least two different solutions $\ell_1 \in \mathbb{Z}$, indeed we have $\max\{0, \ell - \deg(\mathbb{\Lambda}_2)\} < \min\{\deg(\mathbb{\Lambda}_1), \ell\}$ because all the occurring degrees

of Gauss maps are greater than 0 and $\ell < n = \deg(\mathbb{\Lambda}) = \deg(\mathbb{\Lambda}_1) + \deg(\mathbb{\Lambda}_2)$. In conclusion, this shows that there are at least two irreducible components of the form $\mathbb{\Lambda}^{[\ell_1,\ell_2]}$ in $\mathbb{\Lambda}^{[\ell]}$. One easily sees that no two such components are related to each other by a permutation of the factors in $\mathrm{A}^{\ell} \times \mathbb{P}_{\mathrm{A}}$, using again that the fibers of the Gauss maps for $\mathbb{\Lambda}_1$ and for $\mathbb{\Lambda}_2$ are disjoint. Hence, the quotient $\mathbb{\Lambda}^{[\ell]}/\mathfrak{S}_{\alpha}$ has more than one irreducible component, which contradicts our assumption. □

5.6. Weyl orbits for type B

For $\tilde{\mathrm{G}} = \mathrm{Spin}_{2n+1}$, let $\varepsilon_{\pm 1}, \ldots, \varepsilon_{\pm n} \in \tilde{\mathrm{X}}$ be the nontrivial weights of the standard representation of the orthogonal group, with the relations $\varepsilon_{-i} = -\varepsilon_i$. The fundamental weights are

$$\varpi_i = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_i & \text{for } i < n, \\ \frac{1}{2} \cdot (\varepsilon_1 + \cdots + \varepsilon_n) & \text{for } i = n. \end{cases}$$

The first $n-1$ fundamental weights are again highest weights of wedge powers of the standard representation; the last fundamental weight is the highest weight of the spin representation. The dominant integral weights are the weights of the form

$$\alpha = \sum_{\nu=1}^{n} \alpha_\nu \varepsilon_\nu \quad \text{with } \alpha_1 \geq \cdots \geq \alpha_n \geq 0,$$

where $2\alpha_i \in \mathbb{Z}$ are either all even or all odd. Put $\ell = \max\{\nu \mid \alpha_\nu \neq 0\}$, and let $\mathrm{I}(\ell, \pm n)$ be the set of maps

$$\iota \colon \{1, \ldots, \ell\} \hookrightarrow \{\pm 1, \ldots, \pm n\}$$

with the property that the map $\nu \mapsto |\iota(\nu)|$ is still injective. Then α has the Weyl group orbit

$$\mathrm{W}.\alpha = \left\{ \alpha_1 \varepsilon_{\iota(1)} + \cdots + \alpha_\ell \varepsilon_{\iota(\ell)} \;\middle|\; \iota \in \mathrm{I}(\ell, \pm n) \right\}.$$

Each weight in this orbit occurs for precisely $\mathrm{N}(\alpha)$ different choices of ι, where the number $\mathrm{N}(\alpha)$ is defined as above; note that different sign choices will lead to different weights and hence the extra signs do not change the count.

In order to translate this back to geometry, we need to adapt Definition 5.16 to the symmetric case.

Definition 5.19

Let $\mathbb{\Lambda} \in \mathscr{L}(\mathrm{A})$ be a reduced clean cycle with $[-1]_*\mathbb{\Lambda} = \mathbb{\Lambda}$, and let $\mathrm{U} \subset \mathbb{P}_{\mathrm{A}}$ be an open dense subset over which all components of the cycle are finite and flat. For an integer $\ell \geq 1$, let

$$\Delta_\ell := \{(p_1, \dots, p_\ell, v) \mid p_i = p_j \text{ for some } (i,j) \text{ with } i \neq j\} \subset \mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell},$$
$$\Delta_\ell^- := \{(p_1, \dots, p_\ell, v) \mid p_i = -p_j \text{ for some } (i,j) \text{ with } i \neq j\} \subset \mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell}$$

be the big diagonal (resp., antidiagonal) in the fiber product, and consider the Zariski closure

$$\mathbb{\Lambda}_{\mathbb{S}}^{[\ell]} := \overline{\mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell} \smallsetminus (\Delta_\ell \cup \Delta_\ell^-)} \subset \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A}.$$

For $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}^\ell$, we obtain a clean cycle

$$\mathbb{\Lambda}_{\mathbb{S}}^{\alpha} := \frac{1}{\mathrm{N}(\alpha)} \cdot \sigma_{\alpha *}(\mathbb{\Lambda}_{\mathbb{S}}^{[\ell]}) \in \mathscr{L}(\mathrm{A})$$

as the pushforward under the morphism

$$\sigma_\alpha \colon \mathrm{A}^\ell \times \mathbb{P}_\mathrm{A} \longrightarrow \mathrm{A} \times \mathbb{P}_\mathrm{A}, \qquad (p_1, \dots, p_\ell, v) \longmapsto (\alpha_1 p_1 + \cdots + \alpha_\ell p_\ell, v).$$

With this notation, we obtain from Theorem 5.11 the following description of the clean cycles corresponding to the dominant weights.

LEMMA 5.20
Suppose that $\mathrm{G} = \mathrm{G}_\omega(\mathrm{P})$ *is a semisimple group with universal cover* Spin_{2n+1}, *and let* $\alpha \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n \subset \tilde{\mathsf{X}}$ *be a dominant weight. If the clean cycle* $\mathrm{cc}(\mathrm{P}, \alpha)$ *is reduced, then* $\mathbb{\Lambda} := \mathrm{cc}(\mathrm{P}, \varepsilon_1)$ *is reduced, and*

$$\mathrm{cc}(\mathrm{P}, \alpha) = \mathbb{\Lambda}_{\mathbb{S}}^{\alpha},$$

where on the right-hand side we identify $\alpha = \sum_{i=1}^n \alpha_i \varepsilon_i$ *with* $(\alpha_1, \dots, \alpha_n)$.

Proof
Let $v \in \mathbb{P}_\mathrm{A}(\mathbb{C})$ be a very general point, and let $p_{\pm 1}, \dots, p_{\pm n}$ be the projection in $\mathrm{A}(\mathbb{C})$ of the $2n$ points in the fiber of the Gauss map $\mathbb{\Lambda} \to \mathbb{P}_\mathrm{A}$ over v, counted with multiplicities so that

$$\mathbb{\Lambda}_v = [p_1] + \cdots + [p_n] + [p_{-1}] + \cdots + [p_{-n}]$$

inside $\mathbb{Z}[\mathrm{A}(\mathbb{C})]$. By Theorem 5.11, the points in the fiber are precisely the images of the weights in the orbit $\mathrm{W}.d\varepsilon_1 = \{\pm d\varepsilon_1, \dots, \pm d\varepsilon_n\} \subset \mathsf{X}$ under the homomorphism $\varphi_v \colon \mathsf{X} \to \mathrm{A}(\mathbb{C})$. Up to relabeling indices, we may assume that $\varphi_v(\pm d\varepsilon_i) = p_{\pm i} = \pm p_i$ for all i. Then as in the proof of Lemma 5.18 one obtains

$$\mathrm{cc}(\mathrm{P}, \alpha)_v = \frac{1}{\mathrm{N}(\alpha)} \cdot \sum_{\iota \in \mathrm{I}(\ell, \pm n)} [\alpha_1 p_{\iota(1)} + \cdots + \alpha_\ell p_{\iota(\ell)}].$$

For very general v, this 0-cycle contains no multiple points, since we assumed $\mathrm{cc}(\mathrm{P},\alpha)$ to be reduced. The same counting argument as in Lemma 5.18 then shows that for any $\iota,\iota^* \in \mathrm{I}(\ell,\pm n)$ we have the equivalence

$$\sum_{\nu=1}^{\ell} \alpha_\nu p_{\iota(\nu)} = \sum_{\nu=1}^{\ell} \alpha_\nu p_{\iota^*(\nu)} \quad \Longleftrightarrow \quad \exists \tau \in \prod_{i\in\mathbb{Z}} \mathfrak{S}_{\ell_i} : \iota^* = \iota \circ \tau.$$

This forces $p_{\pm 1},\dots,p_{\pm n} \in \mathrm{A}(\mathbb{C})$ to be pairwise distinct. Indeed, assuming that we had

$$p_i = p_j \quad \text{for certain } i,j \in \{\pm 1,\dots,\pm n\} \text{ with } i \neq \pm j,$$

then the implication $\Longrightarrow$ in the above equivalence would fail for any choice of $\iota,\iota^* \in \mathrm{I}(\ell,\pm n)$ with

$$\iota(1) = i = -\iota^*(1), \qquad \iota(2) = -j = -\iota^*(2), \qquad \text{and}$$
$$\iota(\nu) = \iota^*(\nu) \quad \text{for } \nu = 3,\dots,\ell,$$

a contradiction. This shows that the 0-cycle $\mathbb{\Lambda}_v$ is reduced, hence $\mathbb{\Lambda}$ must be reduced as well. For the complement of the big diagonal and antidiagonal, we then get a bijection

$$\mathrm{I}(\ell,\pm n) \longrightarrow \big(\mathbb{\Lambda}_{|\mathrm{U}}^{\times \ell} \smallsetminus (\Delta_\ell \cup \Delta_\ell^-)\big)_v, \qquad \iota \longmapsto (p_{\iota(1)},\dots,p_{\iota(\ell)},v)$$

and can conclude as in Lemma 5.18 that $\mathrm{cc}(\mathrm{P},\alpha) = \mathbb{\Lambda}_{\mathbb{S}}^{\alpha}$. □

5.7. Weyl orbits for type D

For $\tilde{\mathrm{G}} = \mathrm{Spin}_{2n}$, we let $\varepsilon_{\pm 1},\dots,\varepsilon_{\pm n} \in \tilde{\mathsf{X}}$ denote the weights of the standard representation of the orthogonal group, with the relations $\varepsilon_{-i} = -\varepsilon_i$. The fundamental weights are

$$\varpi_i = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_i & \text{for } i < n-1, \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n) & \text{for } i = n-1, \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n) & \text{for } i = n. \end{cases}$$

The first $n-2$ fundamental weights are highest weights of wedge powers of the standard representation; the last two fundamental weights are the highest weights of the two spin representations. The dominant integral weights are the weights of the form

$$\alpha = \sum_{\nu=1}^{n} \alpha_\nu \varepsilon_\nu \quad \text{with } \alpha_1 \geq \cdots \geq \alpha_{n-1} \geq |\alpha_n| \geq 0,$$

where $2\alpha_i \in \mathbb{Z}$ are either all even or all odd. Put $\ell = \max\{\nu \mid \alpha_\nu \neq 0\}$, and let $\mathrm{I}_{\mathrm{even/odd}}(\ell, \pm n)$ be the set of maps

$$\iota\colon \{1,\dots,\ell\} \hookrightarrow \{\pm 1,\dots,\pm n\}$$

with the property that the map $\nu \mapsto |\iota(\nu)|$ is still injective and the number of negative values of ι is even (resp., odd). Then α has the Weyl group orbit

$$\mathrm{W}.\alpha = \left\{\alpha_1\varepsilon_{\iota(1)} + \cdots + \alpha_\ell\varepsilon_{\iota(\ell)} \;\middle|\; \iota \in \mathrm{I}_{\mathrm{even/odd}}(\ell, \pm n)\right\} \quad \text{with} \begin{cases} \text{even} & \text{if } \alpha_n \geq 0, \\ \text{odd} & \text{if } \alpha_n < 0. \end{cases}$$

Each weight in this orbit occurs for precisely $\mathrm{N}(\alpha)$ different choices of ι; again the signs do not matter for this count.

In order to translate this back to geometry, we need to refine Definition 5.19 as follows.

Definition 5.21
Let $\mathbb{\Lambda} \in \mathscr{L}(\mathrm{A})$ be a reduced clean cycle with $[-1]_*\mathbb{\Lambda} = \mathbb{\Lambda}$, and let $\mathrm{U} \subset \mathbb{P}_\mathrm{A}$ be an open dense subset over which all components of the cycle are finite and étale. Labeling the points in a general fiber of the Gauss map $\mathbb{\Lambda}_{|\mathrm{U}} \to \mathrm{U}$ in pairs of opposite points as

$$\gamma_{\mathbb{\Lambda}}^{-1}(u) = \{p_{\pm 1},\dots,p_{\pm n}\} \quad \text{with } p_{-i} = -p_i,$$

we identify the monodromy group of the finite étale cover $\gamma_{\mathbb{\Lambda}|\mathrm{U}}\colon \mathbb{\Lambda}_{|\mathrm{U}} \to \mathrm{U}$ as a subgroup of $(\pm 1)^n \rtimes \mathfrak{S}_n$. We say that the monodromy of the Gauss map is *even* if it is contained in the subgroup

$$(\pm 1)^n_+ \rtimes \mathfrak{S}_n, \quad \text{where } (\pm 1)^n_+ := \left\{(a_1,\dots,a_n) \in (\pm 1)^n \mid a_1\cdots a_n = +1\right\}.$$

Then for $\ell = n$ in Definition 5.19 we obtain that the cycle $\mathbb{\Lambda}^{[n]}_{\mathbb{S}}$ on $\mathrm{A}^n \times \mathbb{P}_\mathrm{A}$ splits as a sum

$$\mathbb{\Lambda}^{[n]}_{\mathbb{S}} = \mathbb{\Lambda}^{[n]}_{\mathbb{S},+} + \mathbb{\Lambda}^{[n]}_{\mathbb{S},-},$$

where $\mathbb{\Lambda}^{[n]}_{\mathbb{S},\pm} \subset \mathbb{\Lambda}^{[n]}_{\mathbb{S}}$ are defined by the condition that their fiber over $u \in \mathrm{U}(k)$ contains an even (resp., odd) number of points with a negative sign; in other words,

$$(\mathbb{\Lambda}^{[n]}_{\mathbb{S},\pm})_u := \left\{(p_{i_1},\dots,p_{i_n},u) \;\middle|\; \prod_{\nu=1}^{n} \mathrm{sgn}(i_\nu) = \pm 1\right\}.$$

Note that this condition depends on the way we have labeled the points in the fiber $\gamma_{\mathrm{Z}}^{-1}(u)$. The labeling by $\pm$ has no intrinsic meaning and is only used as a

notational device to separate the two pieces in the decomposition; none of the two pieces is distinguished. We define

$$\mathbb{\Lambda}^{\alpha}_{\mathbb{S},\pm} := \sigma_{\alpha *}(\Lambda^{[n]}_{\mathbb{S},\pm}) \in \mathscr{L}(\mathrm{A})$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ and the sum morphism σ_α as in Definition 5.19. Note that while there is no intrinsic meaning to the labels $\pm$, it might nevertheless happen that $\mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{\alpha}_{\mathbb{S},-}) \neq \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{\alpha}_{\mathbb{S},+})$. We also note that by symmetry of $\mathrm{Z} \subset \mathrm{A}$ we can assume without loss of generality that $\alpha_n \geq 0$.

LEMMA 5.22
Suppose that $\mathrm{G} = \mathrm{G}_\omega(\mathrm{P})$ *is a semisimple group with universal cover* Spin_{2n}, *and let* $\alpha \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n \subset \tilde{\mathrm{X}}$ *be a dominant weight of length* ℓ. *If* $\mathrm{cc}(\mathrm{P}, \alpha)$ *is reduced, then* $\mathbb{\Lambda} := \mathrm{cc}(\mathrm{P}, \varepsilon_1)$ *is reduced. In this case,*

$$\mathrm{cc}(\mathrm{P}, \alpha) = \begin{cases} \mathbb{\Lambda}^{\alpha}_{\mathbb{S}} & \text{for } \ell < n, \\ \mathbb{\Lambda}^{\alpha}_{\mathbb{S},\varepsilon} & \text{for } \ell = n \text{ and suitable } \varepsilon \in \{+,-\}, \end{cases}$$

where on the right-hand side we identify $\alpha = \sum_{i=1}^{n} \alpha_i \varepsilon_i$ *with* $(\alpha_1, \ldots, \alpha_n)$.

Proof
This is similar to the argument for type B_n. □

6. Simplicity of the Tannaka group

We now take a closer look at the Tannaka group of the perverse intersection complex on a smooth nondivisible subvariety. By Corollary 5.15, the corresponding representation is minuscule; the goal of this section is to show that, under suitable positivity assumptions, the Tannaka group is simple modulo its center.

6.1. The simplicity criterion

For the rest of this section, we assume that k is algebraically closed of characteristic 0. Throughout, we fix a subvariety $\mathrm{X} \subset \mathrm{A}$ and denote by

$$\omega \colon \langle \delta_{\mathrm{X}} \rangle \longrightarrow \mathrm{Vect}(\mathbb{F})$$

a fiber functor on the Tannaka category generated by the perverse intersection complex $\delta_{\mathrm{X}} \in \mathrm{Perv}(\mathrm{A}, \mathbb{F})$. Consider the Tannaka group $\mathrm{G}_{\mathrm{X},\omega} := \mathrm{G}_\omega(\delta_{\mathrm{X}})$, and denote by

$$\mathrm{G}^{*}_{\mathrm{X},\omega} := [\mathrm{G}^{\circ}_{\mathrm{X},\omega}, \mathrm{G}^{\circ}_{\mathrm{X},\omega}]$$

the derived group of its connected component. This is a connected semisimple group; hence, its Lie algebra is a product of simple Lie algebras. Recall that a connected algebraic group is *simple* if its Lie algebra is simple, or equivalently, it does not contain

connected (reduced) normal subgroups. Writing $g := \dim \mathrm{A}$, the goal of this section is the following simplicity criterion.

