Marco Maculan

DIOPHANTINE APPLICATIONS OF GEOMETRIC INVARIANT THEORY

Marco Maculan
Institut Mathématique de Jussieu, Université Pierre et Marie Curie,
4 place Jussieu, 75005 Paris.
E-mail : `marco.maculan@imj-prg.fr`
Url : `http://webusers.imj-prg.fr/~marco.maculan/`

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Abstract. — This text consists of two parts. In the first one we present a proof of Thue-Siegel-Roth's Theorem (and its more recent variants, such as those of Lang for number fields and that "with moving targets" of Vojta) as an application of Geometric Invariant Theory (GIT). Roth's Theorem is deduced from a general formula comparing the height of a semi-stable point and the height of its projection on the GIT quotient. In this setting, the role of the zero estimates appearing in the classical proof is played by the geometric semi-stability of the point to which we apply the formula.

In the second part we study heights on GIT quotients. We generalise Burnol's construction of the height and refine diverse lower bounds of the height of semi-stable points established to Bost, Zhang, Gasbarri and Chen. The proof of Burnol's formula is based on a non-archimedean version of Kempf-Ness theory (in the framework of Berkovich analytic spaces) which completes the former work of Burnol.

Résumé (***Applications diophantiennes de la théorie géométrique des invariants***)

Ce texte est constitué de deux parties. Dans la première nous présentons une preuve du théorème de Thue-Siegel-Roth (et des variantes plus récentes, comme celle de Lang pour le corps de nombres et celle *with moving targets* de Vojta) basée sur la théorie géométrique des invariants (GIT). Le théorème de Roth est déduit d'une formule reliant la hauteur d'un point semi-stable et la hauteur de sa projection dans le quotient GIT. Dans ce cadre, le rôle du « lemme des zéros » présent dans la preuve classique est joué par la semi-stabilité géométrique du point auquel on applique la formule.

Dans la deuxième partie nous étudions la hauteur sur les quotients GIT. Nous généralisons la construction de Burnol de cette hauteur et nous améliorons plusieurs minorations de la hauteur de point semi-stables précédemment établies par Bost, Zhang Gasbarri et Chen. La preuve de la formule de Burnol porte sur une version non-archimédienne de la théorie de Kempf-Ness (dans le langage de la géométrie analytique de Berkovich), qui complète le travail antérieur de Burnol.

CONTENTS

INTRODUCTION

In its original form, Roth's Theorem states that given a real algebraic number $\alpha \in \mathbf{R}$ which is not rational and a real number $\kappa > 2$, there exist only finitely many rational numbers $p/q \in \mathbf{Q}$ such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{|q|^{\kappa}}$$

where p, q are coprime integers.

The general strategy to prove Roth's Theorem stems back to the work of Thue. The main ingredient is the construction of an "auxiliary" polynomial in several variables f which vanishes at high order at $(\alpha, \ldots, \alpha)$: the crucial step is to prove that it does not vanish too much at rational points which "approximate" $(\alpha, \ldots, \alpha)$.

The original argument of Roth (generalizing those of Thue, Siegel and Gel'fond) involves arithmetic considerations about the height of the rational approximations. On the other hand, in the work of Dyson — who proved an earlier version of Roth's Theorem — the non-vanishing result (usually called "Dyson's Lemma") takes place over the complex numbers: being free from arithmetic constraints, it is said to be of geometric nature. The task to generalize Dyson's Lemma from 2 to several variables was accomplished by Esnault-Viehweg [**EV84**]; afterwards Nakamaye [**Nak99**] was able to give a proof of it relying on a variant of Faltings' Product Theorem and "elementary" concepts of intersection theory.

The advantage of having a geometric proof of Dyson's Lemma was exploited by Bombieri in the remarkable paper [**Bom82**]: he showed that these methods lead to new effective results in diophantine approximation available before only through the linear forms of logarithms of Baker.

Using an arithmetic variant of the Product Theorem, Faltings and Wüstholz [**FW94**] gave a new proof of Schmidt's Subspace Theorem, sensibly different from the original one. Their Zero Lemma, as in Roth and Schmidt, is of arithmetic nature. Their proof involves a notion of semi-stability for filtered vector spaces (see also [**Fal95**]). The role played by semi-stability is anyway rather different from the one in the present paper: here it collects all the geometric informations coming from

Dyson's Lemma (hence from the Product Theorem); in their paper it represents a combinatorial assumption that permits to perform an inductive step based on the Product Theorem.

Inspired by work of Osgood [**Osg85**] and Steinmetz [**Ste86**] Vojta proved in [**Voj96**] a generalised version of Roth's Theorem — called "with moving targets" — where the algebraic point can vary along with the rational approximations. Its proof is based on the use of Schmidt's Subspace Theorem. However it has been noticed by Bombieri and Gubler [**BG06**, Theorem 6.5.2 and §6.6] that the techniques employed to prove Roth's Theorem suffice to prove the version "with moving targets" without recurring to Schmidt's Subspace Theorem.

The study of the interplay between Geometric Invariant Theory and height functions (in the context of Arakelov geometry) has started more than twenty years ago with the work of several authors.

Burnol [**Bur92**] defined a height function on the GIT quotient of a projective space by a reductive group and he expressed it in terms as the sum of the height on the projective space and of local error terms.

Bost [**Bos94, Bos96a**], Zhang [**Zha96a**] and Soulé [**Sou95**] proved several lower bounds on the height of semi-stable points (in some explicit representations) and used them to give lower bounds on the height of semi-stable varieties (*e.g.* semi-stable curves, abelian varieties...).

Gasbarri [**Gas00**] was able to free the arguments of Bost and Zhang from the constraint of knowing explicitly the representation of GL_n. Chen [**Che09**] proved an explicit variant of this type of lower bounds and used it — inspired by work of Ramanan-Ramanathan [**RR84**] and Totaro [**Tot96**] — to study the semi-stability of the tensor product of hermitian vector bundles over a ring of integers.

In the first chapter of this text, we show how a simple version of this general lower bound on the height of (geometrically) semi-stable point leads to a general lower bound on the height of suitable families of points $(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n)$ in $\mathbf{P}^1(K)^n$ and $\mathbf{P}^1(K')^n$ respectively to the diverse v-adic distances (where K is a number field and K' is an extension of degree ≥ 2). This lower bound, which constitutes the main result of the present note, has been established in the case $n = 2$ by Bombieri [**Bom82**, Theorem 2], is effective and implies the version of Roth's Theorem we present here.

Let us discuss briefly the content of the chapters. For the precise statement of the results we refer the reader to the first section of each chapter.

In Chapter 1 we introduce the basic tools of Geometric Invariant Theory that are needed in order to deduce Roth's Theorem. The results expounded in this chapter will be refined in Chapter 4, but we preferred to give a succinct and self-containted account for the reader interested to the proof of Roth's Theorem in Chapter 2. It may also serve the reader interested in Chapters 3 and 4 as an introduction to the results to be improved.

In Chapter 2 we prove Roth's Theorem (along with some more recent variants) as a consequence of the Fundamental Formula in Chapter 2 (applied to a suitably chosen "moduli problem").

In Chapter 3 we investigate a variant of the results of Kempf-Ness [**KN79**] for complex and non-archimedean geometry.

In Chapter 4 we study deeply the height on the quotient and prove (some of) the desired refinements of the results of Chapter 1.

An effort has been made in order to keep the different chapters independent one from the other. In Chapter 2 the only references to Chapter 1 are in section 2.3, while in Chapter 4 the needed facts from Chapter 3 are recalled in section 1. We invite to read the chapters separately.

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CHAPTER 0

CONVENTIONS

Here is a list of conventions and definitions that are used throughout the text.

0.1. For a ring A and a positive integer n, let $A^{n\vee} := \mathrm{Hom}_A(A^n, A)$. The projective line $\mathbf{P}^1_A$ over the ring A is the A-scheme

$$\mathbf{P}^1_A = \mathrm{Proj}(\mathrm{Sym}\, A^{2\vee}).$$

Rather generally, if M is an A-module of finite type, then $M^\vee = \mathrm{Hom}_A(M, A)$ and

$$\mathbf{P}(M) := \mathrm{Proj}(\mathrm{Sym}\, M^\vee).$$

0.2. Let A be a ring, M be an A-module and n be a negative integer. Set

$$M^{\otimes n} := M^{\vee \otimes -n} = \mathrm{Hom}_A(M, A)^{\otimes -n}.$$

0.3. Hermitian Norms. — Let E, F be finite dimensional complex vector spaces equipped respectively with hermitian norms $\|\cdot\|_E$, $\|\cdot\|_F$ and associated hermitian forms $\langle -,-\rangle_E$, $\langle -,-\rangle_F$. Let r be a non-negative integer.

- On the tensor power $E \otimes_{\mathbf{C}} F$ let $\|\cdot\|_{E\otimes F}$ be the norm associated to the hermitian form
$$\langle v \otimes w, v' \otimes w' \rangle_{E\otimes F} := \langle v, v' \rangle_E \cdot \langle w, w' \rangle_F$$
where $v, v' \in E$ and $w, w' \in F$.
- On the r-th symmetric power $\mathrm{Sym}^r E$ let $\|\cdot\|_{\mathrm{Sym}^r E}$ be the quotient norm with respect to the canonical surjection $E^{\otimes r} \to \mathrm{Sym}^r E$. If $e_1, \dots, e_n$ denotes an orthonormal basis of E, where $n = \dim_{\mathbf{C}} E$, for every n-uple of non-negative integers $(r_1, \dots, r_n)$ such that $r_1 + \cdots + r_n = r$:
$$\|e_1^{r_1} \cdots e_n^{r_n}\|_{\mathrm{Sym}^r E} = \binom{r}{r_1, \dots, r_n}^{-1/2} := \left(\frac{r!}{r_1! \cdots r_n!}\right)^{-1/2}.$$
This norm is hermitian and it is sub-multiplicative in the following sense: if $f \in \mathrm{Sym}^r E$ and $g \in \mathrm{Sym}^s E$, then
$$\|fg\|_{\mathrm{Sym}^{r+s} E} \le \|f\|_{\mathrm{Sym}^r E} \|g\|_{\mathrm{Sym}^s E}.$$

Let us also mention that the norm $\|\cdot\|_{\mathrm{Sym}^r E}$ is bigger than the sup-norm on the unit ball: for $f \in \mathrm{Sym}^r E$,

$$\|f\|_{\sup} := \sup_{0 \neq x \in E^\vee} \frac{|f(x)|}{\|x\|_{E^\vee}^r} \leq \|f\|_{\mathrm{Sym}^r E}.$$

– On the r-th external power $\bigwedge^r E$ let $\|\cdot\|_{\bigwedge^r E}$ be the norm associated to the hermitian form

$$\langle v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r \rangle_{\bigwedge^r E} = \det\left(\langle v_i, w_j \rangle_E : i, j = 1, \ldots, r\right)$$

where $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ are elements of E. With this notation Hadamard inequality [(1)] reads:

$$\|v_1 \wedge \cdots \wedge v_r\|_{\bigwedge^r E} \leq \prod_{i=1}^r \|v_i\|_E. \tag{0.3.1}$$

The hermitian norm $\|\cdot\|_{\bigwedge^r E}$ is *not* the quotient norm with respect to the canonical surjection $E^{\otimes r} \to \bigwedge^r E$, but it is $\sqrt{r!}$ times the quotient norm (see [**Che09**, Lemma 4.1]).

– For every linear homomorphism $\varphi \colon E \to F$ write φ^* for the adjoint homomorphism (with respect to the hermitian norms $\|\cdot\|_E$ and $\|\cdot\|_F$). On the vector space $\mathrm{Hom}_{\mathbf{C}}(E, F)$ let $\|\cdot\|_{\mathrm{Hom}(E,F)}$ be the hermitian norm associated to the hermitian form

$$\langle \varphi, \psi \rangle_{\mathrm{Hom}(E,F)} := \mathrm{Tr}(\varphi \circ \psi^*)$$

where $\varphi, \psi \in \mathrm{Hom}_{\mathbf{C}}(E, F)$. If $e_1, \ldots, e_n$ is an orthonormal basis of E,

$$\|\varphi\|_{\mathrm{Hom}(E,F)} := \sqrt{\|\varphi(e_1)\|_F^2 + \cdots + \|\varphi(e_n)\|_F^2}.$$

With these conventions the isomorphism $E^\vee \otimes_{\mathbf{C}} F \to \mathrm{Hom}_{\mathbf{C}}(E, F)$ is isometric.

0.4. Non-archimedean norms. — Let K be a field complete with respect to a non-archimedean absolute value and let $\mathfrak{o}$ be its ring of integers. In order to do some computations it is convenient to interpret $\mathfrak{o}$-modules as K-vector spaces endowed with a non-archimedean norm. More precisely, for every torsion free $\mathfrak{o}$-module $\mathcal{E}$ denote by $E := \mathcal{E} \otimes_{\mathfrak{o}} K$ its generic fiber and consider the following norm: for every $v \in E$ set

$$\|v\|_{\mathcal{E}} := \inf\{|\lambda| : \lambda \in K^\times, v/\lambda \in \mathcal{E}\}.$$

The norm $\|\cdot\|_{\mathcal{E}}$ is non-archimedean and its construction is compatible with operations on $\mathfrak{o}$-modules: for instance, if $\varphi \colon \mathcal{E} \to \mathcal{F}$ is an injective homomorphism with flat cokernel (resp. surjective homomorphism) between torsion free $\mathfrak{o}$-modules then the norm $\|\cdot\|_{\mathcal{E}}$ induced on $E := \mathcal{E} \otimes_{\mathfrak{o}} K$ (resp. the norm $\|\cdot\|_{\mathcal{F}}$ induced on $F := \mathcal{F} \otimes_{\mathfrak{o}} K$) is the restriction of the norm $\|\cdot\|_{\mathcal{F}}$ on F (resp. is the quotient norm deduced from $\|\cdot\|_{\mathcal{E}}$ and φ, that is, the norm defined by

$$w \mapsto \inf_{\varphi(v) = w} \|v\|_{\mathcal{E}}$$

1. This inequality is Hadamard's bound of the volume of a basis of a Euclidean space and not Hermite-Hadamard's inequality concerning convex functions.

for every element w of F.)

For a non-negative integer $r \geq 0$, the norm on symmetric powers $\operatorname{Sym}^r \mathcal{E}$ (resp. on exterior powers $\bigwedge^r \mathcal{E}$) is the norm deduced by the one on the r-th tensor power $\mathcal{E}^{\otimes r}$ through the canonical surjection $\mathcal{E}^{\otimes r} \to \operatorname{Sym}^r \mathcal{E}$ (resp. $\mathcal{E}^{\otimes r} \to \bigwedge^r \mathcal{E}$). In particular, it is sub-multiplicative (resp. Hadamard inequality holds).

0.5. Normalisation of places. — For a number field K let $\mathfrak{o}_K$ be its ring of integers and V_K its the set of its places. If v is a place of K, let K_v be the completion of K with respect to v and $\mathbf{C}_v$ the completion of an algebraic closure of K_v (with respect the unique absolute value extending v). A non-archimedean place v extending a p-adic one is normalized by

$$|p|_v = p^{-[K_v:\mathbf{Q}_p]}.$$

0.6. Hermitian vector bundles and Arakelov degree. — Let K be a number field, $\mathfrak{o}_K$ its ring of integers and V_K its set of places. An hermitian vector bundle $\overline{\mathcal{E}}$ is the data of a flat $\mathfrak{o}_K$-module of finite type $\mathcal{E}$ and, for every complex embedding $\sigma\colon K \to \mathbf{C}$, an hermitian norm $\|\cdot\|_{\mathcal{E},\sigma}$ on the complex vector space $\mathcal{E}_\sigma := \mathcal{E} \otimes_\sigma \mathbf{C}$. These hermitian norms are supposed to be compatible to complex conjugation. For a place $v \in \mathrm{V}_K$ let $\|\cdot\|_{\mathcal{E},v}$ be the norm induced on the K_v-vector space $\mathcal{E}_v := \mathcal{E} \otimes_{\mathfrak{o}_K} K_v$.

If $\overline{\mathcal{E}}$, $\overline{\mathcal{F}}$ are hermitian vector bundles over $\mathfrak{o}_K$, a *homomorphism of hermitian vector bundles* $\varphi\colon \overline{\mathcal{E}} \to \overline{\mathcal{F}}$ is a homomorphism of $\mathfrak{o}_K$-modules such that, for every embedding $\sigma\colon K \to \mathbf{C}$, it decreases the norms: that is, for every $v \in \mathcal{E} \otimes_\sigma \mathbf{C}$,

$$\|\varphi(v)\|_{\mathcal{F},\sigma} \leq \|v\|_{\mathcal{E},\sigma}.$$

If $\overline{\mathcal{L}}$ is an hermitian line bundle, that is an hermitian vector bundle of rank 1, its *degree* is

$$\begin{aligned}\widehat{\deg}(\overline{\mathcal{L}}) &:= \log \#(\mathcal{L}/s\mathcal{L}) - \sum_{\sigma\colon K\to\mathbf{C}} \log\|s\|_{\mathcal{L},\sigma} \\ &= -\sum_{\substack{v\in\mathrm{V}_K\\ \text{finite}}} \log\|s\|_{\mathcal{L},v} - \sum_{\substack{v\in\mathrm{V}_K\\ \text{infinite}}} [K_v:\mathbf{R}]\log\|s\|_{\mathcal{L},v}.\end{aligned}$$

where $s \in \mathcal{L}$ is non-zero. It appears from the second expression that this, according to the Product Formula, does not depend on the chosen section s. For a non-zero hermitian vector bundle $\overline{\mathcal{E}}$ one defines

– its *degree*:
$$\widehat{\deg}\,\overline{\mathcal{E}} := \widehat{\deg}(\textstyle\bigwedge^{\operatorname{rk}\mathcal{E}} \overline{\mathcal{E}});$$
– its *slope*:
$$\hat{\mu}(\overline{\mathcal{E}}) := \frac{\widehat{\deg}(\overline{\mathcal{E}})}{\operatorname{rk}\mathcal{E}};$$
– its *maximal slope*:
$$\hat{\mu}_{\max}(\overline{\mathcal{E}}) := \sup_{0\neq\mathcal{F}\subset\mathcal{E}} \hat{\mu}(\overline{\mathcal{F}}),$$
where the supremum is taken on all non-zero sub-modules $\mathcal{F}$ of $\mathcal{E}$ endowed with the restriction of the hermitian metric on $\mathcal{E}$.

***Proposition 0.1* (Slopes inequality, [Bos96b]).** — *Let $\overline{\mathcal{E}}, \overline{\mathcal{F}}$ be $\mathfrak{o}_K$-hermitian vector bundles and let $\varphi\colon \mathcal{E}\otimes_{\mathfrak{o}_K} K \to \mathcal{F}\otimes_{\mathfrak{o}_K} K$ be an injective homomorphism of K-vector spaces. Then,*

$$\hat{\mu}(\overline{\mathcal{E}}) \leq \hat{\mu}_{\max}(\overline{\mathcal{F}}) + \sum_{v\in \mathrm{V}_K} \log \|\varphi\|_{\sup,v},$$

where, for every place $v \in \mathrm{V}_K$,

$$\|\varphi\|_{\sup,v} := \sup_{0\neq s\in\mathcal{E}_v} \frac{\|\varphi(s)\|_{\mathcal{F},v}}{\|s\|_{\mathcal{E},v}}.$$

0.7. Height function with repsect to a hermitian line bundle. — Let K be a number field, $\mathfrak{o}_K$ its ring of integers and $\mathcal{X}$ a projective flat $\mathfrak{o}_K$-scheme.

A *hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_{\mathcal{L},\sigma}\}_{\sigma\colon K\to\mathbf{C}})$ is the data of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and, for $\sigma\colon K\to\mathbf{C}$, a continuous metric $\|\cdot\|_{\mathcal{L},\sigma}$ on the complex line bundle $\mathcal{L}_{|\mathcal{X}_\sigma(\mathbf{C})}$. The data $\{\|\cdot\|_{\mathcal{L},\sigma}\}_{\sigma\colon K\to\mathbf{C}}$ is supposed compatible with complex conjugation.

Let K' be a finite extension of K, $\mathfrak{o}_{K'}$ its ring of integers and P a K'-point of $\mathcal{X}$. By the valuative criterion of properness, the point P induces a morphism of $\mathfrak{o}_K$-schemes $\varepsilon_P\colon \operatorname{Spec}\mathfrak{o}_{K'} \to \mathcal{X}$. The invertible $\mathfrak{o}_{K'}$-module $\varepsilon_P^*\mathcal{L}$ is endowed with norms deduced from the metric on $\mathcal{L}$: the associated hermitian line bundle on $\mathfrak{o}_{K'}$ is denoted $\varepsilon_P^*\mathcal{L}$.

The *height of P with respect to $\overline{\mathcal{L}}$* is

$$h_{\overline{\mathcal{L}}}(P) := \frac{1}{[K':K]}\widehat{\deg}(\varepsilon_P^*\overline{\mathcal{L}}).$$

More concretely, if $s\in H^0(\mathcal{X},\mathcal{L})$ is a global section not vanishing at P,

$$[K':K]h_{\overline{\mathcal{L}}}(P) = \log\#(\varepsilon_P^*\mathcal{L}/s(P)\varepsilon_P^*\mathcal{L}) - \sum_{\sigma\colon K\to\mathbf{C}} \log\|s\|_{\mathcal{L},\sigma}(P).$$

The real number $h_{\mathcal{L}}(P)$ does not depend on the chosen K', thus one has a well-defined function $h_{\overline{\mathcal{L}}}\colon \mathcal{X}(\bar{\mathbf{Q}})\to\mathbf{R}$ called the *height function with respect to $\overline{\mathcal{L}}$*.

CHAPTER 1

GEOMETRIC INVARIANT THEORY AND ARAKELOV GEOMETRY: BASIC RESULTS

In this chapter we introduce the main tool of Geometric Invariant Theory (the Fundamental Formula, see Theorem 1.6) that we shall apply in Chapter 2 to a specific "moduli problem" in order to get the Roth's Theorem: it is a formula relating the height of a semi-stable point with the height of its projection on the GIT quotient. In this general framework we also state and prove a lower bound of the height on the quotient (see Theorem 2.1).

Even though these results will be sharpen in Chapter 4 through some more powerful techniques, we present them here in a very basic form: this will permit to prove them in a very elementary way. We hope that this will help the reader willing to proceed straight to the proof of Roth's Theorem expounded in Chapter 2.

1. The Fundamental Formula

Let K be a number field and $\mathfrak{o}_K$ its ring of integers.

1.1. Let $\mathcal{X}$ be a projective and flat $\mathfrak{o}_K$-scheme endowed with the action of an $\mathfrak{o}_K$-reductive group [1] $\mathcal{G}$ and let $\mathcal{L}$ be a *very ample* $\mathcal{G}$-linearized invertible sheaf on $\mathcal{X}$. The global sections $\mathcal{E} = \Gamma(\mathcal{X}, \mathcal{L})$ are endowed with a linear action of $\mathcal{G}$. Thus the reductive group $\mathcal{G}$ acts on $\mathbf{P}(\mathcal{E}^\vee)$ and the invertible sheaf $\mathcal{O}_{\mathcal{E}^\vee}(1)$ is $\mathcal{G}$-linearized. The closed embedding $j\colon \mathcal{X} \hookrightarrow \mathbf{P}(\mathcal{E}^\vee)$ and the isomorphism $j^*\mathcal{O}_{\mathcal{E}^\vee}(1) \simeq \mathcal{L}$ are $\mathcal{G}$-equivariant.

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1. Over an algebraically closed field k an algebraic group G — *i.e.* a smooth finite type affine k-group scheme — is said to be *reductive* if it is connected and every normal smooth connected unipotent subgroup is trivial. Over an arbitrary scheme S a group scheme G is said to be *reductive* (or G is a *S-reductive group*) if it satisfies the following conditions:

(1) G is affine, smooth and of finite type over S;
(2) for all $s \in S$, the $\overline{s}$-group scheme $G_{\overline{s}} := G \times_S \overline{s}$ is a reductive algebraic group (where $\overline{s}$ is the spectrum of an algebraic closure of the residue field $\kappa(s)$).

Examples of S-reductive groups are $\mathbf{GL}_{n,S}$, $\mathbf{SL}_{n,S}$ and their products. In the first chapter only $\mathfrak{o}_K$-reductive group $\mathbf{SL}^n_{2,\mathfrak{o}_K}$ will be considered. The interested reader can refer to [**Bor91b**, Chapter IV] for the theory over a field and [**GP11, Con**] for the theory over an arbitrary scheme.

1.2. A point $x \in \mathcal{X}$ is said to be *semi-stable* if there exists, for a sufficiently big $d \geq 1$, a $\mathcal{G}$-invariant global section $s \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})$ that does not vanish at x.

Consider the $\mathfrak{o}_K$-graded algebra of finite type $\mathcal{A} := \bigoplus_{d \geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})$. According to a theorem of Seshadri [**Ses77**, II.4, Theorem 4] the graded algebra

$$\mathcal{A}^{\mathcal{G}} = \bigoplus_{d \geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})^{\mathcal{G}}$$

of $\mathcal{G}$-invariants of $\mathcal{A}$ is an $\mathfrak{o}_K$-algebra of finite type and projective scheme $\mathcal{Y} := \operatorname{Proj} \mathcal{A}^{\mathcal{G}}$ is the categorical quotient of the open subset $\mathcal{X}^{\mathrm{ss}}$ of semi-stable points of $\mathcal{X}$ (with respect to the action of reductive group $\mathcal{G}$ and the invertible sheaf $\mathcal{L}$). For this reason it is denoted by $\mathcal{X} /\!\!/ \mathcal{G}$ (or by $(\mathcal{X}, \mathcal{L}) /\!\!/ \mathcal{G}$ to keep track of the polarisation). Let $\pi \colon \mathcal{X}^{\mathrm{ss}} \to \mathcal{Y}$ be the quotient morphism. Since $\mathcal{Y}$ is of finite type, for every sufficiently divisible integer $D \geq 1$, there exists an ample invertible sheaf $\mathcal{M}_D$ on $\mathcal{Y}$ and a $\mathcal{G}$-equivariant isomorphism of invertible sheaves

$$\varphi_D \colon \pi^* \mathcal{M}_D \longrightarrow \mathcal{L}^{\otimes D}_{|\mathcal{X}^{\mathrm{ss}}}.$$

1.3. For every embedding $\gamma \colon K \to \mathbf{C}$ let $\| \cdot \|_{\mathcal{E}, \gamma}$ be an hermitian norm on $\mathcal{E} \otimes_\gamma \mathbf{C}$ which is invariant under the action of a maximal compact subgroup of $\mathcal{G}_\gamma(\mathbf{C})$. Suppose that the family of norms $\{\| \cdot \|_{\mathcal{E},\gamma}\}_{\gamma \colon K \to \mathbf{C}}$ is compatible under complex conjugation.

Let $\| \cdot \|_{\mathcal{O}(1), \gamma}$ be the Fubini-Study metric on the invertible sheaf $\mathcal{O}_{\mathcal{E}^\vee}(1)$ associated to the hermitian norm $\| \cdot \|_{\mathcal{E}^\vee, \gamma}$ and let $\| \cdot \|_{\mathcal{L}, \gamma}$ be its restriction to $\mathcal{L}$. Denote by $\overline{\mathcal{L}}$ the hermitian line bundle on $\mathcal{X}$ obtained endowing $\mathcal{L}$ with the family of metrics $\{\| \cdot \|_{\mathcal{L}, \gamma}\}_{\gamma \colon K \to \mathbf{C}}$. For every $y \in \mathcal{Y}_\gamma(\mathbf{C})$ and every $t \in y^* \mathcal{M}_D$ set

$$\|t\|_{\mathcal{M}_D, \gamma}(y) := \sup_{\substack{x \in \mathcal{X}^{\mathrm{ss}}_\gamma(\mathbf{C}) \\ \pi(x) = y}} \|\varphi_D(\pi^* t)\|_{\mathcal{L}^{\otimes D}, \gamma}(x).$$

Lemma 1.1. — *Let $f \in \Gamma(\mathcal{Y}, \mathcal{M}_D)$ be a global section.*

(1) There exists a unique $\mathcal{G}$-invariant global section $\widetilde{f} \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})$ which vanishes identically on $\mathcal{X} - \mathcal{X}^{\mathrm{ss}}$ and such that $\varphi_D(\pi^* f) = \widetilde{f}_{|\mathcal{X}^{\mathrm{ss}}}$.
(2) For every complex embedding $\gamma \colon K \to \mathbf{C}$,

$$\sup_{y \in \mathcal{Y}_\gamma(\mathbf{C})} \|f\|_{\mathcal{M}_D, \gamma}(y) = \sup_{x \in \mathcal{X}_\gamma(\mathbf{C})} \|\widetilde{f}\|_{\mathcal{L}^{\otimes D}, \gamma}(x).$$

In particular, $\|t\|_{\mathcal{M}_D, \gamma}(y) < +\infty$ for every $t \in y^* \mathcal{M}_D$, thus the function $\| \cdot \|_{\mathcal{M}_D, \gamma}$ defines a metric on the invertible sheaf $\mathcal{M}_D$.

Proof. —

(1) It is a reformulation of the definition of $\mathcal{Y}$ and $\mathcal{M}_D$.
(2) It follows from (1). □

1.4. Let $\overline{\mathcal{M}}_D$ be the associated hermitian invertible sheaf on $\mathcal{Y}$ and $h_{\overline{\mathcal{M}}_D}$ the height function given by $\overline{\mathcal{M}}_D$ (see [**BG06**, 2.7.17]). Define

$$h_{\min}((\mathcal{X},\overline{\mathcal{L}})/\!\!/\mathcal{G}) := \inf_{Q\in\mathcal{Y}(\bar{\mathbf{Q}})} \frac{1}{D} h_{\overline{\mathcal{M}}_D}(Q) \in [-\infty,+\infty[\,.$$

(which is independent of D).

Lemma 1.2. — $h_{\min}((\mathcal{X},\overline{\mathcal{L}})/\!\!/\mathcal{G}) > -\infty$.

Proof. — Let D be such that $\mathcal{M}_D$ is very ample and let $t_1,\ldots,t_N \in \Gamma(\mathcal{Y},\mathcal{M}_D)$ be a set of generators of the global sections. Let K' be a finite extension of K, let Q be a K'-point of $\mathcal{Y}$ and ε_Q the associated $\mathfrak{o}_{K'}$-point of $\mathcal{Y}$ given by the valuative criterion of properness. There exists $i \in \{1,\ldots,N\}$ such that t_i does not vanish at Q. By definition of the height,

$$\begin{aligned}[K':\mathbf{Q}]h_{\overline{\mathcal{M}}_D}(Q) &:= \log\#\left(\varepsilon_Q^*\mathcal{M}_D/(\varepsilon_Q^*t_i\cdot\varepsilon_Q^*\mathcal{M}_D)\right) - \sum_{\gamma\colon K'\to\mathbf{C}} \log\|t_i\|_{\mathcal{M}_D,\gamma}(Q)\\ &\geq -[K':K]\sum_{\gamma\colon K\to\mathbf{C}}\sup_{y\in\mathcal{Y}_\gamma(\mathbf{C})}\log\|t_i\|_{\mathcal{M}_D,\gamma}(y).\end{aligned}$$

It suffices to show that for every $i=1,\ldots,N$ and every $\gamma\colon K\to\mathbf{C}$ the function $\|t_i\|_{\mathcal{M}_D,\gamma}$ is uniformly bounded on $\mathcal{Y}_\gamma(\mathbf{C})$. This follows from Lemma 1.1 (2) and concludes the proof. □

Remark 1.3. — Proving this Lemma would have been unnecessary if knew that the metric $\|\cdot\|_{\mathcal{M}_D,\gamma}$ was continuous. This is actually the case but in an attempt to be self-contained in this chapter we avoided the recourse to such a result. A proof of the continuity is expounded in Chapter 3 (see Theorem 1.1). In this setting it follows from the arguments of Kempf-Ness (see Kirwan [**MFK94**, Chapter 8, §2], Burnol [**Bur92**] and Schwarz [**Sch00**, Chapter 5]). A proof (which is sensibly different from ours) in the general case is given by Zhang [**Zha96b**, Theorem 4.10].

1.5. Instability measure. — Let v be a place of K. If v is non-archimedean let by $\|\cdot\|_{\mathcal{L},v}$ (resp. $\|\cdot\|_{\mathcal{M}_D,v}$) be the continuous and bounded metric induced by the integral model $\mathcal{L}$ (resp. $\mathcal{M}_D$) (2). For a $\mathbf{C}_v$-point $x\in\mathcal{X}(\mathbf{C}_v)$ its *v-adic instability measure* is

$$\iota_v(x) := -\log\sup_{g\in\mathcal{G}(\mathbf{C}_v)}\frac{\|g\cdot s\|_{\mathcal{L},v}(g\cdot x)}{\|s\|_{\mathcal{L},v}(x)} \in [-\infty,0].$$

2. Let x be a $\mathbf{C}_v$-point of $\mathcal{X}$. Since $\mathcal{X}$ is proper, the $\mathbf{C}_v$-point x gives rise to a $\bar{\mathfrak{o}}_v$-point ε_x of $\mathcal{X}$, where $\bar{\mathfrak{o}}_v$ is the ring of integers of $\mathbf{C}_v$. The invertible sheaf $\varepsilon_x^*\mathcal{L}$ is a free $\bar{\mathfrak{o}}_v$-module of rank 1: choose a basis s_0. Every other element $s\in x^*\mathcal{L}$ can be written in a unique way as $s=\lambda s_0$ with $\lambda\in\mathbf{C}_v$. Set

$$\|s\|_{\mathcal{L},v}(x) := |\lambda|_v.$$

This does not depend on the chosen basis s_0 of $\varepsilon_x^*\mathcal{L}$. See also [**BG06**, Example 2.7.20].

where $s \in x^*\mathcal{L}$ is a non-zero section. This does not depend on the chosen section s. If $\hat{x}$ is a generator of the line $j(x) \in \mathbf{P}(\mathcal{E}^\vee)(\mathbf{C}_v)$,

$$\iota_v(x) = \log \inf_{g \in \mathcal{G}(\mathbf{C}_v)} \frac{\|g \cdot \hat{x}\|_{\mathcal{E}^\vee, v}}{\|\hat{x}\|_{\mathcal{E}^\vee, v}}.$$

Proposition 1.4. — *Let v be a place of K. For every $\mathbf{C}_v$-point $x \in \mathcal{X}^{\mathrm{ss}}(\mathbf{C}_v)$ and every non-zero section $t \in \pi(x)^*\mathcal{M}_D$,*

$$\iota_v(x) \geq -\frac{1}{D} \log \frac{\|t\|_{\mathcal{M}_D, v}(\pi(x))}{\|\pi^* t\|_{\mathcal{L}^{\otimes D}, v}(x)}.$$

Proof. — In the archimedean case this is clear by definition of the metric $\|\cdot\|_{\mathcal{M}_D}$ and the $\mathcal{G}$-invariance of π. Suppose that v is non-archimedean. Up to taking a power of $\mathcal{M}_D$ one may assume that $\mathcal{M}_D$ is very ample.

Let $y := \pi(x)$ and let $\varepsilon_y \in \mathcal{Y}(\bar{\mathfrak{o}}_v)$ the unique $\bar{\mathfrak{o}}_v$-valued point of $\mathcal{Y}$ associated to y by the valuative criterion of properness (where $\bar{\mathfrak{o}}_v$ is the ring of integers of $\mathbf{C}_v$). Up to rescaling t one may assume that t is basis of the free $\bar{\mathfrak{o}}_v$-module $\varepsilon_y^*\mathcal{M}_D$ and thus $\|t\|_{\mathcal{M}_D, v}(y) = 1$.

Since $\mathcal{M}_D$ is generated by its global sections, there exists $f \in \Gamma(\mathcal{Y}, \mathcal{M}_D) \otimes \bar{\mathfrak{o}}_v$ such that $\varepsilon_y^* f = t$. According to Proposition 1.1, the rational section $\pi^* f$ extends uniquely to a $\mathcal{G}$-invariant global section $\widetilde{f} \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D}) \otimes \bar{\mathfrak{o}}_v$ which vanishes identically outside $\mathcal{X}^{\mathrm{ss}}$.

Fix $g \in \mathcal{G}(\mathbf{C}_v)$. Since the section $\widetilde{f}$ is integral,

$$\|\pi^* f\|_{\mathcal{L}^{\otimes D}, v}(g \cdot x) = \|\widetilde{f}\|_{\mathcal{L}^{\otimes D}, v}(g \cdot x) \leq 1,$$

and recalling $\|t\|_{\mathcal{M}_D, v}(y) = 1$ this entails $\|\pi^* t\|_{\mathcal{L}^{\otimes D}, v}(g \cdot x) \leq \|t\|_{\mathcal{M}_D, v}(y)$. Taking the supremum over all $g \in \mathcal{G}(\mathbf{C}_v)$,

$$\iota_v(x) = -\frac{1}{D} \log \sup_{g \in \mathcal{G}(\mathbf{C}_v)} \frac{\|\pi^* t\|_{\mathcal{L}^{\otimes D}, v}(g \cdot x)}{\|\pi^* t\|_{\mathcal{L}^{\otimes D}, v}(x)} \geq -\frac{1}{D} \log \frac{\|t\|_{\mathcal{M}_D, v}(y)}{\|\pi^* t\|_{\mathcal{L}^{\otimes D}, v}(x)}. \qquad \square$$

Remark 1.5. — For a non-archimedean place v, it follows from the proof that in the preceding Proposition one has equality if the reduction $\widetilde{x}$ of the point x at the place v is semi-stable, *i.e.* it is a semi-stable $\overline{\mathbf{F}}_v$-point of the scheme $\mathcal{X} \times_{\mathfrak{o}_K} \overline{\mathbf{F}}_v$ under the action of $\mathcal{G}(\overline{\mathbf{F}}_v)$ (where $\overline{\mathbf{F}}_v$ is the residue field of $\mathbf{C}_v$).

In Chapter 3 the converse of the previous assertion is proved: that is, if the equality holds than the point x is residually semi-stable (see also Theorem 1.16).

1.6. Fundamental formula. — Summing up the previous considerations:

***Theorem 1.6* (Fundamental Formula)**. — *Let $P \in \mathcal{X}(K)$ be a semi-stable point. Then for almost all places $v \in \mathrm{V}_K$ the instability measure $\iota_v(P)$ is zero and*

$$h_{\overline{\mathcal{L}}}(P) + \frac{1}{[K : \mathbf{Q}]} \sum_{v \in \mathrm{V}_K} \iota_v(P) \geq \frac{1}{D} h_{\overline{\mathcal{M}}_D}(\pi(P)).$$

In practice one uses Theorem 1.6 through this immediate Corollary:

Corollary 1.7. — *For every semi-stable point $P \in \mathcal{X}^{\mathrm{ss}}(K)$,*

$$h_{\overline{\mathcal{L}}}(P) + \frac{1}{[K:\mathbf{Q}]} \sum_{v \in \mathrm{V}_K} \iota_v(P) \geq h_{\min}((\mathcal{X}, \overline{\mathcal{L}}) /\!\!/ \mathcal{G}).$$

Remark 1.8. — One of the main tasks of Chapters 3 and 4 is to prove that the inequality in the statement of the Fundamental Formula is actually an identity (this is the reason why this result is called "Fundamental Formula"). For details, see Theorem 1.5 in Chapter 4. In the present situation, it had already been proven by Burnol [**Bur92**, Proposition 5] (see also Corollary 1.6 in Chapter 4).

2. Lower bound of the height on the quotient

2.1. Statement of the lower bound. — Let $\overline{\mathcal{E}} = (\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_n)$ be a n-uple of $\mathfrak{o}_K$-hermitian vector bundles of positive ranks. Let $\overline{\mathcal{F}}$ be a $\mathfrak{o}_K$-hermitian vector bundle and

$$\rho\colon\ \mathbf{GL}(\mathcal{E}) := \mathbf{GL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}(\mathcal{E}_n) \longrightarrow \mathbf{GL}(\mathcal{F}),$$

a representation, that is, a morphism of $\mathfrak{o}_K$-group schemes, which is *unitary*, *i.e.*, for every embedding $\sigma\colon K \to \mathbf{C}$, the action of the compact subgroup

$$\mathbf{U}(\mathcal{E})_\sigma := \mathbf{U}(\|\cdot\|_{\mathcal{E}_1,\sigma}) \times \cdots \times \mathbf{U}(\|\cdot\|_{\mathcal{E}_n,\sigma}) \subset \mathbf{GL}(d)_\sigma(\mathbf{C})$$

respects the hermitian norm $\|\cdot\|_{\mathcal{F},\sigma}$.

Theorem 2.1. — *With the notation introduced above, let $b = (b_1, \dots, b_n)$ be n-uple of integers and*

$$\varpi\colon\ \overline{\mathcal{E}}_1^{\otimes b_1} \otimes \cdots \otimes \overline{\mathcal{E}}_n^{\otimes b_n} \longrightarrow \overline{\mathcal{F}},$$

a homomorphism of hermitian vector bundles, generically surjective and $\mathbf{GL}(\mathcal{E})$-equivariant. Then,

$$h_{\min}((\mathbf{P}(\mathcal{F}^\vee), \mathcal{O}_{\overline{\mathcal{F}}^\vee}(1)) /\!\!/ \mathbf{SL}(\mathcal{E})) \geq \sum_{i=1}^n b_i\, \hat{\mu}(\overline{\mathcal{E}}_i) - \frac{1}{2} \sum_{i=1}^n |b_i| \log \operatorname{rk} \mathcal{E}_i,$$

where $\mathcal{O}_{\overline{\mathcal{F}}^\vee}(1)$ is equipped with the Fubini-Study metric given by $\overline{\mathcal{F}}$ and $\mathbf{SL}(\mathcal{E})$ is the $\mathfrak{o}_K$-reductive group $\mathbf{SL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}(\mathcal{E}_n)$.

Here, for $i = 1, \dots, n$ and negative b_i, $\overline{\mathcal{E}}_i^{\otimes b_i}$ is the $\mathfrak{o}_K$-module $\mathcal{E}_i^{b_i} = \mathcal{E}_i^{\vee \otimes -b_i}$ endowed with the $(-b_i)$-th tensor power of the dual norm.

A different proof of this result will be given in Chapter 4 (see Theorem 1.11). The techniques employed therein permit us to get a sharp lower bound in the case when $\operatorname{rk} \mathcal{E}_i = 2$ (which will be the case of our interest when proving Roth's Theorem).

Remark 2.2. — This statement is more general than [**Che09**, Theorem 4.2] in the following sense: with our notation Chen proves that for every semi-stable K-point P of $\mathbf{P}(\mathcal{F}^\vee)$,

$$h_{\mathcal{O}_{\overline{\mathcal{F}}^\vee}(1)}(P) \geq \sum_{i=1}^n b_i\, \hat{\mu}(\overline{\mathcal{E}}_i) - \frac{1}{2} \sum_{i=1}^n |b_i| \log \operatorname{rk} \mathcal{E}_i.$$

Chen's result is deduced from Theorem 2.1 thanks to the inequality given by Corollary 1.7

$$h_{\mathcal{O}_{\overline{\mathcal{F}}^\vee}(1)}(P) \geq h_{\min}((\mathbf{P}(\mathcal{F}^\vee), \mathcal{O}_{\overline{\mathcal{F}}^\vee}(1))/\!\!/\mathbf{SL}(\mathcal{E})).$$

Remark 2.3. — In the proof of Theorem 2.1 one can limit ourselves to consider the case where the integer b_i are non-negative. Indeed, if the integers b_i are not necessarily non-negative, one can consider, for every $i = 1, \dots, n$,

$$\overline{\mathcal{E}}'_i = \begin{cases} \overline{\mathcal{E}}_i & \text{if } b_i \geq 0 \\ \overline{\mathcal{E}}_i^\vee & \text{otherwise.} \end{cases}$$

Set $\mathbf{GL}(\mathcal{E}') := \mathbf{GL}(\mathcal{E}'_1) \times \cdots \times \mathbf{GL}(\mathcal{E}'_n)$. If $\varpi \colon \overline{\mathcal{E}}_1^{\otimes b_1} \otimes \cdots \otimes \overline{\mathcal{E}}_n^{\otimes b_n} \to \overline{\mathcal{F}}$ is a homomorphism of hermitian vector bundles as in the statement of Theorem 2.1, it induces a generically surjective and $\mathbf{GL}(\mathcal{E}')$ homomorphism of hermitian vector bundles

$$\varpi' \colon \overline{\mathcal{E}}_1'^{\otimes |b_1|} \otimes \cdots \otimes \overline{\mathcal{E}}_n'^{\otimes |b_n|} \longrightarrow \overline{\mathcal{F}}.$$

The quotients of $\mathbf{P}(\mathcal{F}^\vee)$ by $\mathbf{SL}(\mathcal{E})$ and $\mathbf{SL}(\mathcal{E}') := \mathbf{SL}(\mathcal{E}'_1) \times \cdots \times \mathbf{SL}(\mathcal{E}'_n)$ are canonically identified and the metrics induced on the polarisation $\mathcal{M}_D$ are the same. In particular,

$$h_{\min}((\mathbf{P}(\mathcal{F}^\vee), \mathcal{O}_{\overline{\mathcal{F}}^\vee}(1))/\!\!/\mathbf{SL}(\mathcal{E})) = h_{\min}((\mathbf{P}(\mathcal{F}^\vee), \mathcal{O}_{\overline{\mathcal{F}}^\vee}(1))/\!\!/\mathbf{SL}(\mathcal{E}')).$$

The remainder of this section is devoted to the proof of Theorem 2.1 when the integers $b_1, \dots, b_N$ are non-negative.

Invariant theory for a product of linear groups. —

2.2. Let k be a field. Let $n \geq 1$ be a positive integer and $E = (E_1, \dots, E_n)$ a n-uple of non-zero k-vector spaces of finite dimension. Define

$$\mathbf{GL}(E) := \mathbf{GL}(E_1) \times_k \cdots \times_k \mathbf{GL}(E_n),$$
$$\mathbf{SL}(E) := \mathbf{SL}(E_1) \times_k \cdots \times_k \mathbf{SL}(E_n).$$

Definition 2.4. — Let F be a non-zero k-vector space of finite dimension. A representation, *i.e.* a morphism of k-group schemes, $\rho \colon \mathbf{GL}(E) \to \mathbf{GL}(F)$ is said to be *homogeneous of weight* $b = (b_1, \dots, b_n) \in \mathbf{Z}^n$ if, for every k-scheme S and all S-points $t_1, \dots, t_n \in \mathbf{G}_m(S)$,

$$\rho(t_1 \cdot \mathrm{id}_{E_1}, \dots, t_n \cdot \mathrm{id}_{E_n}) = t_1^{b_1} \cdots t_n^{b_n} \cdot \mathrm{id}_F\,.$$

Proposition 2.5. — *Let $\rho \colon \mathbf{GL}(E) \to \mathbf{GL}(F)$ be a homogeneous representation of weight $b = (b_1, \dots, b_n)$ and suppose that the subspace of $\mathbf{SL}(E)$-invariant elements of F is non-trivial. Then:*

(1) for every $i = 1, \dots, n$ the dimension e_i of E_i divides the integer b_i;
(2) for every k-scheme S, any S-point $(g_1, \dots, g_n)$ of $\mathbf{GL}(E)$ and any $\mathbf{SL}(E)$-invariant element w of F:

$$\rho(g_1, \dots, g_n) \cdot w = \det(g_1)^{b_1/e_1} \cdots \det(g_n)^{b_n/e_n} \cdot w. \tag{2.2.1}$$

Proof. — This follows from the fact that characters of the general linear group are powers of the determinant. ☐

2.3. For every non-negative integer N denote by $\mathfrak{S}_N$ the group of permutations on N elements (if $N = 0$, then $\mathfrak{S}_0 = \{\mathrm{id}_\emptyset\}$). If E is a k-vector space the group $\mathfrak{S}_N$ acts on the N-th tensor product $E^{\otimes N}$ permuting factors. Explicitly, if $\sigma \in \mathfrak{S}_N$ is a permutation and $x_1, \dots, x_N$ are elements of E,

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_N) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(N)}.$$

Definition 2.6. — For $N \in \mathbf{Z}$ the preceding action defines a homomorphism of non-commutative k-algebras $\mathfrak{S}_{|N|} \to \mathrm{End}_k(E^{\otimes N})$ denoted by $\eta_{E,N}$.

2.4. Let $b = (b_1, \dots, b_n)$ be a n-uple of non-negative integers. The group

$$\mathfrak{S}_b := \mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_n},$$

acts component-wise on the k-vector space $E^{\otimes b} := E_1^{\otimes b_1} \otimes \cdots \otimes E_n^{\otimes b_n}$. The k-group scheme $\mathbf{GL}(E)$ acts by conjugation on the k-vector space

$$\mathrm{End}_k(E^{\otimes b}) = \mathrm{End}_k(E_1^{\otimes b_1}) \otimes_k \cdots \otimes_k \mathrm{End}_k(E_n^{\otimes b_n}).$$

The associated representation $\mathbf{GL}(E) \to \mathbf{GL}(\mathrm{End}(E^{\otimes b}))$ is homogeneous of weight $0 = (0, \dots, 0)$. Proposition 2.5 (2) entails that the invariant elements of $\mathrm{End}(E^{\otimes Db})$ with respect to the action of $\mathbf{GL}(E)$ and to the action of $\mathbf{SL}(E)$ are the same.

Definition 2.7. — The linear action of $\mathfrak{S}_b$ on $E^{\otimes b}$ defines a homomorphism of non-commutative k-algebras $\bigotimes_{i=1}^n k[\mathfrak{S}_{b_i}] \to \mathrm{End}(E^{\otimes b})$ denoted by $\eta_{E,b}$.

The image of $\eta_{E,b}$ is contained in the subspace of invariants of $\mathrm{End}(E^{\otimes b})$. The First Main Theorem of Invariant Theory affirms that in characteristic 0 the converse inclusion holds too (*cf.* [**Wey39**, Chapter III], [**Che09**, Theorem 3.1, Corollary] and [**ABP73**, Appendix 1]):

Theorem 2.8 **(First Main Theorem of Invariant Theory)**
Suppose that the characteristic of k is zero. The subspace of $\mathbf{SL}(E)$-invariant elements of the k-vector space $\mathrm{End}(E^{\otimes b})$ is the image of the homomorphism $\eta_{E,b}$.

Definition 2.9. — With the notation introduced above suppose that e_i divides b_i for every $i = 1, \dots, n$. Consider the homomorphism of k-vector spaces

$$\Phi_{E_i,b_i} \colon \mathrm{End}\,(E_i)^{\otimes b_i} \otimes \det(E_i)^{\otimes b_i/e_i} \longrightarrow E_i^{\otimes b_i}$$

defined as the composition of the following homomorphisms:

$$\begin{array}{ccc}
\mathrm{End}\,(E_i)^{\otimes b_i} \otimes \det(E_i)^{\otimes b_i/e_i} & & E_i^{\otimes b_i} \\
\| & & \| \\
E_i^{\otimes b_i} \otimes (E_i^{\otimes e_i})^{\vee \otimes b_i/e_i} \otimes \det(E_i)^{\otimes b_i/e_i} & \longrightarrow & E_i^{\otimes b_i} \otimes (\det(E_i)^{\vee} \otimes \det(E_i))^{\otimes b_i/e_i}
\end{array}$$

where the horizontal arrow is $\mathrm{id} \otimes \det \otimes \mathrm{id}$. Set $\Phi_{E,b} := \Phi_{E_1,b_1} \otimes \cdots \otimes \Phi_{E_n,b_n}$.

Corollary 2.10. — *Suppose that the characteristic of k is zero. Let F be a non-zero k-vector space of finite dimension and $\rho\colon \mathbf{GL}(E) \to \mathbf{GL}(F)$ be a representation. Let $b = (b_1, \dots, b_n)$ be a n-uple of non-negative integers and*

$$\varphi\colon \bigotimes_{i=1}^{n} E_i^{\otimes b_i} \longrightarrow F$$

be a surjective and $\mathbf{GL}(E)$-equivariant homomorphism of k-vector spaces. The representation ρ is homogeneous of weight $b = (b_1, \dots, b_n)$ and if the subspace of $\mathbf{SL}(E)$-invariant elements of F is non-zero:

(1) For every $i = 1, \dots, n$ the dimension e_i of E_i divides the integer b_i.
(2) The subspace of $\mathbf{SL}(E)$-invariants of F is the image of the homomorphism

$$\varphi \circ \Phi_{E,b} \circ (\eta_{E,b} \otimes \mathrm{id})\colon \bigotimes_{i=1}^{n} k[\mathfrak{S}_{b_i}] \otimes \bigotimes_{i=1}^{n} \det(E_i)^{\otimes b_i/e_i} \longrightarrow F.$$

This is just a combination of the First Main Theorem of Invariant Theory with following:

Remark 2.11. — In characteristic 0 a linear algebraic group is reductive if and only if for every linear representation E of G there exists a *unique* G-equivariant projection $R_E\colon E \to E^G$ (the so-called Reynolds operator). The uniqueness entails the functoriality of the projection on the invariants: for every G-equivariant linear homomorphism $\psi\colon E \to F$ between linear representations of G one has $R_F \circ \psi = \psi \circ R_E$. In particular, if ψ is surjective the induced homomorphism $\varphi\colon E^G \to F^G$ is surjective too. For details, refer to [**MS72**, page 182] and [**MFK94**, Chapter1, §1].

Non-hermitian norms and tensor product. — In this paragraph we briefly discuss norms on tensor products which are not hermitian. The interested reader can refer to [**Gro53**] for the case of two vector spaces and [**Gau08**, Normes tensorielles, page 33] for the present setting.

Let $N \geq 1$ be a positive integer and for every $i = 1, \dots, N$ let V_i be a finite-dimensional complex vector space endowed with a norm $\| \cdot \|_{V_i}$. Let $\| \cdot \|_{V_i^\vee}$ be the operator norm on $V_i^\vee$. Denote by V the tensor product $V_1 \otimes \cdots \otimes V_N$.

Definition 2.12. — The ε-norm and the π-norm on the tensor product V are the norms respectively defined, for $v \in V$, by

$$\|v\|_{V,\varepsilon} := \sup_{\substack{\varphi_i \in V_i^\vee - \{0\} \\ i=1,\dots,N}} \frac{|\varphi_1 \otimes \cdots \otimes \varphi_N(v)|}{\|\varphi_1\|_{V_1^\vee} \cdots \|\varphi_N\|_{V_N^\vee}},$$

$$\|v\|_{V,\pi} := \inf \left\{ \sum_{\alpha=1}^{R} \|v_{\alpha 1}\|_{V_1} \cdots \|v_{\alpha N}\|_{V_N} : v = \sum_{\alpha=1}^{R} v_{\alpha 1} \otimes \cdots \otimes v_{\alpha N} \right\}.$$

The vector space V equipped with the norm $\| \cdot \|_{V,\varepsilon}$ is denoted $V_1 \otimes_\varepsilon \cdots \otimes_\varepsilon V_N$. Analogously, V equipped with $\| \cdot \|_{V,\pi}$ is denoted $V_1 \otimes_\pi \cdots \otimes_\pi V_N$.

If the norms $\| \cdot \|_{V_i}$ are hermitian let $V_1 \otimes_2 \cdots \otimes_2 V_N$ be the vector space V with the hermitian norm on the tensor product.

Proposition 2.13. — *With the notation introduced above:*

(1) The ε-norm $\|\cdot\|_\varepsilon$ (resp. the π-norm $\|\cdot\|_\pi$) is the smallest (resp. the biggest) among the norms $\|\cdot\|$ on V such that, for $i = 1, \dots, N$ and $v_i \in V_i$,

$$\|v_1 \otimes \cdots \otimes v_N\| \geq \|v_1\|_{V_1} \cdots \|v_N\|_{V_N},$$
$$(\text{resp. } \|v_1 \otimes \cdots \otimes v_N\| \leq \|v_1\|_{V_1} \cdots \|v_N\|_{V_N}),$$

and, for $i = 1, \dots, n$ and $\varphi_i \in V_i^\vee$,

$$\|\varphi_1 \otimes \cdots \otimes \varphi_N\|^\vee \leq \|\varphi_1\|_{V_1^\vee} \cdots \|\varphi_N\|_{V_N^\vee},$$
$$(\text{resp. } \|\varphi_1 \otimes \cdots \otimes \varphi_N\|^\vee \geq \|\varphi_1\|_{V_1^\vee} \cdots \|\varphi_N\|_{V_N^\vee}),$$

where $\|\cdot\|^\vee$ denotes the operator norm induced by $\|\cdot\|$ on $V^\vee$.

(2) $\|v\|_{V,\varepsilon} \leq \|v\|_{V,\pi}$ for all $v \in V$.

(3) (Duality) *The isomorphism $V^\vee \simeq V_1^\vee \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} V_N^\vee$ induces the following isometries:*

$$(V_1 \otimes_\varepsilon \cdots \otimes_\varepsilon V_N)^\vee \xrightarrow{\sim} V_1^\vee \otimes_\pi \cdots \otimes_\pi V_N^\vee,$$
$$(V_1 \otimes_\pi \cdots \otimes_\pi V_N)^\vee \xrightarrow{\sim} V_1^\vee \otimes_\varepsilon \cdots \otimes_\varepsilon V_N^\vee.$$

(4) (Fonctoriality) *For every $i = 1, \dots, N$ let W_i be a finite-dimensional complex vector space equipped with a norm $\|\cdot\|_{W_i}$ and let $\varphi_i \colon V_i \to W_i$ be a linear map decreasing the norms. Then, the induced maps*

$$\varphi_1 \otimes \cdots \otimes \varphi_N : V_1 \otimes_\varepsilon \cdots \otimes_\varepsilon V_N \longrightarrow W_1 \otimes_\varepsilon \cdots \otimes_\varepsilon W_N,$$
$$\varphi_1 \otimes \cdots \otimes \varphi_N \colon V_1 \otimes_\pi \cdots \otimes_\pi V_N \longrightarrow W_1 \otimes_\pi \cdots \otimes_\pi W_N,$$

decrease the norms.

(5) Let L be a normed vector space of dimension 1. *Then,*

$$(V_1 \otimes_\varepsilon \cdots \otimes_\varepsilon V_N) \otimes_\varepsilon L = V_1 \otimes_\varepsilon \cdots \otimes_\varepsilon V_N \otimes_\varepsilon L,$$
$$(V_1 \otimes_\pi \cdots \otimes_\pi V_N) \otimes_\pi L = V_1 \otimes_\pi \cdots \otimes_\pi V_N \otimes_\pi L.$$

Sketch of the proof. — (1), (4) and (5) follow from the definitions of the norms.

(2) By bi-duality, for all $i = 1, \dots, N$ and all $v_i \in V_i$ the very definition of the ε-norms entails

$$\|v_1 \otimes \cdots \otimes v_N\|_\varepsilon = \prod_{i=1}^N \|v_i\|_{V_i^{\vee\vee}} \leq \prod_{i=1}^N \|v_i\|_{V_i},$$

and one concludes thanks to (1).

(3) Follows from (1) and (2) by duality. □

Proposition 2.14. — *Let V and W be finite-dimensional normed vector spaces. The operator norm on $\mathrm{Hom}_{\mathbf{C}}(V, W)$ coincides with the ε-norm on $V^\vee \otimes_{\mathbf{C}} W$ through the canonical isomorphism*

$$V^\vee \otimes_{\mathbf{C}} W \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(V, W)$$

Proof. — This is Theorem [**Gro53**, §1.1, Théorème 1] for $E = V$, $F = \mathbf{C}$ and $G = W$. □

Remark 2.15. — Let L be a normed complex vector line. It follows from the preceding Proposition that through the isomorphism $L \otimes L^\vee \simeq \mathbf{C}$ the ε-norm on $L \otimes L^\vee$ induces the absolute value on $\mathbf{C}$.