THEOREM 6.1
Suppose that $g \geq 3$, and let X be a smooth nondivisible subvariety with ample normal bundle. Then the following are equivalent:

(1) *The algebraic group $\mathrm{G}^*_{\mathrm{X},\omega}$ is not simple.*

(2) *There are smooth positive-dimensional subvarieties $\mathrm{X}_1, \mathrm{X}_2 \subset \mathrm{A}$ such that the sum morphism induces an isomorphism*

$$\mathrm{X}_1 \times \mathrm{X}_2 \xrightarrow{\sim} \mathrm{X}.$$

The proof of this result will occupy the rest of this section, but let us first observe that the criterion applies in many cases. First, a smooth projective curve $\mathrm{X} \subset \mathrm{A}$ generating A has ample normal bundle (see [25, Proposition 4.1]), thus $\mathrm{G}^*_{\mathrm{X},\omega}$ is simple as soon as X is nondivisible and $g \geq 3$. More generally, we have the following.

COROLLARY 6.2
Suppose that $g \geq 3$, and let $\mathrm{X} \subset \mathrm{A}$ be a smooth nondivisible subvariety with ample normal bundle. Assume that one of the following conditions holds:

(1) *the image of any Albanese morphism $\mathrm{X} \to \mathrm{Alb}(\mathrm{X})$ is a nondegenerate subvariety of $\mathrm{Alb}(\mathrm{X})$ in the sense of Section 2.3, or*

(2) *the natural map $\mathrm{Alb}(\mathrm{X}) \to \mathrm{A}$ is an isogeny.*

Then the algebraic group $\mathrm{G}^*_{\mathrm{X},\omega}$ is simple.

Proof of Corollary 6.2
For smooth connected subvarieties $\mathrm{X}_1, \mathrm{X}_2 \subset \mathrm{A}$, the image of all Albanese morphisms $\mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{Alb}(\mathrm{X}_1) \times \mathrm{Alb}(\mathrm{X}_2)$ is degenerate if $0 < \dim \mathrm{X}_i < \dim \mathrm{Alb}(\mathrm{X}_i)$ for $i = 1, 2$. Theorem 6.1 therefore shows the statement assuming (1). Now hypothesis (2) implies hypothesis (1). Indeed, $\mathrm{X} \subset \mathrm{A}$ is nondegenerate by Theorem 2.8 and nondegeneracy is invariant under isogenies (see Remark 2.6). □

Remark 6.3
For a smooth integral subvariety $\mathrm{X} \subset \mathrm{A}$, condition (2) holds if the normal bundle of X is the direct sum of vector bundles $\mathscr{V}_1, \ldots, \mathscr{V}_r$ with $r \geq 1$ such that $\mathscr{V}_i$ is d_i-ample[3] and (see [11, Theorem 4.5])

[3] A line bundle $\mathscr{L}$ on an integral variety X is ***d*-*ample*** if there is an integer $n \geq 1$ such that $\mathscr{L}^{\otimes n}$ is globally generated and the fibers of the morphism $\mathrm{X} \to \mathbb{P}(\mathrm{H}^0(\mathrm{X}, \mathscr{L}^{\otimes n})^\vee)$ have dimension $\leq d$. Ordinary ampleness is equivalent to **0**-ampleness. A vector bundle $\mathscr{E}$ on X is ***d*-*ample*** if the line bundle $\mathscr{O}(1)$ on $\mathbb{P}(\mathscr{E}^\vee)$ is d-ample.

$$\dim \mathrm{X} > \max_{i=1,\dots,r} \operatorname{rk} \mathscr{V}_i + d_i .$$

By Theorem 2.8, ampleness of the normal bundle implies nondegeneracy, thus Theorem 6.1 applies to any smooth nondivisible subvariety $\mathrm{X} \subset \mathrm{A}$ such that

- the normal bundle of X is ample and $\dim \mathrm{X} > g/2$, or
- $\dim \mathrm{X} \geq 2$ and X is a complete intersection of ample divisors in A.

6.2. Product decomposition: From geometry to groups

We start by showing how to obtain a product decomposition of the Lie algebra of the Tannaka group starting from a product decomposition of the subvariety. Keeping the notation of Section 6.1, we write $\mathrm{G}^*_{\mathrm{X},\omega} := \mathrm{G}^*_{\omega}(\delta_{\mathrm{X}})$ for any fixed fiber functor $\omega \colon \langle \delta_{\mathrm{X}} \rangle \to \mathrm{Vect}(\mathbb{F})$.

LEMMA 6.4

Let G *be a simple, simply connected algebraic group over* $\mathbb{F}$*, and let* V, W *be nontrivial irreducible representations of* G*. Then* $\mathrm{V} \otimes \mathrm{W}$ *is not minuscule.*

Proof

The minuscule representations of G are given up to isomorphism by Table 1. In particular, the highest weight of any minuscule representation is a fundamental weight. But the highest weight in $\mathrm{V} \otimes \mathrm{W}$ is the sum of the highest weights of V and W; hence, it is a sum of two dominant integral weights and therefore cannot be a fundamental weight. □

PROPOSITION 6.5

Let $\mathrm{X}_1, \mathrm{X}_2 \subset \mathrm{A}$ *be smooth subvarieties such that* $\mathrm{X} := \mathrm{X}_1 + \mathrm{X}_2$ *is nondivisible and the sum morphism* $\sigma \colon \mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{X}$ *is an isomorphism. Then we have an isomorphism of Lie algebras*

$$\operatorname{Lie} \mathrm{G}^*_{\mathrm{X},\omega} \simeq \operatorname{Lie} \mathrm{G}^*_{\mathrm{X}_1,\omega} \oplus \operatorname{Lie} \mathrm{G}^*_{\mathrm{X}_2,\omega}.$$

Proof

Let G be the universal cover of the derived connected component of the Tannaka group of $\delta_{\mathrm{X}_1} \oplus \delta_{\mathrm{X}_2}$. The group G decomposes as a product

$$\mathrm{G} = \mathrm{G}_1 \times \cdots \times \mathrm{G}_n$$

with $\mathrm{G}_1, \dots, \mathrm{G}_n$ simply connected simple (nontrivial) algebraic groups. Both perverse sheaves δ_{X_1} and δ_{X_2} belong to the tensor category $\langle \delta_{\mathrm{X}_1} \oplus \delta_{\mathrm{X}_2} \rangle$ and therefore define representations of G. Since X_1, X_2 are smooth and nondivisible (otherwise X would not be nondivisible), by Corollary 5.15 such representations are minuscule. By seeing

them as representations of the product $\mathrm{G}_1 \times \cdots \times \mathrm{G}_n$ they decompose as an external tensor product,

$$\omega(\delta_{\mathrm{X}_i}) = \mathrm{V}_{i,1} \boxtimes \cdots \boxtimes \mathrm{V}_{i,n}, \quad i = 1, 2,$$

for representations $\mathrm{V}_{i,\ell}$ of G_ℓ. Note that for all i, ℓ the representation $\mathrm{V}_{i,\ell}$ is necessarily minuscule. By hypothesis, the sum morphism $\mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{X}$ is an isomorphism, thus $\delta_{\mathrm{X}} = \delta_{\mathrm{X}_1} * \delta_{\mathrm{X}_2}$ by definition of the convolution product. In particular, δ_{X} belongs to $\langle \delta_{\mathrm{X}_1} \oplus \delta_{\mathrm{X}_2} \rangle$ and as representations of G we have

$$\omega(\delta_{\mathrm{X}}) = \omega(\delta_{\mathrm{X}_1}) \otimes \omega(\delta_{\mathrm{X}_2}) = (\mathrm{V}_{1,1} \otimes \mathrm{V}_{2,1}) \boxtimes \cdots \boxtimes (\mathrm{V}_{1,n} \otimes \mathrm{V}_{2,n}).$$

Again by Corollary 5.15, the representation $\omega(\delta_{\mathrm{X}})$ of G is minuscule because X is smooth and nondivisible. Thus, the representation $\mathrm{V}_{1,\ell} \otimes \mathrm{V}_{2,\ell}$ of G_ℓ is also minuscule for each $1 \le \ell \le n$. Now Lemma 6.4 implies that for $1 \le \ell \le n$ the representation $\mathrm{V}_{i,\ell}$ is trivial for exactly one $i \in \{1, 2\}$. Strictly speaking, Lemma 6.4 gives only the existence of such an i; however, if $\mathrm{V}_{1,\ell}$ and $\mathrm{V}_{2,\ell}$ were both trivial, then G_ℓ would act trivially on $\omega(\delta_{\mathrm{X}_1}) \oplus \omega(\delta_{\mathrm{X}_2})$ contradicting the fact that G_ℓ is a nontrivial simple factor of G. Resuming the proof, for $i = 1, 2$, let $\mathrm{L}_i \subset \{1, \dots, n\}$ be the subset made of those ℓ for which $\mathrm{V}_{i,\ell}$ is nontrivial. Then $\mathrm{L}_1, \mathrm{L}_2 \subset \{1, \dots, n\}$ are complementary subsets and

$$\mathrm{G}^*_{\mathrm{X}_i,\omega} = \mathrm{Im}\Big(\prod_{\ell \in \mathrm{L}_i} \mathrm{G}_\ell \to \mathrm{GL}\big(\omega(\delta_{\mathrm{X}_i})\big) \Big), \quad i = 1, 2,$$

$$\mathrm{G}^*_{\mathrm{X},\omega} = \mathrm{Im}\big(\mathrm{G} \to \mathrm{GL}\big(\omega(\delta_{\mathrm{X}})\big)\big).$$

Rather generally, for a simple, simply connected algebraic group H and a nontrivial minuscule representation W of H, the kernel of $\mathrm{H} \to \mathrm{GL}(\mathrm{W})$ is finite. This gives isomorphisms of Lie algebras

$$\mathrm{Lie}\, \mathrm{G}^*_{\mathrm{X},\omega} \simeq \mathrm{Lie}\, \mathrm{G} \simeq \bigoplus_{\ell \in \mathrm{L}_1} \mathrm{Lie}\, \mathrm{G}_\ell \oplus \bigoplus_{\ell \in \mathrm{L}_2} \mathrm{Lie}\, \mathrm{G}_\ell \simeq \mathrm{Lie}\, \mathrm{G}^*_{\mathrm{X}_1} \oplus \mathrm{Lie}\, \mathrm{G}^*_{\mathrm{X}_2},$$

as desired. □

6.3. Conic maps

In this section, we introduce the notion of conic map, which will turn out useful in dealing with conormal varieties (see Definition 6.7). Recall that the *domain of definition* of a rational map is the maximal open subset of its source on which the map is well defined.

PROPOSITION 6.6

Let $\mathrm{X}, \mathrm{X}' \subset \mathrm{A}$ *be integral subvarieties, and* $\mathrm{F}\colon \mathbb{A}_{\mathrm{X}} \dashrightarrow \mathbb{A}_{\mathrm{X}'}$ *a rational map between*

their conormal varieties. Then there exists a unique rational map $f : X \dashrightarrow X'$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{\Lambda}_X & \overset{F}{\dashrightarrow} & \mathbb{\Lambda}_{X'} \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle pr_{X'}} \\ X & \overset{f}{\dashrightarrow} & X' \end{array}$$

Moreover, the domain of definition of f *contains the smooth locus* $X^{reg} \subset X$.

Proof
Let U be the domain of definition of the rational map $F\colon \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X'}$. For any smooth point $x \in X^{reg}(k)$, the fiber $\mathbb{\Lambda}_{X,x}$ is a projective space. Every rational map from a projective space to an abelian variety is constant (see [45, Corollary 3.9]), so for any point $x \in X^{reg}(k)$ the morphism

$$pr_{X'} \circ F_{|U_x} : U_x := U \cap \mathbb{\Lambda}_{X,x} \longrightarrow X' \subset A$$

must be constant. Therefore, the morphism

$$pr_{X'} \circ F_{|V} : V := U \cap pr_X^{-1}(X^{reg}) \longrightarrow X' \subset A$$

is constant along the fibers of the smooth morphism $pr_X : V \to X^{reg}$. Over the open subset $pr_X(V) \subset X^{reg}$ the morphism pr_X locally has sections, so we have

$$pr_{X'} \circ F = f \circ pr_X$$

for a unique morphism $f : pr_X(V) \to X'$. The latter extends to a morphism $X^{reg} \to X'$ because a rational map from a variety to an abelian variety is defined at every smooth point of the source (see [45, Theorem 3.1]). □

In the above proof, we have not used anything specific about conormal varieties. In fact, the only thing we used was that $pr_X : \mathbb{\Lambda}_X \to X$ is a projective bundle over $X^{reg} \subset X$ and that X' embeds in an abelian variety. However, the conormal geometry will be taken into account by the following notion of a conic map.

Definition 6.7
In the setup of Proposition 6.6, the rational map $f : X \dashrightarrow X'$ is called the *base* of $F\colon \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X'}$. A rational map $F\colon \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X'}$ is said to be *conic* if the diagram

$$\begin{array}{ccc} \mathbb{\Lambda}_X & \xrightarrow{\gamma_X} & \mathbb{P}_A \\ {\scriptstyle F}\vdots & & \| \\ \mathbb{\Lambda}_{X'} & \xrightarrow{\gamma_{X'}} & \mathbb{P}_A \end{array}$$

commutes, that is, if F is compatible with the respective Gauss maps.

Example 6.8
Let $X, X' \subset A$ be integral subvarieties, and let $F\colon \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X'}$ be a conic map whose base is birational. Then

$$\deg \mathbb{\Lambda}_X = \deg \mathbb{\Lambda}_{X'}$$

because the Gauss degree can be computed over any nonempty open subset of $\mathbb{P}_A$.

Note that even when the base $f : X \dashrightarrow X'$ of a conic map is defined everywhere, it is still not clear whether it is the restriction of an endomorphism of the abelian variety A. However, the results about conic maps in the rest of this section will suffice for our purpose.

PROPOSITION 6.9
Let $X, X' \subset A$ *be integral subvarieties, and let* $F\colon \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X'}$ *be a conic map. If the algebraic group* $\mathrm{Stab}(X)$ *is finite, then the rational map* F *is dominant and generically finite.*

Proof
By Theorem 2.8(1), the Gauss map γ_X is generically finite. It follows from the commutative diagram in Definition 6.7 that the rational map F is also generically finite. Since $\dim(\mathbb{\Lambda}_X) = \dim(\mathbb{\Lambda}_{X'}) = g - 1 = \dim(\mathbb{P}_A)$, it follows that F is also dominant, being a generically finite map between varieties of the same dimension. □

6.4. Product decompositions: From groups to geometry
We now explain how to obtain from a product decomposition for the Lie algebra of the Tannaka group a product decomposition for conormal varieties, using the above results. Borrowing notation from Section 6.1, we write $G^*_{X,\omega} := G^*_\omega(\delta_X)$ for any fixed fiber functor $\omega\colon \langle \delta_X \rangle \to \mathrm{Vect}(\mathbb{F})$.

PROPOSITION 6.10
Assume that $X \subset A$ *is smooth nondivisible and that* $G^*_{X,\omega}$ *is not simple. Then there is an integer* $n \geq 1$ *and for* $i = 1, 2$ *there are integral subvarieties* $X_i \subset A$ *with conormal varieties* $\mathbb{\Lambda}_{X_i} \in \langle \mathbb{\Lambda}_X \rangle$ *of Gauss degree* $\deg(\mathbb{\Lambda}_{X_i}) > 1$, *conic maps* $F_i : \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X_i}$ *with the following properties:*

(1) *We have an identity of cycles* $[n]_* \mathbb{\Lambda}_X = \mathbb{\Lambda}_{[n](X)} = \mathbb{\Lambda}_{X_1} \circ \mathbb{\Lambda}_{X_2}$.

(2) *The following square is commutative*

$$\begin{array}{ccc} \mathbb{\Lambda}_X & \xrightarrow{[n]} & \mathbb{\Lambda}_{[n](X)} \\ {\scriptstyle F_1\times F_2}\,\vdots\downarrow & & \| \\ \mathbb{\Lambda} & \xrightarrow{\sigma} & \mathbb{\Lambda}_{X_1}\circ\mathbb{\Lambda}_{X_2} \end{array}$$

where $\mathbb{\Lambda}$ *and* σ *are as in Definition 5.2.*

(3) *The rational maps* $F_i : \mathbb{\Lambda}_X \dashrightarrow \mathbb{\Lambda}_{X_i}$ *are dominant and generically finite.*

(4) *The base of* F_i *is a morphism* $f_i : X \to X_i$ *which is surjective.*

Proof

By Corollary 4.4 and Lemma A.1, we may assume $k = \mathbb{C}$, which will allow us later to use the results about characteristic cycles in Theorem 5.11. By Corollary 3.5, the group $G^*_{X,\omega}$ does not change if we replace X by $X + a$ for any $a \in A(k)$, and it clearly suffices to achieve properties (1)–(4) for any such translate. We will therefore assume that

$$\det(\delta_X) = \delta_0$$

so that the connected component $G := G^\circ_{X,\omega}$ is semisimple by Proposition 3.3. If the group $G^*_{X,\omega}$ is not simple modulo its center, then by the structure theory of semisimple groups there are simply connected semisimple groups $G_1, G_2 \not\simeq \{1\}$ and an isogeny

$$p\colon \tilde{G} := G_1 \times G_2 \twoheadrightarrow G.$$

Then $V := \omega(\delta_X)$ restricts to an irreducible representation of the covering group $\tilde{G}$ and as such it decomposes as

$$V_{|\tilde{G}} \simeq V_1 \otimes V_2 \quad \text{with irreducible } V_i \in \mathrm{Rep}_{\mathbb{F}}(G_i) \subset \mathrm{Rep}_{\mathbb{F}}(\tilde{G}).$$

Note that both factors G_1 and G_2 must act nontrivially on V since otherwise they would not appear in the Tannaka group. In particular, we have $\dim V_i \geq 2$ as any 1-dimensional representation of a connected semisimple group is trivial.