Proposition 2.16. — *Let W be an hermitian vector space and let $r \geq 1$ be a positive integer. Endow the exterior powers $\bigwedge^r W$ with the hermitian norm defined in 0.3 on page 11. Then the canonical map $\det : W^{\otimes_\pi r} \to \bigwedge^r W$ decreases the norms.*

Proof. — For every element $w \in W^{\otimes r}$ and every writing $w = \sum_{\alpha=1}^R w_{\alpha 1} \otimes \cdots \otimes w_{\alpha r}$ the Hadamard inequality (0.3.1) on page 12 yields

$$\begin{aligned}
\langle \det w, \det w \rangle_{\det E_i} &= \sum_{\alpha,\beta=1}^R \langle w_{\alpha 1} \wedge \cdots \wedge w_{\alpha r}, w_{\beta 1} \wedge \cdots \wedge w_{\beta r} \rangle_{\det E_i} \\
&\leq \sum_{\alpha,\beta=1}^R \|w_{\alpha 1}\|_{E_i} \cdots \|w_{\alpha r}\|_{E_i} \|w_{\beta 1}\|_{E_i} \cdots \|w_{\beta r}\|_{E_i} \\
&= \left(\sum_{\alpha=1}^R \|w_{\alpha 1}\|_{E_i} \cdots \|w_{\alpha r}\|_{E_i} \right)^2,
\end{aligned}$$

which concludes the proof. □

Application to the lower bound of the height on the quotient. —

2.5. Let us go back to the proof of Theorem 2.1 in the case when the integers b_i are non-negative. Denote by $\mathcal{Y}$ the quotient of semi-stable points of $\mathbf{P}(\mathcal{F})$ by $\mathbf{SL}(\mathcal{E})$ and, for every sufficiently divisible D, by $\overline{\mathcal{M}}_D$ the hermitian invertible sheaf on $\mathcal{Y}$ induced by $\mathcal{O}_{\overline{\mathcal{F}}}(D)$. Fix D such that $\mathcal{M}_D$ is very ample.

2.6. Application of the First Main Theorem of Invariant Theory. — Assume that $\mathcal{Y}$ is non-empty. Since the characteristic of K is zero and the homomorphism ϖ decreases the norms, the general statement reduces to the case

$$\overline{\mathcal{F}} = \overline{\mathcal{E}}^{\otimes b} := \overline{\mathcal{E}}_1^{\otimes b_1} \otimes_{\mathfrak{o}_K} \cdots \otimes_{\mathfrak{o}_K} \overline{\mathcal{E}}_n^{\otimes b_n}.$$

For every $i = 1, \ldots, n$ denote by E_i the K-vector space $\mathcal{E}_i \otimes_{\mathfrak{o}_K} K$ and by $E^{\otimes b}$ the K-vector space $E_1^{\otimes b_1} \otimes_K \cdots \otimes_K E_n^{\otimes b_n}$.

The subspace of $\mathbf{SL}(E)$-invariant elements of $\mathrm{Sym}^D F$ is non-zero because $\mathcal{M}_D$ is very ample. Therefore for every $i = 1, \ldots, n$ the integer $e_i := \dim_K \mathcal{E}_i$ divides Db_i. Consider the maps $\eta := \eta_{E,Db}$ and $\Phi := \Phi_{E,Db}$ (see Definitions 2.7 and 2.9) and the surjection:

$$\varphi : E^{\otimes Db} \longrightarrow \mathrm{Sym}^D(E^{\otimes b}),$$

where $E^{\otimes Db} := E_1^{\otimes Db_1} \otimes_K \cdots \otimes_K E_n^{\otimes Db_n}$.

Lemma 2.17. — *For every $i = 1, \ldots, n$ let $\delta_i \in \det(E_i)$ be non-zero.*

(1) A set of generators of the $\mathbf{SL}(E)$*-invariant elements of the vector space*

$$\operatorname{Sym}^D E^{\otimes b} = \Gamma(\mathbf{P}(E^{\otimes b}), \mathcal{O}(D))$$

is given by the image through $\varphi \circ \Phi$ *of the elements*

$$f_\sigma := \eta(\sigma) \otimes \left(\delta_1^{\otimes Db_1/e_1} \otimes \cdots \otimes \delta_1^{\otimes Db_n/e_n}\right)$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$ *ranges in* $\mathfrak{S}_{Db} := \mathfrak{S}_{Db_1} \times \cdots \times \mathfrak{S}_{Db_n}$.

(2) Through the identification $\Gamma(\mathcal{Y}, \mathcal{M}_D) \otimes_{\mathfrak{o}_K} K \simeq \Gamma(\mathbf{P}(E^{\otimes b}), \mathcal{O}(D))^{\mathbf{SL}(E)}$*:*

$$h_{\min}((\mathbf{P}(\mathcal{E}^{\otimes b}), \mathcal{O}_{\overline{\mathcal{F}}}(1)) /\!\!/ \mathbf{SL}(\mathcal{E})) \geq -\frac{1}{D} \sup_{\sigma \in \mathfrak{S}_{D|b|}} \left\{ \sum_{v \in \mathrm{V}_K} \log \sup_{\mathcal{Y}(\mathbf{C}_v)} \|(\varphi \circ \Phi)(f_\sigma)\|_{\mathcal{M}_D, v} \right\}.$$

Proof. —

(1) This is a direct consequence of Corollary 2.10 (applied to the representation $\operatorname{Sym}^D E^{\otimes b}$ and the $\mathbf{SL}(V)$-equivariant surjection φ).

(2) Let $Q \in \mathcal{Y}(\bar{\mathbf{Q}})$ be a point defined on a finite extension K' of K. Since $\mathcal{M}_D$ is very ample, according to (1) there exists $\sigma \in \mathfrak{S}_{D|b|}$ such that the $\mathbf{SL}(E)$-invariant polynomial $\varphi \circ \Phi(f_\sigma)$ — seen as a global section of $\mathcal{M}_D$ — does not vanish at Q. By definition of the height:

$$\begin{aligned} h_{\mathcal{M}_D}(Q) &= \sum_{v \in \mathrm{V}_K} -\log \|\varphi \circ \Phi(f_\sigma)\|_{\mathcal{M}_{\mathcal{D}}, v}(Q) \\ &\geq \sum_{v \in \mathrm{V}_K} -\log \sup_{y \in \mathcal{Y}(\mathbf{C}_v)} \|\varphi \circ \Phi(f_\sigma)\|_{\mathcal{M}_{\mathcal{D}}, v}(y), \end{aligned}$$

from which the conclusion of the Lemma follows. □

2.7. Size of the invariants. — Consider the $\mathfrak{o}_K$-module

$$\mathcal{E}^{\otimes Db} = \mathcal{E}_1^{\otimes Db_1} \otimes \cdots \otimes \mathcal{E}_n^{\otimes Db_n},$$

and denote by $\overline{\mathcal{E}}^{\otimes_\varepsilon Db}$ (resp. $\overline{\mathcal{E}}^{\otimes_2 Db}$) the $\mathfrak{o}_K$-module $\mathcal{E}^{\otimes Db}$ endowed for every embedding $\gamma\colon K \to \mathbf{C}$ with the ε-norm $\|\cdot\|_\varepsilon$ on the normed vector space

$$(\overline{\mathcal{E}}_{1,\gamma}^{\otimes_\varepsilon e_1})^{\otimes_\varepsilon Db_1/e_1} \otimes_\varepsilon \cdots \otimes_\varepsilon (\overline{\mathcal{E}}_{n,\gamma}^{\otimes_\varepsilon e_n})^{\otimes_\varepsilon Db_n/e_n},$$

(resp. with the hermitian norm $\|\cdot\|_2$ on tensor product). Consider the $\mathfrak{o}_K$-module $\operatorname{End}_{\mathfrak{o}_K}(\mathcal{E}^{\otimes Db})$ endowed for all embedding $\gamma\colon K \to \mathbf{C}$ with the operator norm $\|\cdot\|_{\varepsilon,2,\gamma}$ on

$$\operatorname{End}_{\varepsilon,2}(\overline{\mathcal{E}}_\gamma^{\otimes Db}) := \operatorname{Hom}(\overline{\mathcal{E}}_\gamma^{\otimes_\varepsilon Db}, \overline{\mathcal{E}}_\gamma^{\otimes_2 Db}).$$

Denote the resulting normed $\mathfrak{o}_K$-module by $\operatorname{End}_{\varepsilon,2}(\overline{\mathcal{E}}^{\otimes Db})$. For every $\gamma\colon K \to \mathbf{C}$ endow the complex vector space $\operatorname{Sym}^D(\mathcal{E}^{\otimes b}) \otimes_\gamma \mathbf{C}$ with the sup-norm on polynomials (see paragraph 0.3 on page 11).

Lemma 2.18. — *With the notation introduced above, the map $\Phi_{E,Db} \circ \varphi$ defines an homomorphism of $\mathfrak{o}_K$-modules*

$$\varphi \circ \Phi \colon \operatorname{End}_{\varepsilon,2}(\overline{\mathcal{E}}^{\otimes Db}) \otimes_\varepsilon \bigotimes_{i=1}^{n}{}_{\varepsilon} (\det \overline{\mathcal{E}}_i)^{\otimes_\varepsilon Db_i/e_i} \longrightarrow \operatorname{Sym}^D(\mathcal{E}^{\otimes b}),$$

which decreases the norms.

Proof. — The fact that the homomorphism $\Phi_{E,Db} \circ \varphi$ is defined at the level of $\mathfrak{o}_K$-module is clear. The map $\varphi \circ \Phi$ is defined as a composition of the following maps:

(1) $\overline{\mathcal{E}}^{\otimes_2 Db} \longrightarrow \operatorname{Sym}^D(\overline{\mathcal{E}}^{\otimes_2 b})$;

(2) $\overline{\mathcal{E}}^{\otimes_2 Db} \otimes_\varepsilon \left(\bigotimes_{i=1}^{n}{}_{\varepsilon} (\det \overline{\mathcal{E}}_i)^{\otimes_\varepsilon Db_i/e_i} \right)^\vee \otimes_\varepsilon \bigotimes_{i=1}^{n}{}_{\varepsilon} (\det \overline{\mathcal{E}}_i)^{\otimes_\varepsilon Db_i/e_i} \xrightarrow{\sim} \overline{\mathcal{E}}^{\otimes_2 Db}$;

(3) $\det \colon \left(\overline{\mathcal{E}}_i^{\otimes_\varepsilon e_i} \right)^\vee \longrightarrow \det \overline{\mathcal{E}}_i^\vee$;

(4) $\operatorname{End}_{\varepsilon,2}(\overline{\mathcal{E}}^{\otimes Db}) \longrightarrow \overline{\mathcal{E}}^{\otimes_2 Db} \otimes_\varepsilon \left(\bigotimes_{i=1}^{n}{}_{\varepsilon} (\det \overline{\mathcal{E}}_i)^{\otimes_\varepsilon Db_i/e_i} \right)^\vee.$

For every $\gamma \colon K \to \mathbf{C}$ each of these maps reduces the norms: indeed, (1) it is a reformulation of the fact that the hermitian norm on polynomials defined in paragraph 0.3 on page 11 is bigger than the sup norm; (2) follows from Propositions 2.13 (5) and Remark 2.15; (3) follows from the isometric isomorphism $(\overline{\mathcal{E}}_i^\vee)^{\otimes_\pi e_i} \simeq (\overline{\mathcal{E}}_i^{\otimes_\varepsilon e_i})^\vee$ given by Proposition 2.13 (3) and Proposition 2.16; (4) follows from (3) and Proposition 2.14. □

Lemma 2.19. — *Let $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_{Db}$. For every $\gamma \colon K \to \mathbf{C}$:*

$$\|\eta(\sigma)\|_{\varepsilon,2,\gamma} \leq \sqrt{e_1^{Db_1} \cdots e_N^{Db_N}}.$$

Proof. — For every $i = 1, \dots, n$ let $v_{i1}, \dots, v_{ie_i}$ be an orthonormal basis of $\overline{\mathcal{E}}_i$. Consider the set $\mathcal{R}$ of indices $R = (r_{i,j} : 1 \leq i \leq n, 1 \leq j \leq D|b_i|)$ with integral entries satisfying $1 \leq r_{i,j} \leq e_i$ for every i, j. For every $R \in \mathcal{R}$ set

$$v_R := \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{Db_i/e_i} \bigotimes_{\alpha=1}^{e_i} v_{ir_{i,je_i+\alpha}} = \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{Db_i/e_i} v_{ir_{i,je_i+1}} \otimes \cdots \otimes v_{ir_{i,je_i+e_i}}.$$

The vectors v_R for $R \in \mathcal{R}$ form an orthonormal basis of $\overline{\mathcal{E}}^{\otimes Db}$. For every $T \in \overline{\mathcal{E}}^{\otimes Db}$ write $T = \sum_{R \in \mathcal{R}} T_R v_R$. With this notation:

$$\|T\|_2^2 = \sum_{R \in \mathcal{R}} |T_R|^2, \qquad \|T\|_\varepsilon \geq \max_{R \in \mathcal{R}} |T_R|.$$

For every $R \in \mathcal{R}$ write $\sigma(R) = (r_{i,\sigma_i(j)})_{i,j}$. By definition of $\eta(\sigma)$ for every T one has $\eta(\sigma)(T) = \sum_{R \in \mathcal{R}} T_R v_{\sigma(R)}$. Therefore $\|\eta(\sigma)(T)\|_2 = \|T\|_2$ and

$$\sup_{T \neq 0} \frac{\|\eta(\sigma)(T)\|_2^2}{\|T\|_\varepsilon^2} = \sup_{T \neq 0} \frac{\|T\|_2^2}{\|T\|_\varepsilon^2} \leq \sup_{T \neq 0} \frac{\displaystyle\sum_{R \in \mathcal{R}} |T_R|^2}{\displaystyle\max_{R \in \mathcal{R}} \left\{|T_R|^2\right\}} = \#\mathcal{R} = e_1^{Db_1} \cdots e_n^{Db_n}$$

(the last supremum is attained for $T = \sum_{R \in \mathcal{R}} v_R$). $\square$

2.8. End of the proof of Theorem 2.1. — For every $i = 1, \ldots, n$ let $\delta_i \in \det(E_i)$ be non-zero. For every $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{S}_{Db}$ consider

$$f_\sigma := \eta(\sigma) \otimes \left(\delta_1^{\otimes Db_1/e_1} \otimes \cdots \otimes \delta_1^{\otimes Db_n/e_n}\right) \in \operatorname{End}(E^{\otimes Db}) \otimes \bigotimes_{i=1}^n \det(E_i)^{\otimes Db_i/e_i}$$

Since the elements η are integral and the map $\varphi \circ \Phi$ is defined at the level of $\mathfrak{o}_K$-modules, for every non-archimedean place v,

$$\sup_{y \in \mathcal{Y}(\mathbf{C}_v)} \|\varphi \circ \Phi(f_\sigma)\|_{\mathcal{M}_D, v}(y) \leq \prod_{i=1}^n \|\delta_i\|_{\det E_i, v}^{Db_i/e_i}.$$

According to Lemmata 2.18 and 2.19 for every embedding $\gamma \colon K \to \mathbf{C}$:

$$\begin{aligned}\sup_{y \in \mathcal{Y}(\mathbf{C})} \|\varphi \circ \Phi(f_\sigma)\|_{\mathcal{M}_D, \gamma}(y) \leq \|\varphi \circ \Phi(f_\sigma)\|_{\sup, \gamma} &\leq \|\eta(\sigma)\|_{\varepsilon, 2} \cdot \prod_{i=1}^n \|\delta_i\|_{\det E_i, \gamma}^{Db_i/e_i} \\ &\leq \sqrt{e_1^{Db_1} \cdots e_N^{Db_N}} \cdot \prod_{i=1}^n \|\delta_i\|_{\det E_i, \gamma}^{Db_i/e_i}.\end{aligned}$$

According to Lemma 2.17,

$$\begin{aligned}&[K : \mathbf{Q}] h_{\min}((\mathbf{P}(\mathcal{E}^{\otimes b}), \mathcal{O}_{\overline{\mathcal{F}}}(1)) /\!\!/ \mathbf{SL}(\mathcal{E})) \\ &\qquad \geq -\sum_{i=1}^n \left(\frac{b_i}{e_i} \sum_{v \in \mathrm{V}_K} \log \|\delta_i\|_{\det E_i, v}\right) - \log \sqrt{e_1^{b_1} \cdots e_N^{b_N}} \\ &\qquad \geq \frac{b_i}{e_i} \widehat{\deg}\, \overline{\mathcal{E}_i} - \frac{1}{2} \sum_{i=1}^n b_i \log \operatorname{rk} \mathcal{E}_i,\end{aligned}$$

and one concludes recalling $\hat{\mu}(\mathcal{E}_i) = \widehat{\deg}(\mathcal{E}_i)/e_i$. $\square$

CHAPTER 2

DIOPHANTINE APPROXIMATION ON $\mathbf{P}^1$ VIA GEOMETRIC INVARIANT THEORY

In this chapter we prove Roth's Theorem (and some variants) thanks to the tools of Geometric Invariant Theory developed in Chapter 1. Let us describe briefly the structure of the present chapter.

In Section 1 we review some material concerning Roth's Theorem and we state the main result of this chapter (the Main Theorem, see Theorem 1.12). More precisely, we show that Roth's Theorem with moving targets is a consequence of an effective statement (the Main Effective Lower Bound, see Theorem 1.7) and how this last result is obtained from the Main Theorem for a suitable choice of parameters.

In Section 2 we introduce the situation of Geometric Invariant Theory that we are interested in. Admitting the semi-stability of the point that we introduce and some intermediate computations, we show that the Fundamental Formula translates into the Main Theorem.

These intermediate computations (upper bounds of the height and the instability measure of the point) are developed in detail in Sections 3 and 4.

Finally, in Section 5, we show the semi-stability of the point, which is the crucial result in order to apply the Fundamental Formula. Our proof is based on the Higher Dimensional Dyson's Lemma by Esnault-Viehweg-Nakamaye (Theorem 2.2). We also give an alternative proof in dimension 2 based on the classical constructions of Wronskians. This argument provides a simple "GIT proof" of the classical Theorem of Dyson-Gelfon'd.

1. Statement of the results

1.1. Roth's Theorem with moving targets and the Main Effective Lower Bound. —

1.1.1. Height and distance on the projective line. — In order to state the results in their most precise way it is convenient to make the following definitions.

Definition 1.1. —

(1) For a K-point $x = (x_0 : x_1)$ of the projective line $\mathbf{P}^1_{\mathbf{Q}}$ its *absolute (logarithmic) height* is

$$h(x) = \frac{1}{[K : \mathbf{Q}]} \sum_{v \in \mathrm{V}_K} \log \|(x_0, x_1)\|_v$$

where V_K denotes the set of places of K and, for every place v,

$$\|(x_0, x_1)\|_v := \begin{cases} \max\{|x_0|_v, |x_1|_v\} & \text{if } v \text{ is non-archimedean} \\ \sqrt{|x_0|_v^2 + |x_1|_v^2} & \text{if } v \text{ is archimedean.} \end{cases}$$

(2) Let K be a number field and $v \in \mathrm{V}_K$ a place of K. The *v-adic spherical distance* on $\mathbf{P}^1$ is defined, for $\mathbf{C}_v$-points $x = (x_0 : x_1)$ and $y = (y_0 : y_1)$, by

$$\mathrm{d}_v(x, y) := \frac{|x_0 y_1 - x_1 y_0|_v}{\|(x_0, x_1)\|_v \|(y_0, y_1)\|_v} \in [0, 1].$$

(3) Let x, y be K-points of $\mathbf{P}^1$. Then $\mathrm{d}_v(x, y) = 1$ for all but finitely many places v of K. For a subset $S \subset \mathrm{V}_K$ (not necessarily finite) set

$$\mathrm{m}_S(x, y) := \sum_{v \in S} -\log \mathrm{d}_v(x, y) \in \mathbf{R}_{\geq 0}.$$

If $S = \{v\}$ is a singleton write $\mathrm{m}_v(x, y)$.

Proposition 1.2 (**[BG06**, Theorem 2.8.21]). — *Let x, y be distinct K-rational points of $\mathbf{P}^1$. Then,*

$$\frac{1}{[K : \mathbf{Q}]} \mathrm{m}_{\mathrm{V}_K}(x, y) = h(x) + h(y).$$

Definition 1.3. — Let a be a point of $\mathbf{P}^1$ defined over a finite extension K' of K, $S \subset \mathrm{V}_K$ a finite subset of V_K and for every $v \in S$ let $\sigma_v \colon K' \to \mathbf{C}_v$ be a K-linear embedding. Denote by $a^{(\sigma_v)}$ the $\mathbf{C}_v$-point of $\mathbf{P}^1$ induced by σ_v and set:

$$\mathrm{m}_S(a, x) := \sum_{v \in S} \mathrm{m}_v(a^{(\sigma_v)}, x).$$

1.1.2. Roth's Theorem with moving targets. — In this paper we prove the following form of Roth's Theorem with moving targets:

Theorem 1.4. — *Let K be a number field, $S \subset \mathrm{V}_K$ a finite subset, K' a finite extension of K and $\kappa > 2$ a real number. For every place $v \in S$ fix an embedding $\sigma_v \colon K' \to \mathbf{C}_v$ which respects K.*

There is no sequence of couples (x_i, a_i) with $i \in \mathbf{N}$ made of a K-rational point x_i of $\mathbf{P}^1$ and a K'-rational point a_i of $\mathbf{P}^1$ distinct from x_i satisfying the following properties:

(1) the sequence $\{h(x_i)\}$ is unbounded;
(2) $h(a_i) = o(h(x_i))$ as i goes to infinity;
(3) for all $i \in \mathbf{N}$ the following inequality is satisfied:

$$\frac{1}{[K : \mathbf{Q}]} \mathrm{m}_S(a_i, x_i) \geq \kappa h(x_i).$$

Vojta's original form of Roth's Theorem with moving targets is more general, in the sense that it allows the target points to be K-rational:

***Theorem 1.5* (*cf.* [**Voj96**, Theorem 1]).** — *Let K be a number field, $S \subset \mathrm{V}_K$ a finite subset, $q \geq 1$ a positive integer and $\kappa > 2$ a real number.*

There is no sequence of $(q+1)$-uples $(x_i, a_i^{(1)}, \ldots, a_i^{(q)})$ ($i \in \mathbf{N}$) made of pairwise distinct[1] *K-rational points of $\mathbf{P}^1$ satisfying the following properties:*

(1) for all $\sigma = 1, \ldots, q$, $h(a_i^{(\sigma)}) = o(h(x_i))$ as i goes to infinity;

(2) for all $i \in \mathbf{N}$ the following inequality is satisfied:

$$\frac{1}{[K : \mathbf{Q}]} \sum_{\sigma=1}^{q} \mathrm{m}_S(a_i^{(\sigma)}, x_i) \geq \kappa h(x_i).$$

Theorem 1.5 implies Theorem 1.4 by means of extending scalars from K to a Galois closure of K' over K and taking the points $a_i^{(\sigma)}$ to be the conjugated points of the points a_i. We ignore at the moment if such a statement can be obtained by the methods expounded in the present paper.

Let us conclude this introduction remarking that for $q = 1, 2$ all these results are a straightforward consequence of Proposition 1.2, which moreover gives an explicit upper bound for height of the points x_i in terms of the height of the points $a_i^{(\sigma)}$. However for $q \geq 3$ this result is ineffective in the sense such that an explicit bound is not known.

1.1.3. Main Effective Lower Bound. — As explained above, there is an intermediate step in the proof of Roth's Theorem which is effective and implies Roth's Theorem through an elementary argument by contradiction that will be reproduced in the next paragraph — this is the principal cause of loss of effectiveness.

This intermediate effective result is a lower bound of the height of K-rational points $x_1, \ldots, x_n$ in terms of their v-adic distances from the algebraic points $a_1, \ldots, a_n$. Although this type of lower bounds plays a crucial role in the seminal work of Bombieri [**Bom82**], it is rarely stated as a stand-alone theorem.

This lower bound is called "Main Effective Lower Bound" and the aim of this paper is to prove it by means of Geometric Invariant Theory. It involves some auxiliary real numbers of geometric nature $r_1, \ldots, r_n$: in the proof they are interpreted as the multi-degree of an invertible sheaf on $(\mathbf{P}^1)^n$.

To state it we need to introduce the crucial concepts that govern the combinatorics in Roth's Theorem:

Definition 1.6. — Let $q, n \geq 1$ be positive integers and let $t \geq 0$ and $\delta \in [0, 1]$ be real numbers.

(1) Consider the set $\Delta_n(t) := \{(\zeta_1, \ldots, \zeta_n) \in [0,1]^n : \zeta_1 + \cdots + \zeta_n < t\}$.

(2) Let $t_{q,n}(\delta) \in [0, n]$ be the unique real number such that

$$1 - q \operatorname{vol} \Delta_n(t_{q,n}(\delta)) = \delta,$$

---

1. *i.e.* for all $i \in \mathbf{N}$ one has $a_i^{(\sigma)} \neq a_i^{(\tau)}$ for every $\sigma \neq \tau$ and one has $x_i \neq a_i^{(\sigma)}$ for every $\sigma = 1, \ldots, q$.

the volume being taken with respect the Lebesgue measure of $\mathbf{R}^n$.

The function $t_{q,n}\colon [0,1] \to [0,n]$ defined in this way is continuous.

(3) Let $R_{q,n}(\delta)$ be the unique positive real number such that

$$\left(1 + \frac{q-1}{R_{q,n}(\delta)}\right)^{n-1} - 1 = \delta \sqrt[n]{\delta}.$$

(4) For a n-uple of positive real numbers $r = (r_1, \dots, r_n)$ write $|r| = r_1 + \cdots + r_n$.

We are now able to state the Main Effective Lower Bound (*cf.* [**Bom82**, Theorem 2] for $n = 2$):

***Theorem 1.7* (Main Effective Lower Bound).** — *Let K' be a finite extension of K of degree $q \geq 2$ and let $S \subset \mathrm{V}_K$ be a finite set of finite places.*

Let $n \geq 2$ be a positive integer, let $0 < \delta < 1/(2 \cdot n!)$ be a real number and let $r = (r_1, \dots, r_n)$ be an n-uple of positive real numbers such that $r_i / r_{i+1} > R_{q,n}(\delta)$ for all $i = 1, \dots, n-1$.

Then, for all $i = 1, \dots, n$ and for all couples (x_i, a_i) made of a K-rational point x_i of $\mathbf{P}^1$ and a K'-rational point a_i of $\mathbf{P}^1$ such that $K(a_i) = K'$, the following inequality holds:

$$\frac{1}{[K:\mathbf{Q}]} t_{q,n}(\delta) \sum_{v \in S} \left(\max_{\sigma\colon K' \to \mathbf{C}_v} \left\{ \min_{i=1,\dots,n} \left\{ r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) \right\} \right\} \right)$$
$$\leq (1 + 2q\sqrt[n]{\delta}) \sum_{i=1}^{n} r_i h(x_i) + \frac{q}{\delta} \sum_{i=1}^{n} r_i h(a_i) + \left(\frac{\log \sqrt{2q}}{\delta} + \log 8 \right) |r|,$$

where, for every place $v \in \mathrm{V}_K$, the embeddings $\sigma\colon K' \to \mathbf{C}_v$ are meant to be K-linear.

Remark 1.8. — We do not claim that the constants appearing in the Main Effective Lower Bound are optimal. For instance, it follows from its proof and Remark 1.13 that the Main Effective Lower Bound holds with $\log 8$ replaced by $\log \sqrt{12}$.

Anyway in order to deduce Roth's Theorem the only thing that matters is the asymptotic behaviour of the right-hand side.

1.1.4. Deducing Roth's Theorem from the Main Effective Lower Bound. — Let us show how the Main Effective Lower Bound (Theorem 1.7) implies Roth's Theorem with moving targets (Theorem 1.4).

Let us begin with the following bound which goes back to the work of Roth and it is based on an explicit version of a phenomenon of concentration of measure (see [**Mil88**]). This is where the number 2 in Roth's Theorem comes from.

Lemma 1.9. — *Let $q, n \geq 1$ be positive integers. Then,*

$$t_{q,n}(0) \geq n/2 - \sqrt{(n \log q)/6}.$$

In particular,

$$\liminf_{n \to \infty} \frac{n}{t_{q,n}(0)} = 2.$$

Proof. — According to [**BG06**, Lemma 6.3.5] for every $0 \leq \varepsilon \leq 1/2$:

$$\operatorname{vol} \Delta_n \left(\left(\frac{1}{2} - \varepsilon \right) n \right) \leq \exp(-6n\varepsilon^2).$$

The result is obtained taking $\varepsilon := 1/2 - t_{q,n}(0)/n$. □

Proof of Theorem 1.4. — Arguing by contradiction suppose that there exists a sequence $\{(x_i, a_i)\}_{i \in \mathbf{N}}$ verifying the conditions in the statement of Theorem 1.4. Up to extracting a subsequence and passing to a sub-extension of K', one may assume $K(a_i) = K'$ for all $i \in \mathbf{N}$ and $q = [K' : K] \geq 2$.

Fix a positive real number $\varepsilon > 0$. By a pigeonhole argument (the so-called "Mahler's Trick", see [**BG06**, 6.4.2]) there exists an infinite subset $I_\varepsilon \subset \mathbf{N}$ such that, for every place $v \in S$ there exists a positive real number $\lambda(\varepsilon, v)$ which verifies, for every $i \in I_\varepsilon$,

$$\lambda(\varepsilon, v) \mathrm{m}_S(a_i, x_i) \leq \mathrm{m}_v(a_i, x_i) \leq \left(\lambda(\varepsilon, v) + \frac{\varepsilon}{\# S} \right) \mathrm{m}_S(a_i, x_i)$$

(the writing of the embeddings σ_v's has been dropped) and

$$1 - \varepsilon \leq \sum_{v \in S} \lambda(\varepsilon, v) \leq 1.$$

Up to renumbering the subsequence $\{(x_i, a_i)\}_{i \in I_\varepsilon}$ one may assume that the previous conditions are satisfied for all $i \in \mathbf{N}$.

Take an integer $n \geq 2$, a positive real number δ and n-uple of positive real numbers r satisfying the conditions in the statement of Theorem 1.7. Applying the latter to the couples (x_i, a_i) for $i = 1, \ldots, n$ and using, for every place $v \in S$,

$$\min_{\sigma \colon K' \to \mathbf{C}_v} \left\{ \min_{i=1,\ldots,n} \left\{ r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) \right\} \right\} \leq \min_{i=1,\ldots,n} \left\{ r_i \mathrm{m}_v(a_i^{(\sigma_v)}, x_i) \right\},$$

one gets

$$\begin{aligned}(1 + 2q \sqrt[n]{\delta}) \sum_{i=1}^n r_i h(x_i) + \frac{1}{\delta} \left(\sum_{i=1}^n r_i \left(q h(a_i) + C \right) \right) \\ \geq \frac{1}{[K : \mathbf{Q}]} t_{q,n}(\delta) \sum_{v \in S} \min_{i=1,\ldots,n} \{ r_i \mathrm{m}_v(a_i, x_i) \} \\ \geq \frac{1}{[K : \mathbf{Q}]} t_{q,n}(\delta)(1 - \varepsilon) \min_{i=1,\ldots,n} \{ r_i \mathrm{m}_S(a_i, x_i) \},\end{aligned}$$

where $C := \log \sqrt{2q} + \log 8$. By hypothesis, for all $i = 1, \ldots, n$,

$$\mathrm{m}_S(a_i, x_i) \geq [K : \mathbf{Q}] \kappa h(x_i),$$

thus,

$$\begin{aligned}\kappa t_{q,n}(\delta)(1 - \varepsilon) \min_{i=1,\ldots,n} \{ r_i h(x_i) \} \\ \leq (1 + 2q \sqrt[n]{\delta}) \sum_{i=1}^n r_i h(x_i) + \frac{1}{\delta} \left(\sum_{i=1}^n r_i \left(q h(a_i) + C \right) \right).\end{aligned}$$

Since the x_i's are infinitely many the ratios of the heights $h(x_{i+1})/h(x_i)$ can be supposed arbitrarily large (larger than $R_{q,n}(\delta)$). Therefore n-uple r can be taken such that

$$r_i h(x_i) = r_j h(x_j),$$

for every $i, j = 1, \ldots, n$. Dividing the preceding inequality by $r_1 h(x_1)$:

$$\kappa t_{q,n}(\delta)(1-\varepsilon) \leq (1 + 2q\sqrt[n]{\delta})n + \frac{q}{\delta}\sum_{i=1}^{n} \frac{h(a_i)}{h(x_i)} + \frac{|r|}{r_1}\frac{C}{\delta h(x_1)}.$$

Since $h(a_i) = o(h(x_i))$ as i goes to infinity one may assume $h(a_i) \leq \delta\sqrt[n]{\delta} h(x_i)$. The ratios r_i/r_{i+1} and the height $h(x_1)$ can be supposed arbitrarily big. Thus,

$$\kappa t_{q,n}(\delta)(1-\varepsilon) \leq (1 + 3q\sqrt[n]{\delta})n.$$

Letting δ and ε go to 0 and n go to infinity, according to Lemma 1.9,

$$\kappa \leq \liminf_{n\to\infty} \frac{n}{t_{q,n}(0)} = 2,$$

which contradicts the hypothesis $\kappa > 2$. □

1.2. Statement of the Main Theorem. —

1.2.1. More combinatorial data. — It is convenient to fix some more notations on the combinatorics appearing in the study.

Definition 1.10. — Let $q, n \geq 1$ be positive integers, $r = (r_1, \ldots, r_n)$ be a n-uple of positive real numbers and $t \geq 0$ be a non-negative integer.

(1) Consider the following subsets of $\mathbf{R}^n$:

$$\square_r := \{(\zeta_1, \ldots, \zeta_n) \in \mathbf{R}^n : 0 \leq \zeta_i \leq r_i \text{ for all } i = 1, \ldots, n\} = \prod_{i=1}^{n} [0, r_i],$$

$$\nabla_r(t) := \left\{(\zeta_1, \ldots, \zeta_n) \in \square_r : \frac{\zeta_1}{r_1} + \cdots + \frac{\zeta_n}{r_n} \geq t\right\},$$

$$\Delta_r(t) := \left\{(\zeta_1, \ldots, \zeta_n) \in \square_r : \frac{\zeta_1}{r_1} + \cdots + \frac{\zeta_n}{r_n} < t\right\} = \square_r - \nabla_r(t).$$

Add $\mathbf{Z}$ in superscript to denote the intersection of these sets with $\mathbf{Z}^n$: write $\square_r^{\mathbf{Z}}$, $\nabla_r^{\mathbf{Z}}(t)$ and $\Delta_r^{\mathbf{Z}}(t)$. For $r = (1, \ldots, 1)$ write $\square_n$, $\nabla_n(t)$ and $\Delta_n(t)$.

(2) If λ_n is the Lebesgue measure on $\mathbf{R}^n$, let $\mu_n \colon [0, n] \to \mathbf{R}$,

$$\mu_n(t) := \int_{\nabla_n(t)} (2\zeta_1 - 1)\, d\lambda_n = \int_{\Delta_n(t)} (1 - 2\zeta_1)\, d\lambda_n.$$

(3) Define:

$$\varepsilon_{q,r} := \prod_{i=1}^{n-1}\left(1 + \max_{i+1 \leq j \leq n}\left\{\frac{r_j}{r_i}\right\}(q-1)\right) - 1.$$

(4) Denote by $u_{q,r}(t)$ the unique real number in $[0, n]$ such that

$$\operatorname{vol}\Delta_n(u_{q,r}(t)) = \min\left\{\max\left\{1 + \varepsilon_{q,r} - q\operatorname{vol}\Delta_n(t), 0\right\}, 1\right\}.$$

Lemma 1.11. — *The function $\mu_n\colon [0,n] \to \mathbf{R}$ is strictly concave*[2]. *Moreover the following properties are satisfied:*

(1) For all $t \in [0,1]$,

$$\mu_n(t) = \frac{t^n}{n!}\left(1 - 2\frac{t}{n+1}\right);$$

(2) $\mu_n(t) \geq 0$ for all $t \in [0,n]$;

(3) $\mu_n(n-t) = \mu_n(t)$ for all $t \in [0,n]$;

(4) The function μ_n is increasing on $[0,n/2]$ and decreasing on $[n/2,n]$.

Proof. — (1) and (3) are easy computations left to the reader. When $n = 1$ statement (1) entails the strict concavity of μ_1. For $n > 1$ arbitrary the strict concavity of μ_n is proved by induction thanks to the relation

$$\mu_n(t) = \int_0^{\min\{t,1\}} \mu_{n-1}(t - \zeta_n)\ d\lambda_1(\zeta_n).$$

(2) Follows from $\mu_n(0) = \mu_n(n) = 0$ and the concavity of μ_n.

(4) Follows from (3) and the strict concavity of μ_n. □

1.2.2. Main Theorem. — Keeping the notation just introduced, the main technical result of the present paper is the following:

***Theorem 1.12* (Main Theorem)**. — *Let K' be a finite extension of K of degree $q \geq 2$ and let $S \subset \mathrm{V}_K$ be a finite subset. Let $n \geq 2$ be an integer, $t_{\boldsymbol{a}}, t_x \geq 0$ non-negative real numbers and let $r = (r_1, \ldots, r_n)$ be an n-uple of positive real numbers. If the following inequality is satisfied,*

$$\mu_n(u_{q,r}(t_{\boldsymbol{a}})) > \mu_n(t_x) + \varepsilon_{q,r}, \tag{SS}$$

then, for all $i = 1, \ldots, n$ and for all couples (x_i, a_i) made of a K-point x of $\mathbf{P}^1$ and K'-point a_i of $\mathbf{P}^1$ such that $K(a_i) = K'$, the following inequality holds:

$$\begin{aligned}\frac{1}{[K:\mathbf{Q}]}(1 - q\operatorname{vol}\Delta_n(t_{\boldsymbol{a}}))t_{\boldsymbol{a}} \sum_{v\in S}\left(\max_{\sigma\colon K'\to\mathbf{C}_v}\left\{\min_{i=1,\ldots,n}\left\{r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i)\right\}\right\}\right)& \\ \leq C^{(1)}_{q,r}(t_{\boldsymbol{a}}, t_x)\sum_{i=1}^n r_i h(x_i) + qC^{(2)}_{q,r}(t_{\boldsymbol{a}}, t_x)\sum_{i=1}^n r_i h(a_i) + C^{(3)}_{q,r}(t_{\boldsymbol{a}}, t_x)|r|&,\end{aligned}$$

where

$$\begin{aligned}C^{(1)}_{q,r}(t_{\boldsymbol{a}}, t_x) &:= \int_{\nabla_n(t_x)} \zeta_1\ d\lambda_n + q\frac{\operatorname{vol}\Delta(u_{q,r}(t_{\boldsymbol{a}})) - \mu_n(t_x)}{2},\\ C^{(2)}_{q,r}(t_{\boldsymbol{a}}, t_x) &:= q\operatorname{vol}\Delta_n(t_{\boldsymbol{a}}) + q\frac{\operatorname{vol}\Delta(u_{q,r}(t_{\boldsymbol{a}})) - \mu_n(t_x)}{2},\\ C^{(3)}_{q,r}(t_{\boldsymbol{a}}, t_x) &:= \operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}}))\log\sqrt{6} + \operatorname{vol}\nabla_n(t_x)\log\sqrt{8} + q\operatorname{vol}\Delta_n(t_{\boldsymbol{a}})\log\sqrt{2q}.\end{aligned}$$

2. That is, for every $t_1 < t_2$ in $[0,n]$ and every $\xi \in\ ]0,1[$,

$$\mu_n(\xi t_1 + (1-\xi)t_2) > \xi\mu_n(t_1) + (1-\xi)\mu_n(t_2).$$

Theorem 1.12 is interesting only when condition (SS) is close to its limit of validity (that is, $\mu_n(u_{q,r}(t_{\boldsymbol{a}})) - \mu_n(t_x) - \varepsilon_{q,r}$ is very small) and $1 - q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}})$ tends to zero. This is the case that leads to Theorem 1.12 in the proof given in the next paragraph.

The fact that this is the only interesting case may be formulated more precisely saying that, as soon as $1 - q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}}) = \delta$, Theorem 1.7 entails Theorem 1.12 with slightly bigger error terms, which are insignificant for applications and arise from simplifications in computations in the proof that follows.

Remark 1.13. — The constant $C^{(3)}_{q,r}(t_{\boldsymbol{a}}, t_x)$ can be slightly sharpen by employing Theorem 1.11 in Chapter 4 instead of Theorem 2.1 in Chapter 1 in the proof of the Main Theorem. Doing like this, one finds that Theorem 1.12 holds with $C^{(3)}_{q,r}(t_{\boldsymbol{a}}, t_x)$ replaced by

$$\operatorname{vol} \Delta_n(u_{q,r}(t_{\boldsymbol{a}})) \log \sqrt{3} + \operatorname{vol} \nabla_n(t_x) \log 2 + q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}}) \log \sqrt{2q},$$

(see also Remark 2.8).

1.3. From the Main Theorem to the Main Effective Lower Bound. — In this section the Main Effective Lower Bound (Theorem 1.7) is deduced from the Main Theorem (Theorem 1.12). Since $r_i/r_{i+1} > R_{q,n}(\delta)$ for every $i = 1, \dots, n-1$, then $\varepsilon_{q,r} < \delta \sqrt[n]{\delta}$.

1.3.1. Choice of the parameters. — The Main Effective Lower Bound is deduced from Theorem 1.12 setting

$$t_{\boldsymbol{a}} := t_{q,n}(\delta).$$

Write also $\widetilde{u}_{q,r}(\delta) := u_{q,r}(t_{q,n}(\delta))$.

Lemma 1.14. — *With the notation introduced above:*

(1) $\operatorname{vol} \Delta_n(\widetilde{u}_{q,r}(\delta)) = \delta + \varepsilon_{q,r} \leq 1/n!$, *hence* $\widetilde{u}_{q,r}(\delta) \leq 1$;

(2) $\displaystyle\int_{\Delta_n(\widetilde{u}_{q,r}(\delta))} \zeta_1 \, d\lambda_n \leq \tfrac{1}{2}(\delta + \varepsilon_{q,r})^{\frac{n+1}{n}}$;

(3) $\mu_n(\widetilde{u}_{q,r}(\delta)) > \varepsilon_{q,r}$;

(4) $\mu_n(\widetilde{u}_{q,r}(\delta)) \leq \varepsilon_{q,r} + \mu_n(\sqrt[n]{n!\delta})$.

Proof. —

(1) By hypothesis $\delta < 1/(2 \cdot n!)$ thus $t_{q,n}(\delta) \leq 1$. Thus,

$$\operatorname{vol} \Delta_n(\widetilde{u}_{q,r}(\delta)) = \delta + \varepsilon_{q,r} \leq 1/n!.$$

Since $\operatorname{vol} \Delta_n(t) = t^n/n!$ for $t \leq 1$ one concludes

$$\widetilde{u}_{q,r}(\delta) = \sqrt[n]{n!(\delta + \varepsilon_{q,r})}.$$

(2) The expression of $\widetilde{u}_{q,r}(\delta)$ found in (1) gives

$$\int_{\Delta_n(\widetilde{u}_{q,r}(\delta))} \zeta_1 \, d\lambda_n = \frac{\widetilde{u}_{q,r}(\delta)^{n+1}}{(n+1)!} = (\delta + \varepsilon_{q,r})^{\frac{n+1}{n}} \frac{\sqrt[n]{n!}}{n+1}.$$

Conclude by noticing $\sqrt[n]{n!}/(n+1) \leq 1/2$ for all $n \geq 1$.

(3) Follow from the explicit expression given by Lemma 1.11 (1),

$$\mu_n(\widetilde{u}_{q,r}(\delta)) = (\delta + \varepsilon_{q,r})\left(1 - \frac{2}{n+1}\sqrt[n]{n!(\delta+\varepsilon_{q,r})}\right),$$

and the hypotheses on δ and $\varepsilon_{q,r}$.

(4) Similar to (3). □

For what concerns the choice of the parameter t_x, roughly speaking, one stresses the validity of condition (SS) to its limit. More precisely, since the function μ_n is strictly decreasing on $[n/2, n]$, there exists a unique real number $w_{q,r}(\delta) \in [n/2, n[$ such that

$$\mu_n(\widetilde{u}_{q,r}(\delta)) = \mu_n(w_{q,r}(\delta)) + \varepsilon_{q,r}.$$

Lemma 1.15. — *With the notation introduced above:*

(1) $\operatorname{vol}\nabla_n(w_{q,r}(\delta)) \leq \delta$*;*

(2) $\displaystyle\int_{\nabla_n(w_{q,r}(\delta))} \zeta_1 \, d\lambda_n \leq \delta$*;*

(3) $\operatorname{vol}\Delta_n(\widetilde{u}_{q,r}(\delta)) - \mu_n(w_{q,r}(\delta)) \leq 3\delta\sqrt[n]{\delta}$.

Proof. —

(1) Lemma 1.14 (3) entails $w_{q,r}(\delta) \geq n - \sqrt[n]{n!\delta}$.

(2) Follows from (1): for every $t \in [n-1, n]$,

$$\int_{\nabla_n(t)} \zeta_1 \, d\lambda_n \leq \operatorname{vol}\nabla_n(t).$$

(3) By definition of $w_{q,r}(\delta)$ and by Definition 1.10 (2):

$$\begin{aligned}\operatorname{vol}\Delta_n(\widetilde{u}_{q,r}(\delta)) - \mu_n(w_{q,r}(\delta)) &= \operatorname{vol}\Delta_n(\widetilde{u}_{q,r}(\delta)) + \varepsilon_{q,r} - \mu_n(\widetilde{u}_{q,r}(\delta))\\ &= 2\int_{\Delta_n(\widetilde{u}_{q,r}(\delta))} \zeta_1 \, d\lambda_n + \varepsilon_{q,r}\\ &\leq (\delta + \varepsilon_{q,r})^{\frac{n+1}{n}} + \varepsilon_{q,r},\end{aligned}$$

where in the last inequality one uses Lemma 1.14 (2). The result follows from the hypotheses $\varepsilon_{q,r} < \delta\sqrt[n]{\delta}$ and $\delta < 1/(2 \cdot n!)$. □

1.3.2. Application of the Main Theorem. — Lemma 1.14 (3) permits us to apply Theorem 1.12 with $t_{\boldsymbol{a}} = t_{q,n}(\delta)$ and $t_x \in]w_{q,r}(\delta), n[$ close enough to $w_{q,r}(\delta)$. Letting t_x tend to $w_{q,r}(\delta)$ and taking in account the estimates given by Lemma 1.15:

$$\begin{aligned}\frac{1}{[K:\mathbf{Q}]}\delta t_{q,n}(\delta)\sum_{v\in S}\left(\max_{\sigma\colon K'\to\mathbf{C}_v}\left\{\min_{i=1,\dots,n}\left\{r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i)\right\}\right\}\right)&\\ \leq \delta\left(1 + \frac{3}{2}q\sqrt[n]{\delta}\right)\sum_{i=1}^n r_i h(x_i) + q\sum_{i=1}^n r_i h(a_i) + |r|C_{q,r}(\delta),&\end{aligned}$$

where $C_{q,r}(\delta) := \delta(1 + \sqrt[n]{\delta})\log\sqrt{6} + \delta\log\sqrt{8} + (1-\delta)\log\sqrt{2q}$. This concludes the proof. □

2. From the Fundamental Formula to the Main Theorem

2.1. Interlude on the index. — Let K be a field of characteristic 0.

2.1.1. Index. — Let $n \geq 1$ be a positive integer and $\mathbf{P} = (\mathbf{P}^1)^n$ be the product of n copies of the projective line over K. For every $i = 1, \ldots, n$ let $\mathrm{pr}_i \colon \mathbf{P} \to \mathbf{P}^1$ be the projection onto the i-th factor.

Let $z = (z_1, \ldots, z_n)$ be a K-point of $\mathbf{P}$ and $b = (b_1, \ldots, b_n)$ be a n-uple of positive real numbers. For every $i = 1, \ldots, n$ let t_i be a local parameter around $z_i \in \mathbf{P}^1(K)$.

Definition 2.1. — Let $f \in \mathcal{O}_{\mathbf{P},z}$ be a non-zero regular function on $\mathbf{P}$ defined on an open neighbourhood of z. The function f develops into power series

$$f = \sum_{\ell = (\ell_1, \ldots, \ell_n) \in \mathbf{N}^n} f_\ell t_1^{\ell_1} \cdots t_n^{\ell_n},$$

with $f_\ell \in K$. The *index of f at z with respect to the weight b* is

$$\mathrm{ind}_b(f, z) := \min \{ b_1 \ell_1 + \cdots + b_n \ell_n : f_\ell \neq 0 \} ;$$

If $f = 0$, then $\mathrm{ind}_b(0, z) := +\infty$.

If $b = (b_1, \ldots, b_n)$ is a n-uple of positive real numbers, the n-uple $(1/b_1, \ldots, 1/b_n)$ is denoted $1/b$ and the index with the respect the weight $1/b$ by $\mathrm{ind}_{1/b}$. The notion of index can be extended to meromorphic sections s of an invertible sheaf L on $\mathbf{P}$ which are regular at z: it suffices to choose a trivialising section s_0 of L around z and set

$$\mathrm{ind}_b(s, z) := \mathrm{ind}_b(s/s_0, z).$$

2.1.2. Higher dimensional Dyson's Lemma. — The main result concerning the index is the Higher Dimensional Dyson's Lemma: the version stated here is due to Nakamaye [**Nak99**]. The original version of Esnault-Viehweg [**EV84**] (which has a slightly bigger error term) would work as well.

Let $r = (r_1, \ldots, r_n)$ be a n-uple of positive integers. Consider the following invertible sheaf on the projective scheme $\mathbf{P}$:

$$\mathcal{O}_{\mathbf{P}}(r) := \mathrm{pr}_1^* \mathcal{O}_{\mathbf{P}^1}(r_1) \otimes \cdots \otimes \mathrm{pr}_n^* \mathcal{O}_{\mathbf{P}^1}(r_n).$$

***Theorem 2.2* (Higher dimensional Dyson's Lemma).** — *Let $z^{(0)}, \ldots, z^{(q)}$ be K-points of $\mathbf{P}$ and $t^{(0)}, \ldots, t^{(q)}$ be non-negative real numbers. Suppose that*

- *$\mathrm{pr}_i(z^{(\sigma)}) \neq \mathrm{pr}_i(z^{(\tau)})$ for every $i = 1, \ldots, n$ and every $\sigma \neq \tau$;*
- *there exists a non-zero global section $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ such that, for every $\sigma = 0, \ldots, q$,*
$$\mathrm{ind}_{1/r}(f, z^{(\sigma)}) \geq t^{(\sigma)}.$$

Then the following inequality is satisfied:

$$\sum_{\sigma=0}^{q} \mathrm{vol}\, \Delta_n(t^{(\sigma)}) \leq 1 + \varepsilon_{q,r},$$

where

$$\varepsilon_{q,r} := \prod_{i=1}^{n-1} \left(1 + \max_{i+1 \leq j \leq n} \left\{ \frac{r_j}{r_i} \right\} \max\{q-1, 0\} \right) - 1.$$

2.1.3. Index at a single point. — Let $z = (z_1, \dots, z_n)$ be a K-point of $\mathbf{P}$, $t \geq 0$ a non-negative real number and $r = (r_1, \dots, r_n)$ a n-uple of positive integers.

Definition 2.3. — Let $Z_{q,r}(z,t)$ be the subscheme of $\mathbf{P}$ defined by the ideal sheaf of regular sections f such that $\operatorname{ind}_{1/r}(f,z) \geq t$. Consider the following linear subspace of $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$:

$$\begin{aligned} K_r(z,t) &:= \operatorname{Ker}\left(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \to \Gamma(Z_r(z,t), \mathcal{O}_{\mathbf{P}}(r))\right) \\ &= \left\{ f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) : \operatorname{ind}_{1/r}(f,z) \geq t \right\}. \end{aligned}$$

Proposition 2.4. — *Keeping the notation introduced above, for every $i = 1, \dots, n$ let T_{i0}, T_{i1} be a basis of $K^{2\vee}$ such that T_{i1} vanishes at z_i.*

(1) The monomials $T_z(\ell) = \bigotimes_{i=1}^n T_{i0}^{r_i - \ell_i} T_{i1}^{\ell_i}$ for $\ell \in \nabla_r^{\mathbf{Z}}(t)$ form a basis of the K-vector space $K_r(z,t)$.

(2) $\dim_K K_r(z,t) = \#\nabla_r^{\mathbf{Z}}(t)$ and

$$\lim_{\alpha \to \infty} \frac{\dim_K K_{\alpha r}(z,t)}{\alpha^n (r_1 \cdots r_n)} = \operatorname{vol} \nabla_n(t).$$

(3) $\dim_K \Gamma(Z_r(z,t), \mathcal{O}_{\mathbf{P}}(r)) = \#\Delta_r^{\mathbf{Z}}(t)$ and

$$\lim_{\alpha \to \infty} \frac{\dim_K \Gamma(Z_{\alpha r}(z,t), \mathcal{O}_{\mathbf{P}}(\alpha r))}{\alpha^n (r_1 \cdots r_n)} = \operatorname{vol} \Delta_n(t).$$

Proof. — Left to the reader as an easy exercise. □

2.1.4. Index at multiple points. — Let $r = (r_1, \dots, r_n)$ be a n-uple of positive integers and let $t \geq 0$ be a non-negative real number and let $N \geq 1$ be a positive integer.

For $\sigma = 1, \dots, q$ let $z^{(\sigma)} = (z_1^{(\sigma)}, \dots, z_n^{(\sigma)})$ be a K-point of $\mathbf{P}$. Suppose $z_i^{(\sigma)} \neq z_i^{(\tau)}$ for every $\sigma \neq \tau$ and every $i = 1, \dots, n$.

Definition 2.5. — Consider the q-uple $\boldsymbol{z} = (z^{(1)}, \dots, z^{(q)})$. Consider the closed subscheme of $\mathbf{P}$,

$$Z_{q,r}(\boldsymbol{z},t) := \bigcup_{\sigma=1}^q Z_r(z^{(\sigma)}, t).$$

Consider the following linear subspace of $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$:

$$\begin{aligned} K_{q,r}(\boldsymbol{z},t) &:= \operatorname{Ker}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \to \Gamma(Z_{q,r}(\boldsymbol{z},t), \mathcal{O}_{\mathbf{P}}(r))) \\ &= \left\{ f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) : \operatorname{ind}_{1/r}(f, z^{(\sigma)}) \geq t \text{ for all } \sigma = 1, \dots, q \right\}. \end{aligned}$$

Recall that $u_{q,r}(t)$ is defined as the unique real number belonging to $[0,n]$ and such that

$$\operatorname{vol} \Delta_n(u_{q,r}(t)) = \min \left\{ \max \left\{ 1 + \varepsilon_{q,r} - q \operatorname{vol} \Delta_n(t), 0 \right\}, 1 \right\}.$$

Proposition 2.6. — *Keeping the notation introduced above:*

(1) $\displaystyle \Gamma(Z_{q,r}(\boldsymbol{z},t), \mathcal{O}_{\mathbf{P}}(r)) = \bigoplus_{\sigma=1}^q \Gamma(Z_r(z^{(\sigma)}, t), \mathcal{O}_{\mathbf{P}}(r)).$

(2) $\dim_K K_{q,r}(\boldsymbol{z},t) \geq \prod_{i=1}^{n}(r_i+1) - q\#\Delta_r^{\mathbf{Z}}(t)$. *In particular,*

$$\liminf_{\alpha\to\infty} \frac{\dim_K K_{q,\alpha r}(\boldsymbol{z},t)}{\alpha^n(r_1\cdots r_n)} \geq 1 - q\operatorname{vol}\Delta_n(t). \tag{2.1.1}$$

(3) Suppose $u_{q,r}(t) < n$. *Let* $z^{(0)} \in \mathbf{P}^1(K)$ *be a point such that for* $i = 1,\dots,n$ *and* $\sigma = 1,\dots,q$, *one has* $\operatorname{pr}_i(z^{(0)}) \neq \operatorname{pr}_i(z^{(\sigma)})$. *Then, for* $t^{(0)} > u_{q,r}(t)$,

$$K_{q,r}(\boldsymbol{z},t) \cap K_r(z^{(0)},t^{(0)}) = 0.$$

(4) $\limsup_{\alpha\to\infty} \frac{\dim_K K_{q,\alpha r}(\boldsymbol{z},t)}{\alpha^n(r_1\cdots r_n)} \leq \operatorname{vol}\Delta_n(u_{q,r}(t))$.

Remark that (3) and (4) are consequences of the Higher Dimensional Dyson's Lemma.

Proof. —

(1) This is because the closed subschemes $Z_r(z^{(\sigma)},t)$'s are pairwise disjoint (see [**EV84**, Lemma 2.8]).

(2) Using the definition of $K_{q,r}(\boldsymbol{z},t)$ as $\operatorname{Ker}(\Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)) \to \Gamma(Z_{q,r}(\boldsymbol{z},t),\mathcal{O}_{\mathbf{P}}(r)))$, the preceding point yields

$$\begin{aligned}\dim_K K_{q,r}(\boldsymbol{z},t) &\geq \dim_K \Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)) - \dim_K \Gamma(Z_{q,r}(\boldsymbol{z},t),\mathcal{O}_{\mathbf{P}}(r))\\ &= \dim_K \Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)) - \sum_{\sigma=1}^{q}\dim_K \Gamma(Z_r(z^{(\sigma)},t),\mathcal{O}_{\mathbf{P}}(r))\\ &= \prod_{i=1}^{n}(r_i+1) - q\#\Delta_r^{\mathbf{Z}}(t),\end{aligned}$$

where in the last equality ones uses Proposition 2.4 (3).

(3) By contradiction suppose that there exists a non-zero element f in the intersection $K_{q,r}(\boldsymbol{z},t) \cap K_r(z^{(0)},t^{(0)})$. The Higher Dimensional Dyson's Lemma entails

$$\sum_{\sigma=1}^{q}\operatorname{vol}\Delta_n(t) + \operatorname{vol}\Delta_n(t^{(0)}) \leq 1 + \varepsilon_{q,r},$$

and thus $\operatorname{vol}\Delta_n(t^{(0)}) \leq \operatorname{vol}\Delta_n(u_{q,r}(t))$. This yields $t^{(0)} \leq u_{q,r}(t)$ which contradicts the hypothesis $t^{(0)} > u_{q,r}(t)$.

(4) If $\operatorname{vol}\Delta_n(u_{q,r}(t)) = 1$, which implies $u_{q,r}(t) = n$, the statement is trivial. Assume $u_{q,r}(t) < n$. According to (3), for $t^{(0)} > u_{q,r}(t)$,

$$K_{q,r}(\boldsymbol{z},t) \cap K_r(z^{(0)},t^{(0)}) = 0.$$

Therefore Grassman's formula of dimensions gives

$$\begin{aligned}\dim_K K_{q,r}(\boldsymbol{z},t) &\leq \dim_K \Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)) - \dim_K K_r(z^{(0)},t^{(0)})\\ &= \dim_K \Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)) - \#\nabla_r^{\mathbf{Z}}(t^{(0)}),\end{aligned}$$

where one uses Proposition 2.4 (2) in the last equality. The statement is then obtained by applying this inequality to any positive multiple of r and then letting $t^{(0)}$ tend to $u_{q,r}(t)$. $\square$

2.2. Definition of the "moduli problem". — Let K be a number field and let V_K be its set of places.

2.2.1. Linear actions on grassmannians. — Let $\mathcal{E}$ be a flat $\mathfrak{o}_K$-module of finite rank. For every non-negative integer N let $\mathbf{Grass}_N(\mathcal{E})$ be the grassmannian of subspaces of rank N of $\mathcal{E}$, *i.e.* the $\mathfrak{o}_K$-scheme representing the functor

$$\begin{aligned} \underline{\mathrm{Gr}}_N(\mathcal{E})\colon \{\mathfrak{o}_K\text{-schemes}\} &\longrightarrow \{\text{sets}\} \\ (f\colon X \to \operatorname{Spec} \mathfrak{o}_K) &\longmapsto \left\{ \begin{matrix} \text{locally free sub-}\mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{of } f^*\mathcal{E} \text{ of rank } N \text{ with flat cokernel} \end{matrix} \right\}. \end{aligned}$$

Suppose that an $\mathfrak{o}_K$-group scheme $\mathcal{G}$ acts linearly on the $\mathfrak{o}_K$-module $\mathcal{E}$. Then, for every integer $N \geq 0$, the $\mathfrak{o}_K$-group scheme $\mathcal{G}$ acts on the grassmannian $\mathbf{Grass}_N(\mathcal{E})$ of subspaces of rank N, on the projective space $\mathbf{P}(\bigwedge^N \mathcal{E})$ and in an equivariant way on the invertible sheaf $\mathcal{O}_{\bigwedge^N \mathcal{E}}(1)$. The Plücker embedding $\varpi\colon \mathbf{Grass}_N(\mathcal{E}) \to \mathbf{P}(\bigwedge^N \mathcal{E})$ is $\mathcal{G}$-equivariant.

2.2.2. Back to the Main Effective Lower Bound. — Let K' be a finite extension of K of degree $q \geq 2$. Let $n \geq 1$ be a positive integer. Let $\mathbf{P} = (\mathbf{P}^1_{\mathfrak{o}_K})^n$ be the product of n copies of the projective line over $\mathfrak{o}_K$. Let $r = (r_1, \ldots, r_n)$ be a n-uple of positive integers and let $\mathcal{O}_{\mathbf{P}}(r)$ be the following invertible sheaf on $\mathbf{P}$,

$$\mathcal{O}_{\mathbf{P}}(r) := \operatorname{pr}_1^* \mathcal{O}_{\mathbf{P}^1}(r_1) \otimes \cdots \otimes \operatorname{pr}_n^* \mathcal{O}_{\mathbf{P}^1}(r_n).$$

For all $i = 1, \ldots, n$ let x_i be a K-point of $\mathbf{P}^1_{\mathfrak{o}_K}$ and let a_i be a K'-point of $\mathbf{P}^1_{\mathfrak{o}_K}$ such that $K(a_i) = K'$. Consider the following points of $\mathbf{P}$:

$$\begin{aligned} x &:= (x_1, \ldots, x_n), \\ a &:= (a_1, \ldots, a_n). \end{aligned}$$

Set $\boldsymbol{a} := \{a^{(\sigma)} : \sigma \in \operatorname{Hom}_{K\text{-alg}}(K', \bar{\mathbf{Q}})\}$. Let $t_x, t_{\boldsymbol{a}} \geq 0$ be non-negative real numbers and consider the following K-vector spaces

$$\begin{aligned} K_r(x, t_x) &:= \{f \in \Gamma(\mathbf{P}_K, \mathcal{O}_{\mathbf{P}}(r)) : \operatorname{ind}_{1/r}(f, x) \geq t_x\}, \\ K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) &:= \{f \in \Gamma(\mathbf{P}_K, \mathcal{O}_{\mathbf{P}}(r)) : \operatorname{ind}_{1/r}(f, a) \geq t_{\boldsymbol{a}}\}, \end{aligned}$$

where $\mathbf{P}_K$ denotes the generic fiber of $\mathbf{P}$.[3] Since f is K-rational and a is not, imposing index at a automatically imposes the same index condition at all conjugates of a.

Denote $k_r(t_x)$ and $k_{q,r}(t_{\boldsymbol{a}})$ respectively the dimension of the K-vector spaces $K_r(x, t_x)$ and $K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$. In such a way, these sub-vector spaces of the global sections $\Gamma(\mathbf{P}_K, \mathcal{O}_{\mathbf{P}}(r))$ define the following K-points of grassmannians:

$$[K_r(x, t_x)] \in \mathbf{Grass}_{k_r(t_x)}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))),$$
$$[K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})] \in \mathbf{Grass}_{k_{q,r}(t_{\boldsymbol{a}})}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))).$$

The $\mathfrak{o}_K$-reductive group $\mathbf{SL}^n_{2,\mathfrak{o}_K}$ acts on the product $\mathbf{P} = \left(\mathbf{P}^1_{\mathfrak{o}_K}\right)^n$ and thus on the grassmannians mentioned above. Write

$$\mathcal{F}_r(t_x) := \bigwedge^{k_r(t_x)} \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)),$$
$$\mathcal{F}_{q,r}(t_{\boldsymbol{a}}) := \bigwedge^{k_{q,r}(t_{\boldsymbol{a}})} \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r),$$

and consider the Plücker embeddings, which are equivariant morphisms with respect to the action of $\mathbf{SL}^n_{2,\mathfrak{o}_K}$:

$$\mathbf{Grass}_{k_r(t_x)}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))) \longrightarrow \mathbf{P}(\mathcal{F}_r(t_x)),$$
$$\mathbf{Grass}_{k_{q,r}(t_{\boldsymbol{a}})}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))) \longrightarrow \mathbf{P}(\mathcal{F}_{q,r}(t_{\boldsymbol{a}})).$$

2.2.3. The geometric invariant theory data. — We shall apply the Fundamental Formula to the following situation:

$$\begin{aligned}
P_r &= ([K_r(x, t_x)], [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]),\\
\mathcal{X}_r &= \mathbf{Grass}_{k_r(t_x)}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))) \times_{\mathfrak{o}_K} \mathbf{Grass}_{k_{q,r}(t_{\boldsymbol{a}})}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))),\\
\mathcal{G} &= \mathbf{SL}^n_{2,\mathfrak{o}_K},\\
\mathcal{L}_r &= \text{polarization of } \mathcal{X}_r \text{ given by the Plücker embeddings of the grassmannians,}
\end{aligned}$$

and the closed embedding:

$$j_r \colon \mathcal{X}_r \longrightarrow \mathbf{P}(\mathcal{F}_r(t_x)) \times_{\mathfrak{o}_K} \mathbf{P}(\mathcal{F}_{q,r}(t_{\boldsymbol{a}})) \longrightarrow \mathbf{P}(\mathcal{F}_r(t_x) \otimes_{\mathfrak{o}_K} \mathcal{F}_{q,r}(t_{\boldsymbol{a}})).$$

---

3. Here the index of the section f, which is defined over K, at the point a, which is defined over K', means the index of the extension of f to K'. Alternatively, one may define the $\bar{\mathbf{Q}}$-vector space

$$\overline{K}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) := \{f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \otimes_{\mathfrak{o}_K} \bar{\mathbf{Q}} : \mathrm{ind}_{1/r}(f, a^{(\sigma)}) \geq t_{\boldsymbol{a}} \text{ for all } \sigma \colon K' \to \bar{\mathbf{Q}}\}$$

and notice that it is invariant under Galois action, thus it comes from a K-vector space $K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$. In any case,

$$K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) \otimes_K \bar{\mathbf{Q}} = \bigcap_{\sigma \colon K' \to \bar{\mathbf{Q}}} \overline{K}_r(a^{(\sigma)}, t_{\boldsymbol{a}}),$$

where $\overline{K}_r(a^{(\sigma)}, t_{\boldsymbol{a}}) := \{f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \otimes_{\mathfrak{o}_K} \bar{\mathbf{Q}} : \mathrm{ind}_{1/r}(f, a^{(\sigma)}) \geq t_{\boldsymbol{a}}\}$.

(The first arrow is the Plücker embedding of the Grassmannians and the second one is the Segre embedding). For every embedding $\gamma\colon K \to \mathbf{C}$ the complex vector spaces

$$\mathcal{F}_r(t_x) \otimes_\gamma \mathbf{C} = \bigwedge^{k_r(t_x)} \left(\bigotimes_{i=1}^{n} \operatorname{Sym}^{r_i} \mathbf{C}^{2\vee} \right),$$
$$\mathcal{F}_{q,r}(t_{\boldsymbol{a}}) \otimes_\gamma \mathbf{C} = \bigwedge^{k_{q,r}(t_{\boldsymbol{a}})} \left(\bigotimes_{i=1}^{n} \operatorname{Sym}^{r_i} \mathbf{C}^{2\vee} \right),$$

are respectively equipped with the hermitian norms $\| \cdot \|_{\mathcal{F}_r(t_x),\gamma}$ and $\| \cdot \|_{\mathcal{F}_{q,r}(t_{\boldsymbol{a}}),\gamma}$ obtained by tensor operations (see paragraph 0.3 on page 11). Endow the complex vector space $\mathcal{F}_r(t_x)_\gamma \otimes_{\mathbf{C}} \mathcal{F}_{q,r}(t_{\boldsymbol{a}})_\gamma$ with the tensor norm associated to these norms. The result hermitian norm is invariant under the action of $\mathbf{SU}_2^n$. Denote $\overline{\mathcal{L}}_r$ for the associated hermitian invertible sheaf on $\mathcal{X}_r$.

2.3. Proof of the Main Theorem. —

2.3.1. In this section Theorem 1.12 is deduced admitting a semi-stability result (Theorem 2.7) that is proved in section 5.2 and some intermediate computations (namely Propositions 3.1, 3.2 and 4.1) detailed in sections 3 and 4.

To show Theorem 1.12, by an approximation argument, the n-uple $r = (r_1, \dots, r_n)$ is assumed to be made of positive rational numbers. Even better, r can be taken with integer coefficients as the Main Effective Lower Bound is homogeneous in r.

2.3.2. Semi-stability conditions. — To prove Theorem 1.12 one applies the Fundamental Formula to the point P_r, so one must show that P_r is semi-stable. In Section 5.2 the following is proved:

Theorem 2.7. — *Let $n \geq 1$ be a positive integer and $r = (r_1, \dots, r_n)$ be a n-uple of positive integers. Let $t_x, t_{\boldsymbol{a}} \geq 0$ be real numbers with $t_{\boldsymbol{a}} < t_{q,n}(0)$. If the inequality*

$$\mu_n(u_{q,r}(t_{\boldsymbol{a}})) > \mu_n(t_x) + \varepsilon_{q,r},$$

is satisfied then there exists a positive integer $\alpha_0 = \alpha_0(q, n, r, t_{\boldsymbol{a}}, t_x)$ such that, for every integer $\alpha \geq \alpha_0$, the K-point $P_{\alpha r} \in \mathcal{X}_{\alpha r}(K)$ is semi-stable under the action of $\mathbf{SL}_2^n$ with respect to the polarization given by the Plücker embeddings.

2.3.3. Applying the Fundamental Formula. — The numerical condition appearing in the previous statement is exactly the condition (SS) in Theorem 1.12. Thus according to Theorem 2.7 there exists a positive integer $\alpha_0 = \alpha_0(q, n, r, t_{\boldsymbol{a}}, t_x)$ such that, for every integer $\alpha \geq \alpha_0$, the K-point

$$P_{\alpha r} = ([K_{\alpha r}(x, t_x)], [K_{q,\alpha r}(\boldsymbol{a}, t_{\boldsymbol{a}})])$$

is semi-stable. The Fundamental Formula (or, better, Corollary 1.7 in Chapter 1) applied for every $\alpha \geq \alpha_0$ to the point $P_{\alpha r}$ gives the following inequality:

$$h_{\overline{\mathcal{L}}_{\alpha r}}(P_{\alpha r}) + \frac{1}{[K:\mathbf{Q}]} \sum_{v \in S} \iota_v(P_{\alpha r}) \geq h_{\min}((\mathcal{X}_{\alpha r}, \overline{\mathcal{L}}_{\alpha r}) /\!\!/ \mathcal{G}),$$

where one uses that the instability measures are non-positive. Dividing the previous expression by $\alpha^{n+1}(r_1 \cdots r_n)$ and letting α go to infinity,

$$
\begin{aligned}
(2.3.1) \quad & -\frac{1}{[K:\mathbf{Q}]} \sum_{v \in S} \limsup_{\alpha \to \infty} \frac{\iota_v(P_{\alpha r})}{\alpha^{n+1}(r_1 \cdots r_n)} \\
& \leq \limsup_{\alpha \to \infty} \frac{h_{\overline{\mathcal{L}}_{\alpha r}}(P_{\alpha r})}{\alpha^{n+1}(r_1 \cdots r_n)} - \limsup_{\alpha \to \infty} \frac{h_{\min}\big((\mathcal{X}_{\alpha r}, \overline{\mathcal{L}}_{\alpha r}) /\!\!/ \mathcal{G}\big)}{\alpha^{n+1}(r_1 \cdots r_n)}.
\end{aligned}
$$

In the following paragraphs the terms appearing in the preceding inequality are estimated.

The bound of the term involving the height of the point P_r will make appear the height of the points x_i's and a_i's. It is the counterpart of the classical upper bound of the size of the auxiliary polynomial made by means of Siegel's Lemma. Here it will be a direct consequence of basic definitions in Arakelov geometry.

The term where the instability measure occurs is of local nature and will make intervene the distance between the algebraic and the rational point. In the classical framework this corresponds to the Taylor expansion of the auxiliary polynomial around the algebraic point.

The term involving the lowest height on the quotient will finally play the role of the constant terms.

2.3.4. Upper bound of the height. — The Plücker embeddings give a closed isometric embedding of $\mathcal{X}_r$ into $\mathbf{P}(\mathcal{F}_{q,r}(t_{\boldsymbol{a}})) \times \mathbf{P}(\mathcal{F}_r(t_x))$. Thus:

$$
h_{\overline{\mathcal{L}}_r}(P_r) = h_{\overline{\mathcal{F}}_r(t_x)}([K_r(x, t_x)]) + h_{\overline{\mathcal{F}}_{q,r}(t_{\boldsymbol{a}})}([K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]).
$$

Some elementary estimates of Arakelov degrees (see Propositions 3.1-3.2) give:

$$
\begin{aligned}
h_{\overline{\mathcal{F}}_r(t_x)}([K_r(x, t_x)]) &\leq \sum_{i=1}^{n} \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_x)} \ell_i h(x_i), \\
h_{\overline{\mathcal{F}}_{q,r}(t_{\boldsymbol{a}})}([K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) &\leq \left(\prod_{i=1}^{n} (r_i + 1) - k_{q,r}(t_{\boldsymbol{a}}) \right) \left(q \sum_{i=1}^{n} r_i h(a_i) + |r| \log \sqrt{2q} \right).
\end{aligned}
$$

Applying these estimates to every positive integer multiple of r:

$$
\begin{aligned}
(2.3.2) \quad \limsup_{\alpha \to \infty} \frac{h_{\overline{\mathcal{L}}_{\alpha r}}(P_{\alpha r})}{\alpha^{n+1}(r_1 \cdots r_n)} &\leq \left(\int_{\nabla_n(t_x)} \zeta_1 \, d\lambda \right) \sum_{i=1}^{n} r_i h(x_i) \\
& \quad + q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}}) \left(q \sum_{i=1}^{n} r_i h(a_i) + |r| \log \sqrt{2q} \right).
\end{aligned}
$$

2.3.5. Upper bound of the instability measure. — Let v be a place of K. If the place v is non-archimedean:

$$-\iota_v(P_r) \geq \max_{\sigma\colon K'\to\mathbf{C}_v} \Bigg\{ k_{q,r}(t_{\boldsymbol{a}})t_{\boldsymbol{a}} \min_{i=1,\dots,n} \Big\{ r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) \Big\} + \sum_{i=1}^{n} \Bigg(\sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)} \ell_i - \frac{k_r(t_x)+k_{q,r}(t_{\boldsymbol{a}})}{2} r_i \Bigg) \mathrm{m}_v(a_i^{(\sigma)}, x_i) \Bigg\},$$

whereas in the archimedean case the previous lower bound holds when the error term

$$k_{q,r}(t_{\boldsymbol{a}}) \sum_{i=1}^{n} \log\sqrt{r_i+1} + k_{q,r}(t_{\boldsymbol{a}})|r|\log\sqrt{3} + k_r(t_x)|r|\log 2$$

is subtracted from the right-hand side of the previous lower bound. These bounds are proved in Section 4 (see Proposition 4.1). If v is non-archimedean, then applying these estimates to every positive integer multiple of f, and using Propositions 2.6 (2), 2.4 (2) and 2.6 (4):

$$-\limsup_{\alpha\to\infty} \frac{\iota_v(P_{\alpha r})}{\alpha^{n+1}(r_1\cdots r_n)} \geq \max_{\sigma\colon K'\to\mathbf{C}_v} \Bigg\{ (1-q\operatorname{vol}\Delta_n(t_{\boldsymbol{a}}))t_{\boldsymbol{a}} \min_{i=1,\dots,n} \Big\{ r_i\mathrm{m}_v(a_i^{(\sigma)}, x_i) \Big\} + \frac{\mu_n(t_x) - \operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}}))}{2} \sum_{i=1}^{n} r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) \Bigg\}$$

If v is archimedean, the term $|r|(\operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}}))\log\sqrt{3} + \operatorname{vol}\nabla_n(t_x)\log 2)$ has to be subtracted from the right-hand side. By Definition 1.10 (2),

$$\mu_n((u_{q,r}(t_{\boldsymbol{a}})) \leq \operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}})),$$

thus condition (SS) entails $\mu_n(t_x) - \operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}})) < 0$. Bound from above the term $\mathrm{m}_v(a_i^{(\sigma)}, x_i)$ by

$$\sum_{w|v} \mathrm{m}_w(a_i, x_i),$$

the sum being taken over the places w of K' over v. Taking the sum over the places of S and noticing that by Proposition 1.2,

$$\sum_{v\in S}\sum_{w|v} \mathrm{m}_w(a_i,x_i) \leq \sum_{v\in S}\sum_{w\in \mathrm{V}_{K'}} \mathrm{m}_w(a_i,x_i) = [K':\mathbf{Q}](h(x)+h(a_i)),$$

conclude that the term

$$-\frac{1}{[K:\mathbf{Q}]}\sum_{v\in S} \limsup_{\alpha\to\infty} \frac{\iota_v(P_{\alpha r})}{\alpha^{n+1}(r_1\cdots r_n)},$$

is bounded below by

$$\frac{1}{[K:\mathbf{Q}]}(1-q\operatorname{vol}\Delta_n(t_{\boldsymbol{a}}))t_{\boldsymbol{a}}\sum_{v\in S}\max_{\sigma\colon K'\to\mathbf{C}_v}\left\{\min_{i=1,\dots,n}\left\{r_i\mathrm{m}_v(a_i^{(\sigma)},x_i)\right\}\right\} \\ +\frac{\mu_n(t_x)-\operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}}))}{2}\sum_{i=1}^n r_iq(h(x_i)+h(a_i)) \\ -|r|(\operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}}))\log\sqrt{3}+\operatorname{vol}\nabla_n(t_x)\log 2). \tag{2.3.3}$$

2.3.6. Lower bound of the height on the quotient. — For $i=1,\dots,n$ set $\overline{\mathcal{E}}_i=\mathfrak{o}_K^2$ and $b_i=-r_i(k_{q,r}(t_{\boldsymbol{a}})+k_r(t_x))$. Apply the lower bound given by Theorem 2.1 in Chapter 1 to the representation

$$\mathcal{G}=\mathbf{SL}(\mathcal{E}_1)\times_{\mathfrak{o}_K}\cdots\times_{\mathfrak{o}_K}\mathbf{SL}(\mathcal{E}_n)\longrightarrow\mathbf{GL}(\mathcal{F}_{q,r}(t_{\boldsymbol{a}})\otimes\mathcal{F}_r(t_x)),$$

and to the surjection

$$\varpi\colon\bigotimes_{i=1}^n\mathcal{E}_i^{\otimes b_i}=\bigotimes_{i=1}^n\left(\mathfrak{o}_K^{2\vee}\right)^{\otimes r_i(k_{q,r}(t_{\boldsymbol{a}})+k_r(t_x))}\longrightarrow\mathcal{F}_{q,r}(t_{\boldsymbol{a}})\otimes\mathcal{F}_r(t_x).$$

The hermitian vector bundle $\overline{\mathcal{E}}_i$ is trivial, thus $\hat{\mu}(\overline{\mathcal{E}}_i)=0$ for $i=1,\dots,n$. Through the closed $\mathcal{G}$-equivariant embedding $j_r\colon\mathcal{X}_r\to\mathbf{P}(\mathcal{F}_{q,r}(t_{\boldsymbol{a}})\otimes\mathcal{F}_r(t_x))$ Theorem 2.1 yields

$$h_{\min}((\mathcal{X}_r,\overline{\mathcal{L}}_r)/\!\!/\mathcal{G})\geq h_{\min}\left((\mathbf{P}(\mathcal{F}_{q,r}(t_{\boldsymbol{a}})\otimes\mathcal{F}_r(t_x)),\mathcal{O}(1))/\!\!/\mathcal{G}\right) \\ \geq-(k_{q,r}(t_{\boldsymbol{a}})+k_r(t_x))|r|\log\sqrt{2}-\frac{1}{2}\left(\log k_{q,r}(t_{\boldsymbol{a}})!+\log k_r(t_x)!\right),$$

where the term $-1/2\,(\log k_{q,r}(t_{\boldsymbol{a}})!+\log k_r(t_x)!)$ is due to the ratio between the hermitian norm on the alternating product and the quotient norm with the respect to surjection ϖ (see 0.3 on page 11). Thanks to Stirling's approximation,

$$\lim_{\alpha\to\infty}\frac{\log k_{q,r}(t_{\boldsymbol{a}})!}{\alpha^{n+1}(r_1\cdots r_n)}=0,$$

and similarly for $k_r(t_x)$. The previous estimates, applied to every positive multiple of r, give:

$$-\limsup_{\alpha\to\infty}\frac{h_{\min}((\mathcal{X}_{\alpha r},\overline{\mathcal{L}}_{\alpha r})/\!\!/\mathcal{G})}{\alpha^{n+1}(r_1\cdots r_n)} \\ \leq(\operatorname{vol}\Delta_n(u_{q,r}(t_{\boldsymbol{a}}))+\operatorname{vol}\nabla_n(t_x))\,|r|\log\sqrt{2}. \tag{2.3.4}$$

Remark 2.8. — The refined version of Theorem 2.1 in Chapter 1 given by Theorem 1.11 in Chapter 4, gives

$$h_{\min}((\mathcal{X}_r,\overline{\mathcal{L}}_r)/\!\!/\mathcal{G})\geq-\tfrac{1}{2}(\log k_{q,r}(t_{\boldsymbol{a}})!+\log k_r(t_x)!),$$

thus,

$$-\limsup_{\alpha\to\infty}\frac{h_{\min}((\mathcal{X}_{\alpha r},\overline{\mathcal{L}}_{\alpha r})/\!\!/\mathcal{G})}{\alpha^{n+1}(r_1\cdots r_n)}\leq 0.$$

To conclude the proof of the Main Theorem it suffices to bound the asymptotic terms in (2.3.1) taking in account the inequalities (2.3.2), (2.3.3) and (2.3.4).

3. Upper bound of the height

3.1. Rational point. —

Proposition 3.1. — *With the notation introduced in Section 2.2,*

$$h_{\overline{\mathcal{F}}_r(t_x)}([K_r(x,t_x)]) \leq \sum_{i=1}^{n} \sum_{\ell \in \nabla_n^{\mathbf{Z}}(r,t_x)} \ell_i h(x_i).$$

Proof. — Let T_0, T_1 be the canonical basis of $K^{2\vee}$. For every $i = 1, \ldots, n$ let $(x_{i0}, x_{i1}) \in K^2$ be a generator of the line $x_i \in \mathbf{P}^1(K)$. Suppose that x_{i0} is non-zero. For every n-uple of non-negative integers $\ell \in \square_r$ define

$$T(\ell) := \bigotimes_{i=1}^{n} T_0^{r_i - \ell_i} T_{x_i}^{\ell_i}$$

where $T_{x_i} = x_{i0}T_1 - x_{i1}T_0$. A basis of the K-vector space $K_r(x,t_x)$ is given by the elements $T(\ell)$ while ℓ ranges in $\nabla_r^{\mathbf{Z}}(t_x)$.

Let v be a place of K. The Hadamard inequality (0.3.1) on page 12 gives

$$\log \left\| \bigwedge_{\ell \in \nabla_r^{\mathbf{Z}}(t,x)} T(\ell) \right\|_{\overline{\mathcal{F}}_r(t_x),v} \leq \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_x)} \log \|T(\ell)\|_{\Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)),v}.$$

For every n-uple of non-negative integers $\ell \in \square_r$ the sub-multiplicativity of the norm on symmetric powers gives

$$\begin{aligned}\log \|T(\ell)\|_{\Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)),v} &= \sum_{i=1}^{n} \log \|T_0^{r_i-\ell_i} T_{x_i}^{\ell_i}\|_v \\ &\leq \sum_{i=1}^{n} (r_i - \ell_i) \log \|T_0\|_v + \sum_{i=1}^{n} \ell_i \log \|T_{x_i}\|_v = \sum_{i=1}^{n} \ell_i \log \|x_i\|_v.\end{aligned}$$

Conclude the proof by taking the sum over all places. □

3.2. Target points. —

Proposition 3.2. — *With the notation introduced in Section 2.2,*

$$h_{\overline{\mathcal{F}}_{q,r}(t_{\boldsymbol{a}})}([K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \leq \left(\prod_{i=1}^{n} (r_i + 1) - k_{q,r}(t_{\boldsymbol{a}}) \right) \left(q \sum_{i=1}^{n} r_i h(a_i) + |r| \log \sqrt{2q} \right).$$

The rest of this section is devoted to the proof of this upper bound.

3.2.1. Equip the $\mathfrak{o}_K$-module $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ with the hermitian metric induced by the identification

$$\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) = \bigotimes_{i=1}^{n} \mathrm{Sym}^{r_i} \left(\mathfrak{o}_K^{2\vee} \right).$$

Denote by $\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ the resulting $\mathfrak{o}_K$-hermitian vector bundle. The $\mathfrak{o}_K$-hermitian vector bundle $\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ is not trivial since the basis of $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ given by the elements

$$T(\ell) = \bigotimes_{i=1}^{n} T_0^{r_i - \ell_i} T_1^{\ell_i}$$

is orthogonal but not orthonormal. Anyway, for every place v, the sub-multiplicativity of the norm on symmetric powers gives

$$\log \|T(\ell)\|_{\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)), v} \leq \sum_{i=1}^{n} (r_i - \ell_i) \log \|T_0\|_v + \sum_{i=1}^{n} \ell_i \log \|T_1\|_v = 0.$$

In particular,

$$\hat{\mu}(\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))) \geq - \sum_{v \in \mathrm{V}_K} \sum_{\ell \in \square_r^{\mathbf{Z}}} \log \|T(\ell)\|_{\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)), v} \geq 0. \tag{3.2.1}$$

3.2.2. Endow the K-vector space $K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$ with the structure of $\mathfrak{o}_K$-hermitian vector bundle induced by the one of $\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$. The $\mathfrak{o}_K$-module

$$\mathcal{K}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) = \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \cap K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}),$$

is equipped with the restriction of the hermitian norms on $\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$, and

$$\mathcal{C} = \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) / \mathcal{K}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$$

is endowed it with quotient norms deduced from $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \to \mathcal{C}$. Denote by $\overline{\mathcal{C}}$ the $\mathfrak{o}_K$-hermitian vector bundle obtained in this way. With these choices and according to (3.2.1):

$$\begin{aligned}
[K : \mathbf{Q}] h_{\overline{\mathcal{F}}_{q,r}(t_{\boldsymbol{a}})}([K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) &= - \widehat{\deg}\, \overline{\mathcal{K}}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) \\
&= \widehat{\deg}\, \overline{\mathcal{C}} - \widehat{\deg}\, \overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \leq \widehat{\deg}\, \overline{\mathcal{C}}.
\end{aligned}$$

3.2.3. Denote by E the K-vector space $\Gamma(Z_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}), \mathcal{O}_{\mathbf{P}}(r))$ and let Ω be a Galois closure of K' over K. Endow the Ω-vector space $E \otimes_K \Omega$ with a structure of $\mathfrak{o}_\Omega$-hermitian vector bundle as follows. According to Proposition 2.6:

$$E \otimes_K \Omega = \bigoplus_{\sigma=1}^{q} E^{(\sigma)} \tag{3.2.2}$$

where, for embedding $\sigma \colon K' \to \bar{\mathbf{Q}}$, $E^{(\sigma)} := \Gamma(Z_r(a^{(\sigma)}, t_{\boldsymbol{a}}), \mathcal{O}_{\mathbf{P}_\Omega}(r))$ (4).

For $i = 1, \ldots, n$ let $(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)}) \in \Omega^2$ a generator of the line $a_i^{(\sigma)} \in \mathbf{P}^1(K)$. Since $a^{(\sigma)}$ is not K-rational assume $a_{i0}^{(\sigma)}$ may be taken non-zero and, up to rescaling, $a_{i0}^{(\sigma)} = 1$.

For an embedding $\sigma \colon K' \to \bar{\mathbf{Q}}$, a basis of the Ω-vector space $E^{(\sigma)}$ is given by

$$T_{a^{(\sigma)}}(\ell) = \bigotimes_{i=1}^{n} T_0^{r_i - \ell_i} T_{a_i^{(\sigma)}}^{\ell_i},$$

4. Here the point $a^{(\sigma)}$ is seen as an Ω-point of $\mathbf{P}_\Omega^1 = \mathbf{P}_K^1 \times_K \Omega$ and $Z_r(a^{(\sigma)}, t_{\boldsymbol{a}})$ denotes the subscheme of $\mathbf{P}_\Omega^1$ of index $t_{\boldsymbol{a}}$ on the point $a^{(\sigma)}$.

where $T_{a_i^{(\sigma)}} = T_1 - a_{i1}^{(\sigma)} T_0$ and $\ell = (\ell_1, \dots, \ell_n)$ ranges in the elements of $\Delta_r^{\mathbf{Z}}(t_{\boldsymbol{a}})$. Let $\mathcal{E}^{(\sigma)}$ be the $\mathfrak{o}_\Omega$-submodule of $\subset E^{(\sigma)}$ generated by the elements $T(\ell)$'s. Equip it with the hermitian norm having the elements $T(\ell)$'s as an orthonormal basis. Denote by $\overline{\mathcal{E}}^{(\sigma)}$ the associated $\mathfrak{o}_\Omega$-hermitian vector bundle.

Finally, according with (3.2.2), endow K-vector space $E \otimes_K \Omega$ with the structure of $\mathfrak{o}_\Omega$-hermitian vector bundle given by the orthogonal direct sum of the $\mathfrak{o}_\Omega$-hermitian vector bundles $\overline{\mathcal{E}}^{(\sigma)}$'s. Denote by $\overline{\mathcal{E}}_\Omega$ the so-obtained hermitian vector bundle.

3.2.4. The evaluation homomorphism $\eta\colon \Gamma(\mathbf{P}_K, \mathcal{O}_{\mathbf{P}}(r)) \to E = \Gamma(Z_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}), \mathcal{O}_{\mathbf{P}}(r))$ factors through an injection $\varepsilon\colon \mathcal{C} \otimes_{\mathfrak{o}_K} K \to E$. Applying the slope inequality (Proposition 0.1 on page 14), one gets
(3.2.3)

$$h_{\overline{\mathcal{F}}_{q,r}(t_{\boldsymbol{a}})}([K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \leq \frac{\widehat{\deg}\,\overline{\mathcal{C}}}{[K:\mathbf{Q}]} \leq \frac{\operatorname{rk}\mathcal{C}}{[\Omega:\mathbf{Q}]} \left(\hat{\mu}_{\max}(\overline{\mathcal{E}}_\Omega) + \sum_{v \in \mathrm{V}_\Omega} \log \|\varepsilon\|_{\sup,v} \right)$$

where, for a place $v \in \mathrm{V}_\Omega$, $\|\varepsilon\|_{\sup,v}$ is the v-adic operator norm of ε,

$$\|\varepsilon\|_{\sup,v} := \sup_{0 \neq f \in \mathcal{C} \otimes \Omega} \frac{\|\varepsilon(f)\|_{\mathcal{E},v}}{\|f\|_{\mathcal{C},v}}.$$

This coincides with the operator norm $\|\eta\|_{\sup,v}$ of η. The $\mathfrak{o}_\Omega$-hermitian vector bundle $\overline{\mathcal{E}}$ is trivial hence $\hat{\mu}_{\max}(\overline{\mathcal{E}}_\Omega) = 0$.

3.2.5. It remains to bound the v-adic size of the evaluation homomorphism η. For an embedding $\sigma = 1, \dots, q$ let $\eta^{(\sigma)}\colon \Gamma(\mathbf{P}_\Omega, \mathcal{O}_{\mathbf{P}}(r)) \to E_\sigma$ be the composition of η and the canonical projection $E \otimes_K \Omega \to E^{(\sigma)}$. Denote by $\|\eta^{(\sigma)}\|_{\sup,v}$ the operator norm of $\eta^{(\sigma)}$. With this notation:

– if v non-archimedean: $\|\eta\|_{\sup,v} = \max\limits_{\sigma=1,\dots,q} \|\eta^{(\sigma)}\|_{\sup,v}$;
– if v archimedean: $\|\eta\|_{\sup,v} \leq \sqrt{q} \max\limits_{\sigma=1,\dots,q} \left\{ \|\eta^{(\sigma)}\|_{\sup,v} \right\}$.

For $\sigma = 1, \dots, q$ and $i = 1, \dots, n$ let $\varphi_i^{(\sigma)}$ be the linear automorphism of $\Omega^{2\vee}$ defined by

$$\varphi_i^{(\sigma)} : \begin{cases} T_0 \mapsto T_0 \\ T_1 \mapsto T_{a_i^{(\sigma)}} = T_1 - a_{i1}^{(\sigma)} T_0. \end{cases}$$

Consider the linear automorphism $\varphi_r^{(\sigma)} = \operatorname{Sym}^{r_1} \varphi_1^{(\sigma)} \otimes \cdots \otimes \operatorname{Sym}^{r_n} \varphi_n^{(\sigma)}$ on the Ω-vector space

$$\Gamma(\mathbf{P}_K, \mathcal{O}_{\mathbf{P}}(r)) \otimes_K \Omega = \operatorname{Sym}^{r_1}(\Omega^{2\vee}) \otimes_\Omega \cdots \otimes_\Omega \operatorname{Sym}^{r_n}(\Omega^{2\vee}),$$

where for any $i = 1, \dots, n$ the linear automorphism $\varphi_i^{(\sigma)}$ its acting on the i-th factor through its action on symmetric powers.

With this notation the homomorphism $\eta^{(\sigma)} \circ \varphi_r^{(\sigma)} : \Gamma(\mathbf{P}_\Omega, \mathcal{O}_{\mathbf{P}}(r)) \to E^{(\sigma)}$ coincides with the evaluation morphism at the closed subscheme $Z_r((1:0), t_{\boldsymbol{a}})$:

$$T(\ell) = \bigotimes_{i=1}^n T_0^{r_i - \ell_i} T_1^{\ell_i} \mapsto \begin{cases} T_{a^{(\sigma)}}(\ell) & \text{if } \ell \in \Delta_r^{\mathbf{Z}}(t_{\boldsymbol{a}}) \\ 0 & \text{otherwise.} \end{cases}$$

By definition the elements $T_{a^{(\sigma)}}(\ell)$'s form an orthonormal basis of the trivial $\mathfrak{o}_\Omega$-hermitian vector bundle $\overline{\mathcal{E}}^{(\sigma)}$. Thus one has $\|\eta^{(\sigma)} \circ \varphi_r^{(\sigma)}\|_{\sup,v} \leq 1$ and

$$\|\eta^{(\sigma)}\|_{\sup,v} \leq \|(\varphi_r^{(\sigma)})^{-1}\|_{\sup,v}.$$

By recalling that for an endomorphism ψ of a $\mathfrak{o}_K$-hermitian vector bundle $\overline{\mathcal{V}}$ the sup-norm of ψ is smaller than its norm as an element of $\overline{\mathcal{V}}^\vee \otimes \overline{\mathcal{V}}$,

$$\begin{aligned} \|(\varphi_r^{(\sigma)})^{-1}\|_{\sup,v} &\leq \log \left\| (\varphi_r^{(\sigma)})^{-1} \right\|_{\mathrm{End}(\overline{\Gamma}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))),v} \\ &\leq \sum_{i=1}^n r_i \log \|(\varphi_i^{(\sigma)})^{-1}\|_{\mathrm{End}(\mathfrak{o}_K^{2\vee}),v}. \end{aligned}$$

The archimedean and the non-archimedean cases have to be distinguished. By definition of $\varphi_i^{(\sigma)}$ one has $(\varphi_i^{(\sigma)})^{-1}(T_0) = (1,0)$ and $(\varphi_i^{(\sigma)})^{-1}(T_1) = (a_{i1}^{(\sigma)}, 1)$. Thus,

– if v is non-archimedean:

$$\begin{aligned} \log \|(\varphi_i^{(\sigma)})^{-1}\|_{\mathrm{End}(\mathfrak{o}_\Omega^{2\vee}),v} &= \log \max\{\|(\varphi_i^{(\sigma)})^{-1}(T_0)\|_v, \|(\varphi_i^{(\sigma)})^{-1}(T_1)\|_v\} \\ &= \log \|(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)})\|_v. \end{aligned}$$

– if is v archimedean:

$$\begin{aligned} \log \|(\varphi_i^{(\sigma)})^{-1}\|_{\mathrm{End}(\mathfrak{o}_\Omega^{2\vee}),v} &= \log \sqrt{\|(\varphi_i^{(\sigma)})^{-1}(T_0)\|_v^2 + \|(\varphi_i^{(\sigma)})^{-1}(T_1)\|_v^2} \\ &\leq \log \|(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)})\|_v + \log \sqrt{2}. \end{aligned}$$

Taking the sum over all the places of K:

$$\begin{aligned} \sum_{v \in \mathrm{V}_\Omega} \log \|\eta\|_{\sup,v} &\leq \sum_{v \in \mathrm{V}_\Omega} \max_{\sigma\colon K' \to \bar{\mathbf{Q}}} \left\{ \sum_{i=1}^n r_i \log \|(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)})\|_v \right\} + |r|[\Omega : \mathbf{Q}] \log \sqrt{2q} \\ &\leq \sum_{v \in \mathrm{V}_\Omega} \sum_{\sigma\colon K' \to \bar{\mathbf{Q}}} \sum_{i=1}^n r_i \log \|(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)})\|_v + |r|[\Omega : \mathbf{Q}] \log \sqrt{2q} \\ &= [\Omega : \mathbf{Q}] \left(\sum_{\sigma\colon K' \to \bar{\mathbf{Q}}} \sum_{i=1}^n r_i h(a_i^{(\sigma)}) + |r| \log \sqrt{2q} \right). \end{aligned}$$

Dividing by $[\Omega : \mathbf{Q}]$, writing $h(a_i^{(\sigma)}) = h(a_i)$ for $\sigma\colon K' \to \bar{\mathbf{Q}}$ and according to (3.2.3),

$$h_{\overline{\mathcal{F}}_{q,r}(t_{\boldsymbol{a}})}([K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \leq \operatorname{rk} \mathcal{C} \left(q \sum_{i=1}^n r_i h(a_i) + |r| \log \sqrt{2q} \right).$$

Using $\operatorname{rk}\mathcal{C} = \operatorname{rk}\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) - \operatorname{rk}\mathcal{K}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$ and

$$\operatorname{rk}\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) = \prod_{i=1}^{n}(r_i + 1),$$
$$\operatorname{rk}\mathcal{K}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) = k_{q,r}(t_{\boldsymbol{a}}),$$

one concludes the proof. □

4. Upper bound of the instability measure

4.1. Notations and first reductions. — Let v be a place of K.

Proposition 4.1. — *With the notation introduced in Section 2.2, if the place v is non-archimedean,*

$$\iota_v(P_r) \leq - \max_{\sigma\colon K' \to \mathbf{C}_v} \Bigg\{ k_{q,r}(t_{\boldsymbol{a}}) t_{\boldsymbol{a}} \min_{i=1,\dots,n} \left\{ r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) \right\} + \sum_{i=1}^{n} \Bigg(\sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_x)} \ell_i - \frac{k_r(t_x) + k_{q,r}(t_{\boldsymbol{a}})}{2} r_i \Bigg) \mathrm{m}_v(a_i^{(\sigma)}, x_i) \Bigg\},$$

whereas, in the v archimedean case, the previous inequality holds with

$$k_{q,r}(t_{\boldsymbol{a}}) \sum_{i=1}^{n} \log \sqrt{r_i + 1} + k_{q,r}(t_{\boldsymbol{a}}) |r| \log \sqrt{3} + k_r(t_x) |r| \log 2$$

added on the right-hand side.

Throughout this section fix for $i = 1, \dots, n$ and $\sigma\colon K' \to \mathbf{C}_v$:

(1) a generator $(x_{i0}, x_{i1}) \in K_v^2$ of $x_i \in \mathbf{P}^1(K_v)$ such that $\|(x_{i0}, x_{i1})\|_v = 1$.
(2) a generator $(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)}) \in \mathbf{C}_v^2$ of $a_i^{(\sigma)} \in \mathbf{P}^1(\mathbf{C}_v)$ such that $\|(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)})\|_v = 1$;
(3) a square root $\theta_i^{(\sigma)} \in \mathbf{C}_v$ of $(a_{i0}^{(\sigma)} x_{i1} - a_{i1}^{(\sigma)} x_{i0})^{-1}$.

4.1.1. Elements of $\mathbf{SL}_2$ measuring distances. —

Definition 4.2. — For $i = 1, \dots, n$ and $\sigma = 1, \dots, q$ let $g_i^{(\sigma)}$ be the linear automorphism of $\mathbf{C}_v^2$ given by the matrix:

$$g_i^{(\sigma)} := \left(\begin{pmatrix} a_{i0}^{(\sigma)} & a_{i1}^{(\sigma)} \\ x_{i0} & x_{i1} \end{pmatrix}^{\top} \right)^{-1} = \frac{1}{a_{i0}^{(\sigma)} x_{i1} - a_{i1}^{(\sigma)} x_{i0}} \begin{pmatrix} x_{i1} & -x_{i0} \\ -a_{i1}^{(\sigma)} & a_{i0}^{(\sigma)} \end{pmatrix} \in \mathbf{GL}_2(\mathbf{C}_v).$$

Consider the n-uple $g^{(\sigma)} := (g_1^{(\sigma)}, \dots, g_n^{(\sigma)}) \in \mathbf{GL}_2(\mathbf{C}_v)^n$.

Proposition 4.3. — *With the notation introduced above, for all $i = 1, \dots, n$ and $\sigma = 1, \dots, q$,*

(1) $\det g_i^{(\sigma)} = (a_{i0}^{(\sigma)} x_{i1} - a_{i1}^{(\sigma)} x_{i0})^{-1}$;
(2) $\log |\det g_i^{(\sigma)}|_v = \mathrm{m}_v(a_i^{(\sigma)}, x_i)$;

(3) *For a non-zero vector* $(y_0, y_1) \in \mathbf{C}_v^2$ *such that* $\|(y_0, y_1)\|_v = 1$,

$$\|g_i^{(\sigma)} \cdot (y_0 T_1 - y_1 T_0)\|_v = \begin{cases} \max\left\{\mathrm{d}_v(a_i^{(\sigma)}, [y]), \mathrm{d}_v(x_i, [y])\right\} & v \text{ non-archimedean} \\ \sqrt{\mathrm{d}_v(a_i^{(\sigma)}, [y])^2 + \mathrm{d}_v(x_i, [y])^2} & v \text{ archimedean} \end{cases}$$

where T_0, T_1 *denotes the canonical basis of* $\mathbf{C}_v^{2\vee}$ *and* $[y] \in \mathbf{P}^1(\mathbf{C}_v)$ *the line generated by* (y_0, y_1).

Proof. —

(1) Clear: the points (x_{i0}, x_{i1}) and $(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)})$ are of norm 1.
(2) Clear from (1).
(3) The automorphism $g_i^{(\sigma)}$ of $\mathbf{C}_v^2$ acts on the dual vector space $\mathbf{C}_v^{2\vee}$ through the transposed inverse automorphism, whose matrix (with respect to the canonical basis T_0, T_1) is

$$\begin{pmatrix} a_{i0}^{(\sigma)} & a_{i1}^{(\sigma)} \\ x_{i0} & x_{i1} \end{pmatrix}.$$

The remainder is an elementary computation. □

4.1.2. Proof of Proposition 4.1. — In this paragraph Proposition 4.1 is deduced from Proposition 4.6 and 4.7: the latter are proved in the following sections.

Definition 4.4. — For $h = (h_1, \ldots, h_n) \in \mathbf{GL}_2(\mathbf{C}_v)^n$ define

– $\iota_v(h, [K_r(x, t_x)]) := \log \dfrac{\|h \cdot w_x\|_{\mathcal{F}_r(t_x), v}}{\|w_x\|_{\mathcal{F}_r(t_x), v}}$,
– $\iota_v(h, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) := \log \dfrac{\|h \cdot w_a\|_{\mathcal{F}_{q,r}(t_{\boldsymbol{a}}), v}}{\|w_a\|_{\mathcal{F}_{q,r}(t_{\boldsymbol{a}}), v}}$,

where $w_x \in \mathcal{F}_r(t_x) \otimes K$ (resp. $w_{\boldsymbol{a}} \in \mathcal{F}_{q,r}(t_{\boldsymbol{a}}) \otimes K$) is a non-zero representative of Plücker embedding of the point $[K_r(x, t_x)]$ (resp. $[K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]$).

Definition 4.5. — For $i = 1, \ldots, n$ and $\sigma\colon K' \to \mathbf{C}_v$ consider the linear automorphism $\widetilde{g}_i^{(\sigma)} := g_i^{(\sigma)}/\theta_i^{(\sigma)}$, which is of determinant 1. Set $\widetilde{g}^{(\sigma)} := (\widetilde{g}_1^{(\sigma)}, \ldots, \widetilde{g}_n^{(\sigma)})$.

Employing this notation the instability measure $\iota_v(P_r)$ can be written as

$$\begin{aligned} \iota_v(P_r) &= \inf_{h \in \mathbf{SL}_2(\mathbf{C}_v)^n} \{\iota_v(h, [K_r(x, t_x)]) + \iota_v(h, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})])\} \\ &\leq \min_{\sigma\colon K' \to \mathbf{C}_v} \left\{\iota_v(\widetilde{g}^{(\sigma)}, [K_r(x, t_x)]) + \iota_v(\widetilde{g}^{(\sigma)}, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})])\right\}. \end{aligned}$$

The representations $\mathcal{F}_r(t_x)$ and $\mathcal{F}_r(t_{\boldsymbol{a}})$ of $\mathbf{GL}_{2,\mathfrak{o}_K}^n$ are respectively homogeneous of weights $k_r(t_x)r$ and $k_r(t_{\boldsymbol{a}})r$. By Proposition 4.3 (2), $\log|\theta_i^{(\sigma)}| = \mathrm{m}_v(a_i^{(\sigma)}, x_i)/2$ and

$$\iota_v(\widetilde{g}^{\sigma}, [K_r(x, t_x)]) = \iota_v(g^{(\sigma)}, [K_r(x, t_x)]) + \frac{k_r(t_x)}{2} \sum_{i=1}^n r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i),$$

$$\iota_v(\widetilde{g}^{\sigma}, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) = \iota_v(g^{(\sigma)}, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) + \frac{k_{q,r}(t_{\boldsymbol{a}})}{2} \sum_{i=1}^n r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i).$$

One concludes the proof of Proposition 4.1 applying the following:

Proposition 4.6. — *With the notation introduced above, if v is non-archimedean,*

$$\iota_v(g^{(\sigma)}, [K_r(x, t_x)]) \leq -\sum_{i=1}^{n} \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_x)} \ell_i \mathrm{m}_v(a_i^{(\sigma)}, x_i)$$

whereas, if v is archimedean, the preceding inequality holds with $k_r(t_x)|r|\log 2$ be added on the right-hand side.

Proposition 4.7. — *With the notation introduced above, if v is non-archimedean,*

$$\iota_v(g^{(\sigma)}, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \leq -k_{q,r}(t_{\boldsymbol{a}}) t_{\boldsymbol{a}} \min_{i=1,\dots,n} \left\{ r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) \right\},$$

whereas, if v is archimedean, the preceding inequality holds with

$$\frac{k_{q,r}(t_{\boldsymbol{a}})}{2} \sum_{i=1}^{n} \log(r_i + 1) + k_{q,r}(t_{\boldsymbol{a}})|r| \log \sqrt{3},$$

added on the right-hand side.

4.2. Taylor expansion at the single point: proof of Proposition 4.6. — Keep the notations introduced in Section 4.1.

4.2.1. Let $i \in \{1, \dots, n\}$ and consider the linear form

$$T_{i1} := -x_{i1} T_0 + x_{i0} T_1 \in K_v^{2\vee}.$$

Since the point $(x_{i0}, x_{i1}) \in K_v^2$ is of norm 1 the linear form T_{i1} is of norm 1. If v is non-archimedean let $T_{i0} \in \mathfrak{o}_v^{2\vee}$ be a linear form such that T_{i0}, T_{i1} is a basis of the $\mathfrak{o}_v$-module $\mathfrak{o}_v^{2\vee}$. If v is archimedean let $T_{i0} \in K_v^{2\vee}$ be a linear form such that T_{i0}, T_{i1} is an orthonormal basis of $K_v^{2\vee}$.

Since the linear form T_{i1} vanishes at x_i for every $i = 1, \dots, n$, Proposition 2.4 (1) implies that a basis of the K-vector space $K_r(x, t_x)$ is given by the monomials

$$T(\ell) = \bigotimes_{i=1}^{n} T_{i0}^{r_i - \ell_i} T_{i1}^{\ell_i}$$

where $\ell = (\ell_1, \dots, \ell_n)$ ranges in $\nabla_r^{\mathbf{Z}}(t_x)$. The following vector of $\mathcal{F}_r(t_x) \otimes_{\mathfrak{o}_K} K_v$,

$$w := \bigwedge_{\ell \in \nabla_r^{\mathbf{Z}}(t_x)} T(\ell)$$

is a non-zero representative of the Plücker embedding of $[K_r(x, t_x)]$. If v is non-archimedean the elements $T(\ell)$'s form a basis of the $\mathfrak{o}_v$-module

$$(K_r(x, t_x) \otimes K_v) \cap (\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \otimes \mathfrak{o}_v).$$

Thus

$$\log \|w\|_{\mathcal{F}_r(t_x), v} = 0.$$

If v is archimedean the elements $T(\ell)$'s are orthogonal but they are not of norm 1 and

$$\log\|w\|_{\mathcal{F}_r(t_x),v} = \sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)} \log\|T(\ell)\|_{\Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)),v} = -\frac{1}{2}\sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\sum_{i=1}^{n}\log\binom{r_i}{\ell_i}.$$

Bounding the binomial $\binom{r_i}{\ell_i}$ by 2^{r_i},

$$\begin{aligned}\log\|w\|_{\mathcal{F}_r(t_x),v} &= -\frac{1}{2}\sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\sum_{i=1}^{n}\log\binom{r_i}{\ell_i} \geq -\frac{1}{2}\sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\sum_{i=1}^{n} r_i\log 2\\ &= -k_r(t_x)|r|\log\sqrt{2}.\end{aligned}$$

4.2.2. For any $\ell\in\nabla_r^{\mathbf{Z}}(t)$ the sub-multiplicativity of the norm on symmetric powers yields

$$\log\|g^{(\sigma)}\cdot T(\ell)\|_{\Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)),v} \leq \sum_{i=1}^{n}((r_i-\ell_i)\log\|g_i^{(\sigma)}\cdot T_{i0}\|_v + \ell_i\log\|g_i^{(\sigma)}\cdot T_{i1}\|_v).$$

Therefore applying the Hadamard inequality,

$$\begin{aligned}\log\|g^{(\sigma)}\cdot w\|_{\mathcal{F}_r(t_x),v} &\leq \sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\log\|g^{(\sigma)}\cdot T(\ell)\|_{\Gamma(\mathbf{P},\mathcal{O}_{\mathbf{P}}(r)),v}\\ &\leq \sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\sum_{i=1}^{n}((r_i-\ell_i)\log\|g^{(\sigma)}\cdot T_{i0}\|_v + \ell_i\log\|g^{(\sigma)}\cdot T_{i1}\|_v).\end{aligned}$$

For $i=1,\ldots,n$ Proposition 4.3 (3) entails

– $\|g_i^{(\sigma)}\cdot T_{i0}\|_v \leq \begin{cases}1 & v \text{ non-archimedean}\\ \sqrt{2} & v \text{ archimedean}\end{cases}$

– $\|g_i^{(\sigma)}\cdot T_{i1}\|_v = \mathrm{d}_v(a_i^{(\sigma)},x_i)$.

Summarising if v is non-archimedean:

$$\iota_v(g^{(\sigma)},[K_r(x,t_x)]) = \log\|g^{(\sigma)}\cdot w\|_{\mathcal{F}_r(t_x),v} \leq -\sum_{i=1}^{n}\left(\sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\ell_i\right)\mathrm{m}_v(a_i^{(\sigma)},x_i).$$

If v is archimedean:

$$\begin{aligned}\iota_v(g^{(\sigma)},[K_r(x,t_x)]) &\leq \log\|g^{(\sigma)}\cdot w\|_{\mathcal{F}_r(t_x),v} + k_r(t_x)|r|\log\sqrt{2}\\ &\leq -\sum_{i=1}^{n}\left(\sum_{\ell\in\nabla_r^{\mathbf{Z}}(t_x)}\ell_i\right)\mathrm{m}_v(a_i^{(\sigma)},x_i) + k_r(t_x)|r|\log 2,\end{aligned}$$

which concludes the proof. □

4.3. Taylor expansion at the algebraic points: proof of Proposition 4.7. — Keep the notations introduced in Section 4.1.

4.3.1. If v is non-archimedean let $\mathfrak{o}_v$ be the ring of K_v and let $f_1, \dots, f_{k_{q,r}(t_{\boldsymbol{a}})}$ be a basis of the $\mathfrak{o}_v$-module:

$$(K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) \otimes_K K_v) \cap (\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r) \otimes_{\mathfrak{o}_K} \mathfrak{o}_v) .$$

If v is archimedean let $f_1, \dots, f_{k_{q,r}(t_{\boldsymbol{a}})}$ be an orthonormal basis of $K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) \otimes K_v$. With these notations the vector of $\mathcal{F}_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) \otimes_{\mathfrak{o}_K} K_v$,

$$w := \bigwedge_{\alpha=1}^{k_{q,r}(t_{\boldsymbol{a}})} f_\alpha$$

is a non-zero representative of the Plücker embedding of $[K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]$. In order to simplify notation denote by $\| \cdot \|_v$ the induced norm on the $\mathbf{C}_v$-vector space $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \otimes_{\mathfrak{o}_K} \mathbf{C}_v$. With this notation Hadamard's inequality (0.3.1) on page 12 entails

$$\iota_v(g^{(\sigma)}, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) = \log \|g^{(\sigma)} \cdot w\|_{\mathcal{F}_{q,r}(t_{\boldsymbol{a}}),v} \leq \sum_{\alpha=1}^{k_{q,r}(t_{\boldsymbol{a}})} \log \|g^{(\sigma)} \cdot f_\alpha\|_v.$$

and it remains to prove the following:

Lemma 4.8. — *Let f be a non-zero element of $K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$. With the notation introduced above, if v is non-archimedean,*

$$\log \frac{\|g^{(\sigma)} \cdot f\|_v}{\|f\|_v} \leq t_{\boldsymbol{a}} \max_{i=1,\dots,n} \left\{ r_i \log \mathrm{d}_v(a_i^{(\sigma)}, x_i) \right\}$$

whereas, if v is archimedean, the preceding inequality holds with

$$\frac{1}{2} \sum_{i=1}^{n} \log(r_i + 1) + |r| \log \sqrt{3}$$

added on the right-hand side.

4.3.2. For $i = 1, \dots, n$ consider the linear form:

$$T_{i1} := -a_{i1}^{(\sigma)} T_0 + a_{i0}^{(\sigma)} T_1 \in \mathbf{C}_v^{2\vee}.$$

Since the point $(a_{i0}^{(\sigma)}, a_{i1}^{(\sigma)}) \in \mathbf{C}_v^2$ is of norm 1 the linear form T_{i1} is of norm 1. If v is non-archimedean let $\overline{\mathfrak{o}}_v$ be the ring of integers of $\mathbf{C}_v$ and let $T_{i0} \in \overline{\mathfrak{o}}_v^{2\vee}$ be a linear form such that T_{i0}, T_{i1} is a basis of the $\overline{\mathfrak{o}}_v$-module $\overline{\mathfrak{o}}_v^{2\vee}$.(5) If v is archimedean let $T_{i0} \in \mathbf{C}_v^{2\vee}$ be a linear form such that T_{i0}, T_{i1} is an orthonormal basis of $\mathbf{C}_v^{2\vee}$. For every n-uple of integers $\ell = (\ell_1, \dots, \ell_n) \in \square_r$ define

$$T(\ell) = \bigotimes_{i=1}^{n} T_{i0}^{r_i - \ell_i} T_{i1}^{\ell_i}.$$

5. Such a T_{i0} exists because the $\overline{\mathfrak{o}}_v$-module $\overline{\mathfrak{o}}_v^{2\vee}/T_{i1}\overline{\mathfrak{o}}_v$ is torsion-free, thus free ([**BGR84**, 1.6.1 Proposition 2]). One can avoid using this result by taking T_{i1} defined over a finite extension of K_v. In the latter case the existence of T_{i0} follows from the widely-know fact that a finite type, torsion-free module over a discrete valuation ring is free.

The monomials $T(\ell)$'s form a basis of the $\mathbf{C}_v$-vector space $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \otimes_K \mathbf{C}_v$. If v is non-archimedean the elements $T(\ell)$'s form a basis of the $\bar{\mathfrak{o}}_v$-module $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \otimes_{\mathfrak{o}_K} \bar{\mathfrak{o}}_v$. If v is archimedean the monomials $T(\ell)$'s are orthogonal and for every $\ell \in \square_r$,

$$\|T(\ell)\|_v = \binom{r}{\ell}^{-1/2} := \prod_{i=1}^{n} \binom{r_i}{\ell_i}^{-1/2}.$$

4.3.3. If v is non-archimedean, a computation similar to the one in paragraph 4.2.2 yields

$$\log \|g^{(\sigma)} \cdot T(\ell)\|_v \leq \sum_{i=1}^{n} \ell_i \log \mathrm{d}_v(a_i^{(\sigma)}, x_i) = -\sum_{i=1}^{n} \ell_i \mathrm{m}_v(a_i^{(\sigma)}, x_i),$$

whereas, if v is archimedean, the preceding inequality holds when $k_{q,r}(t_{\boldsymbol{a}})|r| \log \sqrt{2}$ is added to the right-hand side.