Let $n = m \cdot \deg(p)$ for the smallest integer $m \geq 1$ with $m \cdot \Gamma_{cc(\delta_X)} = \{0\}$. Since X is smooth, its perverse intersection complex has characteristic cycle $cc(\delta_X) = \mathbb{\Lambda}_X$. Via the first part of Theorem 5.11, the above decomposition as a tensor product of two representations of the universal covering group yields clean cycles $\mathbb{\Lambda}_1, \mathbb{\Lambda}_2 \in \langle \mathbb{\Lambda}_X \rangle$ such that

$$[n]_* \mathbb{\Lambda}_X = \mathbb{\Lambda}_1 \circ \mathbb{\Lambda}_2,$$

and the second part of the theorem implies that $\deg \mathbb{\Lambda}_i = \dim V_i \geq 2$. Moreover, by Lemma 5.14, the cycles $\mathbb{\Lambda}_i$ are effective. Since we assumed the subvariety $X \subset A$ to

be nondivisible, the morphism $[n]\colon X \to [n](X)$ is dominant and birational, in fact we have $[n]_*\mathbb{\Lambda}_X = \mathbb{\Lambda}_{[n](X)}$ as an identity of cycles by Lemma 2.4. Altogether, it follows that

$$\mathbb{\Lambda}_{[n](X)} = [n]_*\mathbb{\Lambda}_X = \mathbb{\Lambda}_1 \circ \mathbb{\Lambda}_2.$$

The cycle on the left-hand side is reduced and irreducible, so the same must hold for both factors on the right-hand side because the convolution product $\circ$ on cycles is bilinear and the convolution of any two clean effective cycles is again a clean effective cycle. Hence, there exist integral subvarieties $X_i \subset A$ with $\mathbb{\Lambda}_i = \mathbb{\Lambda}_{X_i}$. By definition of $\circ$, we have

$$\mathbb{\Lambda}_1 \circ \mathbb{\Lambda}_2 = \sigma_*(\overline{\mathbb{\Lambda}_{X_1|U} \times_U \mathbb{\Lambda}_{X_2|U}}),$$

where $\sigma\colon A \times A \times \mathbb{P}_A \to A \times \mathbb{P}_A$ denotes the sum morphism and $U \subset \mathbb{P}_A$ is as in Definition 5.2. The multiplicities of the cycle-theoretic pushforward on the right-hand side are given by the degree of the sum morphism

$$\sigma\colon \quad \mathbb{\Lambda} := \overline{\mathbb{\Lambda}_{X_1|U} \times_U \mathbb{\Lambda}_{X_2|U}} \longrightarrow \mathbb{\Lambda}_{[n](X)}$$

on the various components of its source. Since the cycle $\mathbb{\Lambda}_1 \circ \mathbb{\Lambda}_2 = \mathbb{\Lambda}_{[n](X)}$ is integral as observed above, it follows in fact that the fiber product $\mathbb{\Lambda}$ is integral and is mapped birationally onto its image by σ. Consider then the composition of rational maps

$$F_i\colon \quad \mathbb{\Lambda}_X \xrightarrow{[n]\times \mathrm{id}} \mathbb{\Lambda}_{[n](X)} \overset{\sigma^{-1}}{\dashrightarrow} \mathbb{\Lambda} = \overline{\mathbb{\Lambda}_{X_1|U} \times_U \mathbb{\Lambda}_{X_2|U}} \xrightarrow{\mathrm{pr}_i} \mathbb{\Lambda}_{X_i},$$

where pr_i denotes be the projection onto the ith factor. By construction, F_i is a conic map, and by Proposition 6.6 its base $f_i\colon X \dashrightarrow X_i$ is defined on all of X because we assumed X to be smooth. Moreover, by Proposition 6.9, the rational map F_i is dominant and generically finite, so the morphism f_i is surjective. □

6.5. Proof of Theorem 6.1

We can now prove the simplicity criterion for Tannaka groups as follows. Thanks to Proposition 6.5, only the implication (1) $\Longrightarrow$ (2) is left to be shown. Suppose that the algebraic group $G^*_{X,\omega}$ is not simple. According to [39, Lemma 4.6], this is never the case when X is a smooth ample divisor, thus from now we may assume $\dim X < g-1$. Let n, X_i, F_i and $f_i\colon X \to X_i$ be as in Proposition 6.10. For $i = 1, 2$, the Gauss map of X_i is a finite morphism because the one of X is by Theorem 2.8 and $\mathbb{\Lambda}_{X_i} \in \langle \mathbb{\Lambda}_X \rangle$ by construction. It is therefore possible to apply Corollary 5.7 and deduce the equality

$$\dim X = \dim X_1 + \dim X_2$$

from the identity $\mathbb{\Lambda}_{X_1} \circ \mathbb{\Lambda}_{X_2} = \mathbb{\Lambda}_{[n](X)}$ and the hypothesis $\dim X < g - 1$. Moreover, the subvarieties X_1 and X_2 are nondegenerate by Theorem 2.8. Thus, Lemma 2.7 implies that the subvariety

$$X_1 + X_2 \subset A$$

is nondegenerate and the sum morphism $\sigma : X_1 \times X_2 \to X_1 + X_2$ is generically finite. Thus, the conormal cone $\mathbb{\Lambda}_{X_1+X_2}$ appears as a summand with multiplicity $\deg(\sigma)$ in the cycle $\mathbb{\Lambda}_{X_1} \circ \mathbb{\Lambda}_{X_2}$. But $\mathbb{\Lambda}_{X_1} \circ \mathbb{\Lambda}_{X_2} = \mathbb{\Lambda}_{[n](X)}$, hence

$$\deg(\sigma) = 1 \qquad \text{and} \qquad X_1 + X_2 = [n](X).$$

Together with Proposition 6.10(2), this gives the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{[n]} & [n](X) \\ {\scriptstyle f_1 \times f_2}\downarrow & & \| \\ X_1 \times X_2 & \xrightarrow{\sigma} & X_1 + X_2 \end{array}$$

The primitivity assumption on X implies that the morphism $[n] : X \to [n](X)$ is finite birational. This forces $f = (f_1 \times f_2)$ to be finite birational. As X is smooth, this says that f is the normalization morphism. For $i = 1, 2$, let $\tilde{X}_i$ be the normalization of X_i. The morphism $\tilde{f} : X \to \tilde{X}_1 \times \tilde{X}_2$ induced by f is an isomorphism, thus $\tilde{X}_1$ and $\tilde{X}_2$ are smooth. Identifying X with $\tilde{X}_1 \times \tilde{X}_2$ permits us to embed $\tilde{X}_1$ and $\tilde{X}_2$ in A and to write $\mathrm{Alb}(X) = \mathrm{Alb}(\tilde{X}_1) \times \mathrm{Alb}(\tilde{X}_2)$. By suitably embedding X, $\tilde{X}_1$, and $\tilde{X}_2$ in their Albanese variety, we have the following identity:

$$\big(\tilde{X}_1 \times \{0\}\big) + \big(\{0\} \times \tilde{X}_2\big) = X \subset \mathrm{Alb}(X).$$

The commutativity of the following square

$$\begin{array}{ccc} \big(\mathrm{Alb}(\tilde{X}_1) \times \{0\}\big) \times \big(\{0\} \times \mathrm{Alb}(\tilde{X}_2)\big) & \longrightarrow & \mathrm{Alb}(X) \\ \downarrow & & \downarrow \\ A \times A & \longrightarrow & A \end{array}$$

where the horizontal arrows are the sum morphisms and the vertical ones are given by universal property of the Albanese, implies that $X = \tilde{X}_1 + \tilde{X}_2$. □

7. Wedge powers

We now characterize low-dimensional smooth subvarieties whose Tannaka group is the image of a special linear group acting on a nontrivial wedge power of its standard representation. In the case of hypersurfaces, Lawrence and Sawin [39, Lemma 4.10]

show that such groups do not occur using combinatorial properties of Eulerian numbers. In higher codimension, wedge powers do occur, but we show by geometric arguments that they only arise from symmetric powers of curves.

7.1. Statement of the main result

As in the previous section, we assume the field k to be algebraically closed of characteristic 0. Fix a subvariety $\mathrm{X} \subset \mathrm{A}$ and a fiber functor $\omega\colon \langle \delta_{\mathrm{X}} \rangle \to \mathrm{Vect}(\mathbb{F})$ on the Tannaka category generated by the perverse intersection complex of X. As in Section 6.1, put $\mathrm{G}_{\mathrm{X},\omega} := \mathrm{G}_{\omega}(\delta_{\mathrm{X}})$, and denote by

$$\mathrm{G}^{*}_{\mathrm{X},\omega} := [\mathrm{G}^{\circ}_{\mathrm{X},\omega}, \mathrm{G}^{\circ}_{\mathrm{X},\omega}]$$

the derived group of its connected component of the identity. We are interested in the following situation.

Definition 7.1

Let $r \geq 1$ be an integer. We say that $\mathrm{X} \subset \mathrm{A}$ is an *rth wedge power* if we have

$$\mathrm{G}^{*}_{\mathrm{X},\omega} \simeq \mathrm{Alt}^{r}\big(\mathrm{SL}_{n}(\mathbb{F})\big) \quad \text{with the standard action on } \omega(\delta_{\mathrm{X}}) \simeq \mathrm{Alt}^{r}(\mathbb{F}^{n})$$

for some $n \geq 1$. Notice that if r is given, then n is determined by the topological Euler characteristic

$$\chi(\delta_{\mathrm{X}}) = \dim_{\mathbb{F}}\big(\omega(\delta_{\mathrm{X}})\big) = \binom{n}{r}.$$

If X is an rth wedge power, then by duality it is also an $(n-r)$th wedge power.

The typical example of wedge powers arises from symmetric powers of curves, as announced in the introduction.

LEMMA 7.2

Let $\mathrm{C} \subset \mathrm{A}$ be a smooth projective curve, and let $r \geq 2$ be an integer. If the sum morphism

$$s\colon \mathrm{Sym}^{r}\mathrm{C} \longrightarrow \mathrm{X} = \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{A}$$

is an isomorphism onto its image, then this image $\mathrm{X} \subset \mathrm{A}$ is an rth wedge power.

Proof

Replacing A by the abelian subvariety $\langle \mathrm{C} \rangle \subset \mathrm{A}$, we may assume that C generates A. The Abel–Jacobi map $\mathrm{C} \hookrightarrow \mathrm{Alb}(\mathrm{C})$ for some basepoint induces a sum morphism

$$\mathrm{Sym}^r(\mathrm{C}) \longrightarrow \mathrm{Alb}(\mathrm{C}).$$

Let $\mathrm{W}_r \subset \mathrm{Alb}(\mathrm{C})$ denote the image of this sum morphism. The embedding $\mathrm{C} \hookrightarrow \mathrm{A}$ also induces a morphism $\mathrm{Alb}(\mathrm{C}) \to \mathrm{A}$; this last morphism is surjective because we assumed $\mathrm{A} = \langle \mathrm{C} \rangle$. We obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Sym}^r(\mathrm{C}) & \twoheadrightarrow & \mathrm{W}_r & \hookrightarrow & \mathrm{Alb}(\mathrm{C}) \\ \| & & \downarrow & & \downarrow \\ \mathrm{Sym}^r(\mathrm{C}) & \xrightarrow{\sim} & \mathrm{X} & \hookrightarrow & \mathrm{A} \end{array}$$

The square on the left implies that the morphism $\mathrm{W}_r \to \mathrm{X}$ is an isomorphism, therefore W_r is smooth (hence by the Riemann singularity theorem the curve C is not r-gonal, so in particular it is not hyperelliptic if $r \geq 2$). Fixing any $r-2$ points on the curve and varying the remaining two points, the above also implies that the morphisms $\mathrm{Sym}^2(\mathrm{C}) \to \mathrm{W}_2 \subset \mathrm{Alb}(\mathrm{C})$ and $\mathrm{Sym}^2(\mathrm{C}) \to \mathrm{C}+\mathrm{C} \subset \mathrm{A}$ have the same fibers. So we can apply the variant of Larsen's alternative in [37, Section 6] to the perverse sheaf $\delta_{\mathrm{C}} \in \mathrm{Perv}(\mathrm{A}, \mathbb{F})$ to see that

$$\mathrm{G}^*_{\omega}(\delta_{\mathrm{C}}) \simeq \mathrm{SL}_n(\mathbb{F}) \quad \text{with the standard action on } \omega(\delta_{\mathrm{C}}) \simeq \mathbb{F}^n \text{ for } n = \chi(\delta_{\mathrm{C}}).$$

The above diagram then shows that $\mathrm{G}^*_{\mathrm{X},\omega} \simeq \mathrm{Alt}^r(\mathrm{SL}_n(\mathbb{F}))$ and $\omega(\delta_{\mathrm{X}}) \simeq \mathrm{Alt}^r(\mathbb{F}^n)$. □

The goal of this section is to show a converse to the above lemma. More precisely, we obtain the following complete classification of wedge powers for all nondivisible smooth subvarieties $\mathrm{X} \subset \mathrm{A}$ of high codimension whose Gauss map $\gamma_{\mathrm{X}} \colon \mathbb{A}_{\mathrm{X}} \to \mathbb{P}_{\mathrm{A}}$ is finite.

THEOREM 7.3
Let $\mathrm{X} \subset \mathrm{A}$ *be a nondivisible smooth subvariety with ample normal bundle, and suppose that its Euler characteristic is* $\chi(\delta_{\mathrm{X}}) = \binom{n}{r}$ *for some integer* r *with* $1 < r \leq n/2$. *If* $2\dim \mathrm{X} < \dim \mathrm{A} - 1$, *then the following are equivalent:*

(1) *The subvariety* $\mathrm{X} \subset \mathrm{A}$ *is an* r*th wedge power.*

(2) *There is a nondegenerate irreducible smooth projective curve* $\mathrm{C} \subset \mathrm{A}$ *such that*
- $\mathrm{X} = \mathrm{C} + \cdots + \mathrm{C} \subset \mathrm{A}$ *is the sum of* r *copies of* C, *and*
- *the sum morphism* $\mathrm{Sym}^r \mathrm{C} \to \mathrm{X}$ *is an isomorphism.*

In view of Lemma 7.2, we only need to show the implication (1) $\Longrightarrow$ (2), which will take up the rest of this section.

7.2. Structure of the proof

The proof relies on three independent steps. The first step is to show that the structure of wedge powers is reflected by characteristic cycles. To any subvariety $Z \subset A$ and an integer $r \geq 1$ we will attach a clean effective cycle

$$\mathrm{Alt}^r \mathbb{\Lambda}_Z \in \mathscr{L}(A),$$

and we show the following.

THEOREM 7.4

Let $X \subset A$ be a positive-dimensional smooth nondivisible subvariety that is an rth wedge power for some integer $r \geq 1$. Then there is an integral subvariety $Z \subset A$ with $\mathbb{\Lambda}_Z \in \langle \mathbb{\Lambda}_X \rangle$ such that

$$\mathrm{Alt}^r \mathbb{\Lambda}_Z = \mathbb{\Lambda}_{[e](X)}$$

for some integer $e \geq 1$. The Gauss degrees are related by

$$\deg \mathbb{\Lambda}_X = \binom{n}{r}, \quad \text{where } n = \deg \mathbb{\Lambda}_Z,$$

and if the Gauss map $\gamma_X \colon \mathbb{\Lambda}_X \to \mathbb{P}_A$ is finite, then so is $\gamma_Z \colon \mathbb{\Lambda}_Z \to \mathbb{P}_A$.

For the construction of the clean cycle $\mathrm{Alt}^r \mathbb{\Lambda}_Z$ and the proof of the above result, see Section 7.3. Once we have this, the second step in our classification of wedge powers will be to prove a monotonicity statement for the cycles $\mathrm{Alt}^r(\mathbb{\Lambda}_Z)$ as a function of r. Let

$$\mathrm{Alt}^r Z := \mathrm{Im}\big(\mathrm{pr}_A \colon \mathrm{Supp}(\mathrm{Alt}^r \mathbb{\Lambda}_Z) \to A\big) \subset A$$

be the image of the support of the clean effective cycle $\mathrm{Alt}^r \mathbb{\Lambda}_Z \in \mathscr{L}(A)$ under the projection to the abelian variety. Then we will show the following.

THEOREM 7.5

Let $Z \subset A$ be an integral subvariety whose Gauss map is a finite morphism of degree $n = \deg(\gamma_Z)$. Suppose there exists an integer $r \geq 1$ with $r \leq n/2$ such that

$$\dim \mathrm{Alt}^r Z < (\dim A - 1)/2.$$

Then we have

$$r \dim Z < \dim A.$$

The proof of this will be given in Section 7.4. Finally, the last step for our classification of wedge powers will be to show that in the given dimension range, the smoothness of $\mathrm{Alt}^r \mathbb{\Lambda}_Z$ forces Z to be a curve. More precisely, we have the following result.

THEOREM 7.6
Let $X, Z \subset A$ *be integral subvarieties whose Gauss maps are finite morphisms. Suppose moreover that* X *is smooth and nondivisible and that there are integers* $e, r \geq 1$ *such that*

$$\mathrm{Alt}^r \mathbb{\Lambda}_Z = \mathbb{\Lambda}_{[e](X)}.$$

If $r \dim Z < \dim A$*, then* Z *is a curve and*
(1) *the normalization* C *of* Z *embeds in* A*,*
(2) $X = C + \cdots + C \subset A$ *is the sum of* r *copies of* C*, embedded suitably in* A*, and*
(3) *the sum morphism* $\mathrm{Sym}^r C \to X$ *is an isomorphism.*

We will prove this in Section 7.5. Before coming to the details, let us note how the above three results conclude the classification of wedge powers.