4.3.4. Write $f = \sum_\ell f_\ell T(\ell)$ with $f_\ell \in \mathbf{C}_v$. If v is non-archimedean,

$$\|f\|_v = \max\left\{|f_\ell|_v : \ell \in \square_r^{\mathbf{Z}}\right\}.$$

If v is archimedean,

$$\|f\|_v^2 = \sum_{\ell \in \square_r^{\mathbf{Z}}} |f_\ell|_v^2 \binom{r}{\ell}^{-1}.$$

Since the real numbers $\mathrm{m}_v(a_i^{(\sigma)}, x_i)$ are non-negative for every $\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})$:

$$\begin{aligned}\sum_{i=1}^{n} \ell_i \mathrm{m}_v(a_i^{(\sigma)}, x_i) &\geq \left(\sum_{i=1}^{n} \frac{\ell_i}{r_i}\right) \min_{i=1,\dots,n} \left\{r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i)\right\} \\ &\geq t_{\boldsymbol{a}} \min_{i=1,\dots,n} \left\{r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i)\right\}.\end{aligned}$$

By definition the global section f satisfies $\mathrm{ind}_{1/r}(f, a^{(\sigma)}) \geq t_{\boldsymbol{a}}$, that is, $f_\ell = 0$ for every $\ell \in \Delta_r^{\mathbf{Z}}(t_{\boldsymbol{a}})$. In the non-archimedean case this yields:

$$\begin{aligned}\log \|g^{(\sigma)} \cdot f\|_v &\leq \max_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} \left\{\log |f_\ell|_v + \log \|g^{(\sigma)} \cdot T(\ell)\|_v\right\} \\ &\leq -t_{\boldsymbol{a}} \min_{i=1,\dots,n} \left\{r_i \mathrm{m}_v(a_i^{(\sigma)}, x_i)\right\} + \log \|f\|_v,\end{aligned}$$

which concludes the proof in the non-archimedean case.

4.3.5. Suppose henceforth v archimedean. Proposition 4.3 (3) and the triangle inequality give:

$$\begin{aligned}\|g^{(\sigma)} \cdot f\|_v &\leq \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} |f_\ell|_v \|g^{(\sigma)} \cdot T(\ell)\|_v \\ &\leq \max_{i=1,\dots,n} \left\{\mathrm{d}_v(a_i^{(\sigma)}, x_i)^{r_i}\right\}^{t_{\boldsymbol{a}}} \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} |f_\ell|_v \prod_{i=1}^{n} \sqrt{2}^{r_i - \ell_i}. \qquad (4.3.1)\end{aligned}$$

Comparing ℓ^1 and ℓ^2 norms on $\Gamma(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(r))$ thanks to Jensen's inequality:

$$\sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} |f_\ell|_v \prod_{i=1}^n \sqrt{2}^{r_i - \ell_i} \leq \sqrt{\left(\prod_{i=1}^n (r_i+1)\right) \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} |f_\ell|_v^2 \prod_{i=1}^n 2^{r_i - \ell_i}},$$

where one uses $\operatorname{rk} \Gamma(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(r)) = \prod_{i=1}^n (r_i + 1)$. The right term can be compared with the norm of f:

$$\begin{aligned}\sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} |f_\ell|_v^2 \prod_{i=1}^n 2^{r_i-\ell_i} &\leq \max_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} \left\{ \binom{r}{\ell} \prod_{i=1}^n 2^{r_i - \ell_i} \right\} \sum_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} |f_\ell|_v^2 \binom{r}{\ell}^{-1} \\ &= \max_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} \left\{ \binom{r}{\ell} \prod_{i=1}^n 2^{r_i - \ell_i} \right\} \|f\|_v^2.\end{aligned}$$

Using $\sum_{a=0}^b \binom{b}{a} 2^{b-a} = 3^b$,

$$\max_{\ell \in \nabla_r^{\mathbf{Z}}(t_{\boldsymbol{a}})} \left\{ \binom{r}{\ell} \prod_{i=1}^n 2^{r_i - \ell_i} \right\} \leq 3^{|r|}$$

According to (4.3.1) this concludes the proof. □

5. End of the proof: semi-stability in the general case

5.1. Basic facts about the semi-stability of subspaces. —

5.1.1. Instability coefficient. — Let K be a field and let G a K-reductive group acting on a proper K-scheme X equipped with a G-equivariant invertible sheaf L. Let x be a K-point of X. Let $\lambda \colon \mathbf{G}_m \to G$ be a one-parameter subgroup of G (which means that λ is a morphism of algebraic groups) and consider the morphism $\lambda_x \colon \mathbf{G}_m \to X$ given by

$$\lambda_x(\tau) := \lambda(\tau) \cdot x.$$

By properness of X, the morphism λ_x extends in a unique way to a morphism $\overline{\lambda}_x \colon \mathbf{A}^1 \to X$. Denote by x_0 the K-point $\overline{\lambda}_x(0)$. Since it is a fixed point under the action of $\mathbf{G}_m$, then $\mathbf{G}_m$ acts on the K-vector space $x_0^* L$ through a character

$$\tau \mapsto \tau^{-\mu_L(\lambda, x)}$$

with $\mu_L(\lambda, x) \in \mathbf{Z}$. It is called the *instability coefficient of* x with respect to the one-parameter subgroup λ and the invertible sheaf L. [(6)]

Theorem 5.1 **(Hilbert-Mumford criterion).** — *Suppose that K is perfect and L is ample. With the notation introduced above, the point x is semi-stable if and only if*

$$\mu_L(\lambda, x) \geq 0$$

for every one-parameter subgroup $\lambda \colon \mathbf{G}_m \to G$.

6. We follow here the convention adopted in [**MFK94**, Definition 2.2].

When K is algebraically closed, this theorem has been proved by Mumford [**MFK94**, Theorem 2.1]. The general case has been proved independently by Kempf [**Kem78**, Theorem 4.2] and Rousseau [**Rou81**] (7).

5.1.2. Instability coefficient of linear subspaces. — Let V be a finite dimension K-vector space and r be a positive integer. Consider the grassmannian of r-dimensional subspaces $\mathbf{Grass}_r(V)$ and its Plücker embedding $\varpi\colon \mathbf{Grass}_r(V) \to \mathbf{P}(\bigwedge^r V)$.

Suppose that a K-reductive group G acts linearly on V. Then it acts on the grassmannian $\mathbf{Grass}_r(V)$, on the projective space $\mathbf{P}(\bigwedge^r V)$ and in a equivariant way on the invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(\bigwedge^r V)$. Since the Plücker embedding ϖ is G-equivariant with respect to this action, the ample invertible sheaf $\varpi^*\mathcal{O}(1)$ on $\mathbf{Grass}_r(V)$ is endowed with a G-equivariant action.

Definition 5.2. — Let $\lambda\colon \mathbf{G}_m \to G$ be a one-parameter subgroup.

(1) Let $W \subset V$ be a linear subspace of dimension r. Set
$$\mu(\lambda, [W]) := \mu_{\varpi^*\mathcal{O}(1)}(\lambda, [W])$$
omitting the polarisation $\varpi^*\mathcal{O}(1)$.

(2) For every integer $p \in \mathbf{Z}$ consider the subspace $V_{\lambda,p} := \{v \in V : \lambda(\tau)\cdot v = \tau^p v\}$. Let $p_{\lambda,\min}$ (resp. $p_{\lambda,\max}$) be the smallest (resp. the biggest) integer p such that $V_{\lambda,p}$ is non-zero.

(3) For every integer $p \in \mathbf{Z}$ set $V[p] := \bigoplus_{q\geq p} V_{\lambda,q}$.

Since the action of a torus is diagonalisable, one has $V = \bigoplus_{p\in\mathbf{Z}} V_{\lambda,p}$. In particular,
$$V[p] = \begin{cases} 0 & \text{if } p > p_{\lambda,\max} \\ V & \text{if } p < p_{\lambda,\min}. \end{cases}$$

Proposition 5.3. — *Let $W \subset V$ be a linear subspace of dimension r. For every integer p set $W[p] := W \cap V[p]$.*

(1) The subspaces $W[p]$ form a decreasing filtration of W and
$$\begin{aligned}\mu(\lambda, [W]) &= \sum_{p\in\mathbf{Z}} p\,(\dim_K W[p] - \dim_K W[p+1]) \\ &= -p_{\lambda,\min}\dim_K W - \sum_{p=p_{\lambda,\min}+1}^{p_{\lambda,\max}} \dim_K W[p].\end{aligned}$$

(2) Let $w_1, \ldots, w_r$ be a basis of W. For every $i = 1, \ldots, r$ let $\mu(\lambda, [w_i])$ be the instability coefficient of the point $[w_i] \in \mathbf{P}(V)$. Then the vector w_i writes as
$$\lambda(\tau)\cdot w_i = \tau^{-\mu(\lambda,[w_i])} w_{i,\min} + \text{terms of higher order in } \tau,$$

7. In order to understand that [**Kem78**, Theorem 4.2] translates into Theorem 5.1 it is useful to consult the dictionary between Kempf's and Mumford's notations given in the table in [**MFK94**, Appendix to Chapter 2, section B].

with $w_{i,\min} \in V$. If the elements $w_{1,\min}, \dots, w_{r,\min} \in V$ are linearly independent, then

$$\mu(\lambda, [W]) = \sum_{i=1}^{r} \mu(\lambda, [w_i]).$$

(3) *With the notations of (2), there exists a basis $w_1, \dots, w_r$ of W such that their components of minimal weight $w_{1,\min}, \dots, w_{r,\min} \in V$ are linearly independent.*

Proof. — This is a reformulation of the computations in [**MFK94**, Chapter 4, §4]. See also [**Tot96**, §2, Lemma 2]. □

Proposition 5.4. — *Let W_1, W_2 be subvector spaces of V. Then:*

(1) (Inclusion formula) *If W_1 is contained in W_2, then*

$$\mu(\lambda, [W_1]) \geq \mu(\lambda, [W_2]) - p_{\lambda,\min}(\dim_K W_1 - \dim_K W_2). \tag{5.1.1}$$

(2) (Grassmann formula)

$$\mu(\lambda, [W_1]) + \mu(\lambda, [W_2]) \geq \mu(\lambda, [W_1 + W_2]) + \mu(\lambda, [W_1 \cap W_2]). \tag{5.1.2}$$

Proof. —

(1) Clear.

(2) For every integer p the usual Grassmann formula for linear subspaces gives

$$\dim_K W_1[p] + \dim_K W_2[p] = \dim_K(W_1[p] + W_2[p]) + \dim_K(W_1[p] \cap W_2[p]).$$

Conclude by noticing that $W_1[p] + W_2[p] \subset (W_1 + W_2)[p]$. □

Remark 5.5. — The second statement of the previous Proposition is a generalisation of the following fact: if V writes is the direct sum of W_1 and W_2 then the couple $([W_1], [W_2])$ is semi-stable (as point in a product of suitable grassmannians of V). Indeed, both terms in the right-hand side of (2) vanish.

5.2. Asymptotic semi-stability: proof of Theorem 2.7. —

5.2.1. Go back to the notation introduced in Section 2.2. The construction of invariant elements is compatible with flat base change [**Ses77**, §2 Lemma 2]. It follows that the semi-stability of the points $P_{\alpha r}$ is only a matter of the generic fiber of $\mathcal{X}_{\alpha r}$. From now on we silently work over K (for instance $\mathbf{P}$ will denote the projective scheme $(\mathbf{P}^1_K)^n$).

Theorem 5.6. — *Let $n \geq 1$ be a positive integer and $r = (r_1, \dots, r_n)$ be a n-uple of positive integers. Let $t_x, t_{\boldsymbol{a}} \geq 0$ be real numbers with $t_{\boldsymbol{a}} < t_{q,n}(0)$. If the inequality*

$$\mu_n(u_{q,r}(t_{\boldsymbol{a}})) > \mu_n(t_x) + \varepsilon_{q,r},$$

is satisfied then there exists a positive integer $\alpha_0 = \alpha_0(q, n, r, t_{\boldsymbol{a}}, t_x)$ such that, for every integer $\alpha \geq \alpha_0$, the K-point $P_{\alpha r} \in \mathcal{X}_{\alpha r}(K)$ is semi-stable under the action of $\mathbf{SL}_2^n$ with respect to the polarization given by the Plücker embeddings.

5.2.2. Computation of the instability coefficients. — For every n-uple of positive integers $r = (r_1, \dots, r_n)$, every non-negative real number $t \geq 0$ and every $i \in \{1, \dots, n\}$ set:

$$\mu^{\mathbf{Z}}_{r,i}(t) := \sum_{\ell \in \nabla^{\mathbf{Z}}_r(t)} 2\ell_i - r_i$$

Arguments similar to those in Lemma 1.11 show that $\mu^{\mathbf{Z}}_{r,i}(t)$ is non-negative.

Definition 5.7. — Let $\lambda : \mathbf{G}_m \to \mathbf{SL}^n_{2,K}$ be a one-parameter subgroup.

(1) For $i = 1, \dots, n$, there exist a basis T_{i0}, T_{i1} of $K^{2\vee}$ and a non-negative integer $m_{\lambda,i} \geq 0$ such that

$$\lambda(\tau) \cdot T_{i0} = \tau^{m_{\lambda,i}} T_{i0}, \quad \lambda(\tau) \cdot T_{i1} = \tau^{-m_{\lambda,i}} T_{i1},$$

for every $\tau \in \mathbf{G}_m(K)$.

The n-uple of non-negative integers $m_\lambda = (m_{\lambda,1}, \dots, m_{\lambda,n})$ is called the *weight of* λ and the bases T_{i0}, T_{i1} (for $i = 1, \dots, n$) is called an *adapted basis for* λ.

The integer $m_{\lambda,i}$ does not depend on the choice of an adapted basis. If $m_{\lambda,i}$ is non-zero then the lines $\{T_{i0} = 0\}$ and $\{T_{i1} = 0\}$ are determined by λ.

(2) With the notations introduced above, for every $\bar{\mathbf{Q}}$-point y of $\mathbf{P}$ set

$$\chi_{\lambda,i}(y) := \begin{cases} 1 & \text{if } T_{i0} \text{ vanishes at } y \\ 0 & \text{otherwise.} \end{cases}$$

Denote by y_λ the unique K-point of $\mathbf{P}$ such that $\chi_{\lambda,i}(y_\lambda) = 1$ for all $i = 1, \dots, n$ and call it the *instability point of* λ (with respect to the chosen adapted bases).

***Proposition 5.8* (Instability coefficient at the single point)**

Let $\lambda\colon \mathbf{G}_m \to \mathbf{SL}^n_2$ be a one-parameter subgroup. With the notation introduced above, for $i = 1, \dots, n$,

$$\mu(\lambda, [K_r(x, t_x)]) = \sum_{i=1}^n (-1)^{\chi_{\lambda,i}(x)} m_{\lambda,i} \mu^{\mathbf{Z}}_{r,i}(t_x).$$

***Proposition 5.9* (Instability coefficient at the algebraic point)**

Let δ be a positive real number. Under the assumptions of Theorem 2.7 there exist a positive real number ρ_0 and a positive integer α_0 (the two of them possibly depending on n, q, r, $t_{\boldsymbol{a}}$ and t_x) satisfying the following properties: for every one-parameter subgroup $\lambda\colon \mathbf{G}_m \to \mathbf{SL}^n_2$, every integer $\alpha \geq \alpha_0$ and every real number $0 < \rho < \rho_0$,

$$\mu(\lambda, [K_{q,\alpha r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \geq \sum_{i=1}^n m_{\lambda,i} \left[\mu^{\mathbf{Z}}_{\alpha r,i}(u_{q,r}(t_{\boldsymbol{a}}) + \rho) - \alpha^{n+1} r_i (r_1 \cdots r_n)(\varepsilon_{q,r} + \delta)\right].$$

Proof of Theorem 2.7. — According to the Hilbert-Mumford criterion (Theorem 5.1) it suffices to show that there exists α_0 such that for every $\alpha \geq \alpha_0$ and every one-parameter subgroup $\lambda\colon \mathbf{G}_m \to \mathbf{SL}^n_2$,

$$\mu(\lambda, P_{\alpha r}) = \mu(\lambda, [K_{\alpha r}(x, t_x)]) + \mu(\lambda, [K_{\alpha r}(a, t_{\boldsymbol{a}})]) \geq 0.$$

Let δ be a positive real number. Let α_0, δ_0 and ρ_0 given by Proposition 5.9. Up to increasing α_0 and decreasing δ and ρ_0 assume, for $i = 1, \dots, n$, $\alpha \geq \alpha_0$ and $0 < \rho < \rho_0$:

$$\text{(SS}'\text{)} \qquad \mu^{\mathbf{Z}}_{\alpha r,i}(u_{q,r}(t_{\boldsymbol{a}}) + \rho) > \mu^{\mathbf{Z}}_{\alpha r,i}(t_x) + \alpha^{n+1} r_i(r_1 \cdots r_n)(\varepsilon_{q,r} + \delta),$$

Fix $\alpha \geq \alpha_0$ and $0 < \rho < \rho_0$. Propositions 5.8 and 5.9 yield that $\mu(\lambda, P_{\alpha r})$ is non-negative if:

$$\sum_{i=1}^{n} m_{\lambda,i} \left[\mu^{\mathbf{Z}}_{\alpha r,i}(u_{q,r}(t_{\boldsymbol{a}}) + \rho) - \alpha^{n+1} r_i(r_1 \cdots r_n)(\varepsilon_{q,r} + \delta) + (-1)^{\chi_{\lambda,i}(x)} \mu^{\mathbf{Z}}_{\alpha r,i}(t_x) \right] \geq 0.$$

Since the integers $m_{\lambda,i}$ are supposed to be non-negative, this is satisfied according to (SS$'$). This concludes the proof. □

Proof of Proposition 5.8. — Consider adapted bases T_{i0}, T_{i1} for λ ($i = 1, \dots, n$). Suppose $\chi_{\lambda,i}(x) = 0$ for $i = 1, \dots, n$ (in the other cases the argument is similar).

Under this assumption there exists $\xi_i \in K$ such that $T_{i1} - \xi_i T_{i0}$ vanishes at x_i. According to Proposition 2.4 (1), a basis of the K-vector space $K_r(x, t_x)$ is given by polynomials of the form

$$T(\ell) := \bigotimes_{i=1}^{n} T_{i0}^{r_i - \ell_i}(T_{i1} - \xi_i T_{i0})^{\ell_i}.$$

where $\ell = (\ell_1, \dots, \ell_n) \in \nabla_r(t_x)$. The action of the one-parameter subgroup λ is given by

$$\lambda_i(\tau) \cdot T(\ell) = \bigotimes_{i=1}^{n} \tau^{m_{\lambda,i}(r_i - 2\ell_i)} \left(T_{i0}^{r_i - \ell_i}(T_{i1} - \tau^{2m_{\lambda,i}} \xi_i T_{i0})^{\ell_i} \right)$$

Since the integers $m_{\lambda,i}$ are supposed to be non-negative, the component of $T(\ell)$ of minimal weight is the polynomial multiplied by $\sum_{i=1}^{n} m_{\lambda,i}(r_i - 2\ell_i)$, that is

$$T(\ell)_{\min} = \bigotimes_{i=1}^{n} T_{i0}^{r_i - \ell_i} T_{i1}^{\ell_i}.$$

The elements $T(\ell)_{\min}$ for $\ell \in \nabla^{\mathbf{Z}}_r(t_x)$ are linearly independent, thus Proposition 5.3 (2) yields

$$\mu(\lambda, [K_r(x, t_x)]) = \sum_{i=1}^{n} \sum_{\ell \in \nabla^{\mathbf{Z}}_r(t_x)} m_{\lambda,i}(2\ell_i - r_i) = \sum_{i=1}^{n} m_{\lambda,i} \mu^{\mathbf{Z}}_{r,i}(t_x). \qquad \square$$

5.2.3. Proof of Proposition 5.9. —

Lemma 5.10. — *Let $\lambda \colon \mathbf{G}_m \to \mathbf{SL}_2^n$ be a one-parameter subgroup. Suppose* [8]

$$u_{q,r}(t_{\boldsymbol{a}}) \neq 0, n.$$

8. If $u_{q,r}(t_{\boldsymbol{a}}) = 0, n$ then Condition (SS) in Theorem 2.7 is not satisfied because $\mu_n(0) = \mu_n(n)$ vanishes.

Let $a^{(0)}$ be a K-rational point of $\mathbf{P}^1$. Then, for $\rho > 0$,

$$\mu(\lambda, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) + \mu(\lambda, [K_r(a^{(0)}, u_{q,r}(t_{\boldsymbol{a}}) + \rho)]) \\ \geq (k_{q,r}(t_{\boldsymbol{a}}) + k_r(u_{q,r}(t_{\boldsymbol{a}}) + \rho) - \dim \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))) \sum_{i=1}^{n} m_{\lambda,i} r_i.$$

Proof. — The hypothesis $u_{q,r}(t_{\boldsymbol{a}}) \neq 0, n$ implies, by Definition 1.10 (4),

$$\operatorname{vol} \Delta_n(u_{q,r}(t_{\boldsymbol{a}})) = 1 + \varepsilon_{q,r} - q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}}).$$

Since the point $a^{(0)}$ is K-rational one has $\operatorname{pr}_i(a^{(0)}) \neq \operatorname{pr}_i(a^{(\sigma)})$ for every $i = 1, \ldots, n$ and every $\sigma \colon K' \to \bar{\mathbf{Q}}$. According to Proposition 2.6 (3),

$$K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) \cap K_r(a^{(0)}, u_{q,r}(t_{\boldsymbol{a}}) + \rho) = 0$$

(apply it over $\bar{\mathbf{Q}}$ to deduce it over K). Grassmann's formula for instability coefficients (Proposition 5.4 2) applied to the subspaces $K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})$ and $K_r(a^{(0)}, u_{q,r}(t_{\boldsymbol{a}}) + \rho)$ yields:

$$\mu(\lambda, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) + \mu(\lambda, [K_r(a^{(0)}, u_{q,r}(t_{\boldsymbol{a}}) + \rho)]) \\ \geq \mu(\lambda, [K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) + K_r(a^{(0)}, u_{q,r}(t_{\boldsymbol{a}}) + \rho)]).$$

With the notations introduced in paragraph 5.1.2, the smallest integer b such that the vector space $\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))_b$ is non-zero is $-\sum_{i=1}^{n} m_{\lambda,i} r_i$ (it occurs only for the monomial $T_{10}^{r_1} \otimes \cdots \otimes T_{n0}^{r_n}$). The inclusion formula (Proposition 5.4 1) applied to

$$K_{q,r}(\boldsymbol{a}, t_{\boldsymbol{a}}) + K_r(y, u_{q,r}(t_{\boldsymbol{a}}) + \rho) \subset \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$$

gives the result. □

Lemma 5.11. — *Let δ be a positive real number. Under the assumption of Theorem 2.7 there exist a positive real number ρ_0 and a positive integer α_0 (the two of them possibly depending on n, d, r, $t_{\boldsymbol{a}}$ and t_x) such that, for every integer $\alpha \geq \alpha_0$ and every real number $0 < \rho < \rho_0$:*

(1) $\displaystyle \left| \frac{\dim_K \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(\alpha r))}{\alpha^n (r_1 \cdots r_n)} - 1 \right| < \frac{\delta}{3};$

(2) $\displaystyle \left| \frac{\#\nabla^{\mathbf{Z}}_{\alpha r}(u_{q,r}(t_{\boldsymbol{a}}) + \rho)}{\alpha^n (r_1 \cdots r_n)} - (q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}}) - \varepsilon_{q,r}) \right| < \frac{\delta}{3};$

(3) $\displaystyle \frac{k_{q,\alpha r}(t_{\boldsymbol{a}})}{\alpha^n (r_1 \cdots r_n)} > (1 - q \operatorname{vol} \Delta_n(t_{\boldsymbol{a}})) - \frac{\delta}{3}.$

Proof. —

(1) Clear.
(2) Follows from the definition of $u_{q,r}(t_{\boldsymbol{a}})$.
(3) Follows from Proposition 2.6 (2). □

Proof of Proposition 5.9. — Let δ be a positive real number and let α_0 and ρ_0 given by Lemma 5.11. Take an integer $\alpha \geq \alpha_0$ and a real number $0 < \rho < \rho_0$.

Let $a^{(0)} \in \mathbf{P}(K)$ be the unique point such that $\chi_{\lambda,i}(a^{(0)}) = 1$ for every $i = 1, \dots, n$. Applying Proposition 5.8 to the point $a^{(0)}$:

$$\mu(\lambda, [K_{\alpha r}(a^{(0)}, u_{q,r}(t_{\boldsymbol{a}}) + \rho)]) = -\sum_{i=1}^{n} m_{\lambda,i} \mu^{\mathbf{Z}}_{\alpha r,i}(u_{q,r}(t_{\boldsymbol{a}}) + \rho).$$

According to Lemma 5.11 and Proposition 2.4 (2):

$$k_{q,\alpha r}(t_{\boldsymbol{a}}) + k_{\alpha r}(u_{q,r}(t_{\boldsymbol{a}}) + \rho) - \dim \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(\alpha r)) \geq -\alpha^n (r_1 \cdots r_n)(\varepsilon_{q,r} + \delta),$$

Therefore Lemma 5.10 yields:

$$\mu(\lambda, [K_{q,\alpha r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \geq \sum_{i=1}^{n} m_{\lambda,i} \left[\mu^{\mathbf{Z}}_{\alpha r,i}(u_{q,r}(t_{\boldsymbol{a}}) + \rho)) - \alpha^{n+1} r_i (r_1 \cdots r_n)(\varepsilon_{q,r} + \delta)\right],$$

which concludes the proof. □

6. Semi-stability on $\mathbf{P}^1 \times \mathbf{P}^1$ and the Wronskian determinant

In this section a different proof of the semi-stability is given in the case $n = 2$ emphasizing the role of the Wronskian determinant. Set

$$\varepsilon_{q,r} = (q-1)\frac{\min\{r_1, r_2\}}{\max\{r_1, r_2\}}.$$

When $r_1 \geq r_2$ this coincides with the previous definition.

6.1. The semi-stability statement will be the following one:

Theorem 6.1. — *Let $r = (r_1, r_2)$ be a couple of positive integers such that $r_1 \geq r_2$. Let $t_x, t_{\boldsymbol{a}} \geq 0$ be non-negative real numbers such that*

$$0 \leq 1 - q \operatorname{vol} \Delta_2(t_{\boldsymbol{a}}) + \varepsilon_{q+1,r} \leq \tfrac{1}{2}.$$

If the inequality

$$\mu_2(t_x) < (1 - q \operatorname{vol} \Delta_2(t_{\boldsymbol{a}})) \left(1 - 2\sqrt{2\,(1 - q \operatorname{vol} \Delta_2(t_{\boldsymbol{a}}) + \varepsilon_{q+1,r})}\right) \tag{6.1.1}$$

is satisfied then there exists a positive integer $\alpha_0 = \alpha_0(q, r, t_{\boldsymbol{a}}, t_x)$ such that, for every integer $\alpha \geq \alpha_0$, the K-point $P_{\alpha r} \in \mathcal{X}_{\alpha r}(K)$ is semi-stable under the action of $\mathbf{SL}_2^2$ with respect to the polarization given by the Plücker embeddings.

In particular, given $0 < \delta < 1$ one can apply it with

– $t_{\boldsymbol{a}} = t_{q,2}(\delta) = \sqrt{\dfrac{2}{q}(1 - \delta)}$;

– t_x that tends to the unique real number $w \in [1, 2]$ such that

$$\mu_2(w) = \delta \left(1 - 2\sqrt{2\,(\delta + \varepsilon_{q+1,r})}\right).$$

This is enough to derive the Main Effective Lower Bound in the case $n = 2$.

***Proposition 6.2* (Instability coefficient at the algebraic point)**

Let δ be a positive real number. Under the assumptions of Theorem 6.1 there exists a positive integer α_0 (possibly depending on q, r, $t_{\boldsymbol{a}}$ and t_x) satisfying the following properties: for every one-parameter subgroup $\lambda\colon \mathbf{G}_m \to \mathbf{SL}_{2,K}^2$ and every integer $\alpha \geq \alpha_0$:

$$\mu(\lambda, [K_{q,\alpha r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) \geq \alpha^3 r_1 r_2 \langle m_\lambda, r\rangle \left(1 - q \operatorname{vol}\Delta_2(t_{\boldsymbol{a}}) - \delta\right) \times \\ \times \left(1 - 2\sqrt{2\left(1 - q \operatorname{vol}\Delta_2(t_{\boldsymbol{a}}) + \varepsilon_{q+1,r}\right)}\right),$$

where $\langle -, - \rangle$ denotes the standard scalar product on $\mathbf{R}^2$.

An argument similar to the proof of Theorem 2.7 permits to deduce Theorem 6.1 from Proposition 6.2. The key point in the proof of Proposition 6.2 is the following:

***Proposition 6.3*.** — *Let $f \in K_r(\boldsymbol{a}, t_{\boldsymbol{a}})$ be a non-zero section. If*

$$1 - q \operatorname{vol}\Delta_2(t_{\boldsymbol{a}}) + \varepsilon_{q+1,r} \leq \tfrac{1}{2},$$

then, for every one parameter subgroup $\lambda\colon \mathbf{G}_m \to \mathbf{SL}_{2,K}^2$,

$$\mu(\lambda, [f]) \geq \langle m_\lambda, r\rangle \left(1 - 2\sqrt{2\left(1 - q \operatorname{vol}\Delta_2(t_{\boldsymbol{a}}) + \varepsilon_{q+1,r}\right)}\right),$$

the instability coefficient of f being taken as a point of $\mathbf{P}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)))$ and with respect to the invertible sheaf $\mathcal{O}(1)$.

Proof of Proposition 6.2. — Let δ be a positive real number. Analogously to the proof of Lemma 5.11 one may find a positive integer α_0 such that for every integer $\alpha \geq \alpha_0$:

$$\frac{k_{q,\alpha r}(t_{\boldsymbol{a}})}{\alpha^2 r_1 r_2} > \left(1 - q \operatorname{vol}\Delta_2(t_{\boldsymbol{a}})\right) - \delta. \tag{6.1.2}$$

Let $f_1, \ldots, f_{k_{q,\alpha r}(t_{\boldsymbol{a}})}$ be a basis of $K_{q,\alpha r}(\boldsymbol{a}, t_{\boldsymbol{a}})$ such that their components of minimal weight (with respect to λ) $f_{1,\min}, \ldots, f_{k_{q,\alpha r}(t_{\boldsymbol{a}}),\min}$ are linearly independent (such a basis exists according to Proposition 5.3). Then Proposition 5.3 (2) entails

$$\begin{aligned}\mu(\lambda, [K_{q,\alpha r}(\boldsymbol{a}, t_{\boldsymbol{a}})]) &= \sum_{\ell=1}^{k_{q,\alpha r}(t_{\boldsymbol{a}})} \mu(\lambda, [f_\ell]) \\ &\geq \alpha k_{q,\alpha r}(t_{\boldsymbol{a}}) \langle m_\lambda, r\rangle \left(1 - 2\sqrt{2\left(1 - q \operatorname{vol}\Delta_2(t_{\boldsymbol{a}}) + \varepsilon_{q+1,r}\right)}\right),\end{aligned}$$

where the second inequality follows from Proposition 6.3. Conclude using (6.1.2). □

The result proved in what follows is actually the following version of Dyson's Lemma:

***Theorem 6.4* (*cf.* Corollary 6.15).** — *Let $f \in \Gamma(\mathbf{P}, \mathcal{O}(r))$ be a non-zero global section and let $b = (b_1, b_2)$ be a couple of non-negative real numbers.*

Let y be a $\bar{\mathbf{Q}}$-point of $\mathbf{P}^1$. Let $q \geq 1$ be an integer and for every $\sigma = 1, \ldots, q$ let $z^{(\sigma)}$ be a $\bar{\mathbf{Q}}$-point of $\mathbf{P}$. For every $i = 1, 2$ suppose:

(1) $\mathrm{pr}_i(z^{(\sigma)}) \neq \mathrm{pr}_i(z^{(\tau)})$ *for every* $\sigma \neq \tau$*;*

(2) $\mathrm{pr}_i(z^{(\sigma)}) \neq \mathrm{pr}_i(y)$ *for every* $\sigma = 1, \dots, q$*.*

For every $\sigma = 1, \dots, q$ *suppose* $t_\sigma \leq 1$ *and*

$$1 - \sum_{\sigma=1}^{q} \mathrm{vol}\, \Delta_2(\mathrm{ind}_{1/r}(f, z^{(\sigma)})) + \varepsilon_{q+1,r} < \frac{1}{2}. \tag{6.1.3}$$

Then, $\mathrm{ind}_b(f, y) < \max\{b_i r_i\}$ *and*

$$\mathrm{vol}\, \Delta_2\left(\frac{\mathrm{ind}_b(f, y)}{\max\{b_i r_i\}}\right) \leq 1 - \sum_{\sigma=1}^{q} \mathrm{vol}\, \Delta_2(\mathrm{ind}_{1/r}(f, z^{(\sigma)})) + \varepsilon_{q+1,r}.$$

A similar version of Dyson's Lemma with two different weights is indicated to hold in [**EV84**, page 489]. The hypothesis (6.1.3) makes quantitative the assertion of Esnault-Viehweg that $\mathrm{ind}_b(f, y)$ should be "very small" (see *loc.cit.*).

In order to prove Proposition 6.3 let us link the instability coefficient and index through the following easy fact (whose proof is left to the reader):

Proposition 6.5. — *Let* $\lambda \colon \mathbf{G}_m \to \mathbf{SL}_{2,K}^2$ *be a one parameter subgroup and fix an adapted basis for* λ*. Then, for every non-zero element* $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$*,*

$$\mu(\lambda, [f]) = \langle m_\lambda, r\rangle - 2\, \mathrm{ind}_{m_\lambda}(f, y_\lambda),$$

where y_λ *is the instability point of* λ *(with respect to the chosen adapted bases).*

Proof of Proposition 6.3. — It suffices to apply Theorem 6.4 to the points $z^{(\sigma)} = a^{(\sigma)}$ for $\sigma \colon K' \to \bar{\mathbf{Q}}$, the weight $b = m_\lambda$ and the point $y = y_\lambda$ (the instability point of λ): since $t_{\boldsymbol{a}} \leq 1$ (otherwise (6.1.1) is not satisfied), $\mathrm{vol}\, \Delta_2(t_{\boldsymbol{a}}) = t_{\boldsymbol{a}}^2/2$. □

The rest of this section is therefore devoted to prove Theorem 6.4.

In view of Proposition 6.5, one would like to use this interpretation of the instability measure to apply the usual Dyson's Lemma — *i.e.* Theorem 2.2 when $n = 2$ [9] — in order to derive the semi-stability of the point. Unfortunately, the usual Dyson's Lemma can be applied when the weight of the index is the same at all points: here instead one has to apply it to weight $1/r$ at the points $z^{(\sigma)}$ for $\sigma = 1, \dots, q$ and the weight m_λ at the point y_λ.

The key point in the proof of Theorem 6.4 is that in general the index of a polynomial taken with respect to two different weights are not comparable. Anyway this is the case when the polynomial is a product of polynomials in separate variables:

Proposition 6.6. — *Let* $b = (b_1, b_2)$ *and* $c = (c_1, c_2)$ *be couples of non-negative real numbers. Suppose that is made of positive real numbers.*

For all $i = 1, 2$ *let* $f_i \in \Gamma(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(r_i))$ *be a non-zero section. Then, for all* $\bar{\mathbf{Q}}$*-point* z *of* $\mathbf{P}^1$*,*

$$\mathrm{ind}_b(f_1 \otimes f_2, z) \leq \max\{b_i/c_i\}\, \mathrm{ind}_c(f_1 \otimes f_2, z).$$

9. This is case originally treated by Dyson [**Dys47**], whose proof has then been revisited by several authors (see [**Bom82**], [**Vio85**] and [**Voj89**]).

In a nutshell, in the proof of Theorem 6.4 the Wronskian permits to reduce to the latter case.

6.2. Homogeneous Wronskian. — In this paragraph we introduce the wronskian as an invariant under of $\mathbf{SL}_{2,K}$. We follow the presentation given in [**AC07**, 2.8]. Let r, ρ be non-negative integers such that $\rho \leq r+1$.

Definition 6.7. — Let $f_1, \dots, f_\rho \in \operatorname{Sym}^r K^{2\vee}$ and let T_0, T_1 be the canonical basis of $K^{2\vee}$. The *homogeneous Wronskian* of the polynomials $f_1, \dots, f_\rho$ is:

$$\operatorname{Wr}(f_1, \dots, f_\rho) := \left(\frac{(r-\rho+1)!}{r!}\right)^\rho \cdot \det\left(\frac{\partial^{\rho-1} f_\ell}{\partial T_0^{\rho-j} \partial T_1^{j-1}}\right)_{j,\ell=1,\dots,\rho}.$$

It is an element of $\operatorname{Sym}^{\rho(r-\rho+1)} K^{2\vee}$, that is, a homogeneous polynomial of degree $\rho(r-\rho+1)$ in the variables T_0, T_1 (each entry is a homogeneous polynomial of degree $r-(\rho-1)$).

The reader may consult [**AC07**, 2.9] for the relation with the classical notion of Wronskian. It follows from Wronski's criterion of linear independence [**BG06**, Proposition 6.3.10] that $f_1, \dots, f_\rho$ are linearly independent if and only if $\operatorname{Wr}(f_1, \dots, f_\rho)$ does not vanish.

The Wronskian is an alternating multi-linear map on $\operatorname{Sym}^r K^{2\vee}$ and therefore it can be extended to a linear map

$$\operatorname{Wr}\colon \bigwedge^\rho \operatorname{Sym}^r K^{2\vee} \longrightarrow \operatorname{Sym}^{\rho(r-\rho+1)} K^{2\vee}.$$

Proposition 6.8 ([**AC07**, 2.3, 2.5 and 2.8]). — *The following properties are satisfied:*

(1) If T_0', T_1' is a basis and $\operatorname{Wr}'\colon \bigwedge^\rho \operatorname{Sym}^r K^{2\vee} \to \operatorname{Sym}^{\rho(r-\rho+1)} K^{2\vee}$ is the Wronskian map taken with respect to the latter basis, then as linear maps,

$$\operatorname{Wr}' = \det(T_0', T_1')^{\rho(r-\rho+1)} \operatorname{Wr},$$

where (T_0', T_1') is the linear map sending T_i on T_i' for $i = 0, 1$.

(2) The linear map Wr is equivariant under the action of $\mathbf{SL}_{2,K}$ on the vector spaces $\bigwedge^\rho \operatorname{Sym}^r K^{2\vee}$ and $\operatorname{Sym}^{\rho(r-\rho+1)} K^{2\vee}$.

6.3. Tensorial rank. — Let V_1, V_2 be finite-dimensional K-vector spaces.

Definition 6.9. — The *tensorial rank* of a non-zero vector $v \in V_1 \otimes_K V_2$ is the minimal integer $\rho \geq 0$ such that v can be written in the form $v_{11} \otimes v_{21} + \dots + v_{1\rho} \otimes v_{2\rho}$ with $v_{i\ell} \in V_i$ for $i = 1, 2$ and $\ell = 1, \dots, \rho$. Denote it by $\operatorname{rk}(v)$.

The tensorial rank is invariant under homotheties and under $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$ (acting component-wise). The tensorial rank of v coincides with the rank of the linear map $V_1^\vee \to V_2$ associated to v through the canonical isomorphism

$$V_1 \otimes_K V_2 \simeq \operatorname{Hom}_K(V_1^\vee, V_2)$$

(analogously it is the rank of the dual map $V_2^\vee \to V_1$).

For $i = 1, 2$ let $v_{i1}, \ldots, v_{i\,\mathrm{rk}(v)} \in V_i$ be such that $v = \sum_{\ell=1}^{\mathrm{rk}(v)} v_{1\ell} \otimes v_{2\ell}$. Then, for $i = 1, 2$, $v_{i1}, \ldots, v_{i\,\mathrm{rk}(v)}$ are linearly independent.

6.4. Splitting polynomials through the Wronskian. — Let $r = (r_1, r_2)$ be a couple of positive integers and for $i = 1, 2$ set $V_i := \mathrm{Sym}^{r_i} K^{2\vee}$. With this notation,

$$\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)) \simeq V_1 \otimes V_2.$$

For every non-zero $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ one can consider its tensorial rank $\mathrm{rk}(f)$ with respect to this decomposition. With the notations of [**Bom82**, page 266],

$$s_2(f) = \mathrm{rk}(f) + 1.$$

For every $i = 1, 2$ fix a basis T_{i0}, T_{i1} of $K^{2\vee}$.

Definition 6.10. — Let $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ be a non-zero section and let $\rho = \mathrm{rk}(f)$ be its tensorial rank. For every couple of positive integers $\ell = (\ell_1, \ell_2)$ such that $\ell_i \leq \rho$ for $i = 1, 2$, set:

$$\partial_\ell^{2(\rho-1)} f := \frac{\partial^{2(\rho-1)} f}{\partial T_{10}^{\rho-\ell_1} \partial T_{11}^{\ell_1 - 1} \partial T_{20}^{\rho-\ell_2} \partial T_{21}^{\ell_2-1}},$$

which is a global section of $\mathcal{O}_{\mathbf{P}}(r_1 - (\rho-1), r_2 - (\rho-1))$. The *homogeneous Wronskian* $\mathrm{Wr}(f)$ is the determinant

$$\mathrm{Wr}(f) := \left[\prod_{i=1}^{2} \left(\frac{(r_i - \rho + 1)!}{r_i!}\right)^{\rho}\right] \cdot \det\left(\partial_\ell^{2(\rho-1)} f\right)_{\ell_1, \ell_2 = 1, \ldots, \rho},$$

seen as a global section of $\mathcal{O}_{\mathbf{P}}(r_1 - (\rho - 1), r_2 - (\rho - 1))^{\otimes \rho}$.

Let $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ be a non-zero section and let $\rho = \mathrm{rk}(f)$ be its tensorial rank. Write $f = \sum_{\ell=1}^{\rho} f_{1\ell} \otimes f_{2\ell}$ with $f_{i\ell} \in \Gamma(\mathbf{P}^1, \mathcal{O}(r_i))$ for all $i = 1, 2$ and all $\ell = 1, \ldots, \rho$. For $i = 1, 2$ consider the homogeneous Wronskian

$$\mathrm{Wr}_i(f) := \mathrm{Wr}(f_{i1}, \ldots, f_{i\rho}),$$

computed with respect to the basis T_{i0}, T_{i1}. An elementary computation shows:

Proposition 6.11. — *With the notations introduced above,*

$$\mathrm{Wr}(f) = \mathrm{Wr}_1(f) \otimes \mathrm{Wr}_2(f).$$

6.5. Index of the Wronskian. — In this section the index of Wronskian is linked with the one of the original polynomial.

Proposition 6.12. — *Let $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ be a non-zero section and let $b = (b_1, b_2)$ be a couple of non-negative real numbers. Then, for every $\bar{\mathbf{Q}}$-point z of $\mathbf{P}$,*

$$\mathrm{ind}_b(\mathrm{Wr}(f), z) \geq \max\{b_i r_i\} \left((\mathrm{rk}(f) - 1) \left(2 - \frac{\mathrm{rk}(f) - 1}{r_2} \right) \mathrm{vol}\, \Delta_2(t) - \mathrm{rk}(f) \varepsilon_{2,r} \right),$$

where $t := \min\{1, \inf_b(f, z) / \max\{b_i r_i\}\}$.

Proof. — Since the Wronskian does not depend (up to a non-zero scalar factor) on the chosen basis (Proposition 6.8), then for $i = 1, 2$ one can chose a basis T_{i0}, T_{i1} of $K^{2\vee}$ such that $T_{i0}(\mathrm{pr}_i(z)) \neq 0$ and $T_{i1}(\mathrm{pr}_i(z)) = 0$.

Let $t := \mathrm{ind}_b(f, z)$ be the index of f at z. Let $\rho = \mathrm{rk}(f)$ be the tensorial rank of f and, up to permuting the coordinates suppose $r_1 \geq r_2$. Since deriving with respect to T_{10} and T_{20} does not affect the index on z, for every $\ell_1, \ell_2 = 1, \ldots, \rho$,

$$\begin{aligned}
\mathrm{ind}_b\left(\partial^{2(\rho-1)}_{(\ell_1,\ell_2)} f, z\right) &\geq \max\left\{0, t - \langle b, (\ell_1 - 1, \ell_2 - 1)\rangle\right\} \\
&\geq \max\left\{0, t - \frac{\ell_2 - 1}{r_2} \max\{b_i r_i\}\right\} - \varepsilon_{2,r} \max\{b_i r_i\}
\end{aligned}$$

where,

$$\langle b, (\ell_1 - 1, \ell_2 - 1)\rangle \leq \max\{b_i r_i\} \langle 1/r, (\ell_1 - 1, \ell_2 - 1)\rangle \leq \max\{b_i r_i\} \left(\varepsilon_{2,r} + \frac{\ell_2 - 1}{r_2}\right),$$

(recall $1/r = (1/r_1, 1/r_2)$ and use $\ell_1 - 1 \leq \rho - 1 \leq r_2$). Let $\mathfrak{S}_\rho$ be the permutation group on $\{1, \ldots, \rho\}$. Since the index is a valuation,

$$\begin{aligned}
\mathrm{ind}_b(\mathrm{Wr}(f), z) &\geq \min_{\pi \in \mathfrak{S}_\rho} \left\{\sum_{\ell=1}^{\rho} \mathrm{ind}_b\left(\partial^{2(\rho-1)}_{(\pi(\ell),\ell)} f, z\right)\right\} \\
&\geq \min_{\pi \in \mathfrak{S}_\rho} \left\{\sum_{\ell=1}^{\rho} \max\left\{0, t - \frac{\ell - 1}{r_2} \max\{b_i r_i\}\right\}\right\} - \rho\varepsilon_{2,r} \max\{b_i r_i\}.
\end{aligned}$$

Writing $t' := t / \max\{b_i r_i\}$ and $u = \min\{(\rho - 1)/r_2, t'\}$,

$$\sum_{\ell=1}^{\rho} \max\left\{0, t - \frac{\ell - 1}{r_2} \max\{b_i r_i\}\right\} = \max\{b_i r_i\} \left(\sum_{\ell=0}^{r_2 u} \left(t' - \frac{\ell}{r_2}\right)\right).$$

Finally,

$$\sum_{\ell=0}^{r_2 u} \left(t' - \frac{\ell}{r_2}\right) = (r_2 u + 1)\left(t' - \frac{u}{2}\right) \geq r_2 u \left(t' - \frac{u}{2}\right).$$

Conclude by:

Lemma 6.13. — *With the notations introduced above, let* $\widetilde{t} := \min\{t', 1\}$. *Then,*

$$u\left(t' - \frac{u}{2}\right) \geq \frac{\rho - 1}{r_2}\left(2 - \frac{\rho - 1}{r_2}\right) \mathrm{vol}\, \Delta_2(\widetilde{t}).$$

Proof of Lemma 6.13. — Two cases have to be considered:

(1) Suppose $u = t'$. Then, $u(t' - u/2) = t'^2/2$ and

$$\frac{\rho - 1}{r_2}\left(2 - \frac{\rho - 1}{r_2}\right) \leq 1,$$

because $\rho - 1 \leq r_2$.

(2) Suppose $u = (\rho - 1)/r_2$. Then,

$$u\left(t' - \frac{u}{2}\right) \geq u\left(\widetilde{t} - \frac{u}{2}\right).$$

The function

$$\frac{\xi\left(\widetilde{t} - \xi/2\right)}{\xi\,(2-\xi)} = \frac{\widetilde{t} - \xi/2}{2-\xi},$$

is decreasing for $\xi \in [0,1]$ because $\widetilde{t} \leq 1$. By assumption, $u \leq \widetilde{t}$, therefore applying this consideration with $\xi = u = (\rho - 1)/r_2$,

$$\frac{u\left(\widetilde{t} - u/2\right)}{u(2-u)} \geq \frac{\widetilde{t}^2/2}{\widetilde{t}(2-\widetilde{t})} \geq \widetilde{t},$$

where in the last inequality one uses again the inequality $\widetilde{t}(2-\widetilde{t}) \leq 1$. This terminates the proof of the lemma. □

This concludes the proof of the Proposition. □

Proposition 6.14. — *Let $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ be non-zero and $b = (b_1, b_2)$ a couple of non-negative real numbers.*

Let $q \geq 1$ be an integer and for every $\sigma = 0, \ldots, q$ let $z^{(\sigma)}$ be a $\bar{\mathbf{Q}}$-point of $\mathbf{P}$. For every $i = 1, 2$ suppose $\mathrm{pr}_i(z^{(\sigma)}) \neq \mathrm{pr}_i(z^{(\tau)})$ for every $\sigma \neq \tau$. Then,

$$\sum_{\sigma=0}^{q} \mathrm{vol}\,\Delta_2(t^{(\sigma)}) \leq 1 + \varepsilon_{q+1,r},$$

where

$$t^{(\sigma)} = \begin{cases} \min\left\{1, \mathrm{ind}_{1/r}(f, z^{(\sigma)})\right\} & \text{if } \sigma = 1, \ldots, q \\ \min\left\{1, \mathrm{ind}_b(f, z^{(0)})/\max\left\{b_i r_i\right\}\right\} & \text{if } \sigma = 0. \end{cases}$$

Proof. — Suppose $\mathrm{rk}(f) > 1$. The proof is done bounding from above and from below the index $\mathrm{ind}_b(\mathrm{Wr}(f), z^{(0)})$.

Upper bound. Borrow the notations from Proposition 6.6. Since

$$\mathrm{Wr}(f) = \mathrm{Wr}_1(f) \otimes \mathrm{Wr}_2(f),$$

Proposition 6.6 applied to the weight $c = 1/r$ gives

$$\mathrm{ind}_b(\mathrm{Wr}(f), z^{(0)}) \leq \max\{b_i r_i\}\,\mathrm{ind}_{1/r}(\mathrm{Wr}(f), z^{(0)}).$$

It remains to estimate $\mathrm{ind}_{1/r}(\mathrm{Wr}(f), z^{(0)})$. Set $\rho := \mathrm{rk}(f)$. Using the definition of the index, the fact that $\mathrm{Wr}_i(f)$ is a section of $\mathcal{O}(\rho(r_i - \rho + 1))$ on $\mathbf{P}^1$ and the hypothesis that the projection of the points $z^{(\sigma)}$ are pairwise distinct:

$$\begin{aligned}\operatorname{ind}_{1/r}(\mathrm{Wr}(f), z^{(0)}) &= \sum_{i=1}^{2} \frac{1}{r_i} \operatorname{mult}(\mathrm{Wr}_i(f), \mathrm{pr}_i(z^{(0)})) \\ &\leq \sum_{i=1}^{2} \frac{1}{r_i} \left(\rho(r_i - \rho + 1) - \sum_{\sigma=1}^{q} \operatorname{mult}(\mathrm{Wr}_i(f), \mathrm{pr}_i(z^{(\sigma)})) \right) \\ &= \rho \left(2 - \sum_{i=1}^{2} \frac{\rho - 1}{r_i} \right) - \sum_{\sigma=1}^{q} \operatorname{ind}_{1/r}(\mathrm{Wr}(f), z^{(\sigma)}),\end{aligned}$$

For $\sigma = 1, \ldots, q$, Proposition 6.12 (applied to $z = z^{(\sigma)}$ and $b = 1/r$) entails:

$$\operatorname{ind}_{1/r}(\mathrm{Wr}(f), z^{(\sigma)}) \geq (\rho - 1) \left(2 - \frac{\rho - 1}{r_2} \right) \operatorname{vol} \Delta_2(t^{(\sigma)}),$$

where $t^{(\sigma)} = \min\left\{1, \operatorname{ind}_{1/r}(f, z^{(\sigma)})\right\}$. Summing up, the index $\operatorname{ind}_b(\mathrm{Wr}(f), z^{(0)})$ is bounded above by

$$\max\{b_i r_i\} \left[\rho \left(2 - \sum_{i=1}^{2} \frac{\rho - 1}{r_i} + \varepsilon_{q+1,r} \right) - (\rho - 1) \left(2 - \frac{\rho - 1}{r_2} \right) \left(\sum_{\sigma=1}^{q} \operatorname{vol} \Delta_2(t^{(\sigma)}) \right) \right],$$

where $q\varepsilon_{2,r} = \varepsilon_{q+1,r}$.

Lower bound. Proposition 6.12 applied to the point $z = z^{(0)}$ and to the weight b gives:

$$\operatorname{ind}_b(\mathrm{Wr}(f), y) \geq \max\{b_i r_i\} (\rho - 1) \left(2 - \frac{\rho - 1}{r_2} \right) \operatorname{vol} \Delta_2(t^{(0)}),$$

where $t^{(0)} := \min\left\{1, \inf_b(f, z^{(0)}) / \max\{b_i r_i\}\right\}$.

Combining the lower bound and the upper bound of $\operatorname{ind}_b(f, z^{(0)})$:

$$(\rho - 1) \left(2 - \frac{\rho - 1}{r_2} \right) \left(\sum_{\sigma=0}^{q} \operatorname{vol} \Delta_2(t^{(\sigma)}) \right) \leq \rho \left(2 - \frac{\rho - 1}{r_2} \right) + \rho\varepsilon_{q+1,r},$$

(in the right-hand side $-(\rho-1)/r_1$ has been neglected). Dividing by $(\rho-1)\left(2 - \frac{\rho-1}{r_2}\right)$ and using $2 - (\rho - 1)/r_2 \geq 1$:

$$\begin{aligned}\sum_{\sigma=0}^{q} \operatorname{vol} \Delta_2(t^{(\sigma)}) &\leq \frac{\rho}{\rho - 1} + \frac{\rho}{\rho - 1} \left(2 + \frac{\rho - 1}{r_2} \right)^{-1} \varepsilon_{q+1,r} \\ &\leq 1 + \varepsilon_{q+1,r} + \frac{1}{\rho - 1} \left(1 + \varepsilon_{q+1,r} \right).\end{aligned}$$

Taking powers of f and multiplying by suitable linear polynomials, one can show that ρ can be taken arbitrarily large (even though it could be small compared to r_2) — see [**Bom82**, Lemma 2 and II.4] for more details. This concludes the proof in the case $\mathrm{rk}(f) > 1$.

The same argument shows that one can suppose $\mathrm{rk}(f) > 1$. □

Corollary 6.15. — *Under the assumptions of Proposition 6.14, suppose*

$$\mathrm{ind}_{1/r}(f, z^{(\sigma)}) \leq 1$$

for $\sigma = 1, \dots, q$ *and*

$$1 - \sum_{\sigma=1}^{q} \mathrm{vol}\,\Delta_2(\mathrm{ind}_{1/r}(f, z^{(\sigma)})) + \varepsilon_{q+1,r} < \frac{1}{2}. \tag{6.5.1}$$

Then, $\mathrm{ind}_b(f, z^{(0)}) < \max\{b_i r_i\}$ *and*

$$\mathrm{vol}\,\Delta_2\left(\frac{\mathrm{ind}_b(f, z^{(0)})}{\max\{b_i r_i\}}\right) \leq 1 - \sum_{\sigma=1}^{q} \mathrm{vol}\,\Delta_2(\mathrm{ind}_{1/r}(f, z^{(\sigma)})) + \varepsilon_{q+1,r}.$$

Proof. — If $\mathrm{ind}_b(f, y) \geq \max\{b_i r_i\}$, Proposition 6.14 entails

$$\frac{1}{2} \leq 1 - \sum_{\sigma=1}^{q} \mathrm{vol}\,\Delta_2(\mathrm{ind}_{1/r}(f, z^{(\sigma)})) + \varepsilon_{q+1,r},$$

which contradicts (6.5.1). □

6.6. Wronskian as a covariant. — Let us conclude with a final remark. Fix a positive integer $\rho \geq 1$. The Wronskian furnishes a "covariant" for the action of $\mathbf{SL}^2_{2,K}$, *i.e.* a rational $\mathbf{SL}^2_{2,K}$-equivariant map

$$\mathrm{Wr} : \mathbf{P}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))) \dashrightarrow \mathbf{P}(\mathrm{Sym}^{\rho(r_1-\rho+1)} K^{2\vee} \otimes_K \mathrm{Sym}^{\rho(r_2-\rho+1)} K^{2\vee}),$$

which is defined on the open subset $U_\rho \subset \mathbf{P}(\Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r)))$ of lines generated by non-zero sections $f \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(r))$ of tensorial rank $\geq \rho$. The Wronskian map Wr moreover induces a $\mathbf{SL}^2_{2,K}$-equivariant isomorphism of line bundles $\mathrm{Wr}^* \mathcal{O}(1) \simeq \mathcal{O}(\rho)_{|U_\rho}$. In the early stages of the present work, this constituted for us one of the main evidences that the proof of Roth's theorem was connected with Geometric Invariant Theory.

To make this intuition more precise, for such a morphism one has

$$\mu_{\mathcal{O}(1)}(\lambda, [f]) \geq \frac{1}{\rho} \mu_{\mathcal{O}(1)}(\lambda, [\mathrm{Wr}(f)]),$$

for every global section f of $\mathcal{O}_{\mathbf{P}}(r)$ of tensorial rank $\geq \rho$ and every one-parameter subgroup $\lambda \colon \mathbf{G}_m \to \mathbf{SL}^2_{2,K}$. If y_λ is the instability point associated to the choice of admissible bases for λ, Proposition 6.5 leads to the lower bound

$$\mathrm{ind}_{m_\lambda}(\mathrm{Wr}(f), y_\lambda) \geq \rho\, \mathrm{ind}_{m_\lambda}(f, y_\lambda) - \frac{\rho(\rho-1)}{2}(m_{\lambda,1} + m_{\lambda,2}).$$

As explained before, we want to apply Theorem 6.4 to the point $y = y_\lambda$ and the weight $b = m_\lambda$. Unfortunately, this lower bound is not sharp enough to deduce the semi-stability of the point P_r (for $\mu_2(t_x)$ small enough). Instead we had to use the lower bound given by Proposition 6.12 in the proof of Proposition 6.14.

CHAPTER 3

KEMPF-NESS THEORY IN NON-ARCHIMEDEAN GEOMETRY

In this chapter we investigate an analogue in non-archimedean geometry of a classical result of Kempf and Ness [**KN79**] concerning the behaviour of hermitian norms on a (finite dimensional) representation of a complex reductive group. Even though the aim is to study the problem in the non-archimedean framework, the techniques that we employ are well-suited to work both in the complex and non-archimedean case at the same time. Therefore we decided to treat the two cases on the same footing, hoping that this will ease the task of the reader.

In Section 1 we recall the original result of Kempf and Ness and we show how one has to modify the statements in order to port them in the context of non-archimedean geometry. This sections serves as an introduction to the results of this chapter and some of which will be necessary for the developments of Chapter 4.

In Section 2 we introduce the three objects that will play the main role in the proof: analytic spaces (in the sense Berkovich in the non-archimedean case), maximal compact subgroups and plurisubharmonic functions. Complying with the choice of developing the complex and the non-archimedean case at the same time, these concepts are presented trying to stress the analogies between these two realms.

Concerning maximal compact subgroups and plurisubhamonic functions, the analogies could have been pushed further. We deliberately opted to give naive definitions which are enough for our purposes instead of being brought into far-reaching results on these subjects (namely the theory of Bruhat-Tits buildings and Thuillier's potential theory on curves which are not rational).

Section 3 is the core of the chapter: we prove the main result concerning the behaviour of invariant plurisubharmonic functions on the orbit of a point and its closure (see Theorem 3.3). This leads to the understanding of the analytic topology on the GIT quotient (which is new in the non-archimedean case, see Theorem 1.6). The proof of the latter relies ultimately on the local compactness of the analytic spaces use and, in the non-archimedean case, recurring to Berkovich spaces seems unavoidable.

In the last section we translate the previous results into the continuity of the metric on the GIT quotient. We also take profit of the occasion to prove the compatibility of the construction this metric with integral models, where the original result of Burnol takes place. These latter results will be crucial for Chapter 4.

1. Statement of the main results

1.1. A result of Kempf and Ness. — Let G be a complex (connected) reductive group and let V be a (finite dimesional) representation of G endowed with an hermitian norm $\| \cdot \| \colon V \to \mathbf{R}_+$. Suppose that the hermitian norm $\| \cdot \|$ is invariant under the action of a maximal compact subgroup $\mathbf{U}$ of G.

Let v be a vector in V. Kempf and Ness studied in their celebrated paper [**KN79**] the properties of the function $p_v \colon G \to \mathbf{R}_+$ defined by

$$p_v(g) := \|g \cdot v\|^2.$$

Among the results therein presented, the following are of particular interest for us:

Theorem 1.1. — *With the notations introduced above:*

(1) The function p_v obtains its minimum value if and only if the orbit of v is closed.
(2) Any critical point of p_v is a point where p_v obtains its minimum.
(3) If p_v obtains its minimum value, then the set where p_v obtains this value consists of a single $\mathbf{U}$ - G_v coset (here G_v is the stabiliser of v in G).

1.2. Interpretation via the moment map. — As discovered by Guillemin-Sternberg and Mumford, these results permit to link the Geometric Invariant Theory of Kähler varieties with the concept of moment map in symplectic geometry.

In the present situation a moment map $\mu \colon \mathbf{P}(V) \to (\operatorname{Lie} \mathbf{U})^\vee$ for the action of G on V is defined as follows. For every non-zero vector $v \in V$, consider the linear map $\mu_{[v]} \colon \operatorname{Lie} \mathbf{U} \to \mathbf{R}$ defined for every $a \in \operatorname{Lie} \mathbf{U}$ by [1]

$$\mu_{[v]}(a) := \frac{1}{\boldsymbol{i} 2\pi} \cdot \frac{\langle \operatorname{ad}(a, v), v \rangle}{\|v\|^2}.$$

Here $\langle -, - \rangle$ denotes the hermitian form associated to the norm $\| \cdot \|$, $\boldsymbol{i}$ denotes a square root of -1 and $\operatorname{ad} \colon \operatorname{Lie} \mathbf{U} \times V \to V$ denotes the adjoint action.

Say that $v \in V$ is minimal [2] if $p_v(g) \geq p_v(e)$ for every $g \in G$ and denote by $\mathbf{P}(V)^{\min}$ the set of points having a non-zero representative which is minimal.

Proposition 1.2. — *A non-zero vector $v \in V$ is minimal if and only if the linear map $\mu_{[v]}$ is identically zero.*

(For a proof the reader can consult the proof of [**MFK94**, Theorem 8.3]). With this notation statement (2) in Theorem 1.1 is translated into the equality:

$$\mu^{-1}(0) = \mathbf{P}(V)^{\min}.$$

1. Other conventions on the scalar factor of μ can be found in the literature.
2. The name minimal refers to the fact that the minimum of the norm on the orbit $G \cdot v$ is attained on v.

Moreover consider the open subset $\mathbf{P}(V)^{\mathrm{ss}}$ of semi-stable points of $\mathbf{P}(V)$ with respect to G and $\mathcal{O}(1)$. Let Y be categorial quotient of $\mathbf{P}(V)^{\mathrm{ss}}$ by G. Then the map

$$\mu^{-1}(0)/\mathbf{U} \longrightarrow Y(\mathbf{C})$$

is a homeomorphism [**MFK94**, Theorem 8.3]. When the action of $\mathbf{U}$ is free, the quotient $\mu^{-1}(0)/\mathbf{U}$ is called the Marsden-Weinstein reduction or symplectic quotient. The interested reader can refer to the original papers of Guillemin-Sternberg [**GS82a, GS82b, GS84**], or the more introductory accounts of Kirwan [**MFK94**, Chapter 8] and Woodward [**Woo10**].

1.3. Present setting. — In this text we study what happens when one replaces:

- the field $\mathbf{C}$ by a field k complete with respect to an absolute value;
- the vector space V by a k-affine scheme X endowed with an action of a reductive k-group G;
- the norm $\|\cdot\|$ by a plurisubharmonic function $u\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ (see Definition 2.29) invariant under a maximal compact subgroup $\mathbf{U}$ of G (see Definitions 2.20 and 2.21).

Azad-Loeb [**AL93**] studied the case when X is a complex smooth affine scheme (or more generally a smooth Stein space) and $u\colon X(\mathbf{C}) \to \mathbf{R}$ is a $\mathbf{U}$-invariant strongly plurisubharmonic function which is twice differentiable. Statement (1) and (3) in Theorem 1.1 are not longer valid in general when the function is not strongly plurisubharmonic:

Example 1.3. — Consider the action of the multiplicative group $\mathbf{C}^\times$ on $\mathbf{C}^2$ given by

$$t \cdot (x, y) = (tx, y).$$

The ℓ^∞ norm $\|(x, y)\|_\infty = \max\{|x|, |y|\}$ is plurisubharmonic and invariant under the action of the maximal compact subgroup $\mathbf{U}(1)$. However the orbit of $(1, 1)$ is given by the points of the form $(t, 1)$ with $t \in \mathbf{C}^\times$, thus

$$\|(t, 1)\|_\infty \geq 1 = \|(1, 1)\|_\infty$$

for every $t \in \mathbf{C}^\times$. Therefore the point $(1, 1)$ is "minimal" in its orbit but its orbit is not closed. Moreover every point of the form $(t, 1)$ with $|t| \leq 1$ is "minimal" and they do not belong to the same orbit under $\mathbf{U}(1)$.

In order to discuss what is a right analogue of the result of Kempf-Ness in this new context, let us first go back to the classical algebraic framework of Geometric Invariant Theory.

1.4. Algebraic setting. — Let k be a field. Let G be a reductive k-group acting on an affine k-scheme $X = \operatorname{Spec} A$ of finite type. Denote by Y the spectrum of the subalgebra of invariants A^G and by $\pi\colon X \to Y$ the morphism induced by the inclusion $A^G \subset A$.

The fundamental theorem of Geometric Invariant Theory in the affine case can be stated as follows (see [**MFK94**, Theorem 1.1 and Corollay 1.2] for characteristic 0, [**Hab75**] on positive characteristic and [**Ses77**, Theorem 3] over more general bases).

Theorem 1.4. — *The k-scheme Y is of finite type and the morphism π satisfies the following properties:*

(1) π is surjective and G-invariant;
(2) let K be a field extension of k and $\pi_K \colon X_K := X \times_k K \to Y_K := Y \times_k K$ be the morphism obtained extending scalars to K; then for all points $x, x' \in X(K)$,

$$\pi_K(x) = \pi_K(x') \quad \textit{if and only if} \quad \overline{G_K \cdot x} \cap \overline{G_K \cdot x'} \neq \emptyset,$$

the orbits being taken in X_K.
(3) for every G-stable closed subset $F \subset X$ its image $\pi(F) \subset Y$ is closed;
(4) the structural morphism $\pi^\sharp \colon \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ induces an isomorphism

$$\pi^\sharp \colon \mathcal{O}_Y \xrightarrow{\sim} (\pi_* \mathcal{O}_X)^G.$$

In particular Y is the categorical quotient of X by G in the category of k-schemes, *i.e.* every G-invariant morphism $\pi' \colon X \to Y'$ factors in a unique way through Y. For this reason, for the rest of this paper Y is called the *quotient* of X by G and π the *quotient morphism* or the *projection (on the quotient)*.

1.5. Analytic setting. — Suppose moreover that the field k is complete with respect to an absolute value $|\cdot|$. Keeping the notations introduced above denote by G^{an} (resp. X^{an}, resp. Y^{an}) the k-analytic space obtained by analytification of the k-affine scheme G (resp. X, resp. Y). Here, a real analytic space is the quotient of a complex analytic space by an anti-holomorphic involution; non-archimedean analytic spaces are taken in the sense of Berkovich. We summarised the needed material on the construction of the analytification in Section 2.1: the reader can refer to that section for the definitions.

The k-analytic group G^{an} acts on the k-analytic space X^{an} and the morphism of k-analytic spaces $\pi \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ induced by the canonical projection (still denoted π) is surjective and G^{an}-invariant.

Let $\sigma \colon G \times_k X \to X$ be the morphism of k-schemes defining the action of G on X.

Definition 1.5. — The *orbit* of a point $x \in X^{\mathrm{an}}$ is the subset of X^{an} defined by

$$G^{\mathrm{an}} \cdot x := \sigma^{\mathrm{an}}(\mathrm{pr}_1^{-1}(x)).$$

A subset $F \subset X^{\mathrm{an}}$ is said to be G^{an}-*stable* (resp. G^{an}-*saturated*) if for every point $x \in F$, its orbit $G^{\mathrm{an}} \cdot x$ (resp. the closure $\overline{G^{\mathrm{an}} \cdot x}$ of its orbit) is contained in F.

In the complex case these are just the usual notions. In any case, for two points $x, y \in X^{\mathrm{an}}$,

$$y \in G^{\mathrm{an}} \cdot x \Longleftrightarrow x \in G^{\mathrm{an}} \cdot y,$$

(in the non-archimedean case the statement is not obvious because x and y may have different complete residue field; see [**Ber90**, Proposition 5.1.1]).

1.6. Analytic topology of the GIT quotient. — Our first main result is the analogue of points (1)-(3) in Theorem 1.4 in the setting of k-analytic spaces:

***Theorem 1.6* (*cf.* Propositions 3.1 and 3.9).** — *With the notation introduced above, the morphism $\pi^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ satisfies the following properties:*

(1) π^{an} is surjective and G^{an}-invariant;

(2) for every $x, x' \in X^{\mathrm{an}}$:

$$\pi^{\mathrm{an}}(x) = \pi^{\mathrm{an}}(x') \quad \textit{if and only if} \quad \overline{G^{\mathrm{an}} \cdot x} \cap \overline{G^{\mathrm{an}} \cdot x'} \neq \emptyset;$$

(3) if F is a G^{an}-stable closed subset of X^{an}, then its projection $\pi^{\mathrm{an}}(F)$ is a closed subset of Y^{an}.

In the complex case statements (1) and (2) are deduced from their “algebraic” version (Theorem 1.4 (1)-(2)). In order show (2), the crucial observation is that the orbit $G \cdot x$ of a point $x \in X(\mathbf{C})$ is a constructible subset of X: its closure with respect to the complex topology coincide with its Zariski closure. This theorem is already known as a consequence of the results of Kempf and Ness. Another proof has been given also by Neeman [**Nee85**].

Theorem 1.6 permits to derive formally the following consequences, whose proof is left to the reader:

Corollary 1.7. — *With the notation introduced above, the following properties are satisfied:*

(1) for every point $x \in X^{\mathrm{an}}$ there exists a unique closed orbit contained in $\overline{G^{\mathrm{an}} \cdot x}$;

(2) for every G^{an}-saturated subsets $F, F' \subset X^{\mathrm{an}}$:

$$\pi^{\mathrm{an}}(F) \cap \pi^{\mathrm{an}}(F') \neq \emptyset \quad \textit{if and only if} \quad F \cap F' \neq \emptyset;$$

(3) a subset $V \subset Y^{\mathrm{an}}$ is open if and only $(\pi^{\mathrm{an}})^{-1}(V) \subset X^{\mathrm{an}}$ is open;

(4) let U be an open subset of X^{an}; then U is G^{an}-saturated if and only if

$$U = (\pi^{\mathrm{an}})^{-1}(\pi^{\mathrm{an}}(U));$$

if U satisfies one of this two equivalent properties, then its projection $\pi^{\mathrm{an}}(U)$ is an open subset of Y^{an}.

In particular the topological space Y^{an} is separated and is the categorical quotient in the category of T_1 topological spaces [(3)] of X^{an} by the equivalence relation:

$$x \mathrel{\mathcal{R}_G} x' \iff G^{\mathrm{an}} \cdot x = G^{\mathrm{an}} \cdot x'.$$

In the complex case, the isomorphism $\pi^\sharp : \mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is known to hold also at the level of holomorphic functions, that is, the homomorphism of sheaves,

$$\pi^\sharp : \mathcal{O}_Y^{\mathrm{an}} \longrightarrow (\pi_* \mathcal{O}_X^{\mathrm{an}})^{G^{\mathrm{an}}},$$

is an isomorphism. This can be shown either by general techniques of Stein spaces (see for instance [**Sch00**, Chapter 5, Proposition 4 (3)]) or as an application of Luna's Slice Theorem (in both cases, one averages functions on a maximal compact subgroup).

3. A topological space S is said to be T_1 if the points of S are closed.

Now, neither of the two approaches is available in non-archimedean geometry — the reasons being that Stein spaces are far to be understood, étale morphisms are not local isomorphisms and one cannot interpret (in characteristic 0) the Reynolds operator as an average on maximal compact subgroup.

Question. — In the non-archimedean case, is the homomorphism of sheaves,

$$\pi^\sharp : \mathcal{O}_Y^{\mathrm{an}} \longrightarrow (\pi_* \mathcal{O}_X^{\mathrm{an}})^{G^{\mathrm{an}}},$$

an isomorphism?

1.7. A variant of the result of Kempf-Ness. —

Definition 1.8. — A function $u\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ is said to be invariant under a maximal compact subgroup of G if there exists a maximal compact subgroup $\mathbf{U} \subset G^{\mathrm{an}}$ with the following property: for every point $t \in G^{\mathrm{an}} \times_k X^{\mathrm{an}}$ such that $\mathrm{pr}_1(t) \in \mathbf{U}$,

$$u(\sigma^{\mathrm{an}}(t)) = u(\mathrm{pr}_2(t)),$$

where $\sigma\colon G \times_k X \to X$ is the morphism defining the action of G on X.

***Theorem 1.9* (*cf.* Theorem 3.3)**. — *Let $u\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ be a plurisubharmonic function which is invariant under the action of a maximal compact subgroup of G. For every point $x \in X^{\mathrm{an}}$,*

$$\inf_{\pi^{\mathrm{an}}(x')=\pi^{\mathrm{an}}(x)} u(x') = \inf_{x' \in G^{\mathrm{an}} \cdot x} u(x').$$

Definition 1.10. — Let $u\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ be a function. A point $x \in X^{\mathrm{an}}$ is said to be:

- *u-minimal on π-fibre* if $u(x) \le u(x')$ for all $x' \in X^{\mathrm{an}}$ such that $\pi(x) = \pi(x')$;
- *u-minimal on G-orbit* if $u(x) \le u(x')$ for all $x' \in G^{\mathrm{an}} \cdot x$.

The set of u-minimal points on π-fibres (resp. u-minimal points on G-fibres) is denoted by $X_\pi^{\min}(u)$ (resp. $X_G^{\min}(u)$).

***Corollary 1.11* (*cf.* Corollary 3.4)**. — *Let $u\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ be a plurisubharmonic function which is invariant under the action of a maximal compact subgroup of G. Then,*

1. *a point x is u-minimal on π-fibre if and only if it is u-minimal on G-orbit;*
2. *$X_\pi^{\min}(u) = X_G^{\min}(u)$;*
3. *if u is moreover continuous, the set of u-minimal points on π-fibres $X_\pi^{\min}(u)$ is closed.*

In order to understand better the relation with the result of Kempf and Ness remark the following consequence of Theorem 1.9:

Corollary 1.12. — *With the notation introduced above, let u be topologically proper. Let $x \in X^{\mathrm{an}}$ be a u-minimal point on its G-orbit (thus on its π-fibre). Then there exists a point $x_0 \in \overline{G^{\mathrm{an}} \cdot x}$ such that its orbit is closed and $u(x_0) = u(x)$.*

Proof. — Indeed let $x' \in \overline{G^{\mathrm{an}} \cdot x}$ be a point whose orbit is closed. It suffices to take a minimal point x_0 in the orbit of x' (this exists because u is topologically proper). □

The techniques employed to prove Theorem 1.9 permit to analyse the positivity conditions that a **U**-invariant function has to satisfy in order to obtain a statement generalizing the one of Kempf and Ness. This aspect is discussed in Section 3.5.

1.8. Metric on GIT quotients. — Let X be a projective k-scheme acted upon by a reductive group G. Let L be a G-linearized ample invertible sheaf. Suppose that L is endowed with an extended metric $\|\cdot\|_L$: this is the data, for every analytic open subset $U \subset X^{\mathrm{an}}$ and every section $s \in \Gamma(U, L^{\mathrm{an}})$, of a function $\|s\|_{L,U} \colon U \to \mathbf{R}_+$ satisfying the following properties for all $x \in U$:

- $\|s\|_{L,U}(x) = 0$ if and only if $s(x) = 0$;
- $\|\lambda s\|_{L,U}(x) = |\lambda| \|s\|_{L,U}(x)$ for all $\lambda \in k$;
- for an open subset $V \subset U$, $\|s\|_{L,V} = \|s\|_{L,U|V}$.

Note that in the complex case this notion coincide with the usual notion of metric on the line bundle L.

Let X^{ss} be the open subset of semi-stable points of X and the let Y be the categorical quotient of X^{ss} by G. Let $\pi \colon X^{\mathrm{ss}} \to Y$ be the quotient map. For every $D \geq 1$ divisible enough there exist an ample invertible sheaf M_D on Y and an isomorphism of invertible sheaves,

$$\varphi_D \colon \pi^* M_D \longrightarrow L^{\otimes D}_{|X^{\mathrm{ss}}},$$

compatible with the action of G (see [**MFK94**, Theorem 1.10]). Define an extended metric on M_D as follows: for a point $y \in Y^{\mathrm{an}}$ and a section t of M_D defined on a open neighbourhood of y, set

$$\|t\|_{M_D}(y) := \sup_{\pi(x)=y} \|\pi^* t\|_{L^{\otimes D}}(x).$$

One checks that this is actually a metric, *i.e.* the right-hand side is not $+\infty$ (see Proposition 4.5).

Theorem 1.13 **([Zha96a**, Theorem 4.10]**, *cf.* Theorem 4.6 in Chapter 3)**
Make the following assumption:

- *the metric $\|\cdot\|_L$ is invariant under the action of a maximal compact subgroup of G;*
- *for every analytic open subset $U \subset X^{\mathrm{an}}$ and every section $s \in \Gamma(U, L^{\mathrm{an}})$ that does not vanish on U, the function $-\log \|s\|_L \colon U \to \mathbf{R}$ is plurisubharmonic.*

Then, the metric $\|\cdot\|_{M_D}$ is continuous.

Remark 1.14. — In the complex case, if the metric $\|\cdot\|_{\mathcal{L},\sigma}$ is the restriction of a Fubini-Study metric this result follows from the results of Kempf-Ness. Zhang shows that the general case can be led back to the case of a Fubini-Study metric thanks to an approximation result due to Tian and to an argument of extension of sections of small size (see [**Zha94**, Theorem 2.2] and [**Bos04**, Appendix A]). The latter argument permits to show that the Kähler form of the metric $\|\cdot\|_{\mathcal{M}_D,\sigma}$ is semi-positive [**Zha95**, Theorem 2.2].

Remark 1.15. — In the non-archimedean case, the main example of a metric with such properties is given by metrics coming from integral models. More precisely, let $\mathcal{X}$ be a projective k°-scheme endowed with an action of a reductive k°-group $\mathcal{G}$. Suppose that $\mathcal{X}$ comes equipped with a $\mathcal{G}$-linearized ample invertible sheaf $\mathcal{L}$. Let X, G and L be respectively the generic fibre of $\mathcal{X}$, $\mathcal{G}$ and $\mathcal{L}$.

The continuous and bounded metric associated to the integral model $\mathcal{L}$ extends to an extended metric $\| \cdot \|_{\mathcal{L}}$ (see paragraph 4.1.3). Then, the extended metric $\| \cdot \|_{\mathcal{L}}$ is invariant under the action of the maximal compact subgroup of G associated to $\mathcal{G}$ (see Definition 2.21) and, since $\mathcal{L}$ is ample, the function $-\log \|s\|_{\mathcal{L}}$ is plurisubharmonic for every analytic open subset $U \subset X^{\mathrm{an}}$ and every invertible section $s \in \Gamma(U, L^{\mathrm{an}})$ (see Corollary 4.4).

***Theorem 1.16* (*cf.* Theorem 4.6)**. — *Under the assumptions of Theorem 1.13 suppose that k is non-trivially valued and algebraically closed. Let $x \in X^{\mathrm{ss}}(k)$ be a semi-stable k-point of X and $t \in \pi(x)^* \mathcal{M}_D$ be a non-zero section. Then,*

$$\sup_{\pi(x') = \pi(x)} \|\pi^* t\|_{L^{\otimes D}}(x') = \sup_{g \in G(k)} \|\pi^* t\|_{L^{\otimes D}}(g \cdot x),$$

(where the supremum on the left-hand side is ranging on k-points x' in the fibre of $\pi(x)$).

Suppose that k is either a finite extension of $\mathbf{Q}_p$ or of the form $\mathbf{F}(\!(t)\!)$ for a field $\mathbf{F}$. With the notations of Remark 1.15 let $\mathcal{X}^{\mathrm{ss}}$ be the open subset of semi-stable points of $\mathcal{X}$ and $\mathcal{Y}$ the categorical quotient of $\mathcal{X}^{\mathrm{ss}}$. For every $D \geq 1$ divisible enough there exist an ample invertible sheaf $\mathcal{M}_D$ on $\mathcal{Y}$ and an isomorphism of invertible sheaves on $\mathcal{X}^{\mathrm{ss}}$,

$$\varphi_D \colon \pi^* \mathcal{M}_D \xrightarrow{\sim} \mathcal{L}^{\otimes D}_{|\mathcal{X}^{\mathrm{ss}}}.$$

Let $\| \cdot \|_{\mathcal{M}_D}$ be the continuous and bounded metric associated to $\mathcal{M}_D$.

***Theorem 1.17* (*cf.* Theorem 4.11)**. — *Let k be either a finite extension of $\mathbf{Q}_p$ or of the form $\mathbf{F}(\!(t)\!)$ for a field $\mathbf{F}$ and let K be the completion of an algebraic closure of k. Let y be a K-point of Y and let $t \in y^* M_D$ be a section. Then,*

$$\|t\|_{\mathcal{M}_D}(y) = \sup_{\pi(x) = y} \|\pi^* t\|_{\mathcal{L}^{\otimes D}}(x),$$

where the supremum on the right-hand side is ranging on K-points x of $\mathcal{X}$ in the fibre of y.

2. Preliminaries to the local part

Let k be a field complete with respect to an absolute value $| \cdot |_k$.

2.1. Analytic spaces. —

2.1.1. Overview. — Our framework will be that of analytic spaces over k. There are three cases: [4]

(1) The complex case: a $\mathbf{C}$-analytic space will be a complex analytic space in the usual sense.
(2) The real case: an $\mathbf{R}$-analytic space will be a $\mathbf{R}$-locally ringed space isomorphic to a quotient X/ι where X is a complex analytic space and $\iota\colon X \to X$ is an anti-holomorphic involution. [5]
(3) The non-archimedean case: if the field k is complete with respect to a non-archimedean absolute value (possibly trivial) k-analytic spaces in the sense of Berkovich are considered. References for the latter theory are the foundational papers of Berkovich [**Ber90, Ber93**]; a self-contained introduction is given in [**RTW10**, §1.2], while a reference linking other approaches to non archimedean analytic geometric to Berkovich's one may be [**Con08**].

We are interested in analytic spaces obtained from algebraic k-schemes. Instead of giving the general definitions, we present a construction of the analytification of a finite type k-scheme X following Berkovich [**Ber90**, §1.5, §3.4 and §3.5], Poineau [**Poi13a**], Nicaise [**Nic14**, §2], which works for all three cases.

2.1.2. Underlying topological space. — Let X be a k-scheme of finite type.

Definition 2.1. — The topological space X^{an} underlying the analytification of X is the set couples $(x, |\cdot|)$ made of a point $x \in X$ (not necessarily closed) and of an absolute value $|\cdot|\colon \kappa(x) \to \mathbf{R}_+$ such that its restriction to k coincides with the original absolute on k (here $\kappa(x)$ denotes the residue field at x).

The set $|X^{\mathrm{an}}|$ is endowed with the coarsest topology such that, for every open subset $U \subset X$,

(1) the subset $|U^{\mathrm{an}}| = \{(x, |\cdot|) \in |X^{\mathrm{an}}| : x \in U\}$ is open in $|X^{\mathrm{an}}|$;
(2) for every function $f \in \Gamma(U, \mathcal{O}_X)$, the map $|f|\colon |U^{\mathrm{an}}| \to \mathbf{R}_+$ defined by
$$|f|\colon (x, |\cdot|) \mapsto |f(x)|,$$
is continuous.

This topology is called the *analytic topology* of X.

Not to burden notation denote a point $(x, |\cdot|)$ of $|X^{\mathrm{an}}|$ simply by x.

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4. The fields $\mathbf{R}$ and $\mathbf{C}$ are always assumed to be endowed with the usual archimedean absolute value.

5. Namely the quotient X/ι is the $\mathbf{R}$-locally ringed space $(|X/\iota|, \mathcal{O}_{X/\iota})$ defined as follows:
- the topological space $|X/\iota|$ is the quotient X/ι endowed with the quotient topology;
- if $\pi\colon X \to |X/\iota|$ denotes the canonical projection, for every open subset $U \subset |X/\iota|$ the sections of the structural sheaf $\mathcal{O}_{X/\iota}$ are defined by
$$\Gamma(U, \mathcal{O}_{X/\iota}) = \Gamma(\pi^{-1}(U), \mathcal{O}_X)^{\iota} = \{f \in \Gamma(U, \mathcal{O}_X) : \iota^\sharp(f) = f\},$$
where $\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_X$ is the anti-holomorphic homomorphism of sheaves of $\mathbf{R}$-algebras associated to the involution ι.

Theorem 2.2. — *If X is non-empty, the topological space $|X^{\mathrm{an}}|$ is non-empty, locally separated and locally compact. Moreover,*

(1) it is Hausdorff if and only if X is separated over k;
(2) it is compact if and only if X is proper over k.

Proof. — The proof of the local compactness can be found in [**Ber90**, §1.5]. (1) and (2) are respectively statements (i) and (ii) in [**Ber90**, Theorems 3.4.8 and 3.5.3]. □

Definition 2.3. — Let $(x, |\cdot|)$ be point of X^{an}. The *complete residue field* $\hat{\kappa}(x)$ at x is the completion of the residue field $\kappa(x)$ with respect to the absolute value $|\cdot|$.

This notation differs from the one that usually occurs in the literature, where the complete residue field is denoted by $\mathcal{H}(x)$.

The topological space underlying the analytification of a scheme is functorial on the scheme: that is, if $f\colon X \to Y$ is a morphism between finite type k-schemes, then f induces a continuous map $|f^{\mathrm{an}}|\colon |X^{\mathrm{an}}| \to |Y^{\mathrm{an}}|$.

Forgetting the absolute value gives rise to a continuous map $\alpha_X \colon |X^{\mathrm{an}}| \to X$, where X is the topological space underlying the scheme X.

Remark 2.4. — The pre-image by α_X of a closed point of x is a singleton: since $\kappa(x)$ is a finite extension, there is a unique absolute value on $\kappa(x)$ extending $|\cdot|_k$. Distinguish three cases:

(1) In the complex case, a theorem of Gel'fand-Mazur affirms that a complete field containing $\mathbf{C}$ (isometrically) coincides with $\mathbf{C}$. The map α_X induces a homeomorphism $\alpha_X \colon |X^{\mathrm{an}}| \to X(\mathbf{C})$ where $X(\mathbf{C})$ is endowed with the complex topology.
(2) In the real case, the map α_X gives a homeomorphism
$$\alpha_X \colon |X^{\mathrm{an}}| \longrightarrow X(\mathbf{C})/\operatorname{Gal}(\mathbf{C}/\mathbf{R}).$$
(3) In the non-archimedean case, this is the topological space underlying the analytification of X in the sense of Berkovich. In this case the map α_X is surjective. [6]

Proposition 2.5. — *If k be an algebraically closed field and the absolute value on k non-trivial, then the set of k-points $X(k)$ is dense in X^{an}.*

An important feature for us will be the behaviour of the closure with respect to the Zariski and analytic topology:

Proposition 2.6. — *Let X be a k-scheme of finite type. For every constructible set $Z \subset X$,*
$$\overline{\alpha_X^{-1}(Z)} = \alpha_X^{-1}(\overline{Z}),$$
where on the left-hand side the closure is taken with respect the analytic topology and on the right-hand side to the Zariski one.

6. For instance, when $X = \mathbf{A}^1_k$, the *Gauss norm* on polynomials $\|\cdot\|\colon k[t] \to \mathbf{R}_+$, defined by $\sum a_i t^i \mapsto \max |a_i|$, is multiplicative and one can extend it to $k(t)$. If η denotes the generic point of the affine line, the couple $(\eta, \|\cdot\|)$ is a point of $|\mathbf{A}^{1,\mathrm{an}}_k|$.

Proof. — See [**Gro71**, Exp. XII, Corollaire 2.3] for the complex case and [**Ber90**, Proposition 3.4.4] for the non-archidemedean one. The real case is deduced from the complex case thanks to the homeomorphism $X^{\mathrm{an}} \simeq X(\mathbf{C})/\operatorname{Gal}(\mathbf{C}/\mathbf{R})$. □

2.1.3. Structural sheaf. — Let us introduce the concept of analytic function on X^{an}. Begin with the case $X = \mathbf{A}^n_k$ for a non-negative integer n.

Definition 2.7. — Let $U \subset |X^{\mathrm{an}}|$ be an open subset. An *analytic function* over U is a map $f \colon U \to \bigsqcup_{x \in U} \hat{\kappa}(x)$ such that for every $x \in U$:

(1) $f(x) \in \hat{\kappa}(x)$;
(2) for every $\varepsilon > 0$ there exists an open neighbourhood U_ε of x in U and a rational function $g_\varepsilon \in k(t_1, \dots, t_n)$ without poles in U_ε such that, for every $y \in U_\varepsilon$, one has $|f(y) - g_\varepsilon(y)| < \varepsilon$ (here $t_1, \dots, t_n$ are the coordinate functions on $\mathbf{A}^n_k$).

The k-algebra of analytic functions on U is denoted by $\mathcal{O}^{\mathrm{an}}_X(U)$.

The correspondence $U \rightsquigarrow \mathcal{O}^{\mathrm{an}}_X(U)$ gives rise to a sheaf of k-algebras on the topological space X^{an}. For every point $x \in X^{\mathrm{an}}$ the stalk at x is a local ring.

Definition 2.8. — The *n-dimensional analytic affine space* is the locally k-ringed space

$$\mathbf{A}^{n,\mathrm{an}}_k := (|\mathbf{A}^{n,\mathrm{an}}_k|, \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^n}).$$

Remark 2.9. — In the complex case, the locally $\mathbf{C}$-ringed space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{C}}$ is the topological space $\mathbf{C}^n$ equipped with the sheaf of holomorphic functions. In real case, the locally $\mathbf{R}$-ringed space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{R}}$ is the topological space $\mathbf{C}^n/\operatorname{Gal}(\mathbf{C}/\mathbf{R})$ equipped with the sheaf of holomorphic functions f on $\mathbf{C}^n$ verifying $f(\bar{z}) = \overline{f(z)}$. In the non-archimedean case it is the analytical n-dimensional affine space in the sense of Berkovich. See for instance [**Ber90**, §1.5] and [**Poi13a**].

Even though holomorphic functions will be used only on $\mathbf{A}^{1,\mathrm{an}}_k$, let us sketch how to define the structural sheaf on the analytification of a k-scheme X of finite type.

(1) If X is affine, fix a closed immersion $j \colon X \to \mathbf{A}^n_k$ for a suitable n. Let $I \subset \mathcal{O}_{\mathbf{A}^n}$ be the ideal sheaf defining X and let $I^{\mathrm{an}} \subset \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^n}$ be the ideal sheaf generated by I. Consider the sheaf of k-algebras on X^{an},
$$\mathcal{O}^{\mathrm{an}}_X := j^{\mathrm{an}\,-1}(\mathcal{O}^{\mathrm{an}}_{\mathbf{A}^n}/I^{\mathrm{an}}).$$
One can show that the sheaf $\mathcal{O}^{\mathrm{an}}_X$ does not depend on the choice of the closed embedding j.
(2) For an arbitrary k-scheme X choose a covering $X = \bigcup_{i=1}^N X_i$ by affine open subsets. The sheaves $\mathcal{O}^{\mathrm{an}}_{X_i}$ on $|X^{\mathrm{an}}_i|$ then glue to a sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ on $|X^{\mathrm{an}}|$. One can show that $\mathcal{O}^{\mathrm{an}}_X$ does not depend on the chosen covering.

Definition 2.10. — The locally k-ringed space $X^{\mathrm{an}} := (|X^{\mathrm{an}}|, \mathcal{O}^{\mathrm{an}}_X)$ is called the *analytification of X*.

To simplify the notation we do not distinguish X^{an} and its underlying topological space $|X^{\mathrm{an}}|$.

A morphism $f\colon Y \to X$ between k-schemes of finite type induces a morphism of k-analytic spaces $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$. If no confusion seems to arise write f instead of f^{an}.

2.1.4. Extension of scalars. —

Definition 2.11. — An *analytic extension* K of k is a field complete with respect to an absolute value $|\cdot|_K$ equipped with an isometric embedding $k \to K$.

Let K be an analytic extension of k. Let X be a k-scheme of finite type and let $X_K := X \times_k K$ the K-scheme obtained extending scalars to K. Let X_K^{an} be the K-analytic space obtained by analytification of the K-scheme X_K.

Definition 2.12. — The morphism of base change $X_K \to X$ gives rise to a morphism of locally k-ringed space

$$\mathrm{pr}_{X,K/k}\colon X_K^{\mathrm{an}} \longrightarrow X^{\mathrm{an}},$$

called the *extension of scalars map.* If no confusion arises, we will omit to write the dependence on the scheme X.

Proposition 2.13. — *The map* $\mathrm{pr}_{X,K/k}$ *is surjective and topologically proper. Since the topological spaces* X^{an} *and* $|X_K^{\mathrm{an}}|$ *are locally compact, the map* $\mathrm{pr}_{X,K/k}$ *is closed.*

Proof. — In the archimedean case, when $k = \mathbf{R}$ and $K = \mathbf{C}$, the map $\mathrm{pr}_{X,\mathbf{C}/\mathbf{R}}$ is just the quotient map by the Galois action. In the non-archimedean case the reference is [**Ber93**, §1.4]. □

The extension of scalars to K is functorial: for a morphism of k-schemes of finite type $f\colon Y \to X$ denote by $f_K^{\mathrm{an}} : Y_K^{\mathrm{an}} \to X_K^{\mathrm{an}}$ the morphism of K-analytic spaces induced by f.

Definition 2.14. — A *K-point* of X is a couple (x, ε_x) made of a point $x \in X^{\mathrm{an}}$ and of an isometric embedding $\varepsilon_x\colon \hat{\kappa}(x) \to K$.

Let $x \in X^{\mathrm{an}}$ be a point of X^{an} and let $\hat{\kappa}(x) \to K$ be an isometric embedding. The point x may be viewed as a $\hat{\kappa}(x)$-point of X. Let x_K be the K-point of X_K which factors the composite map $\operatorname{Spec} K \to \operatorname{Spec} \hat{\kappa}(x) \to X$ through the K-scheme X_K.

Definition 2.15. — The couple $(x_K, |\cdot|_K)$ (where $|\cdot|_K$ is the absolute value on K) is a point of X_K^{an} called the *point associated to x and the embedding $\hat{\kappa}(x) \to K$.*

2.1.5. Fibres. — Let $f\colon Y \to X$ be a morphism between k-schemes of finite type. Let $x \in X^{\mathrm{an}}$ be a point, $K = \hat{\kappa}(x)$ be its complete residue field and x_K the point of X_K^{an} associated to x.

Proposition 2.16. — *Keep the notations just introduced. Then:*

(1) The map of scalars extension $\mathrm{pr}_{Y,K/k}\colon Y_K^{\mathrm{an}} \to Y^{\mathrm{an}}$ *induces a homeomorphism*

$$\mathrm{pr}_{Y,K/k}\colon (Y_K \times_{X_K} \{x_K\})^{\mathrm{an}} \longrightarrow (f^{\mathrm{an}})^{-1}(x);$$

(2) For every analytic extension K' of k and every point $x' \in X^{\mathrm{an}}_{K'}$ such that $\mathrm{pr}_{X,K'/k}(x') = x$, the map induced by $\mathrm{pr}_{K',K/k}$,

$$\mathrm{pr}_{Y,K'/k} : (f^{\mathrm{an}}_{K'})^{-1}(x') \longrightarrow (f^{\mathrm{an}})^{-1}(x)$$

is surjective.

Proof. —

(1) In the complex case this is clear and the real case is deduced from the complex one by Galois action. In the non-archimedean case, see [**Ber93**, §1.4].

(2) Let $\Omega = \hat{\kappa}(x')$ be the completed residue field of x' and let x'_Ω be the point of X^{an}_Ω associated to x'. The point x'_Ω coincides with the point x_Ω associated to the point x and the embedding $\hat{\kappa}(x) \to \Omega = \hat{\kappa}(x')$ (the latter is given by the fact that x' projects on x). Therefore the composite map

$$(f^{\mathrm{an}}_\Omega)^{-1}(x'_\Omega) \overset{\mathrm{pr}_{Y,\Omega/K'}}{\longrightarrow} (f^{\mathrm{an}}_{K'})^{-1}(x') \overset{\mathrm{pr}_{Y,K'/k}}{\longrightarrow} (f^{\mathrm{an}})^{-1}(x),$$

coincides with the map

$$(f^{\mathrm{an}}_\Omega)^{-1}(x_\Omega) \overset{\mathrm{pr}_{Y,\Omega/K}}{\longrightarrow} (f^{\mathrm{an}}_{K})^{-1}(x_K) \overset{\mathrm{pr}_{Y,K/k}}{\longrightarrow} (f^{\mathrm{an}})^{-1}(x),$$

where $K = \hat{\kappa}(x)$ is the completed residue field at x and x_K is the point of X^{an}_K associated to x. The latter composite map is surjective: indeed, the second one is a homeomorphism according to (1); the first one is the map of scalar extension

$$(Y_\Omega \times_{X_\Omega} \{x_\Omega\})^{\mathrm{an}} \longrightarrow (Y_K \times_{X_K} \{x_K\})^{\mathrm{an}}.$$

This implies that $\mathrm{pr}_{Y,K'/k} : (f^{\mathrm{an}}_{K'})^{-1}(x') \to (f^{\mathrm{an}})^{-1}(x)$ is surjective. □

Proposition 2.17. — *With the notations introduced above, let $y_1, y_2 \in Y^{\mathrm{an}}$ be points such that $f(y_1) = f(y_2)$.*

Then, there exists an analytic extension Ω of k and Ω-points $x_{1\Omega}$, $x_{2\Omega} \in X^{\mathrm{an}}_\Omega$ such that:

(1) $\mathrm{pr}_{\Omega/k}(y_{i\Omega}) = y_i$ for $i = 1, 2$;

(2) $f^{\mathrm{an}}_\Omega(y_{1\Omega}) = f^{\mathrm{an}}_\Omega(y_{2\Omega})$.

Proof. — The proof is made in two steps.

First step. Suppose that $X = \operatorname{Spec} k$ is just made of a k-rational point. The result in this case is clear: it suffices to take Ω to be an analytic extension endowed with isometric embedding $\hat{\kappa}(y_i) \to \Omega$ for $i = 1, 2$ and $y_{1\Omega}$, $y_{2\Omega}$ be the points of X^{an}_Ω associated to y_1 and y_2.

Second step. Let $x \in X^{\mathrm{an}}$ be the point $f(y_1) = f(y_2)$ and let $K = \hat{\kappa}(x)$ be its residue field. Let $x_K \in X^{\mathrm{an}}_K$ be point associated to x. According to Proposition 2.16 the map

$$\mathrm{pr}_{Y,K/k} \colon (Y_K \times_{X_K} \{x_K\})^{\mathrm{an}} \longrightarrow (f^{\mathrm{an}})^{-1}(x),$$

is a bijection. Therefore there exists $y_{1K}, y_{2K} \in Y^{\mathrm{an}}_K$ such that

(1) $\mathrm{pr}_{K/k}(y_{iK}) = y_i$ for $i = 1, 2$;

(2) $f^{\mathrm{an}}_K(y_{1K}) = f^{\mathrm{an}}_K(y_{2K})$.

Conclude applying the first step to the K-schemes $Y' = Y_K \times_{X_K} \{x_K\}$, $X' = \{x_K\}$ and the morphism induced by $f_K \colon Y_K \to X_K$. □

2.2. Maximal compact subgroups. — Let k be complete field.

2.2.1. Subgroups. — Let G be a k-algebraic group (*i.e.* a smooth k-group scheme of finite type). Let $m \colon G \times_k G \to G$ be the multiplication map and $\mathrm{inv} \colon G \to G$ be the inverse.

Definition 2.18. — A subset $H \subset G^{\mathrm{an}}$ is said to be a *subgroup* if the following conditions are satisfied:

(1) the image through m^{an} of the subset $\mathrm{pr}_1^{-1}(H) \cap \mathrm{pr}_2^{-1}(H) \subset G^{\mathrm{an}} \times_k G^{\mathrm{an}}$ is contained in H;
(2) the image of H through $\mathrm{inv}^{\mathrm{an}}$ is contained in H;
(3) the neutral element $e \in G(k)$ belongs to H.

A subgroup H is said to be *compact* if it is compact as a subset of G^{an}.

Let K be an analytic extension of k and let $H \subset G^{\mathrm{an}}$ be a subgroup. Then the subset $H_K := \mathrm{pr}_{G,K/k}^{-1}(H) \subset G_K^{\mathrm{an}}$ is a subgroup of G_K^{an}.

Let G act on a k-scheme of finite type and let $\sigma \colon G \times_k X \to X$ be the morphism defining the action.

Definition 2.19. — Let $H \subset G^{\mathrm{an}}$ be a subgroup and let $x \in X^{\mathrm{an}}$ be a point. The *H-orbit of x*, denoted $H \cdot x$, is the image through σ^{an} of the subset

$$\mathrm{pr}_1^{-1}(H) \cap \mathrm{pr}_2^{-1}(x) \subset G^{\mathrm{an}} \times_k X^{\mathrm{an}}.$$

2.2.2. Archimedean definition. — Let $k = \mathbf{R}, \mathbf{C}$ and let G be a (connected) reductive k-group.

Definition 2.20. — If $k = \mathbf{C}$ a *maximal compact subgroup* of G is a compact subgroup $\mathbf{U}$ of $G(\mathbf{C})$ which is maximal among the compact subgroups of $G(\mathbf{C})$.

If $k = \mathbf{R}$ a *maximal compact subgroup* of G is a compact subgroup $\mathbf{U} \subset G^{\mathrm{an}}$ such that $\mathrm{pr}_{\mathbf{C}/\mathbf{R}}^{-1}(\mathbf{U})$ is a maximal compact subgroup of $G(\mathbf{C})$.

Over the complex numbers a connected affine algebraic group H is reductive if and only if $H(\mathbf{C})$ contains a compact subgroup which is Zariski-dense. If this is the case:

- a compact subgroup of $H(\mathbf{C})$ is Zariski-dense if and only if it is maximal;
- all the maximal compact subgroups of $H(\mathbf{C})$ are conjugated.

If $\mathbf{U}$ is a maximal compact subgroup of G, then there exist a real algebraic group $\mathcal{U}$ and an isomorphism of complex algebraic groups $\alpha \colon G \simeq \mathcal{U} \times_{\mathbf{R}} \mathbf{C}$ such that $\alpha(\mathbf{U}) = \mathcal{U}(\mathbf{R})$. A torus $T \subset G$ is defined over $\mathbf{R}$ (that is, it comes from a torus of $\mathcal{U}$) if and only if $T \cap \mathbf{U}$ is the maximal compact subgroup of $T(\mathbf{C})$.

2.2.3. Non-archimedean definition. — Let k be non-archimedean and G a reductive k-group. Consider only compact subgroups of G^{an} associated to reductive models of G. A thorough study of maximal bounded subgroups of $G(k)$ can be found in [**BT72, BT84**], while compact subgroups of G^{an} are considered in [**Ber90**, Chapter 5] and [**RTW10, RTW11**].

Let $\mathcal{H}$ be an affine k°-group scheme of finite type and let $H = \mathcal{H} \times_{k^\circ} k$ be its generic fibre. Consider the compact subset

$$\mathbf{U}_{\mathcal{H}} = \{h \in H^{\mathrm{an}} : |f(g)| \leq 1 \text{ for every } f \in k^\circ[\mathcal{H}]\}$$

where $k^\circ[\mathcal{H}]$ is the k°-algebra of regular functions on $\mathcal{H}$.

Definition 2.21. — A subset $H \subset G^{\mathrm{an}}$ is said to be a *maximal compact subgroup* if it is of the form $H = \mathbf{U}_{\mathcal{G}}$ for a reductive k°-group $\mathcal{G}$ and an isomorphism of k-group schemes $\varphi \colon \mathcal{G} \times_{k^\circ} k \to G$.

The subset $\mathbf{U}_{\mathcal{G}}$ earns the name of maximal compact subgroup because it is a subgroup (in the sense of Definition 2.18), it is compact and it can be shown that it is maximal among the compact subgroups of G^{an}. The latter property will be of no use for us.

Proposition 2.22. — *With the notations introduced above:*

1. *the set of k-rationals points $\mathbf{U}_{\mathcal{G}}(k) := \mathbf{U}_{\mathcal{G}} \cap G(k)$ coincides with the set of k°-points $\mathcal{G}(k^\circ)$;*
2. *for every analytic extension K of k,*
$$\mathrm{pr}_{K/k}^{-1} \mathbf{U}_{\mathcal{G}} = \mathbf{U}_{\mathcal{G} \otimes_{k^\circ} K^\circ}$$
as subsets of G_K^{an}.
3. *if k is algebraically closed and non-trivially valued, the set $\mathbf{U}_{\mathcal{G}}(k)$ is dense in $\mathbf{U}_{\mathcal{G}}$.*

Proof. —

1. Let $\varphi_g \colon A \to k$ be the homomorphism of k-algebras induced by $g \in G(k)$. The point g belongs to $\mathbf{U}_{\mathcal{G}}$ if and only if $|\varphi_g(f)| \leq 1$ for every $f \in k^\circ[\mathcal{G}]$, which means that φ_g restricts to a homomorphism $\varphi_g \colon k^\circ[\mathcal{G}] \to k^\circ$.
2. Let $f_1, \dots, f_N$ be generators of the k°-algebra $k^\circ[\mathcal{G}]$. For every point $g \in G^{\mathrm{an}}$,
$$|f(x)| \leq 1 \text{ for every } f \in k^\circ[\mathcal{G}] \iff |f_i(x)| \leq 1 \text{ for every } i = 1, \dots, N.$$
The statement follows from this and noticing that $f_1, \dots, f_N$ are also generators of the K°-algebra $K^\circ[\mathcal{G}]$.
3. This is true because the compact subset $\mathbf{U}_{\mathcal{G}}$ is strictly affinoid in the sense of Berkovich. Thus this can be found in [**Ber90**, Proposition 2.1.15]. □

The main result of [**Dem65**] and [**GP11**] is that, up to a finite separable extension, all reductive groups comes by base change from $\mathbf{Z}$. More precisely:

Theorem 2.23. — *Let G be a reductive k-group. Then, there exist a finite separable extension k' of k, a $\mathbf{Z}$-reductive group scheme $\mathcal{G}$ and an isomorphism of k'-group schemes*

$$G \times_k k' \simeq \mathcal{G} \times_{\mathbf{Z}} k'.$$

This is the combination of Corollary 3.1.5 and Theorems 3.6.5-3.6.6 in [**Dem65**].

2.3. Plurisubharmonic functions. — In this section we discuss plurisubharmonic functions.

In the complex case we consider the usual plurisubharmonic functions and, in the real case, complex plurisubharmonic functions invariant under complex conjugation.

In the non-archimedean case, subharmonic function on curves $\mathbf{P}^1$ are by now well understood thanks to work of Rumely [**Rum89, Rum93**], Rumely and Baker [**BR10**], Kani [**Kan89**], Favre et Jonsson [**FJ04**] and Thuillier [**Thu05**] (who studied systematically the theory of subharmonic functions also on curves of higher genus). The comparison between these notions can be found in [**Thu05**, Chapitre 5]. Moving to higher dimension, we say that a function is plurisubharmonic if the restriction to the image of any every open subset of $\mathbf{P}^1$ is subharmonic. This does not give a sensible theory of plurisubharmonic functions: for instance, in order to get the Maximum Principle one needs to test subharmonicity on curves of higher genus. However, this definition will be enough for our purposes. Other approaches to plurisubharmonic functions have been studied by Chambert-Loir et Ducros [**CLD12**] and Boucksom, Favre et Jonsson [**BFJ12**].

2.3.1. Harmonic functions. — Let k be a complete field and let $\Omega \subset \mathbf{A}_k^{1,\mathrm{an}}$ be an open subset.

Definition 2.24. — A real-valued function $h\colon \Omega \to \mathbf{R}$ is said to be *harmonic* if for every $x \in \Omega$ there exist an open neighbourhood U of x in Ω, a positive integer N and for every $i = 1, \dots, N$ an invertible analytic function $f_i \in \Gamma(U, \mathcal{O}_U^{\mathrm{an}})^\times$ and a real number $\alpha_i \in \mathbf{R}$ such that

$$h_{|U} = \sum_{i=1}^{N} \alpha_i \log |f_i|.$$

Note that the in the complex case one can always take $N = 1$ thanks to the exponential map, which gives the usual notion of harmonic function. In the real case one finds the notion of harmonic function on the associated open set of $\mathbf{C}$ invariant under conjugation. In the non-archimedean case one recovers the notion of harmonic function of Thuillier (see [**Thu05**, Définition 2.31] taking in account [*loc.cit.*, Théorème 2.3.21]).

Proposition 2.25. — *Let Ω be an open subset of the analytic affine line $\mathbf{A}_k^{1,\mathrm{an}}$. The following properties are satisfied:*

1. *Harmonic functions give rise to a sheaf of $\mathbf{R}$-vector spaces on $\mathbf{A}_k^{1,\mathrm{an}}$.*
2. *If f is an invertible analytic function on Ω then $\log|f|$ is an harmonic function.*
3. *Let $f\colon \Omega' \to \Omega$ be an analytic map between open subsets of $\mathbf{A}_k^{1,\mathrm{an}}$; for every harmonic map h on Ω the composite map $h \circ f$ is harmonic on Ω'.*

(4) Let K be an analytic extension of k and $\Omega_K := \mathrm{pr}_{K/k}^{-1}(\Omega)$. For every harmonic function $h\colon \Omega \to \mathbf{R}$ composite function $h \circ \mathrm{pr}_{K/k}\colon \Omega_K \to \mathbf{R}$ is harmonic.

(5) (Maximum Principle) *If the open set Ω is connected, then an harmonic function h on Ω attains a global maximum if and only if it is constant.*

(6) If Ω is connected, every non-constant harmonic function $h\colon \Omega \to \mathbf{R}$ is an open map.

Proof. — Statements (1) - (4) are straightforward consequence of the definition.

(5) The Maximum Principle it is well-known in the complex case (which imply the real one) [**Dem**, Chapter I, 4.14]; in the non-archimedean case it can be found in the proof of Proposition 2.3.13 in [**Thu05**].

(6) It is sufficient to show that the image of Ω is open. The image of Ω is an interval $I \subset \mathbf{R}$ (possibly unbounded) and one has to show that it does not contain its endpoints.

Treat the case of the right endpoint as follows: if I is unbounded on the right, then we are done; if $b \in \mathbf{R}$ is the right endpoint of I, then h cannot take the value b because of the Maximum Principle. The case of the left endpoint goes similarly. □

2.3.2. Subharmonic functions. — Let k be non-trivially valued and Ω an open subset of $\mathbf{A}_k^{1,\mathrm{an}}$.

Definition 2.26. — A function $u\colon \Omega \to [-\infty, +\infty[$ is said to be *subharmonic* if it is upper semi-continuous and for every connected open subset $\Omega' \subset \Omega$ and every harmonic function h on Ω' the function $u_{|\Omega'} - h$ satisfies the maximum principle, that is, it attains a global maximum if and only if it is constant.

In the complex case this is equivalent to the usual notion of subharmonic function. Thus in the real case giving a subharmonic function on Ω is equivalent to give a subharmonic function on $\Omega_{\mathbf{C}}$ invariant under complex conjugation. In the non-archimedean case one finds the notion of subharmonic function in the sense of Thuillier (see [**Thu05**, Définition 3.1.5], taking in account the characterisation [*loc.cit.*, Corollaire 3.1.12] and compatibility to analytic extensions [*loc.cit.*, Corollaire 3.4.5]).

Proposition 2.27. — *Let Ω be an open subset of the analytic affine line $\mathbf{A}^{1,\mathrm{an}}$. The following properties are satisfied:*

(1) Harmonic functions are subharmonic.

(2) If u, v are subharmonic functions on Ω and α, β are non-negative real numbers, then $\alpha u + \beta v$ and $\max\{u, v\}$ are subharmonic functions.

(3) If f is an analytic function on Ω then $\log|f|$ is subharmonic.

(4) Let K be an analytic extension of k and $\Omega_K := \mathrm{pr}_{K/k}^{-1}(\Omega)$. For every subharmonic function $u\colon \Omega \to \mathbf{R}$ composite function $u \circ \mathrm{pr}_{K/k}\colon \Omega_K \to \mathbf{R}$ is subharmonic.

(5) Let $f\colon \Omega' \to \Omega$ be an analytic map between open subsets of $\mathbf{A}_k^{1,\mathrm{an}}$; for every subharmonic map u on Ω the composite map $u \circ f$ is subharmonic on Ω'.

(6) (Maximum Principle) *If the open set Ω is connected, then a subharmonic function h on Ω attains a global maximum if and only if it is constant.*
(7) *If $\{u_i\}_{i\in I}$ is a locally bounded family of subharmonic functions on Ω, the its regularised upper envelope*[7] *is subharmonic.*
(8) *Let $u_1, \ldots, u_n$ be subharmonic functions on Ω and $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ be a convex function that is non-decreasing in each variable. Extend φ by continuity into a function*
$$\widetilde{\varphi}\colon\ [-\infty, +\infty[^n \longrightarrow [-\infty, +\infty[.$$
Then the function $\widetilde{\varphi}\circ(u_1, \ldots, u_n)\colon \Omega \to [-\infty, +\infty[$ is subharmonic.

Proof. —

(1) Follows from the definitions.
(2) See [**Thu05**, Proposition 3.1.8].
(3) Follows from the definitions.
(4) In the real case it follows from the definition with the mean value inequality [**Dem**, Chapter I, Theorem 4.12]. In the non archimedean case the compatibility to extension of scalars is proven in [**Thu05**, Corollaire 3.4.5].
(5) In the archimedean case this is well-known [**Dem**, Chapter I, Theorem 5.11]. In the non archimedean case this is [**Thu05**, Proposition 3.1.14].
(6) Follows from the definitions.
(7) See [**Thu05**, Proposition 3.1.8].
(8) See [**Dem**, Chapter I, Theorem 4.16]. □

Proposition 2.28. — *Let $v\colon \mathbf{R} \to [-\infty, +\infty[$ be a function. The composite map*
$$v \circ \log|t|\colon \mathbf{G}_m^{\mathrm{an}} \to [-\infty, +\infty[\,,$$
is subharmonic if and only if one of the following conditions are satisfied:

– *v is identically equal to $-\infty$;*
– *v is real-valued and convex.*

Proof. —

($\Leftarrow$) If $v = -\infty$ there is nothing to prove. If v is real valued and convex, then the subharmonicity of $v \circ \log|t|$ is similar to (8) in the previous Proposition: one writes
$$v(\xi) = \sup_{i\in I} h_i(\xi)$$
where $h_i(\xi) = a_i\xi + b_i$ is the family of lines supporting the graph of v. For every $i \in I$ the function $h_i(\log|t|) = a_i \log|t| + b$ is (sub)harmonic. Thus according to (7) in the previous Proposition, the function
$$v(\log|t|) = \sup_{i\in I} h_i(\log|t|)$$
is the (regularised) upper envelope of subharmonic functions, thus it is subharmonic.

7. Namely the smallest upper semi-continuous function bigger than u_i for every $i \in I$.

($\Rightarrow$) Suppose that v is not identically $-\infty$. Since $\log|t|$ is a closed map and $v\circ\log|t|$ is upper semi-continuous, then v is upper semi-continuous. Let $a<b$ real numbers let $\varphi(\xi)=\lambda\xi+\mu$ be an affine function such that

$$\begin{cases} v(a)\le\varphi(a) \\ v(b)\le\varphi(b)\end{cases}$$

One has to show $v(\xi)\le\varphi(\xi)$ for every $\xi\in\,]a,b[$. Since the interval $[a,b]$ is compact and the function $v-\varphi$ is upper semi-continuous, it attains a maximum on a point $\xi_0\in[a,b]$.

By contradiction suppose $v(\xi_0)>\varphi(\xi_0)$, thus $\xi_0\in\,]a,b[$. The function $\varphi(\log|t|)=\lambda\log|t|+\mu$ is harmonic on $\mathbf{G}_m^{\mathrm{an}}$ and the open set

$$\Omega=\{t\in\mathbf{G}_m^{\mathrm{an}}: a<\log|t|<b\},$$

is connected [(8)].

According to the subharmonicity of $v\circ\log|t|$, the function $(v-\varphi)\circ\log|t|$ satisfies the Maximum Principle on Ω. Since it attains a global maximum, it is constant. Moreover, by upper semi-continuity of v,

$$v(\xi_0)-\varphi(\xi_0)\le\max\{v(a)-\varphi(a),v(b)-\varphi(b)\}\le 0,$$

which contradicts the hypothesis $v(\xi_0)>\varphi(\xi_0)$. $\square$

2.3.3. Plurisubharmonic functions. — Let X be a k-analytic space.

Definition 2.29. — A map $u\colon X\to[-\infty,+\infty[$ is said to be *plurisubharmonic* if it is upper semi-continuous and for every analytic extension K of k, every open set Ω of $\mathbf{A}_K^{1,\mathrm{an}}$ and every analytic map $\varepsilon\colon\Omega\to X_K$, the composite map $u\circ\varepsilon\colon\Omega\to[-\infty,+\infty[$ is subharmonic on Ω.

In the complex case this is usual notion of plurisubharmonic function; in the real case a plurisubharmonic function is a plurisubharmonic function on the associated complex space invariant under conjugation .

Proposition 2.30. — *Let X be a k-analytic space.*

(1) If X is an open subset of the affine line $\mathbf{A}_k^{1,\mathrm{an}}$ the u is plurisubharmonic on X if and only if it is subharmonic.

(2) If u,v are plurisubharmonic functions on X and α,β are non-negative real numbers, then $\alpha u+\beta v$ and $\max\{u,v\}$ are plurisubharmonic functions.

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8. In the archimedean case this is trivial. In the non-archimedean case the open subset Ω can be written as the following increasing union $\Omega=\bigcup_{0<\varepsilon<e^{b/a}}C_\varepsilon$ where

$$C_\varepsilon=\{x\in\mathbf{A}_k^{1,\mathrm{an}}: a+\varepsilon/2\le\log|t(x)|\le b-\varepsilon/2\}.$$

For all non-negative real numbers $0\le\alpha\le\beta$, the compact subset

$$C(\alpha,\beta)=\{x\in\mathbf{A}_k^{1,\mathrm{an}}:\alpha\le|t(x)|\le\beta\}$$

is path connected. Therefore Ω is path-connected, thus connected. The fact that $C(\alpha,\beta)$ is path-connected can be shown by hands, and it is a basic, instructive exercise. Otherwise this follows from the fact the $C(\alpha,\beta)$ is a normal k-analytic space, thus connected [**Ber90**, Proposition 3.1.8], hence path-connected [**Ber90**, Theorem 3.2.1].