Proof of Theorem 7.3
Suppose that $X \subset A$ satisfies the assumptions of the theorem and that it is an rth wedge power for some $r \in \mathbb{N}$ with $1 < r \leq n/2$ where $\deg \gamma_X = \binom{n}{r}$. By Theorem 7.4, then

$$\mathrm{Alt}^r \mathbb{\Lambda}_Z = \mathbb{\Lambda}_{[e](X)},$$

for some integer $e \geq 1$ and some integral subvariety $Z \subset A$. Since $[e]\colon A \to A$ is an isogeny, we have $\dim [e](X) = \dim X < (\dim A - 1)/2$ by our dimension assumption on X. Theorem 7.5 then shows that

$$r \dim Z < \dim A,$$

and hence Theorem 7.6 gives the desired result. □

7.3. Characteristic cycles of wedge powers
We now explain how to compute characteristic cycles of wedge powers. Let $Z \subset A$ be a subvariety with dominant Gauss map $\gamma_Z\colon \mathbb{\Lambda}_Z \to \mathbb{P}_A$. For an integer $r \geq 1$, we consider as in Definition 5.16 the Zariski closure

$$\mathbb{\Lambda}_Z^{[r]} := \overline{\mathbb{\Lambda}_{Z|U}^{\times r} \smallsetminus \Delta_r} \subset A^r \times \mathbb{P}_A,$$

where $U \subset \mathbb{P}_A$ is any open dense subset over which the Gauss map γ_Z is finite and flat. Let $\sigma\colon A^r \times \mathbb{P}_A \to A \times \mathbb{P}_A$, $(z_1, \dots, z_r, \xi) \mapsto (z_1 + \cdots + z_r, \xi)$ be the sum morphism, and put

$$\operatorname{Alt}^r \mathbb{\Lambda}_Z := \frac{1}{r!}\sigma_*(\mathbb{\Lambda}_Z^{[r]}) \in \mathscr{L}(A),$$

which is the special case $\alpha = (1, \dots, 1) = (1^r)$ of the cycle in Definition 5.16.

Proof of Theorem 7.4
By Corollary 4.4 and Lemma A.1, we may reduce to the case $k = \mathbb{C}$ and $\mathbb{F} = \mathbb{C}$ so we can use the results about characteristic cycles from Section 5. Replacing X by a translate, we may assume by Corollary 3.5 that the connected component of the group $G_\omega(\delta_X)$ is semisimple. Its universal cover is then isomorphic to $\tilde{G} \simeq \mathrm{SL}_n(\mathbb{C})$ for some $n \geq 1$, so the setup in Section 5.5 applies to $P = \delta_X$. By assumption,

$$\omega(P) \simeq \operatorname{Alt}^r(\mathbb{C}^n).$$

Since $\dim \omega(P) > 1$ for any non-negligible perverse sheaf P which is not a skyscraper sheaf (see [53]), we have $1 < r < n$. The highest weight of the above wedge power representation of $\mathrm{SL}_n(\mathbb{C})$ is $\alpha = \varepsilon_1 + \cdots + \varepsilon_r$. With notation as in Definition 5.12, the Weyl group orbit $W.\alpha$ consists precisely of the weights in the representation $\xi(P^\circ)$, so $\xi^{-1}[W.\alpha] = [P^\circ]$ in the Grothendieck ring $K(\langle P^\circ \rangle)$. It follows that

$$\begin{aligned}
\operatorname{cc}(P, \alpha) &= \operatorname{cc}\big(\xi^{-1}[W.d\alpha]\big) && \text{with notation as in Definition 5.12}\\
&= [d]_* \operatorname{cc}(P^\circ) && \text{since } \xi^{-1}[W.\alpha] = [P^\circ]\\
&= [e]_* \operatorname{cc}(P) && \text{for } e := dm \text{ and } P^\circ := [m]_* P\\
&= [e]_* \mathbb{\Lambda}_X && \text{since } \operatorname{cc}(\delta_X) = \mathbb{\Lambda}_X \text{ for smooth X}\\
&= \mathbb{\Lambda}_{[e](X)} && \text{by Lemma 2.4, since X is nondivisible.}
\end{aligned}$$

In particular, the cycle $\operatorname{cc}(P, \alpha)$ is integral. So by Lemma 5.18 the cycle $\operatorname{cc}(P, \varepsilon_1)$ is integral as well, hence of the form $\operatorname{cc}(P, \varepsilon_1) = \mathbb{\Lambda}_Z$ for an integral subvariety $Z \subset A$, and we have

$$\mathbb{\Lambda}_{[e](X)} = \operatorname{cc}(P, \alpha) = \mathbb{\Lambda}_Z^{\alpha} = \operatorname{Alt}^r \mathbb{\Lambda}_Z.$$

Then $\deg \mathbb{\Lambda}_X = \deg \operatorname{cc}(P, \alpha) = \dim \omega(P) = \binom{n}{r}$. For the finiteness of the Gauss map, recall from Remark 5.13 that $\mathbb{\Lambda}_Z = \operatorname{cc}(P, \varepsilon_1)$ lies in the subring $\langle \mathbb{\Lambda}_X \rangle \subset \mathscr{L}(A)$. As such, it appears in some convolution power of the cycle $\mathbb{\Lambda}_X$. Recalling the definition of the convolution product, this implies that if the Gauss map for γ_X is finite, then so is the one for γ_Z. □

Remark 7.7
The discussion in the proof of Theorem 7.4 also shows that for any reduced subvariety $Z \subset A$ with dominant Gauss map of degree $n = \deg(\gamma_Z)$ and any integer $r \geq 1$, we have the following:

- The projection $\gamma_{Z,r} \colon \mathbb{A}_Z^{[r]} \to \mathbb{P}_A$ is dominant (and hence generically finite) if and only if $r \leq n$. In that case,

$$\deg \gamma_{Z,r} = r! \binom{n}{r}.$$

- If $\gamma_Z \colon \mathbb{A}_Z \to \mathbb{P}_A$ is a finite morphism, then so is $\gamma_{Z,r} \colon \mathbb{A}_Z^{[r]} \to \mathbb{P}_A$.

7.4. Dimension estimates for wedge powers
We will deduce Theorem 7.5 from a monotonicity property of wedge powers. To state this, recall that for $Z \subset A$ and an integer $i \geq 1$ we put $\mathrm{Alt}^i Z := \mathrm{pr}_A(\mathrm{Supp}(\mathrm{Alt}^i \mathbb{A}_Z))$. We are interested in its dimension

$$d_Z(i) := \dim \mathrm{Alt}^i Z.$$

Example 7.8
Let Z be a smooth projective curve of genus $g > 1$, embedded via the Abel–Jacobi map in its Jacobian variety $A = \mathrm{Alb}(Z)$. When Z is suitably translated in A, by [32, Examples 3.1 and 4.1],

$$d_Z(i) = \begin{cases} i & \text{for } 1 \leq i \leq g-1, \\ g-1-i & \text{for } g-1 \leq i \leq 2g-2. \end{cases}$$

Here d_Z is not monotonous, but it is so between zero and $g - 1 = \deg \gamma_Z / 2$.

The following monotonicity result shows that the behavior in the above example is typical for wedge powers in small degrees.

PROPOSITION 7.9
Let $Z \subset A$ *be an integral nondegenerate subvariety, and let* $r \geq 1$ *be an integer such that* $r \dim Z < \dim A$.
(1) *Let* $Y := Z + \cdots + Z \subset A$ *be the sum of* r *copies of* Z. *Then the sum morphism*

$$\sigma \colon \mathrm{Sym}^r Z = Z^r / \mathfrak{S}_r \longrightarrow Y$$

is generically finite, and the subvariety $Y \subset A$ *is nondegenerate.*

(2) *The cycle* $\mathrm{Alt}^r \mathbb{\Lambda}_Z$ *contains* $\mathbb{\Lambda}_Y$ *with multiplicity* $d = \deg(\sigma) \geq 1$, *and we have*

$$\mathrm{Alt}^r Z = Y.$$

(3) *For general* $z = (z_1, \ldots, z_r) \in Z^r(k)$, *the fiber of* $\mathbb{\Lambda}_Y$ *over* $y = \sigma(z)$ *is*

$$\mathbb{\Lambda}_{Y,y} = \mathbb{\Lambda}_{Z,z_1} \cap \cdots \cap \mathbb{\Lambda}_{Z,z_r} \subset \mathbb{P}_A.$$

Proof

The image $Y = Z + \cdots + Z$ of the sum morphism $\sigma\colon \mathrm{Sym}^r Z \to Y$ is a proper subvariety of A because $\dim Y \leq \dim \mathrm{Sym}^r Z = r \dim Z < \dim A$. As we assumed $Z \subset A$ to be nondegenerate, it then follows by Lemma 2.7 that the morphism σ is generically finite and that $Y \subset A$ is again nondegenerate.

This last property implies that the Gauss map $\gamma_Y\colon \mathbb{\Lambda}_Y \to \mathbb{P}_A$ is generically finite and dominant (see Theorem 2.8). Let $W \subset Y^{\mathrm{reg}}$ be a nonempty open subset such that $V := \sigma^{-1}(W)$ is contained in $(Z^{\mathrm{reg}})^r \smallsetminus \Delta_r$ and the sum morphism $\sigma\colon V \to W$ is finite étale. If we view the tangent spaces as subspaces of $\mathrm{Lie}(A)$, then for every point $z = (z_1, \ldots, z_r) \in V(k)$ and $y = \sigma(z)$ the tangent map

$$T_z(\sigma)\colon T_{z_1}(Z) \times \cdots \times T_{z_r}(Z) \longrightarrow T_y(Y)$$

is induced by the sum map $\mathrm{Lie}(A)^r \to \mathrm{Lie}(A)$. In particular, $T_{z_i}(Z) \subset T_y(Y)$ and hence $\mathbb{\Lambda}_{Y,y} \subset \mathbb{\Lambda}_{Z,z_i}$ for all i.

Let $U \subset \mathbb{P}_A$ be a nonempty open subset over which the Gauss maps γ_Y and γ_Z are finite étale, and such that $\mathrm{pr}_Y(\gamma_Y^{-1}(U)) \subset W$. For any $\xi \in U(k)$ and $z \in V(k)$ with $\sigma(z) \in \gamma_Y^{-1}(\xi)$, we have

$$(z_1, \ldots, z_r, \xi) \in \mathbb{\Lambda}_{Z|U}^{\times r} \smallsetminus \Delta_r \subset \mathbb{\Lambda}_Z^{[r]}.$$

So the image of $\mathrm{pr}_{Z,r}\colon \mathbb{\Lambda}_Z^{[r]} \to Z^r$ contains an open dense subset of Z^r. Hence by properness, the morphism $\mathrm{pr}_{Z,r}$ is surjective and then $\mathrm{Alt}^r Z = Y$ by definition. In particular, $\dim \mathrm{Alt}^r Z = \dim Y = r \dim Z$, where the last equality follows from the generic finiteness of the sum morphism $\sigma\colon \mathrm{Sym}^r Z \to Y$. The statement about the general fiber of $\mathrm{pr}_Y\colon \mathrm{Alt}^r \mathbb{\Lambda}_Z \to Y$ follows from the fact that for general $(z_1, \ldots, z_r)$ we have

$$T_y(Y) = T_{z_1}(Z) \oplus \cdots \oplus T_{z_r}(Z)$$

because the summands on the right-hand side span $T_y(Y)$ and their dimension adds up to $r \dim Z = \dim Y = \dim T_y(Y)$; passing to the corresponding normal spaces gives $\mathbb{\Lambda}_{Y,y} = \mathbb{\Lambda}_{Z,z_1} \cap \cdots \cap \mathbb{\Lambda}_{Z,z_r}$ and the claim follows since $\mathrm{Supp}(\mathrm{Alt}^r \mathbb{\Lambda}_Z)_y = \mathbb{\Lambda}_{Y,y}$ for $y \in Y(k)$ general. Finally, it is also clear from the above discussion that the clean cycle $\mathrm{Alt}^r \mathbb{\Lambda}_Z$ contains the component $\mathbb{\Lambda}_Y$ with multiplicity $d = \deg(\sigma)$. □

COROLLARY 7.10
Let $\mathrm{Z} \subset \mathrm{A}$ *be an integral nondegenerate subvariety of positive dimension. Then*

$$\deg(\gamma_{\mathrm{Z}}) \geq 2\frac{\dim \mathrm{A}}{\dim \mathrm{Z}} - 2.$$

Proof
Set $r_0 := \max\{r \mid r \dim \mathrm{Z} < \dim \mathrm{A}\}$. Then Proposition 7.9 shows that the function $r \mapsto d_{\mathrm{Z}}(r) := \dim \mathrm{Alt}^r \mathrm{Z}$ is strictly increasing on the interval $\{0, 1, \ldots, r_0\}$. On the other hand, we have

$$\mathrm{Alt}^r \mathbb{A}_{\mathrm{Z}} = \mathrm{Alt}^{n-r} \mathbb{A}_{p-\mathrm{Z}},$$

where $n = \deg(\gamma_{\mathrm{Z}})$ and where $p \in \mathrm{A}(k)$ is defined by $\mathrm{Alt}^n \mathbb{A}_{\mathrm{Z}} = \mathbb{A}_{\{p\}}$. It follows that

$$\dim \mathrm{Alt}^r \mathrm{Z} = \dim \mathrm{Alt}^{n-r} \mathrm{Z}.$$

The left-hand side is strictly increasing for $r \in \{0, 1, \ldots, r_0\}$, the right-hand side is strictly decreasing for $r \in \{n - r_0, \ldots, n\}$. This forces $r_0 \leq n - r_0$, hence $n \geq 2r_0$. Then $(r_0 + 1) \dim \mathrm{Z} \geq \dim \mathrm{A}$ implies $(n/2 + 1) \cdot \dim \mathrm{Z} \geq \dim \mathrm{A}$ as desired. □

For smooth subvarieties $\mathrm{Z} \subset \mathrm{A}$, the degree of the Gauss map $\gamma_{\mathrm{Z}} \colon \mathbb{A}_{\mathrm{Z}} \to \mathbb{P}_{\mathrm{A}}$ is equal to the topological Euler characteristic of δ_{Z} (see Section 5.3). So for smooth curves $\mathrm{Z} \subset \mathrm{A}$ with $\langle \mathrm{Z} \rangle = \mathrm{A}$, Corollary 7.10 says that the curve has genus at least $\dim \mathrm{A}$; this follows of course also directly from the fact that for such curves the morphism $\mathrm{Alb}(\mathrm{Z}) \to \mathrm{A}$ must be surjective. Note that here the bound in Corollary 7.10 is sharp.

Proposition 7.9 gives a monotonicity statement for wedge powers, but in order to apply it we need to have an a priori bound on $\dim \mathrm{Z}$. The following result will allow us to start instead from a bound only on some wedge power $\dim \mathrm{Alt}^r \mathrm{Z}$, since the bound will be inherited by all lower wedge powers.

LEMMA 7.11
Let $\mathrm{Z} \subset \mathrm{A}$ *be a reduced subvariety whose Gauss map is a finite morphism of degree* n. *Let* r *be an integer with* $1 \leq r \leq n/2$, *and assume that*

$$\dim \mathrm{Alt}^r \mathrm{Z} < (\dim \mathrm{A} - 1)/2.$$

Then the function $d_{\mathrm{Z}} \colon \{1, \ldots, r\} \to \mathbb{N}$, $i \mapsto d_{\mathrm{Z}}(i) := \dim \mathrm{Alt}^i \mathrm{Z}$ *is nondecreasing; in particular,*

$$\dim \mathrm{Alt}^i \mathrm{Z} < (\dim \mathrm{A} - 1)/2 \quad \textit{for all } i \in \{1, \ldots, r\}.$$

Proof

We only need to show that $d_Z(r-1) \leq d_Z(r)$, because we can then proceed by descending induction. Put

$$\mathbb{\Lambda}_i := (\mathrm{Alt}^i \mathbb{\Lambda}_Z) \circ (\mathrm{Alt}^i \mathbb{\Lambda}_{-Z}) \quad \text{for } i = 1, \ldots, n.$$

The support of this cycle is the closure of the subset of points in $(A \times \mathbb{P}_A)(k)$ of the form $(x_1 + \cdots + x_i - y_1 - \cdots - y_i, \xi)$ where

- $\xi \in \mathbb{P}_A(k)$ is a cotangent direction such that $\mathrm{pr}_A(\gamma_Z^{-1}(\xi)) \subset Z^{\mathrm{reg}}(k)$,
- $\{x_1, \ldots, x_i\}, \{y_1, \ldots, y_i\} \subset \mathrm{pr}_A(\gamma_Z^{-1}(\xi))$ are subsets of cardinality i.

For general $\xi \in \mathbb{P}_A(k)$, the fiber $\gamma_Z^{-1}(\xi)$ consists of $n \geq 2r > 2(r-1)+1$ distinct points, hence for any two subsets $\{x_1, \ldots, x_{r-1}\}$ and $\{y_1, \ldots, y_{r-1}\}$ as above there is a point p in $\mathrm{pr}_A(\gamma_Z^{-1}(\xi))$ which belongs to neither of the two subsets. By writing the point

$$z := x_1 + \cdots + x_{r-1} - y_1 - \cdots - y_{r-1}$$

as $z + p - p$, we find that the point (z, ξ) lies in the support of $\mathbb{\Lambda}_r$. Varying ξ and the chosen subsets of points in the fiber of the Gauss map, we obtain the inclusion $\mathrm{Supp}(\mathbb{\Lambda}_{r-1}) \subset \mathrm{Supp}(\mathbb{\Lambda}_r)$ for the supports. Since both cycles are linear combinations of conormal varieties and hence pure of the same dimension $\dim A - 1$, it follows that we have

$$\mathbb{\Lambda}_r = c\mathbb{\Lambda}_{r-1} + E, \tag{7.1}$$

where $c > 0$ is a rational number and E is an effective cycle with rational coefficients.