(3) *If f is an analytic function on X then* $\log|f|$ *is plurisubharmonic.*
(4) *Let K be an analytic extension of k. For every plurisubharmonic function $u\colon X \to [-\infty,+\infty[$, the composite function $u \circ \mathrm{pr}_{K/k}\colon |X_K| \to [-\infty,+\infty[$ is plurisubharmonic.*
(5) *Let $f\colon X' \to X$ be an analytic map between k-analytic spaces; for every plurisubharmonic map u on X the composite map $u \circ f$ is plurisubharmonic on X'.*
(6) *If $\{u_i\}_{i\in I}$ is a locally bounded family of plurisubharmonic functions on X, the its regularised upper envelope is plurisubharmonic.*
(7) *Let $u_1,\dots,u_n$ be plurisubharmonic functions on X and $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ be a convex function which is non-decreasing in each variable. Extend φ by continuity into a function*
$$\widetilde{\varphi}\colon [-\infty,+\infty[^n \longrightarrow [-\infty,+\infty[\,.$$
Then the function $\widetilde{\varphi}\circ(u_1,\dots,u_n)\colon X \to [-\infty,+\infty[$ is plurisubharmonic.

Proof. — Follows from Proposition 2.27. □

2.3.4. Construction of invariant functions. — Let $n \geq 1$ be a positive integer.

Definition 2.31. — An *extended norm* on $\mathbf{A}_k^{n,\mathrm{an}}$ is a function $\alpha\colon \mathbf{A}_k^{n,\mathrm{an}} \to \mathbf{R}_+$ such that for every analytic extension K of k, the map induced on the K-valued points of $\mathbf{A}_k^n$,
$$u_K\colon \mathbf{A}_k^n(K) = K^n \stackrel{\mathrm{pr}_{K/k}}{\longrightarrow} \mathbf{A}_k^{n,\mathrm{an}} \stackrel{u}{\longrightarrow} \mathbf{R}_+,$$
is a norm on the K-vector space K^n.

If $k = \mathbf{R}, \mathbf{C}$ one says that u is hermitian if the induced norm on $\mathbf{C}^n$ is hermitian. If k is non-archimedean one says that u is non-archimedean if, for every analytic extension K of k, the induced norm on K^n is non-archimedean.

Remark 2.32. —
(1) If $k = \mathbf{C}$ an hermitian extended norm is a norm on $\mathbf{C}^n$ in the usual sense.
(2) If $k = \mathbf{R}$ an hermitian extended norm corresponds to a scalar product on $\mathbf{R}^n$.
(3) If k is non-archimedean an example of non-archimedean extended norm is the function
$$u(x) = \max\{r_1|t_1(x)|,\dots,r_n|t_n(x)|\},$$
where $r_1,\dots,r_n$ are positive real numbers and $t_1,\dots,t_n$ the coordinate functions on $\mathbf{A}^n$.

If α, β are norms on k^n, set
$$d(\alpha,\beta) = \sup_{x\in k^n-\{0\}} \left|\log\frac{\alpha(x)}{\beta(x)}\right|.$$
This is well-defined real number since norms on k^n are all equivalent. The function d defines a distance on the set of norms on k^n: the induced topology is the one of uniform convergence on bounded subsets of k^n.

Proposition 2.33. — *Let k be non-archimedean.*

(1) A non-archimedean norm α on k^n extends to a continuous non-archimedean norm u_α on $\mathbf{A}_k^{n,\mathrm{an}}$ in a way such that, if α, β are norms on k^n, then

$$\sup_{x\in\mathbf{A}_k^{n,\mathrm{an}}-\{0\}} \left|\log\frac{u_\alpha(x)}{u_\beta(x)}\right| = d(\alpha,\beta).$$

(2) A continuous non-archimedean extended norm is a plurisubharmonic function.

Proof. —

(1) The extension is defined in two steps. Suppose first that the norm α is diagonalizable, that is, there exists a basis $v_1,\dots,v_n$ of k^n and positive real numbers $r_1,\dots,r_n$ such that

$$\alpha(x_1v_1+\cdots+x_nv_n) = \max\{r_1|x_1|,\dots,r_n|x_n|\}.$$

Let $\varphi_1,\dots,\varphi_n\colon k^n\to k$ be the dual basis. For $x\in\mathbf{A}_k^{n,\mathrm{an}}$ set

$$u_\alpha(x) := \max\{r_1|\varphi_1(x)|,\dots,r_n|\varphi_n(x)|\}.$$

Then u_α is a continuous non-archimedean extended norm which extends α. The formula for the distance of u_α and u_β follows from [**GI63**, Proposition 2.1] applied to every analytic extension of k. The general case follows by the previous one by approximation (see [**BGR84**, 2.6.2, Proposition 3]).

(2) Let u be a non-archimedean extended norm. In order to prove that it is plurisubharmonic, up to extending k, one may assume that k is algebraically closed and maximally complete. The norm α induced on k^n by u is then diagonalizable: there exists a basis $v_1,\dots,v_n$ of k^n and positive real numbers $r_1,\dots,r_n$ such that

$$\alpha(x_1v_1+\cdots+x_nv_n) = \max\{r_1|x_1|,\dots,r_n|x_n|\},$$

(see [**BGR84**, 2.4.1 Definition 1 and 2.4.4 Proposition 2]). Let $\varphi_1,\dots,\varphi_n$ be the dual basis. Since k^n is dense in $\mathbf{A}_k^{n,\mathrm{an}}$ the preceding equality holds everywhere: for $x\in\mathbf{A}_k^{n,\mathrm{an}}$,

$$u(x) = \max\{r_1|\varphi_1(x)|,\dots,r_n|\varphi_n(x)|\}.$$

It follows that u is plurisubharmonic. □

Proposition 2.34. — *Suppose k algebraically closed. Let G be a reductive k-group and X an affine k-scheme of finite type acted upon by G. Let $\mathbf{U}$ be a maximal compact subgroup of G.*

Then, there exists a continuous, $\mathbf{U}$-invariant, plurisubharmonic and topologically proper function $u\colon X^{\mathrm{an}}\to[-\infty,+\infty[$.

Proof. — Up to embedding X in an affine in a G-equivariant way, one may assume $X=\mathbf{A}^n$ and that the action of G on X is linear. If $k=\mathbf{C}$ it suffices to consider an hermitian norm invariant under the action of $\mathbf{U}$. If k is non-archimedean, let $\mathcal{G}$ be the reductive k°-group associated to the maximal compact subgroup $\mathbf{U}$. Let α be a non-archimedean norm on k^n and define

$$\beta(x) := \sup_{g\in\mathcal{G}(k^\circ)} \alpha(g\cdot x).$$

Since $\mathcal{G}(k^\circ)$ is bounded (that is, it is relatively compact in G^{an}) then $\beta(x)$ is a well-defined real number. The function β is a $\mathcal{G}(k^\circ)$-invariant non-archimedean norm on k^n and by Proposition 2.33 it extends to a unique continuous non-archimedean extended norm u_β on $\mathbf{A}_k^{n,\mathrm{an}}$. The extended norm u_β is continuous, plurisubharmonic and invariant under action of $\mathcal{G}(k^\circ)$. Since $\mathcal{G}(k^\circ)$ is dense in $\mathbf{U}$, u_β is $\mathbf{U}$-invariant by continuity. □

2.4. Minima on fibres and orbits. — In this section we collect some basic facts about the variation of minima and maxima of a function along the fibres of a map of analytic spaces and on the orbits under the action of an analytic group.

2.4.1. Minima on fibres. —

Definition 2.35. — Let $f\colon X \to Y$ be a map of sets and $u\colon X \to [-\infty, +\infty]$ a function. The map of *u-minima on f-fibres* $f_\downarrow u\colon Y \to [-\infty, +\infty]$ is defined for every $y \in Y$ by

$$f_\downarrow u(y) := \inf_{f(x)=y} u(x).$$

Let k be a complete field and let $f\colon X \to Y$ be a morphism of k-schemes of finite type. Let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the morphism of k-analytic spaces induced by f. Let K be an analytic extension of k and let $f_K^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the morphism of K-analytic spaces deduced extending scalars to K.

Proposition 2.36. — *Let $u\colon X^{\mathrm{an}} \to [-\infty, +\infty]$ be a function. With the notations just introduced:*

$$f^{\mathrm{an}}_{K\downarrow}(u \circ \mathrm{pr}_{X,K/k}) = (f^{\mathrm{an}}_\downarrow u) \circ \mathrm{pr}_{Y,K/k}\,.$$

Proof. — This follows from the fact that, for every point $y_K \in Y_K^{\mathrm{an}}$, the map induced by the scalar extension

$$\mathrm{pr}_{Y,K/k}\colon (f_K^{\mathrm{an}})^{-1}(y_K) \to (f^{\mathrm{an}})^{-1}(y),$$

where $y = \mathrm{pr}_{Y,K/k}(y_K)$ is surjective (see Proposition 2.16 (2)). □

2.4.2. Minima on orbits. — Let X be a k-scheme of finite type endowed with an action of a k-algebraic group G.

Definition 2.37. — Let $H \subset G^{\mathrm{an}}$ be a subgroup and $u\colon X^{\mathrm{an}} \to [-\infty, +\infty]$ a function. The map of *u-minima on H-orbits* $u_H\colon X^{\mathrm{an}} \to [-\infty, +\infty]$ is defined, for every $x \in X^{\mathrm{an}}$, as

$$u_H(x) := \inf_{x' \in H \cdot x} u(x').$$

In the case $H = G^{\mathrm{an}}$ write u_G instead of $u_{G^{\mathrm{an}}}$.

Remark 2.38. — Let $\sigma\colon G \times_k X \to X$ be the morphism of k-schemes defining the action of G on X. Let $\sigma^{\mathrm{an}}\colon G^{\mathrm{an}} \times_k X^{\mathrm{an}} \to X^{\mathrm{an}}$ be the induced map of k-analytic spaces. Denote by

$$\sigma_H\colon \mathrm{pr}_1^{-1}(H) \subset G^{\mathrm{an}} \times_k X^{\mathrm{an}} \longrightarrow X^{\mathrm{an}},$$

the map induced by σ^{an}. With this notation, by definition,

$$u_H = \sigma_{H\downarrow}(u \circ \mathrm{pr}_2).$$

Proposition 2.39. — *Let $u\colon X^{\mathrm{an}} \to [-\infty, +\infty]$ be a function. Let K be an analytic extension of k and consider the subgroup $H_K := \mathrm{pr}_{G,K/k}^{-1}(H)$ of G_K^{an}. Then,*

$$(u \circ \mathrm{pr}_{X,K/k})_{H_K} = u_H \circ \mathrm{pr}_{X,K/k}\,.$$

Proof. — This follows from Proposition 2.36 combined with Remark 2.38. □

Proposition 2.40. — *Let $u\colon X^{\mathrm{an}} \to [-\infty + \infty[$ be an upper semi-continuous function. Then,*

(1) the function $u_G\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ is upper semi-continuous;

(2) if u is continuous, the subset

$$X_G^{\min}(u) := \{x \in X^{\mathrm{an}} : u_G(x) - u(x) \geq 0\},$$

is closed.

Proof. —

(1) In the complex case the statement is trivial since u_G is the infimum of the upper semi-continuous functions $x \mapsto u(g \cdot x)$ with $g \in G(\mathbf{C})$.

In the general case, according to Proposition 2.39, the statement is compatible to extension of scalars. Suppose that the absolute value on k is non-trivial and k is algebraically closed. In this case the k-rational points $G(k)$ are dense in G^{an}. According to the upper semi-continuity of u for every point $x \in X^{\mathrm{an}}$,

$$u_G(x) := \inf_{x' \in G\cdot x} u(x) = \inf_{g \in G(k)} u(g \cdot x).$$

Conclude by remarking that the right-hand side is an upper semi-continuous function on X^{an} because it is the infimum of the upper semi-continuous functions $u_g\colon x \mapsto u(g \cdot x)$ with $g \in G(k)$.

(2) Follows from upper semi-continuity of $u_G - u$. □

3. Kempf-Ness theory

Let k be a complete field. From now on simplify notations in the two ways:

– If $f\colon X \to Y$ is a morphism of k-schemes, denote by f the morphism of k-analytic spaces $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ induced by f;
– Let X be a k-scheme of finite type endowed with the action of k-algebraic group G. If $x \in X^{\mathrm{an}}$ is a point denote its orbit by $G \cdot x$ instead of $G^{\mathrm{an}} \cdot x$.

3.1. Set-theoretic properties of the analytification of the quotient. — The aim of this section is to prove assertions (i) and (ii) in Theorem 1.6. Let us go back to the notation introduced in paragraphs 1.4-1.5.

Proposition 3.1. — *With the notation introduced above:*

(1) the morphism $\pi\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is surjective and G-invariant;

(2) for every $x, x' \in X^{\mathrm{an}}$,

$$\pi(x_1) = \pi(x_2) \quad \textit{if and only if} \quad \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset.$$

(3) for every point $x \in X^{\mathrm{an}}$ there exists a unique closed orbit contained in $\overline{G \cdot x}$.

In particular, the image of $x, x' \in X^{\mathrm{an}}$ coincide if and only if the unique closed orbit contained in $\overline{G \cdot x}$ and the unique closed orbit contained in $\overline{G \cdot x'}$ coincide.

Before passing to the proof of Proposition 3.1 remark the following:

Corollary 3.2. — *With the notations introduced above, let X' be a G-stable closed subscheme of X and Y' its categorical quotient by G. Then the induced morphism of k-analytic spaces $Y'^{\mathrm{an}} \to Y^{\mathrm{an}}$ is injective.*

Proof of Proposition 3.1. —

(1) Clear from Proposition 1.4.

(2) Consider two points $x_1, x_2 \in X^{\mathrm{an}}$: one has to show

$$\pi(x_1) = \pi(x_2) \quad \Longleftrightarrow \quad \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset.$$

($\Rightarrow$) Suppose that the closure of the orbits $\overline{G \cdot x_1}$ and $\overline{G \cdot x_2}$ meet in a point y. By continuity and G-invariance of π,

$$\pi(x_1) = \pi(y) = \pi(x_2).$$

($\Leftarrow$) Suppose $\pi(x_1) = \pi(x_2)$. First of all one reduces to the case where x_1, x_2 are k-rational points. According to Proposition 2.17 there exist an analytic extension K of k and K-rational points x_{1K}, $x_{2K} \in X_K^{\mathrm{an}}$ such that:

- $\mathrm{pr}_{K/k}(x_{iK}) = x_i$ for $i = 1, 2$;
- $\pi_K(x_{1K}) = \pi_K(x_{2K})$ (where $\pi_K \colon X_K^{\mathrm{an}} \to Y_K^{\mathrm{an}}$ is the morphism of K-analytic spaces associated to π).

Since the construction of the invariants is compatible to the extension of the base field, one has $A_K^{G_K} = A^G \otimes_k K$ where G_K is the K-reductive group deduced from G by extension of scalars. Thus the affine K-scheme Y_K is the categorical quotient of the affine K-scheme X_K by the K-reductive group G_K. Thus, up to extending scalars to K, one may assume that the points $x_1, x_2 \in X^{\mathrm{an}}$ are k-rational.

Let $x_1, x_2 \in X^{\mathrm{an}}$ be k-rational points and let $\alpha \colon X^{\mathrm{an}} \to X$ be the morphism of locally k-ringed spaces deduced by analytification of X. To avoid confusion momentarily denote:

- $\pi^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ the morphism of k-analytic spaces deduced from the morphism of k-schemes $\pi \colon X \to Y$;
- $G^{\mathrm{an}} \cdot x_i$ the orbit of the k-point $x_i \in X^{\mathrm{an}}$ under the action of the analytic group $G^{\mathrm{an}} \cdot x_i$ $(i = 1, 2)$.

For $i = 1, 2$:

$$\alpha^{-1}(G \cdot \alpha(x_i)) = G^{\mathrm{an}} \cdot x_i.$$

The hypothesis $\pi^{\mathrm{an}}(x_1) = \pi^{\mathrm{an}}(x_2)$ implies $\pi(\alpha(x_1)) = \pi(\alpha(x_2))$. According to Theorem 1.4 the closure of their algebraic orbits meet:

$$\overline{G \cdot \alpha(x_1)} \cap \overline{G \cdot \alpha(x_2)} \neq \emptyset.$$

For $i = 1, 2$ the orbit $G \cdot \alpha(x_i)$ a constructible subset of X thus its closure of Z with respect to analytic topology coincide with its closure with respect to the Zariski topology (Proposition 2.6):

$$\alpha^{-1}(\overline{G \cdot \alpha(x_i)}) = \overline{\alpha^{-1}(G \cdot \alpha(x_i))} = \overline{G^{\mathrm{an}} \cdot x_i}.$$

Since the closure of the algebraic orbits $\overline{G \cdot \alpha(x_1)}$, $\overline{G \cdot \alpha(x_2)}$ meet then the closure of the analytic orbits $\overline{G^{\mathrm{an}} \cdot x_1}$, $\overline{G^{\mathrm{an}} \cdot x_2}$ meet as well.

(3) Follows from (2): consider a point $x \in X^{\mathrm{an}}$ and two points $y_1, y_2 \in \overline{G \cdot x}$. Since π is continuous and G-invariant one has $\pi(y_1) = \pi(y_2)$. If one supposes that the orbits of y_1 and y_2 are closed, statement (2) affirms that $G \cdot y_1$ and $G \cdot y_2$ meet, thus they coincide. □

3.2. Comparison of minima. —

3.2.1. Statements. — Let us go back to the notation introduced in paragraphs 1.4-1.5 and recall the statement of Theorem 1.9:

Theorem 3.3. — *With the notation introduced above, let $u \colon X^{\mathrm{an}} \to [-\infty, +\infty[$ be a plurisubharmonic function which is invariant under the action of a maximal compact subgroup of G. For every point $x \in X^{\mathrm{an}}$,*

$$\inf_{\pi(x')=\pi(x)} u(x') = \inf_{x' \in G \cdot x} u(x').$$

Corollary 3.4. — *Let $u \colon X^{\mathrm{an}} \to [-\infty, +\infty[$ be a plurisubharmonic function which is invariant under the action of a maximal compact subgroup of G. Then,*

1. *a point x is u-minimal on π-fibres if and only if it is u-minimal on G-orbits;*
2. *$X^{\min}_{\pi}(u) = X^{\min}_{G}(u)$;*
3. *if u is moreover continuous, the set of u-minimal points on π-fibres $X^{\min}_{\pi}(u)$ is closed.*

Proof of the Corollary. — Clear from the definitions of u-minimal point on π-fibre and u-minimal point on G-orbit. (3) follows from Proposition 2.40 (2). □

In order to prove Theorem 3.3 we show:

Theorem 3.5. — *With the notation previously introduced, for every point $x \in X^{\mathrm{an}}$ there exists a point x_0 that belongs to the unique closed orbit contained in $\overline{G \cdot x}$ and such that $u(x_0) \leq u(x)$.*

Let us show how it entails Theorem 3.3.

Proof of Theorem 3.3. — For every point $x \in X^{\mathrm{an}}$,

$$\inf_{\pi(y)=\pi(x)} u(y) \leq \inf_{y \in G \cdot x} u(y).$$

It remains to prove the converse inequality. Let $x, x' \in X^{\mathrm{an}}$ be such that $\pi(x') = \pi(x)$. Applying Theorem 3.5 to the point x', there exists a point x'_0 that belongs to the unique closed orbit contained in $\overline{G \cdot x'}$ and such that $u(x'_0) \leq u(x')$. By continuity,

$$\pi(x'_0) = \pi(x') = \pi(x),$$

hence $G \cdot x'_0$ is the unique closed orbit contained in $\overline{G \cdot x}$. Thus,

$$u(x') \geq u(x'_0) \geq \inf_{y \in G \cdot x'_0} u(y) \geq \inf_{y \in \overline{G \cdot x}} u(y) = \inf_{y \in G \cdot x} u(y),$$

where the last equality comes from the upper semi-continuity of the function u. Since x' is arbitrary,

$$\inf_{\pi(y)=\pi(x)} u(y) \geq \inf_{y \in G \cdot x} u(y),$$

which concludes the proof of Theorem 3.3. □

The rest of this section is hence devoted to the proof of Theorem 3.5.

3.2.2. Parabolic subgroups containing destabilizing one-parameter subgroups. — Drop for the moment the general notation.

Let k be an algebraically closed field and consider the action of a reductive k-group G on an affine k-scheme X of finite type. Let $S \subset X$ be a closed G-stable subset of X. The following result has been established by Kempf during his proof of the existence of a rational destabilizing one-parameter subgroup (see [**Kem78**, Theorem 3.4]):

Theorem 3.6. — *Let $x \in X(k)$ be k-point of X such that*

$$\overline{G \cdot x} \cap S \neq \emptyset.$$

Then, there exists a parabolic subgroup $P = P(S, x)$ of G satisfying the following property: for every maximal torus $T \subset P$ there exists a one-parameter subgroup $\lambda_T \colon \mathbf{G}_m \to T$ such that the limit

$$\lim_{t \to 0} \lambda(t) \cdot x$$

exists[(9)] *and belongs to S.*

3.2.3. Destabilizing one-parameter subgroups: archimedean case. — Let G be a complex (connected) reductive group and let $\mathbf{U} \subset G(\mathbf{C})$ a maximal compact subgroup. Then, there exists an $\mathbf{R}$-group scheme U such that $U \times_{\mathbf{R}} \mathbf{C} = G$ and $U(\mathbf{R}) = \mathbf{U}$. Moreover, a torus $T \subset G$ is defined over $\mathbf{R}$ if and only if $T(\mathbf{C}) \cap \mathbf{U}$ is the maximal compact subgroup of $T(\mathbf{C})$.

Let $X = \operatorname{Spec} A$ be a complex affine scheme of finite type endowed with an action of G. Let $S \subset X$ be a G-stable Zariski closed subset.

Lemma 3.7. — *Let $x \in X(\mathbf{C})$ be a point such that $\overline{G \cdot x}$ meets S. Then, there exists a one-parameter subgroup $\lambda \colon \mathbf{G}_m \to G$ satisfying the following properties:*

9. Namely the morphism of k-schemes $\lambda_x \colon \mathbf{G}_m \to X$, $t \mapsto \lambda(t) \cdot x$ extends to a morphism of k-schemes $\overline{\lambda}_x \colon \mathbf{A}^1 \to X$ and by definition,

$$\lim_{t \to 0} \lambda(t) \cdot x := \overline{\lambda}_x(0).$$

– *the limit point* $\lim_{t\to 0} \lambda(t)\cdot x$ *exists and belongs to* S*;*
– *the image of* $\mathbf{U}(1)$ *is contained in* $\mathbf{U}$.

This statement is implicitly proven in [**KN79**] when $X = \mathbf{A}^n_{\mathbf{C}}$ is a linear representation of G and $S = \{0\}$. It can be deduced from this case by means of G-equivariant morphism $f\colon X \to \mathbf{A}^n$ such that $f^{-1}(0) = S$.

Proof. — We reproduce here the argument of Kempf-Ness. According to Theorem 3.6 there exists a parabolic subgroup $P \subset G$ with the following property: for every maximal torus $T \subset P$ there exists a one-parameter subgroup $\lambda_T\colon \mathbf{G}_m \to T$ such that the limit point

$$\lim_{t\to 0} \lambda_T(t)\cdot x$$

exists and belongs to S.

Let $\overline{P}$ be the conjugated parabolic subgroup under the real structure of G given by U. Let T be a maximal torus of the subgroup $P \cap \overline{P}$ which is defined over $\mathbf{R}$. As a maximal torus in the intersection of two parabolic subgroups T is a maximal torus of the whole group G.

Thus by Theorem 3.6 there exists a one-parameter subgroup $\lambda\colon \mathbf{G}_m \to T$ which satisfies the required properties. □

3.2.4. Destabilizing one-parameter subgroups: non-archimedean case. — Let k be a field complete with respect to a non-archimedean absolute value and k° its ring of integers. Suppose k algebraically closed.

Let $\mathcal{G}$ be a reductive k°-group and G its generic fibre. A one-parameter subgroup $\lambda\colon \mathbf{G}_{m,k^\circ} \to \mathcal{G}$ induces a map $\lambda^{\mathrm{an}}\colon \mathbf{G}_m^{\mathrm{an}} \to G^{\mathrm{an}}$ which sends $\mathbf{U}(1)$ into the maximal compact subgroup $\mathbf{U}$ associated to $\mathcal{G}$.

Let $X = \operatorname{Spec} A$ be an affine k-scheme of finite type endowed with an action of G and let $S \subset X$ be a G-stable closed subset.

Lemma 3.8. — *Let* $x \in X(k)$ *be a point such that* $\overline{G\cdot x}$ *meets* S*. Then, there exists a one-parameter subgroup* $\lambda\colon \mathbf{G}_{m,k^\circ} \to \mathcal{G}$ *such that the limit point on the generic fibre*

$$\lim_{t\to 0} \lambda(t)\cdot x$$

exists and belongs to S.

Proof. — By Theorem 3.6 there exists a parabolic subgroup $P \subset G$ with the following property: for every maximal torus T contained in G there exists a one-parameter subgroup $\lambda_T\colon \mathbf{G}_m \to T$ such that the limit point

$$\lim_{t\to 0} \lambda_T(t)\cdot x$$

exists and belongs to S.

Denote by $\mathrm{Par}(\mathcal{G})$ the scheme parametrizing the parabolic subgroups of $\mathcal{G}$: it is proper over k° [**GP11**, Exposé XXVI, Théorème 3.3-Corollaire 3.5]. By the valuative criterion of properness there exists a unique parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ with generic fibre P. Let $\mathcal{T}$ be a maximal torus of $\mathcal{P}$ and let T be its generic fibre. [10]

Let $\lambda \colon \mathbf{G}_m \to T$ be the one-parameter subgroup given by Theorem 3.6. Since k is algebraically closed, the torus $\mathcal{T}$ is split and the one-parameter subgroup λ lifts in a unique way to a one-parameter subgroup $\lambda \colon \mathbf{G}_{m,k^\circ} \to \mathcal{T}$ which satisfies the required properties. □

3.2.5. End of proof of Theorem 3.5. — Let $x \in X^{\mathrm{an}}$ and $\mathbf{U}$ a maximal compact subgroup of G which fixes the function u. Up to extending of k one may assume:

– *archimedean case:* $k = \mathbf{C}$;

– *non-archimedean case:* k is algebraically closed and the point x is k-rational.

Let S be the unique closed orbit contained in $\overline{G \cdot x}$. According to Lemmata 3.7-3.8 there exists a one-parameter subgroup $\lambda \colon \mathbf{G}_m \to G$ with the following properties:

– the limit point $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ exists and belongs to S;

– the image of $\mathbf{U}(1)$ is contained in $\mathbf{U}$.

Let us show $u(x_0) \leq u(x)$. The map $t \mapsto \lambda(t) \cdot x$ extends to a morphism of k-schemes $\lambda_x \colon \mathbf{A}^1_k \to X$ such that $\lambda_x(0) = x_0$. The function

$$\begin{aligned} u_x \colon \mathbf{A}^{1,\mathrm{an}}_k &\longrightarrow [\infty, +\infty[\\ t &\longmapsto \begin{cases} u(x_0) & \text{if } t = 0, \\ u(\lambda(t) \cdot x) & \text{otherwise,} \end{cases} \end{aligned}$$

is subharmonic (it can be written as $u \circ \lambda_x$) and $\mathbf{U}(1)$-invariant. By the Maximum Principle,

$$\limsup_{t \to 0} u_x(t) = u_x(0) = u(x_0).$$

According to Proposition 2.28 the function $v_x \colon \mathbf{R} \to [-\infty, +\infty[$ defined by the condition $v_x(\log |t|) = u_x(t)$ is either identically equal to $-\infty$ or it is real-valued and convex. In both cases,

$$\limsup_{\xi \to \infty} v_x(\xi) = \limsup_{t \to 0} u_x(t) = u(x_0) < +\infty,$$

hence v_x has to be non-decreasing. In particular,

$$u(x_0) = \limsup_{\xi \to \infty} v_x(\xi) \leq v_x(0) = u(x)$$

10. By [**GP11**, Exposé XII, Théorème 1.7] the existence (locally for the étale topology) of a maximal torus is equivalent to the locally constance of the reductive rank, that is the function $\mathrm{redrk} \colon S \to \mathbf{N}$ defined for every point $s \in S$ by

$$\mathrm{redrk}(s) := \text{dimension of a maximal torus of } G \times_S \operatorname{Spec} \overline{\kappa(s)}$$

where $\overline{\kappa(s)}$ denotes an algebraic closure of the residue field $\kappa(s)$ at s. In general a maximal torus of a parabolic subgroup Q of a reductive group H is a maximal torus of H [**Bor91a**, Corollary 11.3]: in particular, the reductive rank of Q is equal to the reductive rank H. On the other side, the reductive rank of a reductive group is locally constant [**GP11**, Exp. XIX, Corollaire 2.6]. Therefore the reductive rank of the parabolic subgroup $\mathcal{P}$ is (locally) constant on $\operatorname{Spec} k^\circ$: since k is algebraically closed $\mathcal{P}$ has a maximal torus.

which concludes the proof of Theorem 3.5. □

3.3. Analytic topology of the quotient. —

3.3.1. Statement. — In this section we prove assertion (3) in Theorem 1.6:

Proposition 3.9. — *Let $F \subset X^{\mathrm{an}}$ be a closed G-stable subset of X^{an}. Then its projection $\pi(F)$ is closed in Y^{an}.*

Combining it with Corollary 3.2:

Corollary 3.10. — *Let X' be a G-stable closed subscheme of X and let Y' be its categorical quotient by G. The induced morphism of k-analytic spaces $Y'^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a homeomorphism onto a closed subset of Y^{an}.*

The rest of this section is devoted to the proof of Proposition 3.9.

3.3.2. Minimal points on affine cones. — The proof of Proposition 3.9 is based on a elementary fact concerning minimal points on fibres of a homogeneous map between affine cones. Drop momentarily the general notation. Let

$$A = \bigoplus_{d\geq 0} A_d, \qquad B = \bigoplus_{d\geq 0} B_d,$$

be (positively) graded k-algebras of finite type such that the k-algebras A_0, B_0 are finite (*i.e.* finite dimensional as k-vector spaces). Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be their spectra. Let $\varphi\colon A \to B$ be homogeneous homomorphism of degree $D \geq 1$ of graded k-algebras, that is a homomorphism of k-algebras such that for every $d \geq 0$,

$$\pi(B_d) \subset A_{dD}.$$

The homomorphism φ induces a morphism of k-schemes $f\colon X \to Y$.

Let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the morphism of k-analytic spaces induced by f.

Definition 3.11. — Let $u\colon X^{\mathrm{an}} \to \mathbf{R}_+$ be a map.

(1) A point $x \in X^{\mathrm{an}}$ is said to be *u-minimal on f-fibre* if for every point x' such that $f^{\mathrm{an}}(x') = f^{\mathrm{an}}(x)$ one has $u(x') \leq u(x)$. The subset of u-minimal points on f-fibres is denoted by $X^{\min}_f(u)$.

(2) Let $h\colon \mathbf{A}^{1,\mathrm{an}} \times_k X^{\mathrm{an}} \to X^{\mathrm{an}}$ denote the morphism of multiplication by scalars induced by the grading of A.

The function $u\colon X^{\mathrm{an}} \to \mathbf{R}_+$ is said to be *1-homogeneous* if for every point $z \in \mathbf{A}^{1,\mathrm{an}} \times_k X^{\mathrm{an}}$,

$$u(h(z)) = |\operatorname{pr}_1(z)| \cdot u(\operatorname{pr}_2(z)),$$

where pr_1, pr_2 are the projections on $\mathbf{A}^{1,\mathrm{an}} \times_k X^{\mathrm{an}}$.

Proposition 3.12. — *With the notation introduced above, let $u\colon X^{\mathrm{an}} \to \mathbf{R}_+$ be a continuous, 1-homogeneous and topologically proper function. If $X^{\min}_f(u)$ is closed in X^{an}, then the restriction of f to $X^{\min}_f(u)$,*

$$f^{\mathrm{an}}\colon X^{\min}_f(u) \longrightarrow Y^{\mathrm{an}},$$

is topologically proper.

Proof of Proposition 3.12. — The statement is compatible with extension of scalars, thus the absolute value $|\cdot|\colon k \to \mathbf{R}_+$ can be supposed to be surjective. (11)

Choosing homogeneous generators $b_1, \dots, b_n$ of B with $\deg b_\alpha = \delta_\alpha$, one may replace Y by the weighted affine space

$$\mathbf{A}^n_{(\boldsymbol{\delta})} = \operatorname{Spec} k[t_1, \dots, t_n]_{(\boldsymbol{\delta})},$$

where the $k[t_1, \dots, t_n]_{(\boldsymbol{\delta})}$ is the k-algebra of polynomials $k[t_1, \dots, t_n]$ where the grading is given by $\deg t_\alpha = \delta_\alpha$. To ease notation denote f^{an} by f and $X^{\min}_f(u)$ by $X^{\min}$.

Arguing by contradiction, suppose that the restriction of f to $X^{\min}$ is not topologically proper. Then there exists a sequence of points $\{x_i\}_{i \in \mathbf{N}}$ in $X^{\min}$ such that the images $\{f(x_i)\}_{i \in \mathbf{N}}$ are contained in a compact subset of $\mathbf{A}^{n,\mathrm{an}}_{(\boldsymbol{\delta})}$ while $u(x_i) \to \infty$ as $i \to \infty$. One may suppose $u(x_i) \neq 0$ for every $i \in \mathbf{N}$. Since the absolute value of k is surjective, for every $i \in \mathbf{N}$ there exists $\lambda_i \in k^\times$ such that $|\lambda_i| = u(x_i)$. Define a new sequence in X^{an} setting

$$\widetilde{x}_i := \frac{x_i}{\lambda_i}.$$

By homogeneity of u the points $\widetilde{x}_i$ are minimal on the fibres of f. Moreover the points x_i's are contained in the compact subset $\{x : u(x) = 1\}$. By sequential compactness (12), one may assume that the sequence $\{\widetilde{x}_i\}$ converges to a point $\widetilde{x}_\infty$. By construction:

- $\widetilde{x}_\infty$ is u-minimal on f-fibre;
- $u(\widetilde{x}_\infty) = 1$.

These two properties are contradictory. Indeed the map $f\colon X \to \mathbf{A}^n_{(\boldsymbol{\delta})}$ is given by some polynomials $f_1, \dots, f_n$ of degree $\deg f_\alpha = D\delta_\alpha$ (recall that the homomorphism φ is of degree D). Thus, for every $i \in \mathbf{N}$ and $\alpha = 1, \dots, n$,

$$|f_\alpha(\widetilde{x}_i)| = \frac{|f_\alpha(x_i)|}{|\lambda_i|} = \frac{|f_\alpha(x_i)|}{u(x_i)}.$$

The points $f_\alpha(x_i)$ are contained in a compact set, so the real numbers $|f_\alpha(x_i)|$ are bounded independently of i and α. By hypothesis $u(x_i) \to \infty$ as $i \to \infty$, thus

$$\lim_{i \to \infty} \max_{\alpha = 1, \dots, n} |f_\alpha(\widetilde{x}_i)| = \lim_{i \to \infty} \max_{\alpha = 1, \dots, n} \frac{|f_\alpha(x_i)|}{u(x_i)} = 0,$$

which gives $f(\widetilde{x}_\infty) = 0$. Since $\widetilde{x}_\infty$ is a u-minimal point on f-fibre, the latter fact implies that it must belong to the vertex $\operatorname{Spec} A_0$ of X.

The homogeneity of u implies $u(\widetilde{x}_\infty) = 0$ which contradicts $u(\widetilde{x}_\infty) = 1$. □

11. An analytic extension K of k such that the absolute value $|\cdot|_K\colon K \to \mathbf{R}_+$ is surjective can be constructed by means of transfinite induction. This is not really necessary for the proof: this assumption just makes the exposition clearer. One can adapt the proof in the case when the absolute value $|\cdot|\colon k \to \mathbf{R}_+$ is dense. Another way to circumvent it is to add only the real numbers $u(x_i)$ to the value group of k.

12. In the archimedean case analytic spaces are locally metrizable topological spaces; in the non-archimedean case this is not the case and sequential compactness has been proven in [**Poi13b**].

3.3.3. Reducing to the case of the affine spaces. — Go back to the proof of Proposition 3.9 and to the general notation introduced in paragraphs 1.4-1.5.

One reduces first to the case where X is an affine space $\mathbf{A}^n_k$. Let $X_1 = \operatorname{Spec} A_1$ and $X_2 = \operatorname{Spec} A_2$ be k-affine schemes (of finite type) endowed with an action of the reductive k-group G and let $i : X_1 \to X_2$ be a closed G-equivariant embedding. For $\alpha = 1, 2$ let Y_α be the categorical quotient of X_α by G and let $\pi_\alpha \colon X_\alpha \to Y_\alpha$ be the quotient map. The following diagram of k-schemes

$$\begin{tikzcd} X_1 \arrow[r, "i"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ Y_1 \arrow[r, "j"] & Y_2 \end{tikzcd}$$

is commutative, where j is the morphism induced between categorical quotients. Corollary 3.2 affirms that $j \colon Y_1^{\mathrm{an}} \to Y_2^{\mathrm{an}}$ is set-theoretically injective. In particular, for a subset F of X_1^{an},

$$\pi_1(F) = j^{-1}(\pi_2 \circ i(F)).$$

Suppose that the conclusion of Proposition 3.9 is true for the k-analytic space X_2^{an}. If F is a closed G-stable subset of X_1^{an}, then $i(F)$ is a closed G-stable subset of X_2^{an} and its projection $\pi_1(i(F))$ is closed in Y_2^{an}. Thus $\pi_1(F) = j^{-1}(\pi_2 \circ i(F))$ is closed in Y_1^{an}. One reduces to the case of an affine space taking a closed G-equivariant embedding $i \colon X \to \mathbf{A}^n_k$.

Let X be a linear representation $\mathbf{A}^n_k = \operatorname{Spec} k[t_1, \dots, t_n]$ of G. Since the action of G on X is linear, the action of G on the k-algebra of polynomials $A := k[t_1, \dots, t_n]$ respects its grading. In particular, the subalgebra of G-invariants A^G is graded and the inclusion $A^G \subset A$ is a homogeneous homomorphism of degree 1 of k-graded algebras.

3.3.4. Using Kempf-Ness theory. — The statement of Proposition 3.9 is compatible to extending scalars to an analytic extension of k: in the archimedean case one can take $k = \mathbf{C}$ and in the non-archimedean one can suppose that the reductive k-group G is the generic fibre of a reductive k°-group $\mathcal{G}$.

Let $\mathbf{U}$ be a maximal compact subgroup and take a function $u \colon X^{\mathrm{an}} \to \mathbf{R}_+$ which is continuous, topologically proper, 1-homogeneous, plurisubharmonic and $\mathbf{U}$-invariant (it exists by Proposition 2.34).

To apply Proposition 3.12 to the function u one has to show that the subset of u-minimal points on π-fibres $X_\pi^{\min}(u)$ is closed. Since u is continuous, plurisubharmonic and invariant under a maximal compact subgroup, this is true because, according to Corollary 3.4, the subset of u-minimal points on π-fibres coincide with the set of u-minimal point on G-orbits, which is a closed subset. Now Proposition 3.12 tells that the restriction

$$\pi \colon X_\pi^{\min}(u) \longrightarrow Y^{\mathrm{an}}$$

is topologically proper. As u is topologically proper, it is also surjective. The topological spaces $X_\pi^{\min}(u)$ and Y^{an} are locally compact, thus the restriction of π to $X_\pi^{\min}(u)$ is a closed map.

One can now conclude the proof: for a closed G-stable subset $F \subset X^{\mathrm{an}}$,

$$\pi(F \cap X_\pi^{\min}(u)) = \pi(F). \tag{3.3.1}$$

Together with the fact that $\pi\colon X_\pi^{\min}(u) \to Y^{\mathrm{an}}$ is closed, this conclude the proof.

Let us show (3.3.1). ($\subset$) Clear. ($\supset$) One has to show that for every point $x \in F$ there exists a u-minimal point on π-fibre $x' \in F$ such that $\pi(x) = \pi(x')$. Since F is a closed and G-stable subset, it contains the closure of the orbit $\overline{G \cdot x}$ of the point x. Let $x' \in \overline{G \cdot x}$ a u-minimal point on G-orbit: it exists because the function u is topologically proper. Since u-minimal points on G-orbits and on π-fibres coincide, the point x' is u-minimal on π-fibre; since it belongs to the closure of the orbit of x one has $\pi(x') = \pi(x)$, which concludes the proof of (3.3.1). □

3.4. Continuity of minima on the quotient. — Let X be an affine k-scheme endowed with an action of reductive k-group G. Let Y be the categorical quotient of X by G and let $\pi\colon X \to Y$ be quotient map. Let $u\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ be a plurisubharmonic function which is invariant under the action of a maximal compact subgroup of G.

Consider the *function of u-minima on π-fibres* $\pi_\downarrow u : Y^{\mathrm{an}} \to [-\infty, +\infty[$ defined for every $y \in Y^{\mathrm{an}}$ as

$$\pi_\downarrow u(y) := \inf_{\pi(x)=y} u(x).$$

Proposition 3.13. — *The map $\pi_\downarrow u\colon Y^{\mathrm{an}} \to [-\infty, +\infty[$ is upper semi-continuous. If the function u is continuous and topologically proper, then:*

(1) the restriction of π to $X_\pi^{\min}(u) = X_G^{\min}(u)$ is topologically proper and surjective onto Y^{an};

(2) the function $\pi_\downarrow u$ is continuous on Y^{an}.

Proof. — To prove the upper semi-continuity of $\pi_\downarrow u$, one has to show that for every real number α the subset $V_\alpha := \{y \in Y^{\mathrm{an}} : \pi_\downarrow u(y) < \alpha\}$ is open. Theorem 3.3 implies, for every point $x \in X^{\mathrm{an}}$,

$$\pi_\downarrow u(\pi(x)) := \inf_{\pi(x')=\pi(x)} u(x') = u_G(x) := \inf_{x' \in G \cdot x} u(x').$$

In particular for every real number α:

$$\begin{aligned}\pi^{-1}(V_\alpha) &:= \pi^{-1}\left(\{y \in Y^{\mathrm{an}} : \pi_\downarrow u(y) < \alpha\}\right)\\ &= \{x \in X^{\mathrm{an}} : \pi_\downarrow u(\pi(x)) < \alpha\}\\ &= \{x \in X^{\mathrm{an}} : u_G(x) < \alpha\}.\end{aligned}$$

The function $u_G\colon X^{\mathrm{an}} \to [-\infty, +\infty[$, $x \mapsto \inf_{x' \in G \cdot x} u(x')$ is upper semi-continuous (see Proposition 2.40). The preceding equality implies that $U_\alpha := \pi^{-1}(V_\alpha)$ is a G-saturated open subset of X^{an}. By Corollary 1.7 (4), $V_\alpha = \pi(U_\alpha)$ is an open subset of Y^{an}.

Suppose that u is moreover continuous and topologically proper.

(1) The surjectivity of $\pi\colon X_\pi^{\min}(u) \to Y^{\mathrm{an}}$ follows from the topological properness of the function u. It remains to show that $\pi\colon X_\pi^{\min}(u) \to Y^{\mathrm{an}}$ is topologically proper. Let K be a compact subset of Y^{an}. The function $\pi_\downarrow u$ is upper semi-continuous, thus it is bounded on K: set
$$\alpha := \sup_{y\in K} \pi_\downarrow u(y) < +\infty.$$
The inverse image $\pi^{-1}(K)$ is a closed subset of $\{x \in X^{\mathrm{an}} : \pi_\downarrow u(\pi(x)) \leq \alpha\}$, hence it suffices to show that the subset
$$\{x \in X^{\mathrm{an}} : \pi_\downarrow u(\pi(x)) \leq \alpha\} \cap X_\pi^{\min}(u)$$
is compact. By definition of minimal point on π-fibre the functions $\pi_\downarrow u$ and u coincide on $X_\pi^{\min}(u)$, hence
$$\{x \in X_\pi^{\min}(u) : \pi_\downarrow u(\pi(x)) \leq \alpha\} = \{x \in X_\pi^{\min}(u) : u(x) \leq \alpha\}.$$
The right hand side is a compact subset because the subset $X_\pi^{\min}(u)$ is closed and u is topologically proper.

(2) The topological space $X_\pi^{\min}(u)$ is locally compact because it is a closed subset of X^{an}, thus the map $\pi\colon X_\pi^{\min}(u) \to Y^{\mathrm{an}}$ is closed (it is topologically proper). The equality
$$u_{|X_\pi^{\min}(u)} = (\pi^*\pi_\downarrow u)_{|X_\pi^{\min}(u)}$$
implies that $\pi_\downarrow u$ is continuous. □

3.5. Comparison with the result of Kempf-Ness. —

3.5.1. Special plurisubharmonic functions. — In this section we work over the complex numbers. We show how the techniques employed to prove Theorem 1.9 permit actually to find the result of Kempf-Ness for broader class of functions, called here special plurisubharmonic. Let X be a complex analytic space.

Definition 3.14. — A function $u\colon X \to [-\infty, +\infty[$ is said *special plurisubharmonic* if it is plurisubharmonic and for every non-constant holomorphic map $\varepsilon\colon \mathbf{D} \to X$, where $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is the unit disk, the function $u \circ \varepsilon$ is non-constant.

Proposition 3.15. — *Special plurisubhamornic functions enjoy the following properties:*

(1) if $\alpha > 0$ is a positive real number and u is a special plurisubharmonic function u, αu is special plurisubharmonic; if u, v are special plurisubharmonic, then $u + v$ is special plurisubharmonic;

(2) if X is a connected analytic curve and f is a non-constant holomorphic function then $\log|f|$ *is a special (pluri)subharmonic function;*

(3) if $f\colon X' \to X$ is a holomorphic map with discrete fibres and u is a special plurisubharmonic function on X, then f^*u is special plurisubharmonic on X';

(4) strongly plurisubharmonic functions are special plurisubharmonic.

Example 3.16. —

(1) The converse to (4) is false: for every $p > 1$ the logarithm of the ℓ^p-norm,
$$\log \|x\|_{\ell^p} = \log \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$
is special plurisubharmonic on $\mathbf{C}^n$ but it is not strongly plurisubharmonic on the radial direction. If $p \neq 2$ (and $n \geq 2$) the Kähler form of the metric induced on $\mathcal{O}_{\mathbf{P}^{n-1}}(1)$ is not positive definite.

(2) The logarithm of the ℓ^∞-norm $\log \|x\|_{\ell^\infty} = \log \max\{|x_1|, \ldots, |x_n|\}$ is not special plurisubharmonic.

Let G be a complex reductive group acting on a complex affine scheme X of finite type.

Theorem 3.17. — *Let $u \colon X(\mathbf{C}) \to [-\infty, +\infty[$ be a special plurisubharmonic function invariant under the action of a maximal compact subgroup $\mathbf{U}$ of G. Let $x \in X(\mathbf{C})$ be a point which is u-minimal on its G-orbit. Then,*

(1) the orbit $G \cdot x$ is closed;

(2) let G_x be the stabilizer of x; the inclusion
$$\{kg : k \in \mathbf{U}, g \in G_x(\mathbf{C})\} \subset \{g \in G(\mathbf{C}) : g \cdot x \in X_G^{\min}(u)\}$$
is an equality.

In other words, minimal points contained in a closed G-orbit form a single $\mathbf{U}$-orbit.

Corollary 3.18. — *Let $u \colon X(\mathbf{C}) \to [-\infty, +\infty[$ be a continuous topologically proper, special plurisubharmonic function, invariant under the action of a maximal compact subgroup $\mathbf{U}$ of G. Then, the continuous map induced by π,*
$$X_\pi^{\min}(u)/\mathbf{U} \longrightarrow Y^{\mathrm{an}}$$
is a homeomorphism.

Proof. —

(1) By contradiction suppose that the orbit of x is not closed and let S be the unique closed orbit contained in $\overline{G \cdot x}$. According to Lemma 3.7 there exists a one-parameter subgroup $\lambda \colon \mathbf{G}_m \to G$ with the following properties:
   - the limit point $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ exists and belongs to S;
   - the image of $\mathbf{U}(1)$ is contained in $\mathbf{U}$.

   Let us show $u(x_0) < u(x)$. The morphism $t \mapsto \lambda(t) \cdot x$ extends to a morphism $\lambda_x \colon \mathbf{A}^1_k \to X$ which is finite because the point x is not fixed under λ. Consider the function on $u_x \colon \mathbf{C} \to [-\infty, +\infty[$,
$$u_x(t) := \begin{cases} u(x_0) & \text{if } t = 0 \\ u(\lambda(t) \cdot x) & \text{otherwise.} \end{cases}$$
   The function u_x is special (pluri)subharmonic and $\mathbf{U}(1)$-invariant. According to the Maximum Principle,
$$\limsup_{t \to 0} = u_x(0) = u(x_0).$$

Proposition 2.28 implies that the function $v_x\colon \mathbf{R} \to [-\infty, +\infty[$ defined by the condition $v_x(\log|t|) = u_x(|t|)$ is either identically equal to $-\infty$ or convex. As u_x is special plurisubharmonic, given an open interval I, v_x is not constant on I. Since

$$\limsup_{\xi\to-\infty} v_x(\xi) = \limsup_{t\to 0} u_x(t) = u(x_0) < +\infty$$

the function v_x has to be increasing. Therefore

$$u(x_0) = v_x(-\infty) < v_x(0) = u(x),$$

which contradicts the minimality of x.

(2) Suppose that the reductive group G is a torus T. According to (1) the orbit $T \cdot x$ of x is closed. Replacing X with $T \cdot x$ and T with T/T_x (where T_x is the stabilizer of x) one may assume that the stabilizer of x is finite, hence the morphism

$$\begin{array}{rcl} \sigma_x\colon T & \longrightarrow & X \\ t & \longmapsto & t \cdot x, \end{array}$$

is finite. The function $u_x(t) := u(t \cdot x)$ is special plurisubharmonic on $T(\mathbf{C})$ and it is invariant under the action of $\mathbf{U}$. Identify $T(\mathbf{C})/\mathbf{U}$ with $\mathbf{R}^n$ (where n is the dimension of T) through logarithmic coordinates:

$$\begin{array}{rcl} T(\mathbf{C})/\mathbf{U} = (\mathbf{C}^\times/\mathbf{U}(1))^n & \xrightarrow{\sim} & \mathbf{R}^n \\ (z_1, \dots, z_n) & \longmapsto & (\log|z_1|, \dots, \log|z_n|), \end{array}$$

Since u_x is invariant under action of $\mathbf{U}$, it descends (through the above identification) on a continuous function $v : \mathbf{R}^n \to \mathbf{R}$ which is convex according to the plurisubhamornicity of u_x. Moreover, since u_x is special plurisubharmonic, v is non-constant on every segment contained in $\mathbf{R}^n$.

The hypothesis of x being u-minimal reads into the fact that v has a global minimum in the origin $0 \in \mathbf{R}^n$. To conclude the proof one has to show that the minimum is not obtained elsewhere: this is true because, if the global minimum was obtained on $\xi \in \mathbf{R}^n - \{0\}$, by convexity the function v would be constant on the segment $[0, \xi] = \{t\xi : t \in [0, 1]\}$.

Let us go back to the case of an arbitrary complex reductive group G. Let $g \in G(\mathbf{C})$ be such that $g \cdot x$ is a u-minimal point on the G-orbit. By Cartan's decomposition there exist elements $k \in \mathbf{U}$ and $t \in T(\mathbf{C})$ such that $g = kt$ and T is a maximal torus of G such that $T(\mathbf{C}) \cap \mathbf{U}$ is the maximal compact subgroup of T. Since the function u is $\mathbf{U}$-invariant,

$$u(g \cdot x) = u(kt \cdot x) = u(t \cdot x),$$

hence $t \cdot x$ is again a u-minimal point on the G-orbit. By the case of a torus there exists $k' \in \mathbf{U} \cap T(\mathbf{C})$ and $t' \in T_x(\mathbf{C})$ such that $t = k't'$. Thus

$$g = kt = (kk')t' \in \{kg : k \in \mathbf{U}, g \in G_x(\mathbf{C})\}$$

which concludes the proof. □

4. Metric on GIT quotients

4.1. Extended metrics. —

4.1.1. Definition. — Let X be a k-scheme of finite type and L be an invertible sheaf on it. Consider the total space of L over X,

$$\mathbf{V}(L) = \mathbf{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X} L^\vee).$$

For a k-scheme S the S-valued points of $\mathbf{V}(L)$ are in functorial bijection with the set of couples (x, s) made of an S-valued point $x \in X(S)$ and a global section $s \in \Gamma(S, x^*L)$.

Consider the k-analytic spaces X^{an} and $\mathbf{V}(L)^{\mathrm{an}}$ associated respectively to X and $\mathbf{V}(L)$.

Definition 4.1. — A map $\| \cdot \|_L \colon \mathbf{V}(L)^{\mathrm{an}} \to \mathbf{R}_+$ is an *extended metric on* L if for every analytic extension K of k, the composite map

$$\| \cdot \|_{L,K} : \mathbf{V}(L, K) \longrightarrow \mathbf{V}(L)^{\mathrm{an}} \xrightarrow{\| \cdot \|_L} \mathbf{R}_+,$$

is a norm on the fibres of L: for every K-point $x \in X(K)$ the map

$$s \in x^*L \mapsto \|s\|_L(x) := \|(x, s)\|_{L,K}$$

is a norm on the K-vector space x^*L. An extended metric is said to be continuous if it is continuous as a map on $\mathbf{V}(L)^{\mathrm{an}}$.

Remark 4.2. — Let $\| \cdot \|_L$ be an extended metric on L. For every analytic open subset $U \subset X^{\mathrm{an}}$ one can consider the function $\|s\|_L \colon U \to \mathbf{R}_+$, $x \mapsto \|s\|_L(x)$.

On the other hand, consider the data, for every analytic open subset $U \subset X^{\mathrm{an}}$ and every section $s \in \Gamma(U, L^{\mathrm{an}})$, of a function $\|s\|_{L,U} \colon U \to \mathbf{R}_+$ satisfying the following properties for all $x \in U$:

- $\|s\|_{L,U}(x) = 0$ if and only if $s(x) = 0$;
- $\|\lambda s\|_{L,U}(x) = |\lambda| \|s\|_L(x)$ for all $\lambda \in k$;
- for an open subset $V \subset U$,

$$\|s\|_{L,U_{|V}} = \|s\|_{L,V}.$$

Then the collection of maps $\{\|s\|_{L,U}\}$ defines an extended metric on L.

In the complex case, the notion of extended metric is the same of the notion of metric on a line bundle. In the real case, an extended metric is a metric on the associated complex line bundle invariant under complex conjugation.

4.1.2. Constructions. — Usual constructions on metrics (dual, tensor powers...) are available also for extended metrics. For instance the dual metric and the tensor powers of a metric are defined as follows. Let $\| \cdot \|_L$ be an extended metric on L, K an analytic extension of k, $x \in X(K)$ a K-valued point of X and $s \in x^*L$. For a section φ of $x^*L^\vee$ and a section $s \in x^*L$, set

$$\|\varphi\|_{L^\vee,K}(x) := \sup_{t \in x^*L - \{0\}} \frac{|\varphi(t)|}{\|t\|_{L,K}(x)},$$

$$\|s^{\otimes n}\|_{L^{\otimes n},K}(x) := \|s\|_{L,K}(x)^n.$$

The real number $\|\varphi\|_{L^\vee,K}(x)$ depends only on the image of (x,φ) in $\mathbf{V}(L^\vee)^{\mathrm{an}}$: the so-obtained extended metric on $L^\vee$ is called the dual metric. Analogously one gets an extended metric on $L^{\otimes n}$ called the n-th tensor power of $\|\cdot\|_L$.

Let K be an analytic extension of k. If $\|\cdot\|_L$ is an extended metric on L then the function

$$\|\cdot\|_L \circ \mathrm{pr}_{\mathbf{V}(L),K/k} \colon \mathbf{V}(L)^{\mathrm{an}}_K \longrightarrow \mathbf{R}_+$$

is an extended metric on the pull-back L_K of L to $X_K := X \times_k K$.

4.1.3. Extended metric associated to integral models. — Suppose k non-archimedean and X proper. Let $\mathcal{X}$ be a proper k°-model of X, that is, a proper k°-scheme together with an isomorphism of k-schemes $\alpha\colon X \simeq \mathcal{X} \times_{k^\circ} k$. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ together with an isomorphism $\beta\colon \alpha^*(\mathcal{L}_{|\alpha(X)}) \simeq L$. The construction that follows depends on the isomorphisms α and β this dependence will not be indicated in order not to burden notation.

Define an extended metric $\|\cdot\|_{\mathcal{L}}$ in the following way. Let K be an analytic extension of k and let $x \in X(K)$ be a K-valued point of X. The ring of integers K° of K is a valuation ring and the valuative criterion properness implies that x lifts to a K°-valued point $\varepsilon_x\colon \operatorname{Spec} K^\circ \to \mathcal{X}$. The K°-module $\varepsilon_x^*\mathcal{L}$ is a K°-module free of rank 1: fix a generator s_0 of $\varepsilon_x^*\mathcal{L}$. Every section $s \in x^*L \simeq \varepsilon_x^*\mathcal{L} \otimes_{K^\circ} K$ writes as $s = \lambda s_0$ for a unique $\lambda \in K$. Set:

$$\|s\|_{\mathcal{L},K}(x) := |\lambda|_K.$$

The real number $\|s\|_{\mathcal{L},K}(x)$ only depends on the image of (x,s) in $\mathbf{V}(L)^{\mathrm{an}}$ and the induced map

$$\|\cdot\|_{\mathcal{L}}\colon \mathbf{V}(L)^{\mathrm{an}} \longrightarrow \mathbf{R}_+,$$

is a continuous extended metric called the extended metric associated to $\mathcal{L}$.

The construction of the extended metric is compatible with the operations on invertible sheaves: for instance the dual of the extended metric $\|\cdot\|_{\mathcal{L}^\vee}$ is the extended metric associated to the dual invertible sheaf $\mathcal{L}^\vee$. If K is analytic extension then the extended metric $\|\cdot\|_{\mathcal{L}} \circ \mathrm{pr}_{K/k}$ is the extended metric associated to the pull-back $\mathcal{L}_{K^\circ}$ of $\mathcal{L}$ to $\mathcal{X}_{K^\circ} = \mathcal{X} \times_{k^\circ} K^\circ$.

Suppose k trivially or discretely valued (thus its ring of integers k° is noetherian) and that L is very ample. The map

$$\theta\colon \mathbf{V}(L^\vee) = \mathbf{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X} L) \longrightarrow \hat{X} = \operatorname{Spec} \bigoplus_{d\geq 0} \Gamma(X, L^{\otimes d})$$

is surjective, proper and it induces an isomorphism of the complementary of the zero section in $\mathbf{V}(L^\vee)$ with $\hat{X} - \operatorname{Spec}\Gamma(X,\mathcal{O}_X)$. For a point $x \in \hat{X}^{\mathrm{an}}$ set

$$u_{\mathcal{L}}(x) := \sup_{f\in\Gamma(\mathcal{X},\mathcal{L})} |f(x)|.$$

For a basis $f_1,\dots,f_n$ of the k°-module $\Gamma(\mathcal{X},\mathcal{L})$,

$$u_{\mathcal{L}}(x) = \max_{i=1,\dots,n} |f_i(x)|.$$

Proposition 4.3. — *Suppose $\mathcal{L}$ generated by its global sections and L very ample. With the notations introduced above,*

$$\|\cdot\|_{\mathcal{L}^\vee} = u_{\mathcal{L}} \circ \theta.$$

In particular the function $\log\|\cdot\|_{\mathcal{L}^\vee} \colon \mathbf{V}(L^\vee)^{\mathrm{an}} \to [-\infty, +\infty[$ is a continuous, topologically proper plurisubharmonic function on $\mathbf{V}(L^\vee)^{\mathrm{an}}$.

Proof. — Let K be an analytic extension and K° its ring of integers. Let x be a K-valued point of X and ε_x the unique K°-valued point of $\mathcal{X}$ which lifts x. Since $\mathcal{L}$ is generated by its global sections, there exists a global section $f_0 \in \Gamma(\mathcal{X}, \mathcal{L})$ such that $\varepsilon_x^* f_0$ is a basis of the K°-module $\varepsilon_x^* \mathcal{L}$. Consider the generator s_0 of $\varepsilon_x^* \mathcal{L}^\vee$ defined by the condition $s_0(\varepsilon_x^* f_0) = 1$.

For $\lambda \in K$ let $s = \lambda s_0$: by definition $\|s\|_{\mathcal{L},K}(x) = |\lambda|_K$. Then,

$$|f_0(x, s)|_K = |\lambda|_K |f_0(x, s_0)|_K = |\lambda|_K = \|s\|_{\mathcal{L}^\vee,K}(x).$$

For a global section $f \in \Gamma(\mathcal{X}, \mathcal{L})$,

$$|f(x, s)|_K = |\lambda|_K |f(x, s_0)|_K \leq |\lambda|_K = \|s\|_{\mathcal{L}^\vee,K}(x),$$

since $f(x, s_0)$ belongs to k°. This concludes the proof. □

Corollary 4.4. — *If $\mathcal{L}$ is ample, then the continuous map*

$$\log\|\cdot\|_{\mathcal{L}^\vee} \colon \mathbf{V}(L^\vee)^{\mathrm{an}} \longrightarrow [-\infty, +\infty[$$

is plurisubharmonic.

4.2. Extended metric on the quotient. —

4.2.1. Definition of the extended metric. — Let X be a projective k-scheme endowed with an action of a reductive k-group G and a G-linearized ample invertible sheaf L. The graded k-algebra of G-invariants

$$A^G := \bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d})^G \subset A := \bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d}),$$

is of finite type. Denote by X^{ss} the open subset of semi-stable points. The inclusion of A^G in A induces a G-invariant morphism of k-schemes

$$\pi \colon X^{\mathrm{ss}} \longrightarrow Y := \operatorname{Proj} A^G,$$

which makes Y the categorical quotient of X^{ss} by G. Since A^G is of finite type, the k-scheme Y is projective: for every positive integer $D \geq 1$ divisible enough there exist an ample invertible sheaf M_D on Y and an isomorphism of invertible sheaves

$$\varphi_D \colon \pi^* M_D \xrightarrow{\sim} L^{\otimes D}_{|X^{\mathrm{ss}}}$$

compatible with the equivariant action of G. The isomorphism φ_D induces a surjective morphism of k-schemes

$$\pi_D \colon \mathbf{V}(L^{\otimes D}_{|X^{\mathrm{ss}}}) \longrightarrow \mathbf{V}(M_D).$$

Let $\|\cdot\|_L$ be a continuous extended metric on L and for a point $t \in \mathbf{V}(M_D)^{\mathrm{an}}$ set

$$\|t\|_{M_D} := \sup_{\pi_D(s) = t} \|s\|_{L^{\otimes D}} \in [0, +\infty],$$

where the supremum ranges on the points $s \in \mathbf{V}(L^{\otimes D}_{|X^{\mathrm{ss}}})^{\mathrm{an}}$.

Proposition 4.5. — *With the notation introduced here above, the function $\|\cdot\|_{M_D}$ is an extended metric on M_D.*

Proof. — Let K be an analytic extension of k which is algebraically closed and non-trivially valued. Let $y \in Y(K)$ be a K-point of Y and $t \in y^*M_D$ a section over y. Since $Y(K)$ is dense in Y_K^{an} and the extended metric $\|\cdot\|_L$ is continuous,

$$\|t\|_{M_D,K}(y) = \sup_{\substack{x\in X^{\mathrm{ss}}(K)\\ \pi(x)=y}} \|\pi^* t\|_{L^{\otimes D},K}(x) \in [0,+\infty].$$

If one shows that the function $\|\cdot\|_{M_D}$ does not take the value $+\infty$, it is clear from the previous formula that $\|\cdot\|_{M_D}$ is an extended metric. Up to taking a power of M_D, the line bundle M_D can be assumed to be generated by its global sections. It suffices to prove that for every global section $s \in \Gamma(Y, M_D)$ every point $y \in Y^{\mathrm{an}}$,

$$\|t\|_{M_D}(y) < +\infty.$$

The crucial point is that every global section $t \in \Gamma(Y, M_D)$ of M_D corresponds through the isomorphism φ_D to a G-invariant global section $\widetilde{t} \in \Gamma(X, L^{\otimes D})^G$ of $L^{\otimes D}$ which vanishes identically on the set of unstable points $X - X^{\mathrm{ss}}$. For every point $y \in Y^{\mathrm{an}}$,

$$\|t\|_{M_D}(y) \leq \sup_{x\in X^{\mathrm{an}}} \|\widetilde{t}\|_{L^{\otimes D}}(x)$$

and the right-hand is a real number according to the compactness of X^{an} and the continuity of $\|\cdot\|_L$. □

Theorem 4.6. — *Suppose that:*

(1) the extended metric $\|\cdot\|_L$ is invariant under a maximal compact subgroup of G;
(2) the dual extended metric $\|\cdot\|_{L^\vee} : \mathbf{V}(L^\vee)^{\mathrm{an}} \to \mathbf{R}_+$ is a plurisubharmonic function.

Then the extended metric $\|\cdot\|_{M_D}$ is continuous.

4.2.2. Passing to the affine cones. — In order to prove the theorem it is convenient to introduce some further notation. The statement is compatible with taking powers of $\mathcal{L}$ and $\mathcal{M}_D$. Suppose D such that $\mathcal{L}^{\otimes D}$ and $\mathcal{M}_D$ are very ample. Consider the following graded k-algebras of finite type:

$$A_D := \bigoplus_{d\geq 0} \Gamma(X, L^{\otimes dD}),$$
$$A_D^G := \bigoplus_{d\geq 0} \Gamma(X, M_D^{\otimes d}) = \bigoplus_{d\geq 0} \Gamma(X, L^{\otimes dD})^G.$$

The k-schemes X and Y are still identified with the homogeneous spectrum respectively of A_D and A_D^G. The inclusion of A_D^G in A_D induces a morphism of k-schemes,

$$\hat{\pi} \colon \hat{X} := \operatorname{Spec} A_D \longrightarrow \hat{Y} := \operatorname{Spec} A_D^G,$$

which makes $\hat{Y}$ the categorical quotient of $\hat{X}$ under the action of G (see [**Ses77**, Theorem 3]). The morphsim $\hat{\pi}$ also fits into the following commutative diagram:

$$\begin{tikzcd}
\mathbf{V}(L^{\vee\otimes D}_{|X^{\mathrm{ss}}}) \arrow[r, hook] \arrow[d, "\pi_D"'] & \mathbf{V}(L^{\vee\otimes D}) \arrow[r, "\theta_{L\otimes D}"] & \hat{X} \arrow[d, "\hat{\pi}"] \\
\mathbf{V}(M_D^{\vee}) \arrow[rr, "\theta_{M_D}"] & & \hat{Y}
\end{tikzcd}$$

where $\theta_{L\otimes D}$ and θ_{M_D} are the morphisms. The morphisms $\theta_{L\otimes D}$ and θ_{M_D} are surjective and proper, and they induce an open immersion outside the zero section of $\mathbf{V}(L^{\vee\otimes D})$ and $\mathbf{V}(M_D^{\vee})$. Therefore the extended metrics $\|\cdot\|_{L^{\vee\otimes D}}$ and $\|\cdot\|_{M_D^{\vee}}$ descend on functions $u_{L\otimes D}$ and u_{M_D} respectively on $\hat{X}^{\mathrm{an}}$ and $\hat{Y}^{\mathrm{an}}$. By definition of the extended metric $\|\cdot\|_{M_D}$, for every $y \in \hat{Y}^{\mathrm{an}}$,

$$u_{M_D}(y) = \inf_{\hat{\pi}(x)=y} u_{L^{\vee\otimes D}}(x) =: \hat{\pi}_{\downarrow} u_{L\otimes D}(y). \tag{4.2.1}$$

(passing to the dual metrics switches the supremum with the infimum).

Proof of Theorem 4.6. — The function $u_{L\otimes D}$ inherits all the properties of the function $\|\cdot\|_{L^{\vee\otimes D}}$: it is continuous, topologically proper, plurisubharmonic and invariant under a maximal compact subgroup of G. According to Proposition 3.13 the function u_{M_D} is continuous, hence the extended metric $\|\cdot\|_{M_D}$ is continuous too. □

4.3. Compatibility with entire models. —

4.3.1. Notation and statements. — Suppose that k is a non-archimedean complete field which is discretely or trivially valued (thus its ring of integers k° is noetherian). Let $\mathcal{G}$ be a reductive k°-group acting on a flat and projective k°-scheme $\mathcal{X}$ equipped with an ample $\mathcal{G}$-linearized invertible sheaf $\mathcal{L}$.

Here the technical hypothesis to make Seshadri's theorem work is to assume that the ring of integers k° is universally japanese.

Definition 4.7. — An integral domain A is said to be *japanese* if for every finite extension K' of its fractions field $K = \mathrm{Frac}(A)$ the integral closure of A in K' is an A-module of finite type (*i.e.* a finite A-algebra). A ring A us said to be *universally japanese* if every integral A-algebra of finite type is japanese.

For instance, the ring of integers of k is universally japanese when k is a finite extension of $\mathbf{Q}_p$ or when $k = \mathbf{F}(\!(t)\!)$ for some field $\mathbf{F}$ [**Gro64**, Corollaire 7.7.4].

Then the fundamental result of Seshadri [**Ses77**, Theorem 2] holds: the graded k°-algebra of $\mathcal{G}$-invariants

$$\mathcal{A}^{\mathcal{G}} := \bigoplus_{d\geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})^{\mathcal{G}} \subset \mathcal{A} := \bigoplus_{d\geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})$$

is of finite type. Denote by $\mathcal{X}^{\mathrm{ss}}$ the open subset of semi-stable points, by $\mathcal{Y}$ its categorical quotient and $\pi\colon \mathcal{X}^{\mathrm{ss}} \to \mathcal{Y}$ the canonical projection. For every D divisible enough let $\mathcal{M}_D$ be the ample line bundle on $\mathcal{Y}$ deduced from $\mathcal{L}^{\otimes D}$ and

$$\varphi_D : \pi^*\mathcal{M}_D \longrightarrow \mathcal{L}^{\otimes D}_{|\mathcal{X}^{\mathrm{ss}}},$$

the $\mathcal{G}$-equivariant isomorphism of invertible sheaves.

Denote with straight capital letters the k-schemes obtained as generic fibre of the k°-schemes introduced previously (for instance $\mathcal{X} \times_{k^\circ} k$ will be denoted by X). Let $\|\cdot\|_{\mathcal{L}}$ be the continuous extended metric on L associated to $\mathcal{L}$.

Definition 4.8. — With the notations introduced above, let Ω be an analytic extension of k which is algebraically closed and non-trivially valued. A semi-stable point $x \in \mathcal{X}(\Omega)$ is:

(1) *minimal* if for a non-zero section $s \in x^*\mathcal{L}^\vee$ and for every $g \in \mathcal{G}(\Omega)$, then

$$\|s\|_{\mathcal{L}^\vee,\Omega}(x) \leq \|g \cdot s\|_{\mathcal{L}^\vee,\Omega}(g \cdot x).$$

This does not depend on the chosen section s.

(2) *residually semi-stable* if the reduction [13] $\widetilde{x} \in \mathcal{X}(\widetilde{\Omega})$ of x is a semi-stable point of the $\widetilde{\Omega}$-scheme $\mathcal{X} \times_{k^\circ} \widetilde{\Omega}$ under the action of the $\widetilde{\Omega}$-reductive group $\mathcal{G} \times_{k^\circ} \widetilde{\Omega}$.

Let $x \in X^{\mathrm{an}}$ be a semi-stable point and Ω be the completion of an algebraic closure of $\hat{\kappa}(x)$. The point x is minimal (resp. residually semi-stable) if the associated Ω-point $x_\Omega \in \mathcal{X}(\Omega)$ is minimal (resp. residually semi-stable.)

Theorem 4.9. — *Suppose that k° is universally japanese. With the notations introduced above, for every semi-stable point $x \in X^{\mathrm{an}}$ the following are equivalent:*

(1) x is minimal;
(2) x is residually semi-stable.

Corollary 4.10. — *Under the hypotheses of Theorem 4.9, let Ω be an analytic extension of k which is algebraically closed and non-trivially valued. Let $x \in X(\Omega)$ be a semi-stable point.*

Then, there exists a semi-stable minimal point $x_0 \in X(\Omega)$ lying in the closure of the orbit of x and whose orbit is closed (in X^{ss}).

In the case of a projective space and k is a finite extension of $\mathbf{Q}_p$, this result was proven by Burnol [**Bur92**, Proposition 1]. We just adapt the argument of Burnol to the framework of Berkovich spaces.

This Theorem and its Corollary will be proved in the next section. As a consequence, consider the following metric on M_D:

(1) the extended metric $\|\cdot\|_{\mathcal{M}_D}$ associated to the integral model $\mathcal{M}_D$;
(2) the extended metric $\|\cdot\|_{M_D}$ defined in the previous section (see paragraph 4.2.1).

13. Since $\mathcal{X}$ is projective, by the valuative criterion of properness the point x lifts to a Ω°-valued point of $\varepsilon_x\colon \operatorname{Spec}\Omega^\circ \to \mathcal{X}$. The *reduction of* x, denoted $\widetilde{x}$, is the reduction of ε_x modulo the maximal ideal of Ω°.

Theorem 4.11. — *Suppose that k° is universally japanese. With the notation introduced above, the metrics $\|\cdot\|_{M_D}$ and $\|\cdot\|_{\mathcal{M}_D}$ coincide.*

In particular, for every analytic extension Ω of k which is algebraically closed and non trivially valued,

$$\|t\|_{\mathcal{M}_D,\Omega}(y) = \sup_{\pi(x)=y} \|\pi^* t\|_{\mathcal{L}^{\otimes D},\Omega}(x)$$

where the supremum is ranging on the semi-stable Ω-points of X.

4.3.2. Some more notations. — Suppose that the integer D be such that the invertible sheaves $\mathcal{L}^{\otimes D}$ and $\mathcal{M}_D$ are very ample. Borrow the notations from paragraph 4.2.2. According to (4.2.1), for every $y \in \hat{Y}^{\mathrm{an}}$,

$$u_{M_D}(y) = \hat{\pi}_\downarrow u_{L^{\otimes D}}(y) := \inf_{\hat{\pi}(x)=y} u_{L^{\otimes D}}(x).$$

Consider the real-valued functions $u_{\mathcal{L}^{\otimes D}}$, $u_{\mathcal{M}_D}$ defined respectively for $x \in \hat{X}^{\mathrm{an}}$ and $y \in \hat{Y}^{\mathrm{an}}$ by

$$u_{\mathcal{L}^{\otimes D}}(x) = \sup_{f \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})} |f(x)|,$$
$$u_{\mathcal{M}_D}(y) = \sup_{g \in \Gamma(\mathcal{Y}, \mathcal{M}_D)} |g(y)|.$$

Since $\mathcal{L}^{\otimes D}$ and $\mathcal{M}_D$ are supposed very ample, according to Proposition 4.3, one has $\|\cdot\|_{\mathcal{L}^{\otimes D}} = u_{\mathcal{L}^{\otimes D}} \circ \theta_{L^{\otimes D}}$ and $\|\cdot\|_{\mathcal{M}_D} = u_{\mathcal{M}_D} \circ \theta_{M_D}$. The identification

$$\Gamma(\mathcal{Y}, \mathcal{M}_D) \simeq \Gamma(\mathcal{X}, \mathcal{L})^{\mathcal{G}},$$

yields an inclusion $\Gamma(\mathcal{Y}, \mathcal{M}_D) \subset \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})$. Thus, for $x \in \hat{X}^{\mathrm{an}}$,

$$u_{\mathcal{M}_D}(\hat{\pi}(x)) \le u_{\mathcal{L}^{\otimes D}}(x). \tag{4.3.1}$$

Lemma 4.12. — *With the notations introduced above, let $x \in \hat{X}^{\mathrm{an}}$ be a point that does not belong to the analytification of vertex $\mathbf{O}_X$ of the affine cone $\hat{X}$,*

$$\mathbf{O}_X = \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \subset \hat{X} = \operatorname{Spec} \bigoplus_{d \ge 0} \Gamma(X, L^{\otimes dD}).$$

Let $[x]$ be the associated point of X^{an}. The following are equivalent:

(1) the point $[x]$ is residually semi-stable;
(2) $u_{\mathcal{L}^{\otimes D}}(x) = u_{\mathcal{M}_D}(\hat{\pi}(x))$.

Proof. — Up to rescaling x one may assume $u_{\mathcal{L}^{\otimes D}}(x) = 1$. Let Ω be the completion of an algebraic closure of $\hat{\kappa}(x)$ and let $\varepsilon_x \colon \operatorname{Spec} \Omega^\circ \to \mathcal{X}$ be the morphism associated to the point $[x]$ by the valuative criterion of properness.

(1) $\Rightarrow$ (2) Since $\mathcal{M}_D$ is supposed very ample there exists a $\mathcal{G}$-invariant global section $f \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})$ such that $\varepsilon_x^* f$ is a basis of the invertible Ω°-module $\varepsilon_x^* \mathcal{L}^{\otimes D}$. In other words, the element $f(x) \in \Omega^\circ$ is a unit. This gives

$$u_{\mathcal{M}_D}(\hat{\pi}(x)) = 1 = u_{\mathcal{L}^{\otimes D}}(x).$$

(2) $\Rightarrow$ (1) The equality $u_{\mathcal{M}_D}(\hat{\pi}(x)) = 1$ implies there exists a $\mathcal{G}$-invariant global section $f \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})$ such that $f(x) \in \Omega^\circ$ is a unit, thus its reduction in $\widetilde{\Omega}$ is non-zero. In particular the reduction $\widetilde{x}$ of x is semi-stable. □

Proof of Theorem 4.11. — Since the construction of the extended metric $\|\cdot\|_{\mathcal{L}}$, $\|\cdot\|_{\mathcal{M}_D}$ and $\|\cdot\|_{M_D}$ are compatible with taking powers of $\mathcal{L}$ and $\mathcal{M}_D$, one may the line bundles $\mathcal{L}^{\otimes D}$ and $\mathcal{M}_D$ very ample.

The equality of metrics $\|\cdot\|_{M_D} = \|\cdot\|_{\mathcal{M}_D}$ is equivalent to the equality of functions $u_{\mathcal{M}_D} = u_{M_D}$. For all $y \in \hat{Y}^{\mathrm{an}}$, the inequality (4.3.1) entails

$$u_{\mathcal{M}_D}(y) \le u_{M_D}(y) := \inf_{\hat{\pi}(x)=y} u_{\mathcal{L}^{\otimes D}}(x).$$

It remains to prove the converse inequality. Let $y \in \hat{Y}^{\mathrm{an}}$ be a point. Since the function $u_{\mathcal{L}^{\otimes D}}$ on $\hat{X}^{\mathrm{an}}$ is topologically proper, it attains a mininum on a point x in the fibre $\hat{\pi}^{-1}(y)$. According to Theorem 4.9 the projection $[x]$ of the point x in X^{an} is residually semi-stable. Lemma 4.12 (2) implies

$$u_{\mathcal{L}^{\otimes D}}(x) = u_{\mathcal{M}_D}(\hat{\pi}(x)) = u_{\mathcal{M}_D}(y).$$

In particular,

$$u_{\mathcal{M}_D}(y) \ge \inf_{\hat{\pi}(x')=y} u_{\mathcal{L}^{\otimes D}}(x'). \qquad \square$$

Proof of Theorem 4.9. — The implication (1) $\Rightarrow$ (2) follows from inequality (4.3.1) and Lemma 4.12 (2).

(2) $\Rightarrow$ (1) Denote by $\hat{\mathcal{X}}$ the affine cone over the projective k°-scheme $\mathcal{X}$ with the respect to the very ample invertible sheaf $\mathcal{L}^{\otimes D}$, that is, the spectrum of the graded k°-algebra

$$\mathcal{A}_D := \bigoplus_{d \ge 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes dD}).$$

Up to rescaling the point x suppose $u_{\mathcal{L}^{\otimes D}}(x) = 1$. This is equivalent to say that x comes from a k°-valued point of $\hat{\mathcal{X}}$ whose reduction $\widetilde{x} \in \hat{\mathcal{X}}(\widetilde{k})$ does not belong to the vertex $\mathbf{O}_{\mathcal{X}} = \operatorname{Spec}\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of $\hat{\mathcal{X}}$.

Arguing by contradiction suppose that the point $[x]$ is not residually semi-stable. This means that its reduction $[\widetilde{x}]$ is not a semi-stable point of the $\widetilde{K}$-scheme $\mathcal{X} \times_{k^\circ} \widetilde{K}$. Applying the Hilbert-Mumford criterion of semi-stability to the point $[\widetilde{x}]$, there exist a finite extension Ω of K and a one-parameter subgroup

$$\widetilde{\lambda} \colon \mathbf{G}_{m,\widetilde{\Omega}} \longrightarrow \mathcal{G} \times_{k^\circ} \widetilde{\Omega},$$

that destabilizes the point $[\widetilde{x}]$: in other words, if $\mathbf{O}_{\mathcal{X}} = \operatorname{Spec}\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ denotes the vertex of the affine cone $\hat{\mathcal{X}}$,

$$\lim_{t \to 0} \widetilde{\lambda}(t) \cdot \widetilde{x} \in \mathbf{O}_{\mathcal{X}} \times_{k^\circ} \widetilde{\Omega}.$$

According to [**GP11**, Exp. XI, Théorème 4.1] the k°-scheme that parametrizes the subgroups of multiplicative type of the k°-group scheme $\mathcal{G}$ is smooth over k°. Since the valuation ring Ω° is henselian, by "Hensel's Lemma" [**GP11**, Exp. XI, Corollaire 1.11] the one-parameter subgroup $\widetilde{\lambda}$ lifts to a one-parameter subgroup

$$\lambda \colon \mathcal{T} \longrightarrow \mathcal{G} \times_{k^\circ} \Omega^\circ,$$

where $\mathcal{T}$ is a subgroup of multiplicative type (necessarily a torus). Thus, up to replacing Ω by a finite extension, one may assume that the torus $\mathcal{T}$ is the multiplicative group $\mathbf{G}_{m,\Omega^\circ}$. The associated morphism of Ω-analytic spaces $\lambda \colon \mathbf{G}_{m,\Omega}^{\mathrm{an}} \to G_\Omega^{\mathrm{an}}$ sends the subgroup $\mathbf{U}(1)$ into the maximal compact subgroup of G_Ω^{an} associated to the Ω°-reductive group $\mathcal{G} \times_{k^\circ} \Omega^\circ$. Consider the map $\varphi_x \colon \mathbf{G}_{m,\Omega}^{\mathrm{an}} \to \mathbf{R}_+$ defined by

$$\varphi_x(t) := u_{\mathcal{L}^{\otimes D}}(\lambda(t) \cdot x).$$

The function φ_x is continuous and invariant under the action of the subgroup $\mathbf{U}(1)$. Define the function $\psi_x \colon \mathbf{R} \to \mathbf{R}$ by the condition: for every point $t \in \mathbf{G}_{m,\Omega}^{\mathrm{an}}$,

$$\psi_x(\log|t|) = \log \varphi_x(t).$$

The function ψ_x is continuous and, since the point x is supposed $u_{\mathcal{L}^{\otimes D}}$-minimal on the G-orbit, it has a global minimum on 0:

$$\psi_x(0) = \log u_{\mathcal{L}^{\otimes D}}(x).$$

Since $u_{\mathcal{L}^{\otimes D}}(x) = 1$ one has $\psi_x(0) = 0$. To conclude the proof it suffices to prove that the function ψ_x takes negative values, contradicting the minimality of the point x.

The group $\mathbf{G}_{m,\Omega^\circ}$ acts linearly on the Ω°-module $\mathcal{E} := \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D}) \otimes \Omega^\circ$ through the one-parameter subgroup λ. Thus $\mathcal{E}$ may be decomposed in its isotypical components,

$$\mathcal{E} = \bigoplus_{m \in \mathbf{Z}} \mathcal{E}_m,$$

where, for every integer $m \in \mathbf{Z}$, $\mathcal{E}_m = \{f \in \mathcal{E} : \lambda(t) \cdot f = t^m f\}$. For $m \in \mathbf{Z}$ set

$$u_m(x) = \sup_{f \in \mathcal{E}_m} |f(x)|.$$

The preceding decomposition gives, for every point $t \in \mathbf{G}_{m,\Omega}^{\mathrm{an}}$,

$$\varphi_x(t) = u_{\mathcal{L}^{\otimes D}}(\lambda(t) \cdot x) = \sup_{m \in \mathbf{Z}} \left\{ |t|^m u_m(x) \right\}.$$

Taking the logarithm of the last expression and writing $\xi = \log|t|$,

$$\psi_x(\xi) = \sup_{\substack{m \in \mathbf{Z} \\ u_m(x) \neq 0}} \left\{ m\xi + \log u_m(x) \right\}.$$

By assumption $u_{\mathcal{L}^{\otimes D}}(x) = 1$ for every integer m, thus $\log u_m(x) \leq 0$. Furthermore, for every negative integer $m \leq 0$, one has $\log u_m(x) < 0$ because the special fibre $\widetilde{\lambda}$ of λ destabilises the point $\widetilde{x}$. Summing up these considerations for every negative real number $\xi < 0$:

- if $m > 0$ then $m\xi + \log u_m(x) \leq m\xi < 0$;
- if $m = 0$ then $m\xi + \log u_m(x) = \log u_m(x) < 0$;
- if $m < 0$ then $m\xi + \log u_m(x) < 0$ if and only if $\xi > -\log u_m(x)/m$ because $-\log u_m(x)/m$ is negative.

Therefore $\psi_x(\xi)$ is negative for every real number ξ belonging to the interval

$$\left]\max_{m<0}\left\{-\frac{\log u_m(x)}{m}\right\},0\right[.$$

This conclude the proof of Theorem 4.9 thus of Theorem 1.17. □

Proof of Corollary 4.10. — Up to extending the scalars, suppose $k=\Omega$ and k algebraically closed and non-trivially valued. Moreover, the orbit of x may be supposed closed in X^{ss}. Consider the Zariski scheme-theoretic closure $\mathcal{Z}$ of $G\cdot x$ in $\mathcal{X}$, which is a flat scheme over k° and the structural morphism $\mathcal{Z}\to\operatorname{Spec}k^\circ$ is surjective.

The closed subscheme $\mathcal{Z}$ is stable under the action of $\mathcal{G}$. Indeed, it coincides with the scheme-theoretic closure of the image of the morphism

$$\mathcal{G}\overset{(\mathrm{id},\varepsilon_x)}{\longrightarrow}\mathcal{G}\times_{k^\circ}\mathcal{X}\overset{\sigma}{\longrightarrow}\mathcal{X},$$

where $\sigma\colon\mathcal{G}\times_{k^\circ}\mathcal{X}\to\mathcal{X}$ is the morphism defining the action of $\mathcal{G}$ on $\mathcal{X}$ and $\varepsilon_x\colon\operatorname{Spec}\Omega^\circ\to\mathcal{X}$ is the morphism induced by x given by the valuative criterion of properness. The intersection $\mathcal{X}^{\mathrm{ss}}\cap\mathcal{Z}$ is an open subset of $\mathcal{Z}$ hence a flat scheme over k°.

Claim. — *The structural morphism $\alpha\colon\mathcal{X}^{\mathrm{ss}}\cap\mathcal{Z}\to\operatorname{Spec}k^\circ$ is surjective.*

Proof of the Claim. — Take a representative $\hat{x}\in\hat{X}(k)$ of x (that does not belong to the vertex $\mathbf{O}_X$ of $\hat{X}$). Since the function $u_{\mathcal{L}\otimes D}$ is topologically proper and the orbit of $\hat{x}$ is closed, the function $u_{\mathcal{L}\otimes D}$ attains its minimum on a point $\hat{y}\in G^{\mathrm{an}}\cdot\hat{x}$ (whose completed residue field can *a priori* be a huge analytic extension of k). The image $y\in X^{\mathrm{an}}$ of $\hat{y}$ is therefore a minimal point in the sense of Definition 4.8, thus, according to Theorem 4.9, residually semi-stable. In other words, the morphism $\varepsilon_y\colon\operatorname{Spec}\hat{\kappa}(y)^\circ\to\mathcal{X}$ given by the valuative criterion of properness factors through $\mathcal{X}^{\mathrm{ss}}\cap\mathcal{Z}$,

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{ss}}\cap\mathcal{Z} \arrow[r, hook] & \mathcal{X} \arrow[d] \\
\operatorname{Spec}\hat{\kappa}(y)^\circ \arrow[u, "\varepsilon_y"] \arrow[r] & \operatorname{Spec}k^\circ
\end{tikzcd}$$

In particular the morphism $\alpha\colon\mathcal{X}^{\mathrm{ss}}\cap\mathcal{Z}\to\operatorname{Spec}k^\circ$ is surjective. □

Since the morphism α is flat and surjective, it admits a section (recall that k is algebraically closed). This section gives the residually semi-stable (thus minimal according to Theorem 4.9) point that one was looking for. □

The fact that α admits a section can be found in [**Gro67**, 17.6.2 and 18.5.11 (c')] or, in a more elementary way, proved as follows:

Lemma 4.13. — *Let k be a complete non-archimedean field, which is non-trivially valued and algebraically closed. Let $\mathcal{S}$ be a flat scheme of finite type over k°.*

If the structural morphism $\varpi\colon\mathcal{S}\to\operatorname{Spec}k^\circ$ is surjective, then there exists a section $s\colon\operatorname{Spec}k^\circ\to\mathcal{S}$ of ϖ.

Proof. — It suffices to show when $\mathcal{S}$ is affine, that is $\mathcal{S} = \operatorname{Spec}\mathcal{A}$ where $\mathcal{A}$ is a flat k°-algebra of finite type. Consider the k-algebra of finite type $A := \mathcal{A} \otimes_{k^\circ} k$. Since $\mathcal{A}$ is torsion free, identify it with its image through the canonical homomorphism $\mathcal{A} \to A$. For every $f \in A$ set:

$$\|f\|_{\mathcal{A}} := \inf\{|\lambda| : f/\lambda \in \mathcal{A}, \lambda \in k^\times\}.$$

The surjectivity of the structural morphism $\varpi \colon \mathcal{S} \to \operatorname{Spec} k^\circ$ translates into the fact that $\|\cdot\|_{\mathcal{A}}$ is not identically zero on A. Thus $\|\cdot\|_{\mathcal{A}}$ is a sub-multiplicative semi-norm on A. Let $\hat{A}$ be the completion of A with the respect to $\|\cdot\|_{\mathcal{A}}$.

Let S be the generic fibre of $\mathcal{S}$ and let S^{an} be its analytification. Then the *spectrum* of the Banach k-algebra $\hat{A}$ (see [**Ber90**, §1.2]) is given by

$$\mathcal{M}(\hat{A}) := \{s \in S^{\mathrm{an}} : |f(s)| \leq \|f\|_{\mathcal{A}} \text{ for all } f \in A\}.$$

Since $\hat{A}$ is not reduced to 0, according to [**Ber90**, Theorem 1.2.1], the topological space $\mathcal{M}(\hat{A})$ is non-empty and compact. Moreover, the Banach k-algebra $\hat{A}$ is strictly affinoid in the sense of Berkovich (see [**RTW10**, 1.2.4]): since k is algebraically closed, the k-points $\mathcal{M}(\hat{A}) \cap S(k)$ are dense in $\mathcal{M}(\hat{A})$. In particular, there is at least one such a point. □

CHAPTER 4

HEIGHTS ON GIT QUOTIENTS: FURTHER RESULTS

In this chapter we prove some finer results concerning the height on GIT quotients, namely the Fundamental Formula (that here is an identity, not an inequality as in Chaapter 1), a compatibility with twists by principal bundles and a lower bound refining the one presented in Chapter 1. This chapter is organised as follows.

Section 1 is an introduction to this chapter: we state here its main results, we collect some facts from Chapter 3 and we take the opportunity to deduce the Fundamental Formula (Theorem 1.5) from these.

In Section 2 we present some examples of height on the quotient. Firstly we explicitly compute it in the case of endomorphisms of a vector space: although being elementary, the proof of the lower bound in Section 4 is based on it. Secondly, in view of the of Fundamental Formula, it is interesting to compare the lowest height in the orbit of a semi-stable point and the height of the projection on the GIT quotient. We show through three examples that the situation is more complicate than what we would hope for.

In Section 3 we illustrate the compatibility of the construction of the GIT quotient with respect to the twist of the initial data by a hermitian principal bundle. From this compatibility we draw a canonical isomorphism between quotients which is the geometric reason underlying the lower bounds proved by Bost, Gasbarri and Zhang. Hopefully, this should make more explicit the geometrical content of this lower bound and its relationship with the former work of Bogomolov.

In Section 4 we end up the global part proving an explicit version of this lower bound which generalises and improves a result of Chen and the lower bound given in Chapter 1 (see Theorem 2.1). The proof here is just a reduction to the case of the endomorphisms of a vector space (which is explicitly computed in the examples). In contrast with its simplicity, the lower bound is sometimes optimal (notably for the case of products of $\mathbf{SL}_2$).

1. Statement of the main results

1.1. Notation. — Let K be a number field and $\mathfrak{o}_K$ be its ring of integers. Let $\mathcal{X}$ be a flat and projective $\mathfrak{o}_K$-scheme endowed with the action of a $\mathfrak{o}_K$-reductive group [1] $\mathcal{G}$. Suppose that $\mathcal{X}$ is equipped with a $\mathcal{G}$-linearized ample invertible sheaf $\mathcal{L}$. According to a fundamental result of Seshadri [**Ses77**, Theorem 2] the graded $\mathfrak{o}_K$-algebra of $\mathcal{G}$-invariants,

$$\mathcal{A}^{\mathcal{G}} := \bigoplus_{d\geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})^{\mathcal{G}} \subset \mathcal{A} := \bigoplus_{d\geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d}),$$

is of finite type.

Denote by $\mathcal{X}^{\mathrm{ss}}$ the open subset of semi-stable points, *i.e.* the set of points $x \in \mathcal{X}$ such that there exist an integer $d \geq 1$ and a $\mathcal{G}$-invariant global section s of $\mathcal{L}^{\otimes d}$ that does not vanish at x. The inclusion of $\mathcal{A}^{\mathcal{G}}$ in $\mathcal{A}$ induces a $\mathcal{G}$-invariant morphism of $\mathfrak{o}_K$-schemes,

$$\pi\colon \mathcal{X}^{\mathrm{ss}} \longrightarrow \mathcal{Y} := \operatorname{Proj} \mathcal{A}^{\mathcal{G}},$$

which makes $\mathcal{Y}$ the categorical quotient of $\mathcal{X}^{\mathrm{ss}}$ by $\mathcal{G}$ [**Ses77**, Theorem 4].

Since $\mathcal{A}^{\mathcal{G}}$ is of finite type, the $\mathfrak{o}_K$-scheme $\mathcal{Y}$ is projective: for every positive integer $D \geq 1$ divisible enough there exist an ample invertible sheaf $\mathcal{M}_D$ on $\mathcal{Y}$ and an isomorphism of invertible sheaves,

$$\varphi_D\colon \pi^*\mathcal{M}_D \longrightarrow \mathcal{L}^{\otimes D}_{|\mathcal{X}^{\mathrm{ss}}},$$

compatible with the equivariant action of $\mathcal{G}$.

To complete the "arakelovian" data, for every complex embedding $\sigma\colon K \to \mathbf{C}$ endow the invertible sheaf $\mathcal{L}_{|\mathcal{X}_\sigma(\mathbf{C})}$ with a continuous metric $\|\cdot\|_{\mathcal{L},\sigma}$. Suppose that the following conditions are satisfied:

- (Semi-positivity): the Kähler form of the metric $\|\cdot\|_{\mathcal{L},\sigma}$ is semi-positive (in the sense of distributions); equivalently for every analytic open subset $U \subset \mathcal{X}_\sigma(\mathbf{C})$ and every section $s \in \Gamma(U, \mathcal{L})$ the function $-\log\|s\|_{\mathcal{L},\sigma}$ is plurisubharmonic;
- (Invariance): the metric $\|\cdot\|_{\mathcal{L},\sigma}$ is invariant under the action of a maximal compact subgroup of $\mathcal{G}_\sigma(\mathbf{C})$.

Suppose that the family of metrics $\{\|\cdot\|_{\mathcal{L},\sigma}\colon \sigma\colon K \to \mathbf{C}\}$ is invariant under complex conjugation. Denote by $\overline{\mathcal{L}}$ the corresponding hermitian invertible sheaf.

Let $\sigma\colon K \to \mathbf{C}$ be a complex embedding. Define a metric on $\mathcal{M}_D$ as follows: for every point $y \in \mathcal{Y}_\sigma(\mathbf{C})$ and every section $t \in y^*\mathcal{M}_D$ set

$$\|t\|_{\mathcal{M}_D,\sigma}(y) := \sup_{\pi(x)=y} \|\pi^*t\|_{\mathcal{L}^{\otimes D},\sigma}(x).$$

Theorem 1.1 **(*cf.* Theorem 1.13 in Chapter 3).** — *Under the assumptions made on the metric $\|\cdot\|_{\mathcal{L},\sigma}$ (semi-positivity and invariance under the action of a maximal compact subgroup) the metric $\|\cdot\|_{\mathcal{M}_D,\sigma}$ is continuous.*

---

1. Let S be a scheme. A S-group scheme G is said to *reductive* (or simply a *S-reductive group*) if the following conditions are satisfied:

(1) G is affine, of finite type and smooth over S;

(2) for every geometric point $\overline{s} : \operatorname{Spec}\Omega \to S$ (where Ω is an algebraically closed field) the fibre $G_{\overline{s}} = G \times_S \overline{s}$ is a connected reductive group over Ω.

The family of metric $\{\|\cdot\|_{\mathcal{M}_D,\sigma} : \sigma\colon K \to \mathbf{C}\}$ just defined is invariant under complex conjugation.

Definition 1.2. — With the notations introduced above, denote by $\overline{\mathcal{M}}_D$ the corresponding hermitian invertible sheaf. Consider the function $h_{\overline{\mathcal{M}}}\colon \mathcal{Y}(\bar{\mathbf{Q}}) \to \mathbf{R}$ defined, for every $Q \in \mathcal{Y}(\bar{\mathbf{Q}})$, by

$$h_{\overline{\mathcal{M}}}(Q) := \frac{1}{D} h_{\overline{\mathcal{M}}_D}(Q),$$

which does not depend on D. The height $h_{\overline{\mathcal{M}}}$ is called the *height on the quotient (with respect to $\mathcal{X}$, $\overline{\mathcal{L}}$ and $\mathcal{G}$).*

1.2. Instability measure. — Let $v \in \mathrm{V}_K$ a place of K. If the place v is non archimedean denote $\|\cdot\|_{\mathcal{L},v}$ the continuous and bounded metric induced by the entire model $\mathcal{L}$.

Definition 1.3. — Let x be a $\mathbf{C}_v$ point of $\mathcal{X}$. The *(v-adic) instability measure* is

$$\iota_v(x) := -\log \sup_{g \in \mathcal{G}(\mathbf{C}_v)} \frac{\|g \cdot s\|_{\mathcal{L},v}(g \cdot x)}{\|s\|_{\mathcal{L},v}(x)} \in [-\infty, 0]$$

where $s \in x^*\mathcal{L}$ is a non-zero section. This is independent on the chosen section s.

The point x is said to be *minimal* at the place v (with the respect to the metric $\|\cdot\|_{\mathcal{L},v}$ and the action of $\mathcal{G}$) it its instability measure vanishes, $\iota_v(x) = 0$.

Proposition 1.4. — *Let x be a $\mathbf{C}_v$-point of $\mathcal{X}$. Then,*

(1) the instability measure $\iota_v(x)$ takes the value $-\infty$ if and only if the point x is not semi-stable;

(2) if v is a non-archimedean place over a prime number p, the instability measure $\iota_v(x)$ takes the value 0 if and only if the point x is residually semi-stable, that is, the reduction[2] *$\widetilde{x}$ of x is a semi-stable $\overline{\mathbf{F}}_p$-point of $\mathcal{X} \times_{\mathfrak{o}_K} \overline{\mathbf{F}}_p$ under the action of $\mathcal{G} \times_{\mathfrak{o}_K} \overline{\mathbf{F}}_p$.*

These assertions are respectively consequences of Theorem 1.9 and Theorem 4.9 in Chapter 3.

1.3. Statement and proof of the Fundamental Formula. —

Theorem 1.5 **(Fundamental Formula).** — *Let $P \in \mathcal{X}^{\mathrm{ss}}(K)$ be a semi-stable K-point. Then the instability measures $\iota_v(P)$ are almost all zero and*

$$h_{\overline{\mathcal{L}}}(P) + \frac{1}{[K : \mathbf{Q}]} \sum_{v \in \mathrm{V}_K} \iota_v(P) = h_{\overline{\mathcal{M}}}(\pi(P)).$$

2. Since $\mathcal{X}$ is projective, by the valuative criterion of properness the point x lifts to a $\bar{\mathfrak{o}}_v$-point of $\mathcal{X}$, where $\bar{\mathfrak{o}}_v$ is the ring of integers of $\mathbf{C}_v$. Taking the reduction mod p one gets a $\overline{\mathbf{F}}_p$ point $\widetilde{x}$ of $\mathcal{X}$ which called the *reduction* of x.

Proof. — Let $P \in \mathcal{X}^{\mathrm{ss}}(K)$ be a semi-stable K-point of $\mathcal{X}$ and let $t \in \pi(P)^*\mathcal{M}_D$ be a non zero section. Then,

$$
\begin{aligned}
[K:\mathbf{Q}]h_{\mathcal{M}_D}(\pi(P)) &= \sum_{v \in \mathrm{V}_K} -\log \|t\|_{\mathcal{M}_D,v}(\pi(P)) \\
&= \sum_{v \in \mathrm{V}_K} -\log \sup_{\pi(P')=\pi(P)} \|\pi^* t\|_{\mathcal{L}^{\otimes D},v}(P')
\end{aligned}
\tag{1.3.1}
$$

where in the second equality one uses the very definition of the metric $\|\cdot\|_{\mathcal{M}_D,\sigma}$ for the archimedean places and Theorem 1.17 in Chapter 3 for the non-archimedean ones. According to Theorem 1.16 in Chapter 3:

$$
\begin{aligned}
[K:\mathbf{Q}]h_{\mathcal{M}_D}(\pi(P)) &= \sum_{v \in \mathrm{V}_K} -\log \sup_{g \in \mathcal{G}(\mathbf{C}_v)} \|\pi^* t\|_{\mathcal{L}^{\otimes D},v}(g \cdot P) \\
&= \sum_{v \in \mathrm{V}_K} -\log \sup_{g \in \mathcal{G}(\mathbf{C}_v)} \frac{\|\pi^* t\|_{\mathcal{L}^{\otimes D},v}(g \cdot P)}{\|\pi^* t\|_{\mathcal{L}^{\otimes D},v}(P)}, \\
&\quad + \sum_{v \in \mathrm{V}_K} -\log \|\pi^* t\|_{\mathcal{L}^{\otimes D},v}(P) \\
&= D\left(\sum_{v \in \mathrm{V}_K} \iota_v(P) + [K:\mathbf{Q}]h_{\overline{\mathcal{L}}}(P)\right),
\end{aligned}
$$

where one uses the definition the v-adic instability measure of P (the section $\pi^* t$ is $\mathcal{G}$-invariant, thus $g \cdot \pi^* t = \pi^* t$ for every $g \in \mathcal{G}(\mathbf{C}_v)$). This concludes the proof of the Fundamental Formula. □

1.4. The case of a projective space. — Let $\overline{\mathcal{F}}$ be an hermitian vector bundle over $\mathfrak{o}_K$. Suppose that an $\mathfrak{o}_K$-reductive group $\mathcal{G}$ acts linearly on $\mathcal{F}$ and that, for every embedding $\sigma\colon K \to \mathbf{C}$, the hermitian norm $\|\cdot\|_{\mathcal{F},\sigma}$ is invariant under the action of a maximal compact subgroup $\mathbf{U}_\sigma$ of $\mathcal{G}_\sigma(\mathbf{C})$.

The $\mathfrak{o}_K$-reductive group $\mathcal{G}$ acts on the projective space $\mathcal{X} = \mathbf{P}(\mathcal{F})$ and in an equivariant way on the invertible sheaf $\mathcal{L} = \mathcal{O}(1)$. For every embedding $\sigma\colon K \to \mathbf{C}$ endow the invertible sheaf $\mathcal{O}(1)_{|\mathcal{X}_\sigma(\mathbf{C})}$ with the Fubini-Study metric $\|\cdot\|_{\mathcal{O}(1),\sigma}$ induced by the hermitian norm $\|\cdot\|_{\mathcal{F},\sigma}$. By hypothesis, the metric $\|\cdot\|_{\mathcal{O}(1),\sigma}$ under a maximal compact subgroup of $\mathcal{G}_\sigma(\mathbf{C})$ and its curvature form is positive.

Let $\overline{\mathcal{L}}$ be the so-obtained hermitian line bundle and borrow the general notation introduced in paragraph 1.1.

Let v be a place of K. Let x be a non-zero vector of $\mathcal{F} \otimes_{\mathfrak{o}_K} \mathbf{C}_v$ and $[x]$ the associated $\mathbf{C}_v$-point of $\mathcal{X}$. By definition,

$$
\iota_v([x]) = \log \inf_{g \in \mathcal{G}(\mathbf{C}_v)} \frac{\|g \cdot x\|_{\mathcal{F},v}}{\|x\|_{\mathcal{F},v}},
$$

where the norm $\|\cdot\|_{\mathcal{F},v}$ has been extended to $\mathcal{F} \otimes_{\mathfrak{o}_K} \mathbf{C}_v$ (and denote again by $\|\cdot\|_{\mathcal{F},v}$ its extension). In this framework the Fundamental Formula reads as follows:

Corollary 1.6. — *Let v be a non-zero vector in $\mathcal{F} \otimes_{\mathfrak{o}_K} K$ and let $P = [v]$ be the associated K-point of $\mathcal{X}$. If the point P is semi-stable, then:*

$$\begin{aligned} h_{\overline{\mathcal{M}}}(\pi(P)) &= h_{\mathcal{O}_{\overline{\mathcal{F}}}(1)}([v]) + \sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{G}(\mathbf{C}_v)} \frac{\|g \cdot x\|_{\mathcal{F},v}}{\|x\|_{\mathcal{F},v}} \\ &= \sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{G}(\mathbf{C}_v)} \|g \cdot x\|_{\mathcal{F},v}. \end{aligned}$$

In this case the result was obtained by Burnol [**Bur92**, Proposition 5].

Minimality at archimedean places can be expressed in terms of vanishing of the moment map. Let $\sigma \colon K \to \mathbf{C}$ be a complex embedding of K. A moment map $\mu_\sigma \colon \mathbf{P}(\mathcal{F})_\sigma(\mathbf{C}) \to (\operatorname{Lie} \mathbf{U}_\sigma)^\vee$ for the action of $\mathcal{G}_\sigma(\mathbf{C})$ on $\mathbf{P}(\mathcal{F})_\sigma(\mathbf{C})$ is defined as follows. For a non-zero vector $x \in \mathcal{F} \otimes_\sigma \mathbf{C}$, consider the linear map $\mu_\sigma(x)$ defined by, for $a \in \operatorname{Lie} \mathbf{U}_\sigma$,

$$\mu_\sigma(x).a := \frac{1}{2\pi \boldsymbol{i}} \cdot \frac{\langle \operatorname{ad}(a, x), x \rangle_\sigma}{\|x\|_\sigma^2},$$

where $\langle -, - \rangle_\sigma$ is the hermitian form associated to the hermitian norm $\| \cdot \|_{\mathcal{F},\sigma}$, $\boldsymbol{i}$ is a (fixed) square root of -1 and $\operatorname{ad} \colon \operatorname{Lie} \mathbf{U}_\sigma \times \mathcal{F} \otimes_\sigma \mathbf{C} \to \mathcal{F} \otimes_\sigma \mathbf{C}$ denotes the adjoint action.

Proposition 1.7. — *With the notations introduced above, the point $[x] \in \mathbf{P}(\mathcal{F})_\sigma(\mathbf{C})$ is minimal if and only if the linear map $\mu_\sigma(x)$ is identically zero.*

For a proof the reader can refer to the Proposition 1.2 in Chapter 3 and the references cited therein.

1.5. Lowest height on the quotient. — Since the metric $\| \cdot \|_{\mathcal{M}_D,\sigma}$ is continuous and the invertible sheaf $\mathcal{M}_D$ is ample the height on $\mathcal{Y}$ is uniformly bounded below. Set

$$h_{\min}((\mathcal{X}, \overline{\mathcal{L}}) /\!\!/ \mathcal{G}) := \inf_{Q \in \mathcal{Y}(\bar{\mathbf{Q}})} h_{\overline{\mathcal{M}}}(Q).$$

By definition of $h_{\min}((\mathcal{X}, \overline{\mathcal{L}}) /\!\!/ \mathcal{G})$ one has the following immediate Corollary of the Fundamental Formula which is relevant for the applications.

Corollary 1.8. — *Let $P \in \mathcal{X}^{\mathrm{ss}}(K)$ be a semi-stable K-point. Then the instability measures $\iota_v(P)$ are almost all zero and*

$$h_{\overline{\mathcal{L}}}(P) + \frac{1}{[K : \mathbf{Q}]} \sum_{v \in \mathrm{V}_K} \iota_v(P) \geq h_{\min}((\mathcal{X}, \overline{\mathcal{L}}) /\!\!/ \mathcal{G}).$$

For applications is sometimes important to have an explicit lower bound of the height on the quotient.

1.6. The lower bound of Bost, Gasbarri and Zhang. — Let $N \geq 1$ be a positive integer and let $e_1, \ldots, e_N$ be positive integers. Consider the $\mathfrak{o}_K$-reductive groups

$$\mathcal{G} = \mathbf{GL}_{e_1, \mathfrak{o}_K} \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}_{e_N, \mathfrak{o}_K},$$
$$\mathcal{S} = \mathbf{SL}_{e_1, \mathfrak{o}_K} \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}_{e_N, \mathfrak{o}_K},$$

and for every embedding $\sigma \colon K \to \mathbf{C}$ consider their maximal compact subgroups

$$\mathbf{U}_\sigma = \mathbf{U}(e_1) \times \cdots \times \mathbf{U}(e_N) \subset \mathcal{G}_\sigma(\mathbf{C})$$
$$\mathbf{SU}_\sigma = \mathbf{SU}(e_1) \times \cdots \times \mathbf{SU}(e_N) \subset \mathcal{S}_\sigma(\mathbf{C}).$$

Let $\overline{\mathcal{F}}$ be a hermitian vector bundle over $\mathfrak{o}_K$ and let $\rho \colon \mathcal{G} \to \mathbf{GL}(\mathcal{F})$ be a representation, that is a morphism of $\mathfrak{o}_K$-group schemes, which respects the hermitian structure: this means that for every embedding $\sigma \colon K \to \mathbf{C}$ the norm $\| \cdot \|_{\mathcal{F}, \sigma}$ is fixed under the action of the maximal compact subgroup $\mathbf{U}_\sigma$. Borrow the notation from paragraph 1.4.

Let $\overline{\mathcal{E}} = (\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_N)$ be a N-uple of hermitian vector bundles over $\mathfrak{o}_K$ such that $\operatorname{rk} \mathcal{E}_i = e_i$ for all $i = 1, \ldots, N$. To this data one can associate a hermitian vector bundle $\overline{\mathcal{F}}_{\overline{\mathcal{E}}}$ obtained from $\mathcal{F}$ by "twisting" it by $\overline{\mathcal{E}}$ (see Section 3.1 for the precise definition). The hermitian vector bundle $\overline{\mathcal{F}}_{\overline{\mathcal{E}}}$ comes endowed with a representation,

$$\rho_{\mathcal{E}} \colon \mathcal{G}_{\mathcal{E}} = \mathbf{GL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}(\mathcal{E}_N) \longrightarrow \mathbf{GL}(\mathcal{F}_{\mathcal{E}}),$$

that respects the hermitian structures. Consider:

$$\begin{aligned}
&\mathcal{S}_{\mathcal{E}} = \mathbf{SL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}(\mathcal{E}_N),\\
&\mathcal{X}_{\mathcal{E}} = \mathbf{P}(\mathcal{F}_{\mathcal{E}}),\\
&\overline{\mathcal{L}}_{\overline{\mathcal{E}}} = \mathcal{O}_{\mathcal{F}_{\mathcal{E}}}(1) \text{ endowed with the Fubini-Study metric induced by } \overline{\mathcal{F}}_{\overline{\mathcal{E}}},\\
&\mathcal{Y}_{\mathcal{E}} = \text{categorical quotient of } \mathcal{X}_{\mathcal{E}}^{\mathrm{ss}} \text{ with respect to } \mathcal{S}_{\mathcal{E}} \text{ and } \mathcal{L}_{\mathcal{E}}.
\end{aligned}$$

The representation ρ is said to be *homogeneous of weight* $a = (a_1, \ldots, a_N) \in \mathbf{Z}^N$ if, for every $\mathfrak{o}_K$-scheme T and for every $t_1, \ldots, t_N \in \mathbf{G}_m(T)$,

$$\rho(t_1 \cdot \mathrm{id}_{\mathcal{E}_1}, \ldots, t_N \cdot \mathrm{id}_{\mathcal{E}_N}) = t_1^{a_1} \cdots t_N^{a_N} \cdot \mathrm{id}_{\mathcal{F}}.$$

Theorem 1.9 **(*cf.* Theorem 3.8)**. — *With the notations introduced above, if the representation ρ is homogeneous of weight $a = (a_1, \ldots, a_N) \in \mathbf{Z}^N$ and the subset of semi-stable points $\mathcal{X}^{\mathrm{ss}}$ is not empty, then:*

(1) there exists an isomorphism $\alpha_{\mathcal{E}} \colon \mathcal{Y}_{\mathcal{E}} \to \mathcal{Y}$;

(2) for every $D \geq 0$ divisible enough there exists a canonical isomorphism of hermitian line bundles, that is an isometric isomorphism of line bundles,

$$\beta_{\overline{\mathcal{E}}} \colon \overline{\mathcal{M}}_{D, \overline{\mathcal{E}}} \longrightarrow \alpha_{\mathcal{E}}^* \overline{\mathcal{M}}_D \otimes \bigotimes_{i=1}^N f_{\mathcal{E}}^* (\det \overline{\mathcal{E}}_i)^{\vee \otimes a_i D / e_i},$$

where $f_{\mathcal{E}} \colon \mathcal{Y}_{\mathcal{E}} \to \operatorname{Spec} \mathfrak{o}_K$ is the structural morphism;

(3) $h_{\min}((\mathcal{X}_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}}) /\!\!/ \mathcal{S}_{\mathcal{E}}) = h_{\min}((\mathcal{X}, \mathcal{L}) /\!\!/ \mathcal{S}) - \sum_{i=1}^N a_i \, \hat{\mu}(\mathcal{E}_i).$

Corollary 1.10. — *With the notation of Theorem 3.8, for every K-point P of $\mathcal{X}_{\mathcal{E}}$ which is semi-stable under the action of $\mathcal{S}_{\mathcal{E}}$:*

$$h_{\overline{\mathcal{L}}_{\overline{\mathcal{E}}}}(P) \geq -\sum_{i=1}^{N} a_i\, \hat{\mu}(\mathcal{E}_i) + h_{\min}((\mathcal{X}, \mathcal{L})/\!\!/\mathcal{S}).$$

For $N = 1$ this is the original statement of Gasbarri [**Gas00**, Theorem 1] which in turn was generalisation of results of Bost [**Bos94**, Proposition 2.1] and Zhang [**Zha96a**, Proposition 4.2].

1.7. An explicit lower bound. — In practice, it is useful to have an explicit lower bound of the height on the quotient. Let $N \geq 1$ be a positive integer and let $\overline{\mathcal{E}} = (\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_N)$ be a N-uple of hermitian vector bundles over $\mathfrak{o}_K$ of positive rank. Consider the following $\mathfrak{o}_K$-reductive groups

$$\mathcal{G} = \mathbf{GL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}(\mathcal{E}_N),$$
$$\mathcal{S} = \mathbf{SL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}(\mathcal{E}_N),$$

and for every complex embedding $\sigma\colon K \to \mathbf{C}$ consider the maximal compact subgroups,

$$\mathbf{U}_\sigma = \mathbf{U}(\|\cdot\|_{\mathcal{E}_1,\sigma}) \times \cdots \times \mathbf{U}(\|\cdot\|_{\mathcal{E}_N,\sigma}) \subset \mathcal{G}_\sigma(\mathbf{C}),$$
$$\mathbf{SU}_\sigma = \mathbf{SU}(\|\cdot\|_{\mathcal{E}_1,\sigma}) \times \cdots \times \mathbf{SU}(\|\cdot\|_{\mathcal{E}_N,\sigma}) \subset \mathcal{S}_\sigma(\mathbf{C}).$$

Let $\overline{\mathcal{F}}$ be a hermitian vector bundle over $\mathfrak{o}_K$ and let $\rho\colon \mathcal{G} \to \mathbf{GL}(\mathcal{F})$ be a representation which respects the hermitian structures, that is, for every embedding $\sigma\colon K \to \mathbf{C}$ the norm $\|\cdot\|_{\mathcal{F},\sigma}$ is fixed under the action of the maximal compact subgroup $\mathbf{U}_\sigma$. Consider the induced action of $\mathcal{S}$ on $\mathcal{F}$. Borrow the notation introduced in paragraph 1.4. For an integer $n \geq 1$ write

$$\ell(n) := \frac{\log n!}{n} = \sum_{i=1}^{n} \frac{\log i}{n}.$$

Then $\ell(n) \leq \log n$ and, by Stirling's approximation, $\ell(n) \sim \log n$ as $n \to \infty$.

Theorem 1.11 **(*cf.* Theorem 4.1).** — *With the notations introduced above, let*

$$\varphi\colon \bigotimes_{i=1}^{N} \Big[\mathrm{End}(\overline{\mathcal{E}}_i)^{\otimes a_i} \otimes_{\mathfrak{o}_K} \overline{\mathcal{E}}_i^{\otimes b_i}\Big] \longrightarrow \overline{\mathcal{F}}$$

be a $\mathcal{G}$-equivariant and generically surjective homomorphism of hermitian vector bundles. Then,

$$h_{\min}((\mathbf{P}(\mathcal{F}), \mathcal{O}_{\overline{\mathcal{F}}}(1))/\!\!/\mathcal{S}) \geq -\sum_{i=1}^{N} b_i\, \hat{\mu}(\overline{\mathcal{E}}_i) - \sum_{i:\mathrm{rk}\,\mathcal{E}_i \geq 3} \frac{|b_i|}{2} \ell(\mathrm{rk}\,\mathcal{E}_i)$$

with equality if $b_1, \ldots, b_N = 0$.