The finiteness of the Gauss map $\gamma_Z \colon \mathbb{\Lambda}_Z \to \mathbb{P}_A$ implies that $\mathrm{Alt}^i \mathbb{\Lambda}_Z \to \mathbb{P}_A$ is finite as well. Indeed, $\mathbb{\Lambda}_Z^{[i]}$ is by definition a subvariety inside the i-fold fiber product $\mathbb{\Lambda}_Z \times_{\mathbb{P}_A} \cdots \times_{\mathbb{P}_A} \mathbb{\Lambda}_Z$ and the fiber product of finite morphisms is a finite morphism. By design the sum map $\mathbb{\Lambda}_Z^{[i]} \to \mathrm{Supp}(\mathrm{Alt}^i \mathbb{\Lambda}_Z)$ is surjective and compatible with Gauss maps, implying the desired finiteness. By Lemma 5.5, the finiteness of the Gauss map for $\mathrm{Alt}^i \mathbb{\Lambda}_Z$ allows us to compute the total Segre class of the cycle $\mathbb{\Lambda}_i = \mathrm{Alt}^i \mathbb{\Lambda}_Z \circ \mathrm{Alt}^i \mathbb{\Lambda}_{-Z}$ as a Pontryagin product

$$s(\mathbb{\Lambda}_i) = s(\mathrm{Alt}^i \mathbb{\Lambda}_Z) * s(\mathrm{Alt}^i \mathbb{\Lambda}_{-Z}) \quad \text{in } \mathrm{CH}_{<g}(A) := \mathrm{CH}_\bullet(A)/\mathrm{CH}_g(A).$$

Comparing this with (7.1), we obtain in the truncated Chow ring $\mathrm{CH}_{<g}(A)$ an identity

$$s(\mathrm{Alt}^r \mathbb{\Lambda}_Z) * s(\mathrm{Alt}^r \mathbb{\Lambda}_{-Z}) = c\, s(\mathrm{Alt}^{r-1} \mathbb{\Lambda}_Z) * s(\mathrm{Alt}^{r-1} \mathbb{\Lambda}_{-Z}) + \cdots,$$

where $\cdots$ stands for effective cycle classes (possibly zero). The left-hand side vanishes in all degrees greater than $2d_Z(r)$ since $s_i(\mathrm{Alt}^r \mathbb{\Lambda}_Z) = s_i(\mathrm{Alt}^r \mathbb{\Lambda}_{-Z}) = 0$ for

all $i > d_Z(r)$. On the other hand, Lemma 5.6 says that on the right-hand side the Pontryagin product

$$s(\mathrm{Alt}^{r-1} \mathbb{\Lambda}_Z) * s(\mathrm{Alt}^{r-1} \mathbb{\Lambda}_{-Z})$$

is nonzero and effective in all degrees at most $\min\{2d_Z(r-1), \dim A - 1\}$ because the subvariety $\mathrm{Alt}^{r-1} Z$ has finite Gauss map and hence all its irreducible components are nondegenerate by Theorem 2.8. So a comparison of the left- and right-hand sides yields

$$\min\{2d_Z(r-1), \dim A - 1\} \leq 2d_Z(r) < \dim A - 1,$$

whence $2d_Z(r-1) \leq \dim A - 1$ and $d_Z(r-1) \leq d_Z(r)$. □

Proof of Theorem 7.5
Let $g := \dim A$. Suppose by contradiction $r \dim Z \geq g$, and consider the integer

$$s := \max\{i \in \mathbb{N} \mid i \dim Z < g\} < r.$$

On the one hand, we have

$$\begin{aligned} s \dim Z &= \dim \mathrm{Alt}^s Z && \text{by Proposition 7.9, since } s \dim Z < g \\ &< (g-1)/2 && \text{by Lemma 7.11, since } s < r. \end{aligned}$$

In particular, $\dim Z < (g-1)/2$. But on the other hand,

$$\begin{aligned} s \dim Z &= (s+1)\dim Z - \dim Z \geq g - \dim Z && \text{since } (s+1)\dim Z \geq g \\ &> (g+1)/2 && \text{since } \dim Z < (g-1)/2, \end{aligned}$$

which contradicts the previous displayed inequality. □

7.5. Smooth wedge powers come from curves
It remains to show that the only smooth wedge powers in the dimension range in question are those coming from smooth curves, as announced in Theorem 7.6. In what follows, let $X, Z \subset A$ be integral subvarieties such that

(1) the subvariety $X \subset A$ is smooth and nondivisible,
(2) the Gauss maps γ_X and γ_Z are finite morphisms, and
(3) $\mathrm{Alt}^r \mathbb{\Lambda}_Z = \mathbb{\Lambda}_{[e](X)}$ for some integers $e, r \geq 1$ with $r \dim Z < \dim A$.

We claim that then $Y := [e](X)$ is birational to the rth symmetric power of Z. More precisely, we have the following.

PROPOSITION 7.12
If the above conditions (1)–(3) hold, then Y *is a sum of* r *copies of* Z*, and the sum morphism* $\sigma\colon \mathrm{Z}^r \to \mathrm{Y}$ *is finite and factors through a finite birational morphism*

$$\tau\colon \operatorname{Sym}^r \mathrm{Z} = \mathrm{Z}^r/\mathfrak{S}_r \longrightarrow \mathrm{Y}.$$

Proof
The finiteness of the Gauss map γ_{Z} implies that $\mathrm{Z} \subset \mathrm{A}$ is nondegenerate by Theorem 2.8. Since $r \dim \mathrm{Z} < \dim \mathrm{A}$, Proposition 7.9 says that $\mathrm{pr}_{\mathrm{Z},r}\colon \mathbb{\Lambda}_{\mathrm{Z}}^{[r]} \to \mathrm{Z}^r$ is surjective and the sum morphism

$$\sigma\colon \mathrm{Z}^r \longrightarrow \mathrm{Y} = \mathrm{Z} + \cdots + \mathrm{Z}$$

is generically finite. Moreover, σ induces a morphism $\tau\colon \operatorname{Sym}^r \mathrm{Z} \to \mathrm{Y}$ and our assumption $\operatorname{Alt}^r \mathbb{\Lambda}_{\mathrm{Z}} = \mathbb{\Lambda}_{\mathrm{Y}}$ implies by Proposition 7.9(2) that τ has generic degree $\deg(\tau) = 1$; that is, it is birational. It remains to show that this morphism is finite. Since we have no control on the singularities of Z, we cannot use Proposition B.4. Instead, to show that σ and hence τ is finite, we consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Z}^r & \xleftarrow{\mathrm{pr}_{\mathrm{Z},r}} & \mathbb{\Lambda}_{\mathrm{Z}}^{[r]} & \xrightarrow{\gamma_{\mathrm{Z},r}} & \mathbb{P}_{\mathrm{A}} \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tilde{\sigma}} & & \| \\
\mathrm{Y} & \xleftarrow[\mathrm{pr}_{\mathrm{Y}}]{} & \mathbb{\Lambda}_{\mathrm{Y}} & \xrightarrow[\gamma_{\mathrm{Y}}]{} & \mathbb{P}_{\mathrm{A}}
\end{array}$$

Here, the fibers of pr_{Y} are all of pure dimension $\mathrm{N} = \operatorname{codim}_{\mathrm{A}} \mathrm{Y} - 1$ by Corollary 2.5. Moreover, the morphism $\tilde{\sigma}$ is finite since the rightmost square in the above diagram commutes and since in that square the horizontal arrows $\gamma_{\mathrm{Z},r}$ and γ_{Y} are finite morphisms by our finiteness assumptions on Gauss maps. Hence, it follows that all fibers of $\mathrm{pr}_{\mathrm{Y}} \circ \tilde{\sigma}\colon \mathbb{\Lambda}_{\mathrm{Z},r} \to \mathrm{Y}$ are of dimension N. Since σ is generically finite, it follows from the commutativity of the leftmost square in the above diagram that the generic fiber of the morphism $\mathrm{pr}_{\mathrm{Z},r}$ has dimension N as well. We can now argue by contradiction. Any positive-dimensional fiber of σ would give rise to a fiber of $\mathrm{pr}_{\mathrm{Y}} \circ \tilde{\sigma}$ of dimension at least $\mathrm{N} + 1$, by semicontinuity of dimension of fibers for proper morphisms (see [48, Lemma 0D4I]). This shows that σ is finite. □

Proof of Theorem 7.6
In Proposition 7.12, we have seen that the sum morphism σ factors through a finite birational morphism $\tau\colon \operatorname{Sym}^r \mathrm{Z} \to \mathrm{Y}$. We claim that a similar finite birational morphism from a symmetric product also exists for X rather than for $\mathrm{Y} = [e](\mathrm{X})$. For this, we use the following general remark.

For any integral subvariety $W \subset A$, the scheme-theoretic preimage $[e]^{-1}(W)$ is reduced because it is the preimage of a reduced subvariety under an étale morphism (see [48, Proposition 03PC (8)]). If W' is an irreducible component of $[e]^{-1}(W)$, then any other irreducible component is of the form $W' + x$ for an e-torsion point $x \in A[e]$, and hence the morphism $[e]\colon W' \to Z$ is surjective. In particular, by Remark 2.6 the subvariety $W \subset A$ is nondegenerate if and only if $W' \subset A$ is.

Applying this to $W = Z$, we see that any irreducible component Z' of $[e]^{-1}(Z)$ is nondegenerate. Hence, if we define $X' = Z' + \cdots + Z' \subset A$ to be the sum of r copies of Z', then Lemma 2.7 implies

$$\dim X' = r \dim Z' = r \dim Z = \dim Y.$$

It follows that X' is an irreducible component of $[e]^{-1}(Y)$. On the other hand, we have

$$[e]^{-1}(Y) = [e]^{-1}\big([e](X)\big) = \bigcup_{t \in A[e]} X + t.$$

Therefore, there is an e-torsion point $t \in A[e]$ such that $X = X' + t$ and X is the sum of r copies of

$$\tilde{Z} := Z' + u,$$

where $u \in A(k)$ is such that $ru = t$. Since the stabilizer of $X = \tilde{Z} + \cdots + \tilde{Z}$ is trivial by assumption, it follows that the stabilizer of $\tilde{Z}$ is trivial, so the finite morphism $[e]\colon \tilde{Z} \to Z$ is birational. Then the morphism

$$\operatorname{Sym}^r[e]\colon \operatorname{Sym}^r \tilde{Z} \longrightarrow \operatorname{Sym}^r Z$$

is finite birational by Proposition B.2. Now the sum morphism $\tilde{\sigma}\colon \tilde{Z}^r \to X$ is invariant under the permutation action of $\mathfrak{S}_r$, so it factors through a morphism $\tilde{\tau}$ as shown in the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Sym}^r \tilde{Z} & \xrightarrow{\tilde{\tau}} & X \\ {\scriptstyle \operatorname{Sym}^r[e]}\downarrow & & \downarrow{\scriptstyle [e]} \\ \operatorname{Sym}^r Z & \xrightarrow{\tau} & Y. \end{array} \tag{7.2}$$

The morphisms τ, $[e]$, and $\operatorname{Sym}^r[e]$ in this diagram are finite birational, hence $\tilde{\tau}$ must be finite birational too. On the other hand, the variety X is smooth by hypothesis, which forces $\tilde{\tau}$ to be an isomorphism. In particular, the symmetric power $\operatorname{Sym}^r \tilde{Z}$ is smooth, which for $r > 1$ implies that $\tilde{Z}$ is a smooth curve by Proposition B.3. □

Remark 7.13

Given X, Y, Z as in the proof of Theorem 7.6, it follows immediately from the fact that

Table 2. Half-spin representations.

	Dual of $\mathbb{S}_N^+$	Center of $\mathrm{Spin}_{2n}(\mathbb{F})$
n even	$\mathbb{S}_N^+$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
n odd	$\mathbb{S}_N^-$	$\mathbb{Z}/4\mathbb{Z}$

the morphisms $[e]$ and τ from (7.2) are finite and birational that X is dominated by the normalization W of $\mathrm{Sym}^r \mathrm{Z}$, which is easily seen to be isomorphic to $\mathrm{Sym}^r \tilde{\mathrm{Z}}$ where $\tilde{\mathrm{Z}}$ is the normalization of Z. Smoothness of X implies that in fact $\mathrm{Sym}^r \tilde{\mathrm{Z}} \simeq \mathrm{X}$. The argument we used instead is maybe a bit longer but yields more, namely, a canonical (up to an e-torsion point) embedding of $\tilde{\mathrm{Z}}$ into A.

8. Spin representations

We now show that under suitable assumptions on a smooth subvariety of an abelian variety, its Tannaka group cannot be the image of a spin group acting via a spin representation. For hypersurfaces, this can be done by showing that their topological Euler characteristic is not a power of 2 (see [39, Lemma 4.9]); we here discuss the case of higher codimension, where we do not know the Euler characteristic but consider characteristic cycles as in the previous section.

8.1. Statement of the main result

Recall that for $\mathrm{N} \geq 3$, the group $\mathrm{SO}_\mathrm{N}(\mathbb{F})$ admits a double cover

$$\mathrm{Spin}_\mathrm{N}(\mathbb{F}) \longrightarrow \mathrm{SO}_\mathrm{N}(\mathbb{F})$$

by the spin group. The spin group is a simply connected algebraic group and admits a faithful representation $\mathbb{S}_\mathrm{N} \in \mathrm{Rep}_\mathbb{F}(\mathrm{Spin}_\mathrm{N}(\mathbb{F}))$, the *spin representation* of dimension $\dim \mathrm{S}_\mathrm{N} = 2^n$ where $n = \lfloor \mathrm{N}/2 \rfloor$. The behavior of this representation depends on the Dynkin type (see [21, Section 20]):

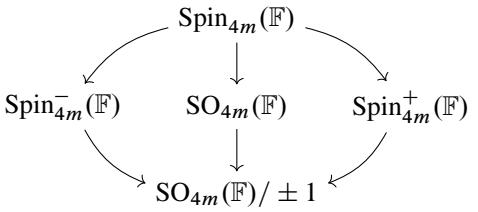
B_n: If $\mathrm{N} = 2n + 1$ is odd, then the spin representation $\mathbb{S}_\mathrm{N}$ is irreducible.

D_n: If $\mathrm{N} = 2n$ is even, then $\mathbb{S}_\mathrm{N} \simeq \mathbb{S}_\mathrm{N}^+ \oplus \mathbb{S}_\mathrm{N}^-$ splits as the direct sum of two irreducible representations called the *half-spin representations*. They both have dimension $\dim \mathbb{S}_\mathrm{N}^+ = \dim \mathbb{S}_\mathrm{N}^- = 2^{n-1}$ but are not isomorphic to each other, they are only related by an outer automorphism of the spin group. The dual of the half-spin representations and the center of the spin group are given by Table 2.

For $n = 2m + 1$ odd, the half-spin representations are faithful. For $n = 2m$ even, the half-spin representation $\mathbb{S}_\pm$ is self-dual and the natural pairing is symmetric if m is even and alternating if m is odd. The images of $\mathrm{Spin}_{4m}(\mathbb{F})$ via the half-spin representations

$$\mathrm{Spin}_{4m}^\pm(\mathbb{F}) \subset \mathrm{GL}(\mathbb{S}_{4m}^\pm)$$

are called the *half-spin groups*. They are isomorphic to each other and fit in the following diagram of isogenies given by dividing out the subgroups of $Z(\mathrm{Spin}_{4m}(\mathbb{F})) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:



We are interested in geometric incarnations of the above.

Definition 8.1
Let $X \subset A$ be a subvariety. Let $G = G^{*}_{X,\omega}$ be the derived group of the connected component of the corresponding Tannaka group, and consider the faithful representation $V = \omega(\delta_X)_{|G} \in \mathrm{Rep}_{\mathbb{F}}(G)$. Let $n \geq 1$ be an integer. We say that $X \subset A$ is

- of *spin type* B_n if $G \simeq \mathrm{Spin}_{2n+1}(\mathbb{F})$ and $V \simeq \mathbb{S}_{2n+1}$, and
- of *spin type* D_n if $G \simeq \mathrm{Spin}^{\varepsilon}_{2n}(\mathbb{F})$ and $V \simeq \mathbb{S}^{\varepsilon}_{2n}$ for some $\varepsilon \in \{+,-\}$.

In both cases, the subvariety X is irreducible because the representation V is so.

Remark 8.2
Suppose that X is of spin type D_n for $n = 2m$. Since half-spin representations are self-dual in this case, the subvariety X is symmetric up to translation. The Poincaré pairing on X is symmetric if $d = \dim X$ is even and alternating if d is odd. By comparison with the natural pairing on $\mathbb{S}^{\pm}_n$, the integers m and d must have the same parity.

The goal of this section is to show that for smooth subvarieties of small dimension this cannot happen. To state our results, let $g = \dim A$.

THEOREM 8.3
Let $X \subset A$ *be a* d*-dimensional nondivisible smooth subvariety with ample normal bundle, and let* $d < (g-1)/2$*. Then for any integer* $n \geq 1$*,*

(1) X *is not of spin type* B_n*;*
(2) *if* X *is of spin type* D_n*, then* $d \geq (g-1)/4$*,* $n = 2m$ *with* $3 \leq m \leq d$ *having the same parity as* d*.*

The list of Dynkin types in the above theorem starts with B_2 and D_3, and at least for these smallest cases the result is optimal in the sense that the dimension bound cannot be weakened.

Example 8.4
Let X be a smooth projective curve of genus $g = 3$, embedded in its Jacobian variety $A = \mathrm{Alb}(X)$. By [37, Theorem 6.1] and [50], there are two cases:

B_2: If X is hyperelliptic, then $G^*_{X,\omega} \simeq \mathrm{Sp}_4(\mathbb{F})$ and $\omega(\delta_X) \simeq \mathbb{F}^4$ is its standard representation. This representation corresponds to the spin representation under the isomorphism $\mathrm{Spin}_5(\mathbb{F}) \simeq \mathrm{Sp}_4(\mathbb{F})$.

D_3: If X is not hyperelliptic, then $G^*_{X,\omega} \simeq \mathrm{SL}_4(\mathbb{F})$ and $\omega(\delta_X) \simeq \mathbb{F}^4$ is its standard representation or its dual. These two representations correspond to the two half-spin representations under the isomorphism $\mathrm{Spin}_6(\mathbb{F}) \simeq \mathrm{SL}_4(\mathbb{F})$.

So spin representations do occur, but in this example $2\dim X = \dim A - 1$.

8.2. Structure of the proof
The proof of Theorem 8.3 relies on three independent steps. The first is to show that the structure of spin representations is reflected by characteristic cycles. We say that a cycle $\mathbb{A} \in \mathscr{L}(A)$ is *symmetric* if $[-1]_*\mathbb{A} = \mathbb{A}$. In this case, if the cycle is reduced and effective of Gauss degree $\deg \mathbb{A} = 2n$, then for any integer $r \geq 1$ we will define via the formalism in Lemma 5.14 a clean effective cycle

$$\mathrm{Alt}^r_{\mathbb{S}}\, \mathbb{A} \in \mathscr{L}(A) \quad \text{of Gauss degree } \deg \mathrm{Alt}^r_{\mathbb{S}}\, \mathbb{A} = 2^r \binom{n}{r}.$$

These cycles are closely related to the wedge powers from Section 7; for instance, we have

$$\mathrm{Supp}\, \mathrm{Alt}^r\, \mathbb{A} = \bigcup_{i=0}^{\lfloor r/2 \rfloor} \mathrm{Supp}\, \mathrm{Alt}^{r-2i}_{\mathbb{S}}\, \mathbb{A}.$$

For $r = n$, the cycle $\mathrm{Alt}^n_{\mathbb{S}}\, \mathbb{A}$ will correspond to the spin representation. Instead, for half-spin representations we consider the monodromy of the Gauss map $\gamma_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{P}_A$. If the Gauss map has even monodromy in the sense of Definition 5.21, we will obtain a decomposition

$$\mathrm{Alt}^n_{\mathbb{S}}\, \mathbb{A} = \mathrm{Alt}^n_{\mathbb{S},+}\, \mathbb{A} + \mathrm{Alt}^n_{\mathbb{S},-}\, \mathbb{A},$$

where $\mathrm{Alt}^n_{\mathbb{S},\pm}\, \mathbb{A} \in \mathscr{L}(A)$ are clean effective cycles of Gauss degree 2^{n-1}. We show the following.

THEOREM 8.5
Let $\mathrm{X} \subset \mathrm{A}$ *be a smooth nondivisible subvariety of spin type* B_n *or* D_n *for some* n. *Then there exist* $a \in \mathrm{A}(k)$, *a reduced symmetric effective cycle* Z *on* A *of Gauss degree* $\deg \mathbb{A}_{\mathrm{Z}} = 2n$, *and an integer* $e \geq 1$ *with the following properties:*

(1) *If* X *is of spin type* B_n, *then* $\mathrm{Alt}^n_{\mathbb{S}} \mathbb{A}_{\mathrm{Z}} = \mathbb{A}_{[e](\mathrm{X}+a)}$.

(2) *If* X *is of spin type* D_n, *then the Gauss map* $\gamma_{\mathbb{A}}$ *has even monodromy and we have*

$$\mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathbb{A}_{\mathrm{Z}} = \mathbb{A}_{[e](\mathrm{X}+a)} \quad \textit{for suitable } \varepsilon \in \{+,-\}.$$

In both cases, if $\gamma_{\mathrm{X}} \colon \mathbb{A}_{\mathrm{X}} \to \mathbb{P}_{\mathrm{A}}$ *is a finite morphism, then so is* $\gamma_{\mathrm{Z}} \colon \mathbb{A}_{\mathrm{Z}} \to \mathbb{P}_{\mathrm{A}}$.

For the precise definition of the clean cycles appearing above and the proof of the theorem, we refer to Section 8.3. The second step will be a dimension estimate for the images

$$\mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathrm{Z} := \mathrm{pr}_{\mathrm{A}}(\mathrm{Supp}(\mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathbb{A}_{\mathrm{Z}}) \subset \mathrm{A},$$

for $\varepsilon \in \{+,-,\varnothing\}$. We show the following.

THEOREM 8.6
Let Z *be a reduced symmetric effective cycle on* A *whose Gauss map is a finite morphism of even degree* $\deg(\gamma_{\mathrm{Z}}) = 2n$.

(1) *If* $\dim \mathrm{Alt}^n_{\mathbb{S}} \mathrm{Z} < (g-1)/2$, *then* $n \dim \mathrm{Z} < g-1$.

(2) *If the Gauss map* γ_{Z} *has even monodromy and there is* $\varepsilon \in \{+,-\}$ *for which the dimension* $d := \dim \mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathrm{Z}$ *satisfies*

$$d < \begin{cases} (g-1)/4 & \textit{if } n = 2m \textit{ is even and } m \leq d+1, \\ (g-1)/2 & \textit{otherwise}, \end{cases}$$

then $n \dim \mathrm{Z} < g-1$.

The proof of this is given in Section 8.4. Finally, the last step for ruling out spin representations will be to show that in the given dimension range, the cycle $\mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathbb{A}_{\mathrm{Z}}$ cannot be smooth and integral. More precisely, we have the following.

THEOREM 8.7
Let $\mathrm{X} \subset \mathrm{A}$ *be a smooth nondivisible subvariety with ample normal bundle, and let* Z *be a reduced symmetric effective cycle on* A *whose Gauss map is finite of even degree* $\deg(\gamma_{\mathrm{Z}}) = 2n \geq 4$. *Suppose that for some integer* $e \geq 1$ *one of the following two conditions holds:*

(1) $\mathbb{\Lambda}_{[e](\mathrm{X})} = \mathrm{Alt}^n_{\mathbb{S}}\, \mathbb{\Lambda}_{\mathrm{Z}}$.
(2) γ_{Z} *has even monodromy and* $\mathbb{\Lambda}_{[e](\mathrm{X})} = \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathbb{\Lambda}_{\mathrm{Z}}$ *for some* $\varepsilon \in \{+,-\}$.
Then $n \dim \mathrm{Z} \geq \dim \mathrm{A}$.

Proof of Theorem 8.3
Let $\mathrm{X} \subset \mathrm{A}$ be a d-dimensional smooth nondivisible subvariety of spin type B_n or D_n for some integer n with $d < (g-1)/2$. After replacing X by a translate, there exists by Theorem 8.5 an integer $e \geq 1$ and a reduced symmetric effective cycle Z on A with $\deg(\gamma_{\mathrm{Z}}) = 2n$ such that

$$\mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathbb{\Lambda}_{\mathrm{Z}} = \mathbb{\Lambda}_{[e](\mathrm{X})} \quad \text{for some } \varepsilon \in \{+,-,\varnothing\}.$$

It follows that

$$\dim \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathrm{Z} = \dim [e](\mathrm{X}) = \dim \mathrm{X}.$$

Note that we have $\varepsilon \neq \varnothing$ only in spin type D_n and in that case γ_{Z} has even monodromy. Moreover, if X is of spin type D_n, $n = 2m$ is even and $m \leq d+1$, assume $d < (g-1)/4$. Then Theorem 8.6 implies $n \dim \mathrm{Z} < \dim \mathrm{A}$, which contradicts Theorem 8.7. The remaining cases are of type D_n with $n = 2m$ even, $d \geq (g-1)/4$ and $m \leq d+1$. Remark 8.2 implies that $d - m$ must be even. Lemmas 2.12 and 2.13 imply that in the current dimension range the absolute value of the topological Euler characteristic of X is never 8, so the case $m = 2$ does not occur. □

8.3. Characteristic cycles of spin representations
We now explain how to compute characteristic cycles for subvarieties of spin type. Consider a symmetric reduced clean effective cycle Z on A with dominant Gauss map $\gamma_{\mathrm{Z}} \colon \mathbb{\Lambda}_{\mathrm{Z}} \to \mathbb{P}_{\mathrm{A}}$. For an integer $r \geq 1$, we put

$$\mathbb{\Lambda}^{[r]}_{\mathrm{Z},\mathbb{S}} := \overline{\mathbb{\Lambda}^{\times r}_{\mathrm{Z}|\mathrm{U}} \smallsetminus (\Delta_r \cup \Delta_r^-)} \subset \mathbb{\Lambda}^{\times r}_{\mathrm{Z}}$$

as in Definition 5.16, where $\mathrm{U} \subset \mathbb{P}_{\mathrm{A}}$ is any open dense subset over which the Gauss map γ_{Z} is finite and étale. For a partition $\alpha = (\alpha_1, \ldots, \alpha_r)$, consider the sum morphism

$$\sigma_\alpha \colon \mathrm{A}^r \times \mathbb{P}_{\mathrm{A}} \longrightarrow \mathrm{A} \times \mathbb{P}_{\mathrm{A}}, \qquad (z_1, \ldots, z_r, \xi) \longmapsto (\alpha_1 z_1 + \cdots + \alpha_r z_r, \xi).$$

We put

$$\mathbb{\Lambda}^{\alpha}_{\mathrm{Z},\mathbb{S}} := \sigma_{\alpha *}(\mathbb{\Lambda}^{[r]}_{\mathrm{Z},\mathbb{S}}).$$

We are mostly interested in the case of the partition $\alpha = (1^r) = (1, \ldots, 1)$ and write

$$\mathrm{Alt}^r_{\mathbb{S}}\,\mathbb{\Lambda}_Z := \mathbb{\Lambda}^{(1^r)}_{Z,\mathbb{S}}$$

in this case. As in Definition 5.21, we say that the Gauss map $\gamma_Z\colon \mathbb{\Lambda}_Z \to \mathbb{P}_A$ has *even monodromy* if its degree is an even integer $\deg(\gamma_Z) = 2n$ and the finite étale cover $\gamma_{Z|U}\colon \mathbb{\Lambda}_{Z|U} \to U$ has as its monodromy group a subgroup of $(\pm 1)^n_+ \rtimes \mathfrak{S}_n$. In this case,

$$\mathbb{\Lambda}^{[n]}_{Z,\mathbb{S}} = \mathbb{\Lambda}^{[n]}_{Z,\mathbb{S},+} + \mathbb{\Lambda}^{[n]}_{Z,\mathbb{S},-}$$

for $\mathbb{\Lambda}^{[n]}_{Z,\mathbb{S},\pm}$ as in Definition 5.21, and for partitions α of length $r = n$ we put

$$\mathbb{\Lambda}^{\alpha}_{Z,\mathbb{S},\pm} = \sigma_{\alpha *}(\mathbb{\Lambda}^{[n]}_{Z,\mathbb{S},\pm}).$$

Again for $\alpha = (1^n)$ we instead write $\mathrm{Alt}^n_{\mathbb{S},\pm}\,\mathbb{\Lambda}_Z := \mathbb{\Lambda}^{(1^n)}_{Z,\mathbb{S},\pm}$.

Proof of Theorem 8.5
Translating $X \subset A$, we may assume $\det(\delta_X) = \delta_0$. As in the proof of Theorem 7.4, but replacing Lemma 5.18 by Lemma 5.20, we then find a reduced symmetric effective cycle Z on A with $\mathbb{\Lambda}_Z \in \langle \mathbb{\Lambda}_X \rangle$ such that

$$\mathbb{\Lambda}_{[e](X)} = \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\,\mathbb{\Lambda}_Z \qquad \text{and} \qquad \deg(\gamma_Z) = 2n$$

for some integer $e \geq 1$ and $\varepsilon \in \{+, -, \varnothing\}$ as claimed. □

8.4. Dimension estimates in the spin case
Let Z be a symmetric reduced effective cycle in A. As before, we put

$$\mathrm{Alt}^r_{\mathbb{S}}\,Z := \mathrm{pr}_A(\mathrm{Supp}\,\mathrm{Alt}^r_{\mathbb{S}}\,\mathbb{\Lambda}_Z) \subset A,$$
$$\mathrm{Alt}^n_{\mathbb{S},\varepsilon}\,Z := \mathrm{pr}_A(\mathrm{Supp}\,\mathrm{Alt}^n_{\mathbb{S},\varepsilon}\,\mathbb{\Lambda}_Z) \subset A.$$

We want to show that under certain assumptions the dimension of these images grows linearly with r in a suitable range. The first step is to relate the cycles associated with the two half-spin representations.

LEMMA 8.8
Let $Z \subset A$ be a symmetric reduced effective cycle whose Gauss map γ_Z is finite of degree $\deg(\gamma_Z) = 2n \geq 4$ and has even monodromy. Suppose that n is odd, or suppose that $n = 2m$ is even and $m - 1 > \min_{\varepsilon\in\{+,-\}} \dim \mathrm{Alt}^n_{\mathbb{S},\varepsilon} Z$. Then

$$\dim \mathrm{Alt}^n_{\mathbb{S},+} Z = \dim \mathrm{Alt}^n_{\mathbb{S},-} Z.$$

Proof
We may suppose $k = \mathbb{C}$. For $\varepsilon \in \{+, -\}$, set

$$\mathbb{\Lambda}_\varepsilon := \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathbb{\Lambda}_{\mathrm{Z}}, \qquad \mathrm{X}_\varepsilon = \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathrm{Z} = \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}_\varepsilon).$$

If $n = 2m + 1$ is odd, then the definitions imply $\mathrm{X}_+ = [-1]_*\mathrm{X}_-$ and the statement follows. The representation-theoretic analogue of the previous identity is that the half-spin representations $\mathbb{S}_+$ and $\mathbb{S}_-$ are dual to each other. Suppose henceforth that $n = 2m$ is even and, up to changing signs, that we have $\dim \mathrm{X}_+ \leq \dim \mathrm{X}_-$. With this notation $m - 1 > d := \dim \mathrm{X}_+$. Similarly, drawing inspiration from the isomorphism $\mathrm{Alt}^2(\mathbb{S}_+) \simeq \mathrm{Alt}^2(\mathbb{S}_-)$ of representations we obtain the following.

CLAIM 8.9
The following identity of Lagrangian cycles holds:

$$\mathbb{\Lambda}_+ \circ \mathbb{\Lambda}_+ - [2]_*\mathbb{\Lambda}_+ = \mathbb{\Lambda}_- \circ \mathbb{\Lambda}_- - [2]_*\mathbb{\Lambda}_- \in \mathscr{L}(\mathrm{A}).$$

Proof of the claim
Pick a general cotangent direction $v \in \mathbb{P}_{\mathrm{A}}(k)$, and denote by

$$\mathbb{\Lambda}_{\mathrm{Z},v} = \{\pm p_1, \ldots, \pm p_n\} \subset \mathrm{A}$$

the corresponding fiber of the Gauss map, which we identify as a 0-cycle on the abelian variety via the projection $\mathrm{pr}_{\mathrm{A}} \colon \mathbb{\Lambda}_{\mathrm{Z}} \to \mathrm{A}$. Let $\{\pm 1\}^n_\varepsilon \subset \{\pm 1\}^n$ be the subset made of n-tuples $a = (a_1, \ldots, a_n)$ such that the sign of $a_1 \cdots a_n$ is ε. As 0-cycles on A, we have

$$(\mathbb{\Lambda}_\varepsilon)_v = \sum_{a \in \{\pm 1\}^n_\varepsilon} [a_1 p_1 + \cdots + a_n p_n] \in \mathrm{Z}_0(\mathrm{A}),$$

$$(\mathbb{\Lambda}_\varepsilon \circ \mathbb{\Lambda}_\varepsilon)_v = \sum_{a,b \in \{\pm 1\}^n_\varepsilon} \big[(a_1 + b_1) p_1 + \cdots + (a_n + b_n) p_n\big] \in \mathrm{Z}_0(\mathrm{A}),$$

where the summation sign refers to the sum as cycles and $[x]$ is the 0-cycle given by a point $x \in \mathrm{A}(k)$. We split the sum in the preceding equation in two. The sum ranging on couples (a, b) with $a = b$ gives the 0-cycle $([2]_*\mathbb{\Lambda}_\varepsilon)_v$. For the remaining couples, notice that we have a bijection

$$\big\{(a, b) \in \big(\{\pm 1\}^n_+\big)^2 \;\big|\; a \neq b\big\} \longrightarrow \big\{(a', b') \in \big(\{\pm 1\}^n_-\big)^2 \;\big|\; a' \neq b'\big\}$$

sending (a, b) to the couple (a', b') obtained by changing the sign of the first entry in which a and b differ. Such a bijection is compatible with sum, that is, $a + b = a' + b'$. Letting vary v gives the desired identity of cycles. □

The identity in the claim can be rewritten as

$$(\mathbb{\Lambda}_+ + \mathbb{\Lambda}_-) \circ (\mathbb{\Lambda}_+ - \mathbb{\Lambda}_-) = [2]_*(\mathbb{\Lambda}_+ - \mathbb{\Lambda}_-).$$

Since $\mathbb{\Lambda}_Z$ has finite Gauss map, so do $\mathbb{\Lambda}_+$ and $\mathbb{\Lambda}_-$ by construction. By Lemma 5.5, we obtain, by passing to Segre classes,

$$s(\mathbb{\Lambda}_+ + \mathbb{\Lambda}_-) * s(\mathbb{\Lambda}_+ - \mathbb{\Lambda}_-) = [2]_* s(\mathbb{\Lambda}_+ - \mathbb{\Lambda}_-) \in \mathrm{CH}_{<g}(\mathrm{A}).$$

Let s and δ be the image of $s(\mathbb{\Lambda}_+)$ and $s(\mathbb{\Lambda}_+ - \mathbb{\Lambda}_-)$ in the homology $\mathrm{H}_\bullet(\mathrm{A}, \mathbb{Z})$, so that

$$(2s + \delta) * \delta = [2]_* \delta \in \mathrm{H}_\bullet(\mathrm{A}, \mathbb{Z}),$$

where $*$ denotes the Pontryagin product in homology. Write $s_i, \delta_i \in \mathrm{H}_{2i}(\mathrm{A}, \mathbb{Z})$ for the pieces of s, δ in degree $2i$. Then in degree $2r$ the previous identity reads as

$$\sum_{i=0}^{r} (2s_i + \delta_i) * \delta_{r-i} = 2^{2r} \delta_r \in \mathrm{H}_{2r}(\mathrm{A}, \mathbb{Z}),$$

because the multiplication by 2 on A acts as multiplying by 2^{2r} on $\mathrm{H}_{2r}(\mathrm{A}, \mathbb{Z})$. Now for $\varepsilon \in \{+, -\}$, the 0th Segre class of $\mathbb{\Lambda}_\varepsilon$ has degree

$$\deg \gamma_{\mathbb{\Lambda}_\varepsilon} = \left| \{\pm 1\}_\varepsilon^n \right| = 2^{2m-1}.$$

Plugging in the identities $s_0 = 2^{2m-1}$ and $\delta_0 = 0$ obtained in this way yields the recursion formula

$$(2^{2r} - 2^{2m}) \delta_r = \sum_{i=1}^{r-1} (2s_i + \delta_i) * \delta_{r-i} \in \mathrm{H}_{2r}(\mathrm{A}, \mathbb{Z}).$$

Then the vanishing of δ_0 inductively implies

$$\delta_r = 0, \quad r = 0, \ldots, m-1.$$

Suppose by contradiction that $\dim \mathrm{X}_- > d = \dim \mathrm{X}_+$. Then

$$\delta_{d+1} = s_{d+1}(\mathbb{\Lambda}_-) - s_{d+1}(\mathbb{\Lambda}_+) = s_{d+1}(\mathbb{\Lambda}_-) \neq 0,$$

where we abusively identified Segre classes with their images in homology. Then the vanishing of δ_r for $r < m$ implies $d + 1 \geq m$, contradicting the assumption $m - 1 > d$. □

The second step is the following analogue of Lemma 7.11 (note that at first we only get a weaker estimate in the spin case since here we only start from a dimension bound on the support of $\mathrm{Alt}^r_{\mathbb{S}}\,\mathrm{Z}$, which a priori might be strictly smaller than the one of $\mathrm{Alt}^r\,\mathrm{Z}$).

PROPOSITION 8.10

Let Z *be a symmetric reduced effective cycle on* A *whose Gauss map has degree* $\deg(\gamma_{\mathrm{Z}}) = 2n \geq 4$.

(1) *If* $\dim \mathrm{Alt}^n_{\mathbb{S}}\,\mathrm{Z} < (g-1)/2$, *then* $\dim \mathrm{Alt}^i_{\mathbb{S}}\,\mathrm{Z} < g-1$ *for* $i = 1, \ldots, n-1$.

(2) *If* γ_{Z} *is finite and has even monodromy and if there is* $\varepsilon \in \{+,-\}$ *such that the dimension* $d := \dim \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\,\mathrm{Z}$ *satisfies*

$$d < \begin{cases} (g-1)/4 & \text{if } n = 2m \text{ is even and } m \leq d+1, \\ (g-1)/2 & \text{otherwise}, \end{cases}$$

then $\dim \mathrm{Alt}^i_{\mathbb{S}}\,\mathrm{Z} < g-1$ *for* $1 \leq i \leq n$ *and* $\dim \mathrm{Alt}^n_{\mathbb{S}}\,\mathrm{Z} < \frac{g-1}{2}$ *if* n *is odd.*

Proof

We may suppose $k = \mathbb{C}$. (1) Put $\mathbb{\Lambda}^{(s)} := \mathrm{Alt}^s_{\mathbb{S}}\,\mathbb{\Lambda}_{\mathrm{Z}}$. Pick a general cotangent direction $v \in \mathbb{P}_{\mathrm{A}}(k)$, and denote by

$$\mathbb{\Lambda}_{\mathrm{Z},v} = \{\pm p_1, \ldots, \pm p_n\} \subset \mathrm{A}$$

the corresponding fiber of the Gauss map, which we identify as usual with a set of points on the abelian variety via the projection $\mathrm{pr}_{\mathrm{A}}\colon \mathbb{\Lambda}_{\mathrm{Z}} \to \mathrm{A}$. Writing $p_{-i} := -p_i$, we have

$$\mathrm{Supp}(\mathbb{\Lambda}^{(s)})_v = \big\{ p_{i_1} + \cdots + p_{i_s} \;\big|\; |i_1|, \ldots, |i_s| \text{ pairwise distinct} \big\},$$

$$\mathrm{Supp}(\mathbb{\Lambda}^{(n)} \circ \mathbb{\Lambda}^{(n)})_v = \Big\{ \sum_{i=1}^{n} (\delta_i + \varepsilon_i)\, p_i \;\Big|\; \delta_1, \ldots, \delta_n, \varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \Big\}.$$

By specializing to the case $\delta_i = \pm\varepsilon_i$ and varying the point $v \in \mathbb{P}_{\mathrm{A}}(k)$, we find

$$\mathrm{Supp}\big([2]_*\mathbb{\Lambda}^{(1)} + [2]_*\mathbb{\Lambda}^{(2)} + \cdots + [2]_*\mathbb{\Lambda}^{(n)}\big) \subset \mathrm{Supp}(\mathbb{\Lambda}^{(n)} \circ \mathbb{\Lambda}^{(n)}).$$

Hence, for the images $\mathrm{Z}^{(s)} := \mathrm{pr}_{\mathrm{A}}(\mathrm{Supp}(\mathbb{\Lambda}^{(s)})) = \mathrm{Supp}(\mathrm{Alt}^s_{\mathbb{S}}\,\mathrm{Z}) \subset \mathrm{A}$ we obtain the inclusions

$$[2](\mathrm{Z} \cup \mathrm{Z}^{(2)} \cup \cdots \cup \mathrm{Z}^{(n)}) \subset \mathrm{pr}_{\mathrm{A}}\big(\mathrm{Supp}(\mathbb{\Lambda}^{(n)} \circ \mathbb{\Lambda}^{(r)})\big) \subset \mathrm{Z}^{(n)} + \mathrm{Z}^{(n)}.$$

Since $[2]\colon \mathrm{A} \to \mathrm{A}$ is an isogeny, for $1 \leq s \leq n$ a look at dimension then shows that we have

$$\dim \mathrm{Z}^{(s)} \leq \dim(\mathrm{Z}^{(n)} + \mathrm{Z}^{(n)}) \leq 2\dim(\mathrm{Z}^{(n)}) < g-1,$$

where the last inequality holds by our dimension assumption. This proves (1).

(2) If $n = 2m+1$ is odd or $n = 2m$ is even with $m > d+1$, then Lemma 8.8 implies

$$d = \dim \mathrm{Alt}^n_{\mathbb{S},+} = \dim \mathrm{Alt}^n_{\mathbb{S},-} = \dim \mathrm{Alt}^n_{\mathbb{S}} \mathrm{Z}$$

so we are done by (1). Now assume $n = 2m$ and $m-1 \leq d < (g-1)/4$. In that case, a set-theoretic look at the fibers of Gauss maps gives

$$\mathrm{Supp}\big([2]_*\mathbb{\Lambda}^{(2)} + [2]_*\mathbb{\Lambda}^{(4)} + \cdots + [2]_*\mathbb{\Lambda}^{(n-2)}\big) \subset \mathrm{Supp}(\mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathbb{\Lambda}_{\mathrm{Z}} \circ \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathbb{\Lambda}_{\mathrm{Z}}).$$

For all $i < m$, then

$$\dim \mathrm{Z}^{(2i)} \leq 2\dim \mathrm{Alt}^n_{\mathbb{S},\varepsilon}\, \mathbb{\Lambda}_{\mathrm{Z}} < (g-1)/2,$$

the second inequality holding by assumption. Since $\mathrm{Alt}^2 \mathrm{Z} = \mathrm{Alt}^2_{\mathbb{S}} \mathrm{Z} + \mathrm{Alt}^0_{\mathbb{S}} \mathrm{Z}$, it follows that $\dim \mathrm{Alt}^2 \mathrm{Z} < (g-1)/2$, and the monotonicity of usual wedge powers in Lemma 7.11 gives

$$\dim \mathrm{Z} = \dim \mathrm{Alt}^1 \mathrm{Z} \leq \dim \mathrm{Alt}^2 \mathrm{Z} < (g-1)/2,$$

since the Gauss map is finite. Then $\mathrm{Supp}\, \mathbb{\Lambda}^{(2i+1)} \subset \mathrm{Supp}\, \mathbb{\Lambda}^{(2i)} \circ \mathbb{\Lambda}^{(1)}$ also implies, for all $i < m$,

$$\dim \mathrm{Z}^{(2i+1)} \leq \dim \mathrm{Z}^{(2i)} + \dim \mathrm{Z}^{(1)} < (g-1)/2 + (g-1)/2 = g-1.$$

Likewise, $\mathrm{Supp}\, \mathbb{\Lambda}^{(n)} \subset \mathrm{Supp}\, \mathbb{\Lambda}^{(n-2)} \circ \mathbb{\Lambda}^{(2)}$ gives $\dim \mathrm{Z}^{(n)} < g-1$. □

Proof of Theorem 8.6

Let Z be a reduced symmetric effective cycle on A whose Gauss map is a finite morphism of degree $\deg(\gamma_{\mathrm{Z}}) = 2n$ for some integer $n \geq 2$. Assume that we are in one of the following two cases:

(1) $\dim \mathrm{Alt}^n_{\mathbb{S}} \mathrm{Z} < (g-1)/2$, or

(2) n is even, the Gauss map γ_{Z} has even monodromy, and for some $\varepsilon \in \{+,-\}$ we have

$$\dim \mathrm{Alt}^n_{\mathbb{S},\varepsilon} < (g-1)/4.$$

By Proposition 8.10, then $\dim \mathrm{Alt}^i_{\mathbb{S}} \mathrm{Z} < g-1$ for all $i \in \{1,\ldots,n\}$, so it will be enough to show that

$$\dim \mathrm{Alt}^i_{\mathbb{S}} \mathrm{Z} = i \dim \mathrm{Z}$$

for all those i. For $i = 1$, there is nothing to show. We now use induction. Suppose that

$$\dim \mathrm{Alt}^i_{\mathbb{S}} \mathrm{Z} = i \dim \mathrm{Z} \quad \text{for all } i \in \{1, 2, \ldots, s-1\},$$

where $s \leq r$ is a positive integer. We want to conclude that $\dim \mathrm{Alt}^s_{\mathbb{S}} \mathrm{Z} = s \dim \mathrm{Z}$.

We may assume $\dim \mathrm{Z} > 0$. For simplicity, put $\mathbb{\Lambda}^\beta := \mathbb{\Lambda}^\alpha_{\mathrm{Z},\mathbb{S}}$ where $\beta = \alpha^t$ denotes the transpose of a partition α, extending our notation from the previous proof. The definitions imply

$$\mathrm{Supp}(\mathbb{\Lambda}^{(s-1)} \circ \mathbb{\Lambda}^{(1)}) = \mathbb{\Lambda}^{(s)} \cup \mathbb{\Lambda}^{(s-1,1)} \cup \mathbb{\Lambda}^{(s-2)}. \tag{8.1}$$

We now compare dimensions. For the last two pieces on the right-hand side, the inclusions $\mathbb{\Lambda}^{(s-1,1)} \subset \mathbb{\Lambda}_{2\mathrm{Z}} \circ \mathbb{\Lambda}^{(s-2)}$ and $\mathbb{\Lambda}^{(s-2)} \subset \mathbb{\Lambda}_{\mathrm{Z}} \circ \cdots \circ \mathbb{\Lambda}_{\mathrm{Z}}$ (with $s-2$ factors) imply

$$\begin{aligned} \dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s-1,1)} \cup \mathbb{\Lambda}^{(s-2)}) &\leq (s-1) \dim \mathrm{Z} \\ &= \dim \mathrm{Alt}^{s-1}_{\mathbb{S}} \mathrm{Z} \quad \text{by our induction assumption} \\ &< g - 1 \quad \text{by Proposition 8.10 as } s - 1 \leq n. \end{aligned}$$

Since $\dim \mathrm{Z} > 0$, it follows that

$$\begin{aligned} \dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s-1,1)} \cup \mathbb{\Lambda}^{(s-2)}) &< \min\{s \dim \mathrm{Z}, g-1\} \\ &= \dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s-1)} \circ \mathbb{\Lambda}^{(1)}), \end{aligned} \tag{8.2}$$

where the last equality holds by Corollary 5.7 since by definition $\mathbb{\Lambda}^{(1)} = \mathbb{\Lambda}_{\mathrm{Z}}$ and by induction $\dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s-1)}) = \dim \mathrm{Alt}^{s-1}_{\mathbb{S}} \mathrm{Z} = (s-1) \dim \mathrm{Z}$. Note that the corollary does not require integrality of the occurring clean cycles; we only need that each irreducible component of their support has finite Gauss map, which follows from our assumption that the Gauss map $\mathbb{\Lambda}_{\mathrm{Z}} \to \mathbb{P}_{\mathrm{A}}$ is a finite morphism. Comparing (8.1) and (8.2) we conclude that

$$\dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s)}) = \min\{s \dim \mathrm{Z}, g-1\}.$$

But $\dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s)}) < g-1$ again by Proposition 8.10 because $s \leq n$. Hence, it follows that

$$\dim \mathrm{pr}_{\mathrm{A}}(\mathbb{\Lambda}^{(s)}) = s \dim \mathrm{Z},$$

which completes the induction step. □

8.5. Spin representations do not occur in small dimension

It remains to show that smooth nondivisible subvarieties of small enough dimension are not of spin type. Recall the setting of Theorem 8.7. We are given an integral subvariety $\mathrm{X} \subset \mathrm{A}$ and a reduced symmetric effective cycle Z on A such that

- the subvariety $X \subset A$ is smooth and nondivisible,
- the Gauss maps γ_X, γ_Z are finite, and $\deg(\gamma_Z) = 2n$ for some $n \geq 2$,
- we have $\mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathbb{A}_Z = \mathbb{A}_{[e](X)}$ for some $e \geq 1$ and suitable $\varepsilon \in \{+, -, \varnothing\}$.

Here we make the convention that the labels $\varepsilon = \pm$ will only be used if γ_Z has even monodromy, so that the corresponding cycles are defined. We want to show that in the above situation $n \dim Z \geq \dim A$.

Proof of Theorem 8.7
Suppose by contradiction that $n \dim Z < \dim A$. Let Z' be an irreducible component of maximal dimension in Z. The finiteness of the Gauss map implies by Theorem 2.8 that $Z' \subset A$ is nondegenerate. Hence, Corollary 7.10 gives

$$\frac{\deg(\gamma_{Z'}) + 2}{2} \dim Z \geq \dim A.$$

This is not quite strong enough to contradict our assumption, but it allows us to reduce to the case of symmetric components. If $Z' \neq -Z'$ were not symmetric, then the effective symmetric cycle Z would contain the two distinct components $\pm Z'$ and these two components clearly have the same Gauss degree, which would lead to the estimate

$$2n = \deg(\gamma_Z) \geq \deg(\gamma_{Z'}) + \deg(\gamma_{-Z'}) = 2\deg(\gamma_{Z'}).$$

The previous inequality then leads to

$$\frac{n+2}{2} \dim Z \geq \dim A$$

and for $n \geq 2$ this contradicts our assumption that $n \dim Z < \dim A$.

So for the remainder of the proof we will assume that $Z' = -Z'$ is symmetric. Since by assumption $n \dim Z' < \dim A$, it follows from Proposition 7.9 that the morphism

$$\mathrm{pr}_{Z',n} : \mathrm{Supp}\, \mathbb{A}^{[n]}_{Z'} \longrightarrow Z'^n$$

is surjective. Let $\mathbb{A}' \subset \mathrm{Supp}\, \mathbb{A}^{[n]}_{Z'}$ be an irreducible component dominating Z'^n. This component is not contained in the union $\Delta \cup \Delta^- \subset A^n \times \mathbb{P}_A$ of the big diagonal and the big antidiagonal, hence

$$\mathbb{A}' \subset \mathrm{Supp}\, \mathbb{A}^{[n]}_{Z,\mathbb{S}}.$$

If the Gauss map has even monodromy, then moreover $\mathbb{A}^{[n]}_{Z,\mathbb{S}} = \mathbb{A}^{[n]}_{Z,\mathbb{S},+} + \mathbb{A}^{[n]}_{Z,\mathbb{S},-}$ and, by irreducibility,

$$\mathbb{A}' \subset \mathrm{Supp}\, \mathbb{A}^{[n]}_{Z,\mathbb{S},\delta} \quad \text{for some } \delta \in \{+, -\}.$$

To adjust the signs, use the involution $\varphi := \mathrm{id}^{n-1} \times [-1] \colon \mathrm{A}^n \to \mathrm{A}^n$ and consider the subvariety

$$\mathbb{\Lambda} \subset \operatorname{Supp} \mathbb{\Lambda}^{[n]}_{\mathrm{Z},\mathbb{S},\varepsilon} \qquad \text{defined by} \qquad \mathbb{\Lambda} := \begin{cases} \varphi_* \mathbb{\Lambda}' & \text{if } \varepsilon = -\delta, \\ \mathbb{\Lambda}' & \text{otherwise.} \end{cases}$$

Since $\mathrm{Z}' = -\mathrm{Z}'$, we still have

$$\mathrm{pr}_{\mathrm{A}^n}(\mathbb{\Lambda}) = (\mathrm{Z}')^{n-1} \times (\pm \mathrm{Z}') = (\mathrm{Z}')^n.$$

By construction, $\mathbb{\Lambda}$ is an irreducible component of $\operatorname{Supp} \mathbb{\Lambda}^{[n]}_{\mathrm{Z},\mathbb{S},\varepsilon}$ and by assumption we have

$$\mathbb{\Lambda}_{\mathrm{Y}} = \mathrm{Alt}^n_{\mathbb{S},\varepsilon} \mathbb{\Lambda}_{\mathrm{Z}} := \frac{1}{\mathrm{N}(\alpha)} \sigma_{\alpha *}(\mathbb{\Lambda}^{[n]}_{\mathrm{Z},\mathbb{S},\varepsilon}) \quad \text{for } \alpha = (1^n),$$

so the irreducibility of the left-hand side forces $\mathbb{\Lambda}_{\mathrm{Y}} = \operatorname{Supp} \sigma_{\alpha *}(\mathbb{\Lambda})$. This gives a commutative diagram

$$\begin{array}{ccccc} (\mathrm{Z}')^n & \xleftarrow{\mathrm{pr}_{\mathrm{Z},r}} & \mathbb{\Lambda} & \xrightarrow{\gamma_{\mathrm{Z},r}} & \mathbb{P}_{\mathrm{A}} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_\alpha} & & \| \\ \mathrm{Y} & \xleftarrow[\mathrm{pr}_{\mathrm{Y}}]{} & \mathbb{\Lambda}_{\mathrm{Y}} & \xrightarrow[\gamma_{\mathrm{Y}}]{} & \mathbb{P}_{\mathrm{A}} \end{array}$$

where σ is the sum morphism, which is generically finite by Lemma 2.7.

In fact, σ must be finite by the same argument as in Proposition 7.12. The fibers of pr_{Y} are of pure dimension $\mathrm{N} = \operatorname{codim}_{\mathrm{A}} \mathrm{Y} - 1$ by Corollary 2.5. Moreover, σ_α is finite since the rightmost square in the above diagram commutes and since in that square $\gamma_{\mathrm{Z},r}$ and γ_{Y} are finite morphisms by our finiteness assumptions on Gauss maps. So all fibers of $\mathrm{pr}_{\mathrm{Y}} \circ \sigma_\alpha \colon \mathbb{\Lambda}_{\mathrm{Z},r} \to \mathrm{Y}$ are of dimension N. Since σ is generically finite, it follows from the commutativity of the leftmost square in the above diagram that the generic fiber of the morphism $\mathrm{pr}_{\mathrm{Z},r}$ has dimension N as well. We can now argue by contradiction. Any positive-dimensional fiber of σ would give rise to a fiber of $\mathrm{pr}_{\mathrm{Y}} \circ \sigma_\alpha$ of dimension at least $\mathrm{N} + 1$, by semicontinuity of dimension of fibers for proper morphisms [48, Lemma 0D4I]. This shows that σ is finite.

But $\mathrm{Z}' = -\mathrm{Z}'$ is symmetric and $\dim \mathrm{Z}' > 0$, hence looking at antidiagonals one sees that the sum morphism σ cannot be finite, which is a contradiction. □

Appendix A. Reduction to the complex case

For reference, we include the following well-known fact about ℓ-adic constructible sheaves with coefficients in $\mathbb{F} = \overline{\mathbb{Q}}_\ell$ on varieties over an algebraically closed field k of characteristic 0.

LEMMA A.1
Let $k \subset \mathrm{K}$ be an extension of algebraically closed fields of characteristic 0, and let X be a variety over k. Then the base change functor

$$(-)_{\mathrm{K}} \colon \mathrm{D}_c^b(\mathrm{X}, \mathbb{F}) \longrightarrow \mathrm{D}_c^b(\mathrm{X}_{\mathrm{K}}, \mathbb{F})$$

is fully faithful. Moreover, for every $\mathrm{P} \in \mathrm{D}_c^b(\mathrm{X}_{\mathrm{K}}, \mathbb{F})$ there exists a subfield $k' \subset \mathrm{K}$ which is the algebraic closure of a finitely generated extension of k such that P is in the essential image of the functor

$$(-)_{\mathrm{K}} \colon \mathrm{D}_c^b(\mathrm{X}_{k'}, \mathbb{F}) \longrightarrow \mathrm{D}_c^b(\mathrm{X}_{\mathrm{K}}, \mathbb{F}).$$

Proof
For the full faithfulness, consider two complexes $\mathrm{P}, \mathrm{P}' \in \mathrm{D}_c^b(\mathrm{X}, \mathbb{F})$. To see the isomorphism $\mathrm{Hom}_{\mathrm{D}_c^b(\mathrm{X},\mathbb{F})}(\mathrm{P}, \mathrm{P}') \simeq \mathrm{Hom}_{\mathrm{D}_c^b(\mathrm{X}_{\mathrm{K}},\mathbb{F})}(\mathrm{P}_{\mathrm{K}}, \mathrm{P}'_{\mathrm{K}})$ take $\mathrm{Q} = \mathrm{R}\mathscr{H}om(\mathrm{P}, \mathrm{P}')$ in the isomorphism

$$\mathrm{H}^{\bullet}(\mathrm{X}, \mathrm{Q}) \xrightarrow{\sim} \mathrm{H}^{\bullet}(\mathrm{X}_{\mathrm{K}}, \mathrm{Q}_{\mathrm{K}})$$

which is obtained by base change (see, e.g., [44, Corollary VI.4.3] for the case of étale torsion sheaves; the case of ℓ-adic sheaf complexes then follows formally).

Now let $\mathrm{P} \in \mathrm{D}_c^b(\mathrm{X}_{\mathrm{K}}, \mathbb{F})$. We want to show that it descends to a subfield $k' \subset \mathrm{K}$ which is the algebraic closure of a finitely generated extension of k. We use induction on the number of nonvanishing cohomology sheaves. Let $m \in \mathbb{Z}$ be maximal with $\mathscr{H}^m(\mathrm{P}) \neq 0$, and consider the triangle

$$\tau_{<m}(\mathrm{P}) \longrightarrow \mathrm{P} \longrightarrow \mathscr{H}^m(\mathrm{P})[-m] \longrightarrow .$$

Rotating the triangle, we obtain

$$\mathrm{P} \simeq \mathrm{cone}\big(\mathscr{H}^m(\mathrm{P})[1-m] \to \tau_{<m}(\mathrm{P})\big).$$

If the source and the target of a morphism descend to a given subfield, then so does the morphism by full faithfulness, and hence also the cone descends to the same subfield. By induction, it therefore suffices to discuss the case where P is a constructible $\overline{\mathbb{Q}}_\ell$-sheaf. Then by [14, Rapport, Proposition 2.5] there is an open dense subset of X_{K} on which P is smooth. Let $k' \subset \mathrm{K}$ be the algebraic closure of a finitely generated extension of k such that the open dense subset has the form $j_{\mathrm{K}} \colon \mathrm{U}_{\mathrm{K}} \to \mathrm{X}_{\mathrm{K}}$ for some open $j : \mathrm{U} \hookrightarrow \mathrm{X}_{k'}$. Looking at the adjunction morphism

$$j_{\mathrm{K}!} j_{\mathrm{K}}^*(\mathrm{P}) \longrightarrow \mathrm{P}$$

and arguing by induction on $\dim \mathrm{Supp}(\mathrm{P})$, it suffices to show that $j_{\mathrm{K}}^*(\mathrm{P}) \simeq \mathrm{L}_{\mathrm{K}}$ for some local system L on U. But this is clear because of the equivalence between ℓ-adic

local systems and representations of the étale fundamental group (see [14, Rapport, Proposition 2.4]) and the invariance of the geometric étale fundamental group (see [24, Exposé XIII, Proposition 4.6]). □

Appendix B. Symmetric powers of varieties

Let X be a variety over an algebraically closed field k of characteristic 0, and fix an integer $n \geq 1$. The *n-fold symmetric product* of X is defined as the categorical quotient

$$\mathrm{Sym}^n \mathrm{X} := \mathrm{X}^n / \mathfrak{S}_n$$

of $\mathrm{X}^n = \mathrm{X} \times \cdots \times \mathrm{X}$ by the permutation action of the symmetric group $\mathfrak{S}_n$. This quotient exists, for instance, if X is quasiprojective (see [7, Section 9.3, p. 253]). In that case, we denote by

$$\pi_{\mathrm{X}} \colon \mathrm{X}^n \longrightarrow \mathrm{Sym}^n \mathrm{X}$$

the quotient morphism. Let $\mathrm{U} \subset \mathrm{X}^n$ be the complement of the big diagonal, that is, the open subset of all n-tuples of pairwise distinct points. Then $\pi_{\mathrm{X}}(\mathrm{U})$ is open in $\mathrm{Sym}^n \mathrm{X}$ and $\pi_{\mathrm{X}} \colon \mathrm{U} \to \pi_{\mathrm{X}}(\mathrm{U})$ is a principal $\mathfrak{S}_n$-bundle (see [23, Exposé V, Théorème 4.1]), in particular,

$$\deg \pi_{\mathrm{X}} = n!.$$

On the other hand, π_{X} is not étale at any point $x \in (\mathrm{X}^n \smallsetminus \mathrm{U})(k)$.

PROPOSITION B.1

Let X *be a quasiprojective variety. If* X *is reduced, irreducible, integral, or normal, then the respective property holds also for* $\mathrm{Sym}^n \mathrm{X}$.

Proof

If X has one of the stated properties, then X^n has the same property since k is algebraically closed. Therefore, the claim follows from the fact that the properties are stable under categorical quotients (see [46, p. 5]). □

PROPOSITION B.2

If $f \colon \mathrm{X}' \to \mathrm{X}$ *is a finite birational morphism between integral quasiprojective varieties, then* $\mathrm{Sym}^n f \colon \mathrm{Sym}^n \mathrm{X}' \to \mathrm{Sym}^n \mathrm{X}$ *is finite birational.*

Proof

In the commutative square

$$\begin{array}{ccc} \mathrm{X}'^n & \xrightarrow{f^n} & \mathrm{X}^n \\ {\scriptstyle \pi_{\mathrm{X}'}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{X}}} \\ \mathrm{Sym}^n \mathrm{X}' & \xrightarrow{\mathrm{Sym}^n f} & \mathrm{Sym}^n \mathrm{X} \end{array}$$

the morphisms f^n, $\pi_{\mathrm{X}'}$, and π_{X} are finite, hence $\mathrm{Sym}^n f$ is also finite. Since f^n is birational, we obtain

$$\deg(\mathrm{Sym}^n f \circ \pi_{\mathrm{X}'}) = \deg(\pi_{\mathrm{X}} \circ f^n) = n! = \deg(\pi_{\mathrm{X}}),$$

which implies that the morphism $\mathrm{Sym}^n f$ has degree 1 as required. □

PROPOSITION B.3
Let X *be a positive-dimensional integral quasiprojective variety, and let* $n \geq 2$ *be an integer. Then*

$$\mathrm{Sym}^n \mathrm{X} \textit{ is smooth} \quad \Longleftrightarrow \quad \mathrm{X} \textit{ is a smooth curve}.$$

Proof
It is well known that for any smooth curve X the symmetric powers $\mathrm{Sym}^n \mathrm{X}$ are smooth (see [7, p. 255]). Conversely, assume that $\mathrm{Sym}^n \mathrm{X}$ is smooth. First, we show that $\dim \mathrm{X} = 1$. For this, we may replace X by its smooth locus and thus assume that X is smooth. By Nagata–Zariski purity (see [24, Théorème X.3.1]), the branch locus B of $\pi_{\mathrm{X}} \colon \mathrm{X}^n \to \mathrm{Sym}^n \mathrm{X}$ is empty or a divisor in $\mathrm{Sym}^n \mathrm{X}$. On the other hand, the morphism π_{X} is ramified at a k-point $x = (x_1, \ldots, x_n)$ of X^n if and only if $x_i = x_j$ for some $i \neq j$, that is, if and only if x lies in the big diagonal Δ_n of X^n. Since $n \geq 2$, the big diagonal is nonempty, so that the branch locus is a divisor and thus

$$n \dim \mathrm{X} - 1 = \dim \mathrm{B} = \dim \Delta_n = (n-1) \dim \mathrm{X},$$

hence $\dim \mathrm{X} = 1$. Having shown that X is a curve, we now verify that it must be smooth. Suppose to the contrary that there exists a singular point $x_1 \in \mathrm{X}(k)$. Pick pairwise distinct points $x_2, \ldots, x_n \in \mathrm{X}(k) \smallsetminus \{x_1\}$. Then the morphism π_{X} is étale at $x = (x_1, x_2, \ldots, x_n)$, so $\pi_{\mathrm{X}}(x)$ is a singular point in $\mathrm{Sym}^n \mathrm{X}$, a contradiction. □

In Section 1.3, we needed a criterion for a birational morphism from a symmetric power of a smooth variety to another smooth variety to be an isomorphism. The proof is naturally cast for certain singularities. To define them, recall that for a coherent sheaf $\mathscr{F}$ on a variety V and an integer $m \geq 1$ we write $\mathscr{F}^{[m]}$ for the reflexive hull of $\mathscr{F}^{\otimes m}$. A reflexive sheaf $\mathscr{F}$ on V of generic rank 1 is a $\mathbb{Q}$-*line bundle* if there is $m \geq 1$ such that $\mathscr{F}^{[m]}$ is a line bundle. When V is proper, such a sheaf $\mathscr{F}$ is *nef* if the line bundle $\mathscr{F}^{[m]}$ is. For V normal, the *canonical sheaf* $\mathscr{K}_{\mathrm{V}}$ is defined as the

pushforward to V of the canonical bundle on V^{reg} and is reflexive of generic rank 1. By definition, V is $\mathbb{Q}$-*Gorenstein* if it is normal and the canonical sheaf is a $\mathbb{Q}$-line bundle. This is the case if V is $\mathbb{Q}$-*factorial*, that is, if V is normal and any reflexive sheaf of generic rank 1 on V is a $\mathbb{Q}$-line bundle. The singularities of V are *terminal* if V is $\mathbb{Q}$-Gorenstein and the pullback of any local section of $\mathscr{K}_{\mathrm{V}}^{[m]}$ for any $m \geq 1$ vanishes along all the components of the exceptional divisor of any resolution of V (see [30, Definition 2.34]). With this terminology we have the following.

PROPOSITION B.4
Let X *and* W *be* $\mathbb{Q}$*-factorial normal, integral projective varieties. Suppose that* W *has terminal singularities. If* $\mathscr{K}_{\mathrm{X}}$ *is nef and* $\dim \mathrm{X} \geq 2$*, then any proper birational morphism* $f\colon \operatorname{Sym}^n \mathrm{X} \to \mathrm{W}$ *is an isomorphism.*

Proof
The symmetric product $\mathrm{S} := \operatorname{Sym}^n \mathrm{X}$ of X is normal by Proposition B.1 and $\mathbb{Q}$-factorial by [30, Lemma 5.16]. The hypothesis $\dim \mathrm{X} \geq 2$ implies that the quotient morphism $\pi\colon \mathrm{X}^n \to \mathrm{S}$ is unramified in codimension 1, hence the natural morphism $(\pi^* \mathscr{K}_{\mathrm{S}})^{\vee\vee} \to \mathscr{K}_{\mathrm{X}^n}$ is an isomorphism. Thus $\mathscr{K}_{\mathrm{S}}$ is nef since $\mathscr{K}_{\mathrm{X}^n}$ is so. To conclude, apply Lemma B.5 below with $\mathrm{V} = \mathrm{S}$. □

To keep track of the arguments that enter the proof, we state the lemma in a generality which is slightly broader than actually needed. To do this, for a proper morphism $f\colon \mathrm{V} \to \mathrm{W}$ a $\mathbb{Q}$-line bundle $\mathscr{L}$ on V is said to be f*-nef* if the restriction of $\mathscr{L}^{[m]}$ to any fiber of f is nef, where $m \geq 1$ is such that $\mathscr{L}^{[m]}$ is a line bundle.

LEMMA B.5
Let $f\colon \mathrm{V} \to \mathrm{W}$ *be a proper birational morphism between normal quasiprojective varieties. Suppose that* V *is* $\mathbb{Q}$*-Gorenstein,* $\mathscr{K}_{\mathrm{V}}$ *is* f*-nef, and* W *is* $\mathbb{Q}$*-factorial with terminal singularities. Then* f *is an isomorphism.*

Proof
By assumption, the varieties V and W are $\mathbb{Q}$-Gorenstein, thus there is an integer $m \geq 1$ such that $\mathscr{K}_{\mathrm{V}}^{[m]}$ and $\mathscr{K}_{\mathrm{W}}^{[m]}$ are line bundles. Write

$$\mathscr{K}_{\mathrm{V}}^{[m]} = f^* \mathscr{K}_{\mathrm{W}}^{[m]} \otimes \mathscr{O}_{\mathrm{V}}(\mathrm{E})$$

for some Cartier divisor E on V. Since W has terminal singularities, the divisor E is effective and its support is exactly the divisorial part of the exceptional locus of f. Moreover, E is f-nef; indeed, for any projective curve $\mathrm{C} \subset \mathrm{V}$ contracted by f we have

$$\mathrm{E}.\mathrm{C} = \mathscr{K}_{\mathrm{V}}^{[m]}.\mathrm{C} - f^{*}\mathscr{K}_{\mathrm{W}}^{[m]}.\mathrm{C} = \mathscr{K}_{\mathrm{V}}^{[m]}.\mathrm{C} \geq 0$$

because $\mathscr{K}_{\mathrm{V}}$ is f-nef. Therefore, $-\mathrm{E}$ is effective by [30, Lemma 3.39(1)], hence trivial because E is effective. It follows that the exceptional set of f has no divisorial part. On the other hand, W is $\mathbb{Q}$-factorial, so the exceptional locus of f is pure of codimension 1 (see [12, Section 1.40]). Thus, f is an isomorphism. □

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Javanpeykar
Institute for Mathematics, Astrophysics and Particle Physics, Radboud University Nijmegen, Nijmegen, Netherlands; apeykar@science.ru.nl

Krämer
Institut für Mathematik, Humboldt Universität zu Berlin, Berlin, Germany; thomas.kraemer@math.hu-berlin.de

Lehn
Fakultät für Mathematik, Ruhr-Universität Bochum, Bochum, Germany; christian.lehn@rub.de

Maculan
Institut de Mathématiques de Jussieu, Sorbonne Université, Paris, France;
marco.maculan@imj-prg.fr