Actually, one would hope for a better lower bound:

Conjecture 1.12. — *Under the same hypotheses of Theorem 1.11,*

$$h_{\min}((\mathbf{P}(\mathcal{F}), \mathcal{O}_{\overline{\mathcal{F}}}(1)) /\!\!/ \mathcal{S}) \geq -\sum_{i=1}^{N} b_i \, \hat{\mu}(\overline{\mathcal{E}}_i).$$

The error terms appearing in Theorem 1.11 are linked to the error terms involved in the upper bound of the maximal slope of the tensor product of hermitian vector bundles over $\mathfrak{o}_K$. The interested reader can refer to [**Che09**] and [**BC13**].

2. Examples of height on the quotient

2.1. Endomorphisms of a vector space. —

2.1.1. Semi-stable endomorphisms. — Let k be an algebraically closed field and E be a k-vector space of (finite) dimension n. Consider the action by conjugation of the reductive k-group $\mathbf{GL}(E)$ on the affine k-scheme

$$X := \mathbf{End}(E) = \operatorname{Spec}(A),$$

where $A = \operatorname{Sym}_k(\operatorname{End}(E)^\vee)$. For every endomorphism $\varphi \colon E \to E$ denote by $P_\varphi(T)$ its characteristic polynomial,

$$P_\varphi(T) := \det(T \cdot \mathrm{id}_E - \varphi) = T^n - \sigma_1(\varphi) T^{n-1} + \cdots + (-1)^n \sigma_n(\varphi).$$

The coefficients $\sigma_1(\varphi), \ldots, \sigma_n(\varphi)$ are polynomials in the coefficients of φ, *i.e.* elements of A, which are invariant under the action of $\mathbf{SL}(E)$.

Proposition 2.1 ([**MS72**, Proposition 2]). — *The affine space $\mathbf{A}_k^n$ together with the map*

$$\begin{array}{rccl} \pi \colon & \mathbf{End}(E) & \longrightarrow & \mathbf{A}_k^n \\ & \varphi & \longmapsto & (\sigma_1(\varphi), \ldots, \sigma_n(\varphi)), \end{array}$$

is the categorical quotient of X by $\mathbf{SL}(E)$.

In particular, the invariants $\sigma_1, \ldots, \sigma_n$ generate the k-algebra of invariants $A^{\mathbf{SL}(E)}$.

Proposition 2.2 ([**MS72**, Proposition 4]). — *Let $\varphi \colon E \to E$ be a linear map. Then:*

1. *the orbit of φ is closed if and only if φ is semi-simple (i.e. it can be diagonalized);*
2. *the closure $\overline{G \cdot \varphi}$ of the orbit of φ contains the orbit of the semi-simple part φ_{ss} of φ.*

Corollary 2.3. — *For every non-zero endomorphism $\varphi \colon E \to E$ the associated k-point $[\varphi] \in \mathbf{P}(\operatorname{End}(E))$ is semi-stable if and only if φ is not nilpotent.*

2.1.2. Arithmetic situation. — Let K be a number field and $\mathfrak{o}_K$ be its ring of integers. Let $\overline{\mathcal{E}}$ be a hermitian vector bundle on $\mathfrak{o}_K$.

Consider the action by conjugation of $\mathcal{S} = \mathbf{SL}(\mathcal{E})$ on $\mathrm{End}(\mathcal{E})$. Endow the $\mathfrak{o}_K$-module $\mathrm{End}(\mathcal{E})$ with the norms on endomorphism (see paragraph 0.3 on page 11). The norm $\|\cdot\|_{\mathrm{End}(\mathcal{E}),\sigma}$ is invariant under the action of the special unitary subgroup $\mathbf{SU}(\|\cdot\|_{\mathcal{E},\sigma})$ of $\mathcal{G}_\sigma(\mathbf{C})$. Borrow notation from paragraph 1.4 (with $\mathcal{F} = \mathrm{End}(\mathcal{E})$ and $\mathcal{G} = \mathcal{S}$).

Theorem 2.4. — *With the notations introduced above, let φ be an endomorphism of the K-vector space $\mathcal{E} \otimes_{\mathfrak{o}_K} K$.*

Suppose that the corresponding K-point $[\varphi]$ of $\mathbf{P}(\mathrm{End}(\mathcal{E}))$ is semi-stable (that is, the endomorphism φ is not nilpotent). Then, $[\Omega : \mathbf{Q}]h_{\overline{\mathcal{M}}}(\pi([\varphi]))$ is equal to

$$\sum_{\substack{v\in \mathrm{V}_\Omega\\ \text{non-arch.}}} \log\max\{|\lambda_1|_v,\dots,|\lambda_n|_v\} + \sum_{\sigma\colon \Omega\to\mathbf{C}} \log\sqrt{|\lambda_1|_\sigma^2+\cdots+|\lambda_n|_\sigma^2},$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of φ (counted with multiplicities) and Ω is a number field containing them.

The categorical quotient $\mathcal{Y}$ of the semi-stable locus of $\mathbf{P}(\mathrm{End}(\mathcal{E}))$ by $\mathbf{SL}(\mathcal{E})$ can be identified with quotient by $\mathbf{P}^{n-1}_{\mathfrak{o}_K}$ by the action of $\mathfrak{S}_n$ permuting coordinates (that is, the weighted projective space $\mathbf{P}(1,2,\dots,n)$ over $\mathfrak{o}_K$).

Consider the "standard" Arakelov height h on $\mathbf{P}^{n-1}_{\mathfrak{o}_K}$, *i.e.* the one defined, for every finite extension Ω of K and every Ω-point $\lambda = (\lambda_1 : \cdots : \lambda_n)$, by

$$\frac{1}{[\Omega:\mathbf{Q}]}\sum_{\substack{v\in \mathrm{V}_\Omega\\ \text{non-arch.}}} \log\max\{|\lambda_1|_v,\dots,|\lambda_n|_v\} + \frac{1}{[\Omega:\mathbf{Q}]}\sum_{\sigma\colon \Omega\to\mathbf{C}} \log\sqrt{|\lambda_1|_\sigma^2+\cdots+|\lambda_n|_\sigma^2}.$$

Then, the preceding theorem says that the height $h_{\overline{\mathcal{M}}}$ is obtained descending h on $\mathcal{Y}$ through the quotient $\mathbf{P}^{n-1}_{\mathfrak{o}_K} \to \mathcal{Y}$ by $\mathfrak{S}_n$.

The remainder of this section is devoted to the proof of Theorem 2.4.

2.1.3. Reduction to local statements. — Let v be a place of K and let E be a $\mathbf{C}_v$-vector space of dimension n. Let $e_1,\dots,e_n$ be a basis of E and equip E with the norm $\|\cdot\|_E$ defined by

$$\|x_1e_1+\cdots+x_ne_n\|_E := \begin{cases} \max\{|x_1|_v,\dots,|x_n|_v\} & \text{if } v \text{ is non-archimedean}\\ \sqrt{|x_1|_v^2+\cdots+|x_n|_v^2} & \text{if } v \text{ is archimedean.}\end{cases}$$

Equip the $\mathbf{C}_v$-vector space $\mathrm{End}(E)$ with the norm $\|\cdot\|_{\mathrm{End}(E)}$ defined by:

$$\|\varphi\|_{\mathrm{End}(E)} := \begin{cases} \sup\limits_{x\neq 0} \dfrac{\|\varphi(x)\|_E}{\|x\|_E} & \text{if } v \text{ is non-archimedean}\\ \sqrt{\|\varphi(e_1)\|_E^2+\cdots+\|\varphi(e_n)\|_E^2} & \text{if } v \text{ is archimedean.}\end{cases}$$

In the non-archimedean case the norm $\|\cdot\|_E$ is the one associated to $\mathfrak{o}$-submodule $\mathfrak{E} = \mathfrak{o}\cdot e_1 \oplus\cdots\oplus \mathfrak{o}\cdot e_n$ of E (where $\mathfrak{o}$ is the ring of integers of $\mathbf{C}_v$). The norm $\|\cdot\|_{\mathrm{End}(E)}$ is then associated to the $\mathfrak{o}$-submodule $\mathrm{End}(\mathfrak{E})$ of $\mathrm{End}(E)$.

In the archimedean case $\|\varphi\|_E^2 = \mathrm{Tr}(\varphi^* \circ \varphi)$ where φ^* is the adjoint endomorphism to φ with respect to the hermitian norm $\|\cdot\|_E$.

Proposition 2.5. — *With the notation introduced above, for every endomorphism φ of E,*

$$\inf_{g\in \mathbf{SL}(E,\mathbf{C}_v)} \|g\varphi g^{-1}\|_{\mathrm{End}(E)} = \begin{cases} \max\{|\lambda_1|_v, \dots, |\lambda_n|_v\} & \text{if } v \text{ is non-archimedean} \\ \sqrt{|\lambda_1|_v^2 + \cdots + |\lambda_n|_v^2} & \text{if } v \text{ is archimedean.} \end{cases}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of φ (counted with multiplicities).

Proof of Theorem 2.4. — It suffices to apply the Fundamental Formula in the form given by Corollary 1.6 and use the expression of the local terms given by Proposition 2.5. □

In order to prove Proposition 2.5, it suffices to show that the endomorphism given by the matrix (with respect to the basis $e_1, \dots, e_n$),

$$\mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

is minimal.

2.1.4. Computing minimal endomorphisms. — In this framework an endomorphism is minimal if and only if

$$\|\varphi\|_{\mathrm{End}(E)} = \inf_{g\in \mathbf{SL}(E,\mathbf{C}_v)} \|g\varphi g^{-1}\|_{\mathrm{End}(E)}.$$

Proposition 2.6. — *Let $\lambda_1, \dots, \lambda_n \in \mathbf{C}_v$. With the notations introduced above, the endomorphism $\varphi = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is minimal.*

Proof of Proposition 2.6: the non-archimedean case. — Let v be a non archimedean place. Proposition 1.4 (2) affirms that a non-zero endomorphism φ is minimal if and only if its reduction $\widetilde{\varphi}$ is a semi-stable $\overline{\mathbf{F}}_v$-point of $\mathbf{P}(\mathrm{End}(\mathfrak{E} \otimes_{\mathfrak{o}} \overline{\mathbf{F}}_p))$.

Let $\lambda_1, \dots, \lambda_n$ be elements of $\mathbf{C}_v$ and suppose that they are not all zero. Up to rescaling the endomorphism $\varphi = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ assume

$$\max\{|\lambda_1|_v, \dots, |\lambda_n|_v\} = 1.$$

The reduction of the point $[\varphi]$ is the $\overline{\mathbf{F}}_p$-point of $\mathbf{P}(\mathrm{End}(\mathfrak{E}))$ associated to the endomorphism $\widetilde{\varphi} = \mathrm{diag}(\widetilde{\lambda_1}, \dots, \widetilde{\lambda_n})$ of the $\overline{\mathbf{F}}_p$-vector space

$$\mathfrak{E} \otimes_{\mathfrak{o}} \overline{\mathbf{F}}_p = \overline{\mathbf{F}}_p \cdot e_1 \oplus \cdots \oplus \overline{\mathbf{F}}_p \cdot e_n,$$

where, for every $i = 1, \dots, n$, $\widetilde{\lambda}_i \in \overline{\mathbf{F}}_p$ denotes the reduction of λ_i. The endomorphism $\widetilde{\varphi}$ is non-zero and semi-simple, hence semi-stable. Thus, according to Proposition 1.4 (3), the endomorphism φ is minimal, which conclude the proof in the non-archimedean case. □

Proof of Proposition 2.6: the archimedean case. — Let $\mathfrak{su}(E)$ be the Lie algebra of the Lie group $\mathbf{SU}(\|\cdot\|_E)$. A moment map $\mu\colon X(\mathbf{C}) \to \mathfrak{su}(E)^\vee$ for this action is defined as follows: for every non-zero endomorphism φ it is the linear map which associates to every skew-hermitian matrix $A \in \mathfrak{su}(E)$ the real number

$$\mu_{[\varphi]}(A) = \frac{1}{2\pi \boldsymbol{i}} \frac{\langle [A,\varphi],\varphi\rangle_{\mathrm{End}(E)}}{\|\varphi\|^2_{\mathrm{End}(E)}},$$

where $[A,\varphi] = A\varphi - \varphi A$ denotes the Lie bracket operation, $\langle -,-\rangle_{\mathrm{End}(E)}$ the hermitian form associated to the norm $\|\cdot\|_{\mathrm{End}(E)}$ and $\boldsymbol{i}$ is a square root of -1.

According to Proposition 1.7, the point φ is minimal if and only if $\mu([\varphi]).A$ vanishes for all $A \in \mathfrak{su}(E)$. This is equivalent to the following condition:

$$\langle A\varphi,\varphi\rangle_{\mathrm{End}(E)} = \langle \varphi A,\varphi\rangle_{\mathrm{End}(E)} \quad \text{for all } A \in \mathrm{End}(E).$$

Lemma 2.7. — *With the notation introduced above, for every endomorphism φ of E the following conditions are equivalent:*

(1) $\langle A\varphi,\varphi\rangle_{\mathrm{End}(E)} = \langle \varphi A,\varphi\rangle_{\mathrm{End}(E)}$ for all $A \in \mathrm{End}(E)$;
(2) $\varphi^*\varphi = \varphi\varphi^*$, where φ^* denotes the adjont endomorphism to φ with respect to the norm $\|\cdot\|_{\mathrm{End}(E)}$;
(3) The endomorphism φ is diagonalisable on a orthonormal basis.

Proof of Lemma 2.7. — The equivalence of (2) and (3) is a well-known fact in linear algebra. The proof of the equivalence of (1) and (2) is the computation that follows.

For all $i,j = 1,\dots,n$ let A_{ij} be endomorphism of E defined by $A_{ij}(e_k) = \delta_{ik} e_j$ for all $k = 1,\dots,n$ (here δ_{ik} is Kronecker's delta). Write $\varphi = \sum_{i,j=1}^n \varphi_{ij} A_{ij}$. With these conventions, for every $i,j = 1,\dots,n$,

$$A_{ij}\varphi = \sum_{k=1}^n \varphi_{jk} A_{ik}, \qquad \varphi A_{ij} = \sum_{k=1}^n \varphi_{ki} A_{kj}.$$

The matrices A_{ij} form an orthonormal basis of $\mathrm{End}(E)$ thus, for $i,j = 1,\dots,n$,

$$\langle A_{ij}\varphi,\varphi\rangle = \sum_{k=1}^n \varphi_{jk}\overline{\varphi_{ik}}, \qquad \langle \varphi A_{ij},\varphi\rangle = \sum_{k=1}^n \varphi_{ki}\overline{\varphi_{kj}}.$$

On the other hand, by definition, $\varphi = \sum_{i,j=1}^n \overline{\varphi_{ji}} A_{ij}$. Therefore

$$\varphi\varphi^* = \sum_{i,j=1}^n \overline{\langle A_{ij}\varphi,\varphi\rangle} A_{ij}, \qquad \varphi^*\varphi = \sum_{i,j=1}^n \overline{\langle \varphi A_{ij},\varphi\rangle} A_{ij}.$$

It follows from these expressions that the two conditions in the statement are equivalent. □

The preceding Lemma concludes the proof: indeed for complex numbers $\lambda_1,\dots,\lambda_n$ the associated diagonal matrix $\varphi = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ is minimal. □

2.1.5. A variant. — Let us stick to the complex case. Instead of considering the norm $\|\cdot\|_{\mathrm{End}(E)}$ endow $\mathrm{End}(E)$ with the operator norm:

$$\|\varphi\|_{\sup} := \sup_{x \neq 0} \frac{\|\varphi(x)\|_E}{\|x\|_E}.$$

It is invariant under the action of $\mathbf{SU}(\|\cdot\|_E)$.

Proposition 2.8. — *With the notation introduced above,*

$$\inf_{g \in \mathbf{SL}(E,\mathbf{C})} \|g\varphi g^{-1}\|_{\sup} = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ are the eigenvalues of φ (counted with multiplicities).

Proof. — For every $i = 1, \ldots, n$ let v_i be an eigenvector with respect to the eigenvalue λ_i: for all $g \in \mathbf{SL}(E,\mathbf{C})$ and all $i = 1, \ldots, n$,

$$\|g\varphi g^{-1}(g \cdot v_i)\|_E = |\lambda_i| \|g \cdot v_i\|_E.$$

In particular, for all $g \in \mathbf{SL}(E,\mathbf{C})$,

$$\|g\varphi g^{-1}\|_{\sup} \geq \max\{|\lambda_1|, \ldots, |\lambda_n\},$$

thus

$$\inf_{g \in \mathbf{SL}(E,\mathbf{C})} \|g\varphi g^{-1}\|_{\sup} \geq \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

Since the endomorphism $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ belongs to the closure of the $\mathbf{SL}(E)$-orbit of φ,

$$\inf_{g \in \mathbf{SL}(E,\mathbf{C})} \|g\varphi g^{-1}\|_{\sup} \leq \|\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\|_{\sup} = \max\{|\lambda_1|, \ldots, |\lambda_n|\},$$

which concludes the proof. □

The same argument holds also in the non-archimedean case, leading to a different proof of Proposition 2.5.

2.2. The lowest height on the orbit is not the height on the quotient. —

2.2.1. The question. — Let us go back to the notation introduced in Section 1.1 and let P be a semi-stable K-point of $\mathcal{X}$. Since the map π is $\mathcal{G}$-invariant:

$$\inf_{g \in \mathcal{G}(\bar{\mathbf{Q}})} h_{\overline{\mathcal{L}}}(g \cdot P) \geq \inf_{\pi(P')=\pi(P)} h_{\overline{\mathcal{L}}}(P'), \tag{2.2.1}$$

where the infimum on the right-hand side is ranging over all semi-stable $\bar{\mathbf{Q}}$-points of $\mathcal{X}$ lying on the fibre $\pi^{-1}(P)$. Since the instability measure $\iota_v(P)$ is a non-positive real number for all $v \in \mathrm{V}_K$, the Fundamental Formula yields $h_{\overline{\mathcal{L}}}(P) \geq h_{\overline{\mathcal{M}}}(\pi(P))$, thus:

$$\inf_{\pi(P')=\pi(P)} h_{\overline{\mathcal{L}}}(P') \geq h_{\overline{\mathcal{M}}}(\pi(P)). \tag{2.2.2}$$

Combining the previous inequalities:

$$\inf_{g \in \mathcal{G}(\bar{\mathbf{Q}})} h_{\overline{\mathcal{L}}}(g \cdot P) \geq h_{\overline{\mathcal{M}}}(\pi(P)). \tag{2.2.3}$$

Question. — When are the inequalities (2.2.1), (2.2.2) and (2.2.3) identities?

Three examples of linear actions of $\mathcal{G} = \mathbf{GL}_{n,\mathbf{Z}}$ on a hermitian vector bundle $\overline{\mathcal{F}}$ are presented:

(1) In the first example (with $n = 1$) we show that inequalities (2.2.2) and (2.2.3) are not identities if the hermitian vector bundle $\overline{\mathcal{F}}$ is not semi-stable [(3)].

(2) In the second one (with $n = 1$ again) we show that even taking $\overline{\mathcal{F}}$ to be the trivial hermitian vector bundle (that is, $\mathcal{F} = \mathbf{Z}^r$ endowed with the standard euclidian norm on $\mathbf{R}^r$) is not sufficient.

The problem seems to arise from the fact that in this example $\mathbf{G}_m$ acts through different weights, that is, the representation $\mathbf{G}_m \to \mathbf{GL}(\mathcal{F})$ is not homogeneous.

(3) In the third one, we consider $\mathbf{GL}_{n,\mathbf{Z}}$ acting on $\mathcal{F} = \mathrm{End}(\mathbf{Z}^n)$ by conjugation. Endow $\mathcal{F}$ with hermitian norm on endomorphisms (see paragraph 0.3 on page 11) deduced from the standard scalar product on $\mathbf{R}^n$.

In this case we show that that the inequalities (2.2.2) and (2.2.3) are indeed equalities for all semi-stable $\bar{\mathbf{Q}}$-points of $\mathbf{P}(\mathcal{F})$ whose orbit is closed (in $\mathbf{P}(\mathcal{F})^{\mathrm{ss}}$). Nonetheless, inequality (2.2.1) is not an equality in general when the orbit of the point is not closed.

2.2.2. Linear action on a non-trivial hermitian vector bundle. — Consider the action of $\mathbf{G}_m$ on $\mathbf{A}^2_{\mathbf{Z}}$ defined by

$$t \cdot (x_0, x_1) = (t^{-1}x_0, tx_1),$$

for every scheme S, every $t \in \mathbf{G}_m(S)$ and every $(x_0, x_1) \in \mathbf{A}^2(S)$.

Consider the induced action of $\mathcal{G} = \mathbf{G}_m$ on $\mathcal{X} = \mathbf{P}^1_{\mathbf{Z}}$ and the $\mathcal{G}$-linearisation of $\mathcal{L} = \mathcal{O}(1)$. For every field k,

$$\mathcal{X}^{\mathrm{ss}}(k) = \mathbf{P}^1(k) - \{0, \infty\}.$$

Let $\mathcal{Y}$ be the categorical quotient of $\mathcal{X}^{\mathrm{ss}}$ (which is canonically identified with $\operatorname{Spec} \mathbf{Z}$) and let $\pi \colon \mathcal{X}^{\mathrm{ss}} \to \mathcal{Y}$ the quotient morphism.

For every couple $r = (r_0, r_1)$ of positive real numbers consider the norm $\| \cdot \|_r$ on $\mathbf{R}^2$ defined by

$$\|(x_0, x_1)\|_r^2 := r_0^2 |x_0|^2 + r_1^2 |x_1|^2.$$

Endow the invertible sheaf $\mathcal{L}$ with the Fubini-Study metric associated to $\| \cdot \|_r$, which is invariant under the action of $\mathbf{U}(1)$. Denote by $\overline{\mathcal{L}}_r$ the hermitian line bundle on $\mathcal{X}$ obtained in this way and by $h_{\overline{\mathcal{M}},r}$ the height on the quotient $\mathcal{Y}$ associated to $\overline{\mathcal{L}}_r$.

Proposition 2.9. — *With the notations introduced above:*

(1) The point $(1 : 1) \in \mathbf{P}^1(\mathbf{Q})$ *is semi-stable and*

$$h_{\overline{\mathcal{M}},r}(\pi(1 : 1)) = \log \sqrt{2 r_0 r_1};$$

(2) For every $t \in \mathbf{G}_m(\bar{\mathbf{Q}})$,

$$h_{\overline{\mathcal{L}}_r}(t^{-1} : t) \geq \log \sqrt{r_0^2 + r_1^2},$$

with equality for $t = 1$.

3. A hermitian vector bundle $\overline{\mathcal{E}}$ on a number field K is said to be semi-stable if $\mu_{\max}(\overline{\mathcal{E}}) = \mu(\overline{\mathcal{E}})$.

In particular, this shows that inequalities (2.2.2) and (2.2.3) are never identities unless $r_0 = r_1$.

The proof of this result is left to the reader as similar arguments appear in the next examples (see Propositions 2.10 and 2.13).

2.2.3. A negative example when the hermitian vector bundle is trivial. — Consider the $\mathbf{Z}$-module $\mathcal{E} = \mathbf{Z}^3$ endowed with the standard euclidian norm

$$\|(x_0, x_1, x_2)\|^2 = |x_0|^2 + |x_1|^2 + |x_2|^2,$$

Let $\mathbf{G}_m$ act on $\mathbf{Z}^3$ by

$$t \cdot (x_0, x_1, x_2) = (t^{-2}x_0, tx_1, t^4 x_2).$$

Consider the induced action of $\mathbf{G}_m$ on $\mathcal{X} = \mathbf{P}^2_{\mathbf{Z}}$ and the linearisation of $\mathcal{O}(1)$. Endow $\mathcal{O}(1)$ with the Fubini-Study metric induced by the norm $\|\cdot\|$.

Proposition 2.10. — *Consider the point $P = (2:2:1)$. With the notations introduced above:*

(1) The point P is semi-stable and

$$h_{\overline{\mathcal{M}}}(\pi(P)) = \log 3 - \log \sqrt[3]{4}.$$

(2) For every $t \in \mathbf{G}_m(\bar{\mathbf{Q}})$,

$$h_{\overline{\mathcal{O}(1)}}(t \cdot P) \geq \log 3,$$

with equality for $t = 1$.

This shows that inequalities (2.2.2) and (2.2.3) are not identities, even though the hermitian vector bundle $\overline{\mathcal{E}}$ is trivial.

Proof. — (1) The semi-stability of the point P is clear. For every prime number $p \neq 2$ the point P is minimal since its reduction

$$\widetilde{P} = (2:2:1) \in \mathbf{P}^2(\mathbf{F}_p)$$

is a semi-stable point of $\mathbf{P}^2_{\mathbf{F}_p}$. It is also an elementary computation to see that the point P is minimal at the unique archimedean place of $\mathbf{Q}$.

On the other hand, for $p = 2$, it is *not* minimal: indeed, its reduction modulo 2 is $(0:0:1)$ which is not a semi-stable point of $\mathbf{P}^2_{\mathbf{F}_2}$. For every $t \in \mathbf{G}_m(\mathbf{C}_2)$,

$$\log \|t \cdot (2,2,1)\|_2 = \max\{-2\log|t|_2 - \log 2, \log|t|_2 - \log 2, 4\log|t|_2\}.$$

The minimum of this function is obtained for $\log|t|_2 = -\log\sqrt[6]{2}$, thus

$$\log \inf_{t \in \mathbf{G}_m(\mathbf{C}_2)} \|t \cdot (2,2,1)\|_2 = -\frac{2}{3}\log 2 = -\log\sqrt[3]{4}.$$

Finally the Fundamental Formula yields

$$h_{\overline{\mathcal{M}}}(\pi(P)) = \sum_{v \in \mathrm{V}_{\mathbf{Q}}} \log \inf_{t \in \mathbf{G}_m(\mathbf{C}_v)} \|t \cdot (2,2,1)\|_v = \log 3 - \log\sqrt[3]{4}.$$

(2) Let K be a number field and let $t \in \mathbf{G}_m(K)$. For every finite place v not dividing 2,

$$\log \|t \cdot (2,2,1)\|_v \geq \max\{-2\log|t|_v, 4\log|t|_v\},$$

whereas if v divides 2,

$$\log \|t \cdot (2,2,1)\|_v \geq \max\{-2\log|t|_v - \log 2, 4\log|t|_v\}.$$

Therefore, summing over all places of K, and thanks to the Product Formula, the height $[K:\mathbf{Q}]h_{\overline{\mathcal{O}(1)}}(t \cdot P)$ is bounded below by

$$2\max\left\{\sum_{\sigma\colon K\to\mathbf{C}} \log|t|_\sigma, \sum_{\sigma\colon K\to\mathbf{C}} -2\log|t|_\sigma\right\} + \frac{1}{2}\sum_{\sigma\colon K\to\mathbf{C}} \log\left(\frac{4}{|t|_\sigma^4} + 4|t|_\sigma^2 + |t|_\sigma^8\right).$$

Lemma 2.11. — *Let $N \geq 1$ be a positive integer. For every $x_1, \ldots, x_N \in \mathbf{R}$,*

$$2\max\left\{\sum_{i=1}^N x_i, \sum_{i=1}^N -2x_i\right\} + \frac{1}{2}\sum_{i=1}^N \log(4e^{-4x_i} + 4e^{2x_i} + e^{8x_i}) \geq N\log 3.$$

Proof of the Lemma. — Let consider the function $\alpha\colon \mathbf{R}^N \to \mathbf{R}$ defined by

$$\alpha(x_1,\ldots,x_N) := 2\max\left\{\sum_{i=1}^N x_i, \sum_{i=1}^N -2x_i\right\} + \frac{1}{2}\sum_{i=1}^N \log(4e^{-4x_i} + 4e^{2x_i} + e^{8x_i}).$$

The function α is convex and is invariant under permutations of coordinates. Therefore its minimum is attained on the diagonal $\mathbf{R} \subset \mathbf{R}^N$, that is, it coincides with the minimum of the function $\beta\colon \mathbf{R} \to \mathbf{R}$ given by

$$\beta(x) := N(2\max\{x, -2x\} + \tfrac{1}{2}\log(4e^{-4x} + 4e^{2x} + e^{8x})).$$

The minimum of β is seen to attained in 0. Thus for all $x \in \mathbf{R}$,

$$\beta(x) \geq \beta(0) = N\log 3,$$

whence the result. □

Let us come back to the proof of Proposition 2.10. Order the complex embeddings $\sigma_1, \ldots, \sigma_N\colon K \to \mathbf{C}$, where $N = [K:\mathbf{Q}]$, and apply the preceding Lemma with $x_i = \log|t|_{\sigma_i}$ for all $i = 1, \ldots, N$. Then,

$$h_{\mathcal{O}(1)}(g \cdot P) \geq \log 3,$$

which concludes the proof. □

2.2.4. Endomorphisms. — Let $n \geq 1$ be a positive integer and consider the hermitian vector bundle $\overline{\mathcal{E}}$ given by the $\mathbf{Z}$-module $\mathcal{E} = \mathbf{Z}^n$ endowed with the standard hermitian norm. Let $e_1, \ldots, e_n$ be the standard basis of $\mathcal{E}$.

As in Section 2.1 consider the action by conjugation of $\mathcal{G} = \mathbf{SL}_{n,\mathbf{Z}}$ on $\mathcal{F} = \mathrm{End}(\mathcal{E})$ and borrow the notations introduced in paragraph 2.1.2.

For every number field K and for every $\lambda_1, \ldots, \lambda_n \in K$, denote by $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ the endomorphism of $\mathcal{E} \otimes K = K^n$ given by the matrix (with respect the standard basis),

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Proposition 2.12. — *Let $\lambda_1, \ldots, \lambda_n \in K$ and suppose that they are not all zero. With the notation introduced above,*

$$h_{\mathcal{O}_{\overline{\mathcal{F}}}(1)}([\operatorname{diag}(\lambda_1, \ldots, \lambda_n)]) = h_{\overline{\mathcal{M}}}(\pi([\operatorname{diag}(\lambda_1, \ldots, \lambda_n)])).$$

In particular, for all non-zero semi-simple endomorphism φ of K^n,

$$\inf_{g \in \mathbf{SL}_n(\bar{\mathbf{Q}})} h_{\mathcal{O}_{\overline{\mathcal{F}}}(1)}(g \cdot [\varphi]) = h_{\overline{\mathcal{M}}}(\pi([\varphi])).$$

This is an immediate consequence of Theorem 2.4. This shows that inequalities (2.2.2) and (2.2.3) are identities for non-zero semi-simple endomorphism, that is, for those points having a closed orbit.

However, inequality (2.2.1) is not an equaility in general. For instance, take $n = 2$ and consider the endomorphism φ of $\mathcal{E}$ given by the matrix (with respect the standard basis),

$$\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proposition 2.13. — *With the notations introduced above:*

(1) The endomorphism φ is semi-stable and

$$h_{\overline{\mathcal{M}}}(\pi([\varphi])) = \log \sqrt{2};$$

(2) For every $g \in \mathbf{SL}_2(\bar{\mathbf{Q}})$,

$$h_{\overline{\mathcal{O}(1)}}(g \cdot [\varphi]) \geq \log \sqrt{3},$$

with equality for $g = \mathrm{id}$.

Proof. — (1) This is a direct consequence of Theorem 2.4. (2) Let K be a number field and let $g \in \mathbf{SL}_2(K)$ be given by the matrix

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

where $a, b, c, d \in K$ are such that $ad - bc = 1$. With this notation,

$$g \varphi g^{-1} = \begin{pmatrix} 1 - ab & a^2 \\ -b^2 & 1 + ab \end{pmatrix}.$$

For every finite place v of K:

$$\|g \varphi g^{-1}\|_v = \max\{|1 - ab|_v, |1 + ab|_v, |a|_v^2, |b|_v^2\} \geq \max\{1, |a|_v^2, |b|_v^2\}.$$

On the other hand, for every complex embedding $\sigma\colon K \to \mathbf{C}$,

$$\|g\varphi g^{-1}\|_\sigma^2 = |1-ab|_\sigma^2 + |1+ab|_\sigma^2 + |a|_\sigma^4 + |b|_\sigma^2 = 2 + \left(|a|_\sigma^2 + |b|_\sigma^2\right)^2.$$

Suppose $a \neq 0$. Since $|b|_v$ is non-negative for all places v of V_K, the previous expressions entail

$$[K:\mathbf{Q}]h_{\overline{\mathcal{O}(1)}}(g\cdot[\varphi]) \geq \sum_{\substack{v\in\mathrm{V}_K\\ \text{finite}}} \log\max\{1,|a|_v^2\} + \frac{1}{2}\sum_{\sigma\colon K\to\mathbf{C}} \log(2+|a|_\sigma^4).$$

Thanks to the Product Formula:

$$\sum_{\substack{v\in\mathrm{V}_K\\ \text{finite}}} \log\max\{1,|a|_v^2\} \geq \max\left\{0, \sum_{\substack{v\in\mathrm{V}_K\\ \text{finite}}} \log|a|_v^2\right\} = \max\left\{0, -\sum_{\sigma\colon K\to\mathbf{C}} \log|a|_\sigma^2\right\}.$$

Putting together the previous lower bounds, the height $[K:\mathbf{Q}]h_{\overline{\mathcal{O}(1)}}(g\cdot[\varphi])$ is bounded below by

$$\max\left\{0, \sum_{\sigma\colon K\to\mathbf{C}} -\log|a|_\sigma^2\right\} + \frac{1}{2}\sum_{\sigma\colon K\to\mathbf{C}} \log(2+|a|_\sigma^4). \tag{2.2.4}$$

Lemma 2.14. — *Let $N \geq 1$ be a positive integer. For every $x_1,\ldots,x_N \in \mathbf{R}$,*

$$\max\left\{0, \sum_{i=1}^N -x_i\right\} + \frac{1}{2}\sum_{i=1}^N \log(2+e^{2x_i}) \geq N\log\sqrt{3}.$$

The proof of the preceding Lemma is similar to the one of Lemma 2.11.

Let us conclude the proof in the case $a \neq 0$. Order the complex embeddings $\sigma_1,\ldots,\sigma_N : K \to \mathbf{C}$, where $N = [K:\mathbf{Q}]$, and apply the preceding Lemma with $x_i = \log|a|_{\sigma_i}^2$ for all $i = 1,\ldots,N$. According to (2.2.4),

$$h_{\mathcal{O}(1)}(g\cdot[\varphi]) \geq \log\sqrt{3}.$$

The case $a = 0$ and $b \neq 0$ is proven similarly. □

3. The approach of Bost, Gasbarri and Zhang

3.1. Twisting by principal bundles. — In order to make more explicit the geometrical content of the approach of Bost, Gasbarri and Zhang and its link with the work of Bogomolov, let us recall a basic construction involving principal G-bundles.

3.1.1. The algebraic construction. — Let G be a group scheme over a non-empty scheme S. Let X be a S-scheme endowed with a (left) action of G and let P be a principal G-bundle [4] (always assumed to be locally trivial for the Zariski topology).

4. A *principal G-bundle* P is a S-scheme endowed with a (right) action $\alpha\colon P\times_S G \to P$ such that:

(1) the morphism $(\mathrm{pr}_1,\alpha)\colon P\times_S G \to P\times_S P$ is an isomorphism of S-schemes;

(2) P is locally trivial for the Zariski topology: there exists an open covering $S = \bigcup_{i\in I} S_i$ and for every $i\in I$ there exists a section $p_i\colon S_i \to P$.

By definition, the *twist of X by P* is the categorical quotient of $X \times_S P$ by the (left) action of G defined as

$$g \cdot (x, p) = (gx, pg^{-1}),$$

for every S-scheme S' and every point $g \in G(S')$ on $(x, p) \in X(S') \times P(S')$. Concretely, X_P is constructed as follows:

(1) Pick a covering $S = \bigcup_{i \in I} S_i$ by open subset on which P is trivial and, for every $i \in I$, let $p_i \colon S_i \to P$ be a section.
(2) Glue the schemes $X \times_S S_i$ along the isomorphisms

$$\begin{array}{ccc} X \times_S S_{ij} & \longrightarrow & X \times_S S_{ij} \\ x & \longmapsto & g_{ij} \cdot x, \end{array}$$

where $S_{ij} = S_i \cap S_j$ and g_{ij} is the unique S_{ij}-point of G sending p_j to p_i.

In particular X_P is isomorphic to X locally on the base S. This construction will be used in the following examples:

(1) Let G acting on itself by conjugation. The twist G_P is a S-group scheme and it acts on X_P. More generally, if H is normal subgroup of G (namely a closed subscheme such that, for every S-scheme S', the set $H(S')$ is a normal subgroup of $G(S')$) then H_P is a normal subgroup of G_P.
(2) Let G act linearly on a vector bundle F over S and let $\mathbf{V}(F)_P$ be the twist by P of the total space $\mathbf{V}(F)$ of F. Consider the sheaf on S defined for every open subset $U \subset S$ by

$$\Gamma(U, F_P) := \mathrm{Mor}_S(U, \mathbf{V}(F)_P).$$

Then F_P is a vector bundle over S, the S-group scheme G_P acts linearly on it and its total space $\mathbf{V}(F_P)$ is identified with $\mathbf{V}(F)_P$. The vector bundle F_P is the twist of F by P.
(3) Let L be a G-linearized line bundle on X and let $\mathbf{V}(L)_P$ be twist by P of its total space $\mathbf{V}(L)$ over X. Consider the sheaf L_P on the twist X_P of X by P defined for every open subset $U \subset X_P$ by

$$\Gamma(U, L_P) := \mathrm{Mor}_S(U, \mathbf{V}(L)_P)$$

Then L_P is G_P-linearized line bundle over X_P and its total space $\mathbf{V}(L_P)$ over X_P is identified with $\mathbf{V}(L)_P$.

Example 3.1. — Let $N \geq 1$ be a positive integer and $e_1, \ldots, e_N$ be positive integers. Consider the following S-group schemes:

$$G = \mathbf{GL}_{e_1,S} \times_S \cdots \times_S \mathbf{GL}_{e_N,S},$$
$$S = \mathbf{SL}_{e_1,S} \times_S \cdots \times_S \mathbf{SL}_{e_N,S}\,.$$

Let $E = (E_1, \ldots, E_N)$ be a N-uple of vector bundles on S such that E_i is of rank e_i for all i. To E one associates the principal G-bundle

$$P_E = \mathbf{F}_S(E_1) \times_S \cdots \times_S \mathbf{F}_S(E_N),$$

where for all i the S-scheme $\mathbf{F}_S(E_i)$ is the *frame bundle* of E_i: for every S-scheme $f\colon S' \to S$,

$$\mathbf{F}_S(E_i)(S') = \mathrm{Iso}_{\mathcal{O}_{S'}\text{-mod}}(\mathcal{O}_{S'}^{e_i}, f^*E_i).$$

The (right) action of G on P_E is given by composing on the right.

Let us be given a vector bundle F on S and a representation $\rho\colon G \to \mathbf{GL}(F)$. Consider the induced action of G on $X = \mathbf{P}(F)$ and the invertible sheaf $L = \mathcal{O}_F(1)$. Denote by F_E the twist of F by P_E. Then:

$$\begin{aligned} G_E &= \mathbf{GL}(E_1) \times_S \cdots \times_S \mathbf{GL}(E_N), \\ S_E &= \mathbf{SL}(E_1) \times_S \cdots \times_S \mathbf{SL}(E_N), \\ X_E &= \mathbf{P}(F_E), \\ L_E &= \mathcal{O}_{F_E}(1), \end{aligned}$$

where one writes G_E, S_E, X_E and L_E instead of G_{P_E}, S_{P_E}, X_{P_E} and L_{P_E}.

3.1.2. The hermitian construction. — Let us work over the complex numbers. Let G be a complex algebraic group and $C_G \subset G(\mathbf{C})$ be a compact subgroup. Denote the couple (G, C_G) by $\overline{G}$.

Definition 3.2. — A *principal hermitian $\overline{G}$-bundle* is a couple (P, C_P) made of a principal G-bundle and of a non-empty compact subset $C_P \subset P(\mathbf{C})$ such that the map induced by the action G,

$$\begin{array}{ccc} C_P \times C_G & \longrightarrow & C_P \times C_P \\ (p,u) & \longmapsto & (p, pu), \end{array}$$

is a bijection.

Let $\overline{X} = (X, C_X)$ be a couple made of a complex scheme of finite type and a compact subset $C_X \subset X(\mathbf{C})$. Suppose that G acts on X and this action induces an action of C_G on C_X.

Definition 3.3. — Let $\overline{P} = (P, C_P)$ be a principal hermitian $\overline{G}$-bundle. The *twist of $\overline{X}$ by $\overline{P}$* is the couple $\overline{X}_{\overline{P}} = (X_P, C_{X_P})$, where X_P is the twist of X by P and C_{X_P} is the image of the map

$$(C_X \times C_P)/C_G \longrightarrow X_P(\mathbf{C}) = (X(\mathbf{C}) \times P(\mathbf{C}))/G(\mathbf{C}).$$

Note that this map is injective by definition of principal hermitian G-bundle.

Let $p \in C_P$ be a point and for every $x \in X(\mathbf{C})$ denote by $[x,p]$ the class of (x,p) in $X_P(\mathbf{C}) = (X(\mathbf{C}) \times P(\mathbf{C}))/G(\mathbf{C})$. Then the map

$$\begin{array}{ccc} X(\mathbf{C}) & \longrightarrow & X_P(\mathbf{C}) \\ x & \longmapsto & [x,p]\,, \end{array}$$

is an isomorphism which identifies the subset C_X with C_{X_P}.

Let $\overline{P}$ be a principal $\overline{P}$-bundle. The examples worked out for principal G-bundles can be now translated in this new context:

(1) Let G acting on itself by conjugation. The twist of $\overline{G} = (G, C_G)$ by $\overline{P}$ is a couple (G_P, C_{G_P}) made of a complex algebraic group G_P and of a compact subgroup C_{G_P} of $G_P(\mathbf{C})$.

Let $\overline{H} = (H, C_H)$ be a couple made of a normal algebraic subgroup H of G and a compact subgroup C_H of $H(\mathbf{C})$ which is stable under conjugation by C_G. Then $\overline{H}_{\overline{P}}$ is a couple (H_P, C_{H_P}) made of the twist of H_P by P (which is a normal algebraic subgroup of G_P) and of a compact subgroup of $H_P(\mathbf{C})$.

(2) Let $\overline{F} = (F, \|\cdot\|_F)$ be a (finite dimensional) hermitian vector space. Suppose that G act linearly on F and that the norm $\|\cdot\|_F$ is invariant under the action of C_G. Consider the couple $\mathbf{V}(\overline{F}) = (\mathbf{V}(F), \mathbf{D}_F)$ where

$$\mathbf{D}_F = \{v \in F : \|v\|_F \leq 1\}.$$

Let $\mathbf{V}(\overline{F})_{\overline{P}} = (\mathbf{V}(F_P), \mathbf{D}_{F,\overline{P}})$ be the twist of $\mathbf{V}(\overline{F})$ by $\overline{P}$. Then there exists a unique hermitian norm $\|\cdot\|_{F_P}$ on F_P such that

$$\mathbf{D}_{F,\overline{P}} = \{v \in F_P : \|v\|_{F_P} \leq 1\}.$$

The hermitian vector space $\overline{F}_{\overline{P}} = (F_P, \|\cdot\|_{F_P})$ is the twist of $\overline{F}$ by $\overline{P}$.

(3) Let X be a proper complex scheme endowed with an action of G together with a G-linearized line bundle L. Suppose that L is equipped with a continuous metric $\|\cdot\|_L$ which is invariant under the action of C_G. Consider the couple $\mathbf{V}(\overline{L}) = (\mathbf{V}(L), \mathbf{D}_L)$ where $\mathbf{V}(L)$ is the total space of L over X and

$$\mathbf{D}_L = \{(x, s) : x \in X(\mathbf{C}), s \in x^*L, \|s\|_L(x) \leq 1\}.$$

Remark since $X(\mathbf{C})$ is compact, then $\mathbf{D}_L$ is a compact subset of $\mathbf{V}(L)(\mathbf{C})$. Moreover it is stable under the action of C_G. Let

$$\mathbf{V}(\overline{L})_{\overline{P}} = (\mathbf{V}(L_P), \mathbf{D}_{L,\overline{P}}),$$

be the twist of $\mathbf{V}(\overline{L})$ by $\overline{P}$. There exists a unique continuous metric $\|\cdot\|_{L_P}$ on L_P such that

$$\mathbf{D}_{L,\overline{P}} = \{(x, s) : x \in X_P(\mathbf{C}), s \in x^*L_P, \|s\|_{L_P}(x) \leq 1\}$$

The hermitian line bundle $\overline{L}_{\overline{P}} = (L_P, \|\cdot\|_{L_P})$ is the twist of the hermitian line bundle $\overline{L} = (L, \|\cdot\|_L)$ by $\overline{P}$.

Example 3.4. — Let $N \geq 1$ be a positive integer and let $e_1, \ldots, e_N$ be positive integers. Consider the complex reductive groups

$$G = \mathbf{GL}_{e_1,\mathbf{C}} \times_{\mathbf{C}} \cdots \times_{\mathbf{C}} \mathbf{GL}_{e_N,\mathbf{C}},$$
$$S = \mathbf{SL}_{e_1,\mathbf{C}} \times_{\mathbf{C}} \cdots \times_{\mathbf{C}} \mathbf{SL}_{e_N,\mathbf{C}},$$

and their maximal compact subgroups

$$C_G := \mathbf{U} = \mathbf{U}(e_1) \times \cdots \times \mathbf{U}(e_N),$$
$$C_S := \mathbf{SU} = \mathbf{SU}(e_1) \times \cdots \times \mathbf{SU}(e_N).$$

Let $\overline{E} = (\overline{E}_1, \ldots, \overline{E}_N)$ be a N-uple of hermitian vector spaces, that is couples $\overline{E}_i = (E_i, \|\cdot\|_{E_i})$ made of a complex vector space E_i and a hermitian norm $\|\cdot\|_{E_i}$.

Suppose $\dim_{\mathbf{C}} E_i = e_i$ for all i. To such a $\overline{E}$ one associates the principal hermitian G-bundle $P_{\overline{E}} = (P_E, C_{\overline{E}})$ defined by

$$P_E = \mathbf{F}_{\mathbf{C}}(E_1) \times_{\mathbf{C}} \cdots \times_{\mathbf{C}} \mathbf{F}_{\mathbf{C}}(E_N),$$
$$C_{\overline{E}} = \mathbf{O}(\overline{E}_1) \times \cdots \times \mathbf{O}(\overline{E}_N),$$

where, for all i, $\mathbf{F}_{\mathbf{C}}(E_i)$ is the frame bundle of E_i and $\mathbf{O}(\overline{E}_i)$ is the *orthonormal frame bundle* of $\overline{E}_i$, *i.e.* the set of linear isometries $\mathbf{C}^{e_i} \to \overline{E}_i$ (here $\mathbf{C}^{e_i}$ is endowed with the standard hermitian norm).

Let $\overline{F} = (F, \|\cdot\|_F)$ be a hermitian vector space and $\rho\colon G \to \mathbf{GL}(F)$ a representation. Suppose that the norm $\|\cdot\|_F$ is invariant under the action of $\mathbf{U}$. Consider the induced action of G on $X = \mathbf{P}(F)$ and the invertible sheaf $L = \mathcal{O}_F(1)$ endowed with the Fubini-Study metric $\|\cdot\|_L$. Denote by $\overline{F}_{\overline{E}} = (F_E, \|\cdot\|_{F_E})$ the twist of $\overline{F}$ by $\overline{P}_{\overline{E}}$. Then:

(1) the twist of the couple $(G, \mathbf{U})$ by $\overline{P}_{\overline{E}}$ is the couple $(G_E, \mathbf{U}_{\overline{E}})$ where

$$G_E = \mathbf{GL}(E_1) \times_S \cdots \times_S \mathbf{GL}(E_N),$$
$$\mathbf{U}_{\overline{E}} = \mathbf{U}(\|\cdot\|_{E_1}) \times \cdots \times \mathbf{U}(\|\cdot\|_{E_N}).$$

(2) the twist of the couple $(S, \mathbf{SU})$ by $\overline{P}_{\overline{E}}$ is the couple $(S_E, \mathbf{SU}_{\overline{E}})$ where

$$S_E = \mathbf{SL}(E_1) \times_S \cdots \times_S \mathbf{SL}(E_N),$$
$$\mathbf{SU}_{\overline{E}} = \mathbf{SU}(\|\cdot\|_{E_1}) \times \cdots \times \mathbf{SU}(\|\cdot\|_{E_N}).$$

(3) the twist of the hermitian line bundle $\overline{L} = (L, \|\cdot\|_L)$ by $\overline{P}_{\overline{E}}$ is the hermitian line bundle $\overline{L}_{\overline{E}} = (L_E, \|\cdot\|_{L_E})$ on $X_E = \mathbf{P}(F_E)$ where

$$L_E = \mathcal{O}_{F_E}(1),$$
$$\|\cdot\|_{L_E} = \text{Fubini-Study metric associated to } \|\cdot\|_{F_E}.$$

3.2. Statement and proof of the result. —

3.2.1. Setup. — Let K be a number field. Let $N \geq 1$ be a positive integer and let $e_1, \ldots, e_N$ be positive integers. Consider the $\mathfrak{o}_K$-reductive groups

$$\mathcal{G} = \mathbf{GL}_{e_1,\mathfrak{o}_K} \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}_{e_N,\mathfrak{o}_K},$$
$$\mathcal{S} = \mathbf{SL}_{e_1,\mathfrak{o}_K} \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}_{e_N,\mathfrak{o}_K},$$

and for every embedding $\sigma\colon K \to \mathbf{C}$ consider their maximal compact subgroups

$$\mathbf{U}_\sigma = \mathbf{U}(e_1) \times \cdots \times \mathbf{U}(e_N) \subset \mathcal{G}_\sigma(\mathbf{C}),$$
$$\mathbf{SU}_\sigma = \mathbf{SU}(e_1) \times \cdots \times \mathbf{SU}(e_N) \subset \mathcal{S}_\sigma(\mathbf{C}).$$

Let $\overline{\mathcal{F}}$ be a hermitian vector bundle over $\mathfrak{o}_K$ and let $\rho\colon \mathcal{G} \to \mathbf{GL}(\mathcal{F})$ be a representation, that is a morphism of $\mathfrak{o}_K$-group schemes, which respects the hermitian structure: this means that for every embedding $\sigma\colon K \to \mathbf{C}$ the norm $\|\cdot\|_{\mathcal{F},\sigma}$ is fixed under the action of the maximal compact subgroup $\mathbf{U}_\sigma$.

The linear action of $\mathcal{G}$ on $\mathcal{F}$ induces an action of $\mathcal{G}$ on $\mathcal{X} = \mathbf{P}(\mathcal{F})$ and a $\mathcal{G}$-linearisation of $\mathcal{L} = \mathcal{O}(1)$. For every embedding $\sigma\colon K \to \mathbf{C}$ endow the invertible sheaf $\mathcal{O}(1)_{|\mathcal{X}_\sigma(\mathbf{C})}$ with the Fubini-Study metric, which invariant under the action of $\mathbf{U}_\sigma$ and whose curvature form is positive.

Consider the open subset of semi-stable points $\mathcal{X}^{\mathrm{ss}}$ with the respect to $\mathcal{S}$ and the categorical quotient $\mathcal{Y} = \mathcal{X} /\!\!/ \mathcal{S}$ of $\mathcal{X}^{\mathrm{ss}}$ by $\mathcal{S}$, namely $\mathcal{Y} = \operatorname{Proj} \mathcal{A}^{\mathcal{S}}$ where

$$\mathcal{A} = \bigoplus_{d \geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d}) = \bigoplus_{d \geq 0} \operatorname{Sym}^d_{\mathfrak{o}_K}(\mathcal{F}^\vee).$$

Let $\pi \colon \mathcal{X}^{\mathrm{ss}} \to \mathcal{Y}$ be quotient morphism. For every integer $D \geq 0$ divisible enough let $\mathcal{M}_D$ be the ample line bundle on $\mathcal{Y}$ induced by $\mathcal{L}^{\otimes D}$. Endow it with the continuous metric defined in Section 1.1.

Since $\mathcal{S}$ is normal in $\mathcal{G}$, for every $d \geq 0$, the sub-$\mathfrak{o}_K$-module of $\mathcal{S}$-invariant global sections of $\mathcal{L}^{\otimes d}$,

$$\Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})^{\mathcal{S}} \subset \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d}),$$

is stable under the linear action of $\mathcal{G}$ on $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})$. This implies that the open subset of semi-stable points $\mathcal{X}^{\mathrm{ss}}$ is stable under the action of $\mathcal{G}$. Moreover $\mathcal{G}$ acts on the quotient $\mathcal{Y}$ and, for every $D \geq 0$ divisible enough, the invertible sheaf $\mathcal{M}_D$ is $\mathcal{G}$-linearized. For every embedding $\sigma \colon K \to \mathbf{C}$ the metric $\| \cdot \|_{\mathcal{M}_D, \sigma}$ is invariant under the action of $\mathbf{U}_\sigma$.

3.2.2. Twisting by hermitian vector bundles. — Let $\overline{\mathcal{E}} = (\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_N)$ be a N-uple of hermitian vector bundles over $\mathfrak{o}_K$ such that $\operatorname{rk} \mathcal{E}_i = e_i$ for every $i = 1, \ldots, N$. Apply the constructions presented in Section 3.1.1 to the representation ρ and the N-uple $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_N)$ of $\mathfrak{o}_K$-modules underlying the hermitian vector bundles of $\overline{\mathcal{E}}$.

Going back to the notations of the Example 3.1, let $\mathcal{F}_{\mathcal{E}}, \mathcal{G}_{\mathcal{E}}, \mathcal{S}_{\mathcal{E}}, \mathcal{X}_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}}$ denote the twist of $\mathcal{F}$, $\mathcal{G}$, $\mathcal{S}$, $\mathcal{X}$, $\mathcal{L}$ by $\mathcal{E}$. Then:

$$\begin{aligned}
\mathcal{G}_{\mathcal{E}} &= \mathbf{GL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}(\mathcal{E}_N), \\
\mathcal{S}_{\mathcal{E}} &= \mathbf{SL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}(\mathcal{E}_N), \\
\mathcal{X}_{\mathcal{E}} &= \mathbf{P}(\mathcal{F}_{\mathcal{E}}), \\
\mathcal{L}_{\mathcal{E}} &= \mathcal{O}_{\mathcal{F}_{\mathcal{E}}}(1).
\end{aligned}$$

The $\mathfrak{o}_K$-reductive group $\mathcal{G}_{\mathcal{E}}$ acts linearly on $\mathcal{F}_{\mathcal{E}}$ and the invertible sheaf $\mathcal{L}_{\mathcal{E}}$ is $\mathcal{G}_{\mathcal{E}}$-linearized. Consider the open subset $\mathcal{X}_{\mathcal{E}}^{\mathrm{ss}}$ of semi-stable points of $\mathcal{X}_{\mathcal{E}}$ with respect $\mathcal{G}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{E}}$. Denote by $\mathcal{X}_{\mathcal{E}} /\!\!/ \mathcal{S}_{\mathcal{E}}$ the categorical quotient of $\mathcal{X}_{\mathcal{E}}^{\mathrm{ss}}$ by $\mathcal{S}_{\mathcal{E}}$ and, for every integer $D \geq 0$ divisible enough, by $\mathcal{M}_{D,\mathcal{E}}$ the ample invertible sheaf associated to $\mathcal{L}_{\mathcal{E}}^{\otimes D}$.

***Proposition 3.5* (Compatibility of GIT quotients to twists)**
With the notations introduced above:

1. *The set of semi-stable points $\mathcal{X}_{\mathcal{E}}^{\mathrm{ss}}$ is the twist of $\mathcal{X}^{\mathrm{ss}}$ by $\mathcal{E}$;*
2. *The quotient $\mathcal{X}_{\mathcal{E}} /\!\!/ \mathcal{S}_{\mathcal{E}}$ is the twist of the quotient $\mathcal{Y} = \mathcal{X} /\!\!/ \mathcal{S}$ by $\mathcal{E}$;*
3. *The invertible sheaf $\mathcal{M}_{D,\mathcal{E}}$ is the twist of $\mathcal{M}_D$ by $\mathcal{E}$.*

Sketch of the proof. — All these assertions follow from the following remark.

Remark 3.6. — Let $\mathcal{V}$ be a vector bundle over $\mathfrak{o}_K$ endowed with a linear action of $\mathcal{G}$. Denote by $\mathcal{V}_{\mathcal{E}}$ its twist by $\mathcal{E}$. Then the subspace $\mathcal{V}_{\mathcal{E}}^{\mathcal{S}_{\mathcal{E}}}$ of invariant elements of $\mathcal{V}_{\mathcal{E}}$ by $\mathcal{S}_{\mathcal{E}}$ coincide with the twist $(\mathcal{V}^{\mathcal{S}})_{\mathcal{E}}$ of $\mathcal{V}^{\mathcal{S}}$ by $\mathcal{E}$.

This is clear: indeed, by construction one has the inclusion $(\mathcal{V}^{\mathcal{S}})_{\mathcal{E}} \subset \mathcal{V}_{\mathcal{E}}^{\mathcal{S}_{\mathcal{E}}}$ and the equality may be checked locally on $\operatorname{Spec} \mathfrak{o}_K$.

To conclude the proof of the Proposition it suffices to apply the preceding remark with $\mathcal{V} = \Gamma(\mathcal{X}, \mathcal{L}^{\otimes d})$ for all $d \geq 0$. $\square$

Let $\sigma \colon K \to \mathbf{C}$ be a complex embedding. By hypothesis, the representation induced by ρ,

$$\rho_\sigma \colon \mathcal{G}_\sigma(\mathbf{C}) \longrightarrow \mathbf{GL}(\mathcal{F})(\mathbf{C}),$$

respects the hermitian structure, that is, the norm $\| \cdot \|_{\mathcal{F},\sigma}$ is invariant under the action of $\mathbf{U}_\sigma$. Therefore one can apply the constructions described in Section 3.1.2. Going back to the notations of the Example 3.4, the complex vector space $\mathcal{F}_{\mathcal{E}} \otimes_\sigma \mathbf{C}$ is endowed with an hermitian norm $\| \cdot \|_{\mathcal{F}_{\mathcal{E}},\sigma}$ which is invariant under the action of the maximal compact subgroup

$$\mathbf{U}_{\overline{\mathcal{E}},\sigma} = \mathbf{U}(\| \cdot \|_{\mathcal{E}_1,\sigma}) \times \cdots \times \mathbf{U}(\| \cdot \|_{\mathcal{E}_N,\sigma}).$$

The invertible sheaf $\mathcal{L}_{\mathcal{E}}$ is endowed with the Fubini-Study metric $\| \cdot \|_{\mathcal{L}_{\mathcal{E}},\sigma}$ associated to the norm $\| \cdot \|_{\mathcal{F}_{\mathcal{E}},\sigma}$. The metric $\| \cdot \|_{\mathcal{L}_{\mathcal{E}},\sigma}$ is invariant under the maximal compact subgroup

$$\mathbf{SU}_{\overline{\mathcal{E}},\sigma} = \mathbf{SU}(\| \cdot \|_{\mathcal{E}_1,\sigma}) \times \cdots \times \mathbf{SU}(\| \cdot \|_{\mathcal{E}_N,\sigma}) \subset \mathcal{S}_\sigma(\mathbf{C}).$$

Let $\| \cdot \|_{\mathcal{M}_D,\mathcal{E},\sigma}$ be the continuous metric on the invertible sheaf $\mathcal{M}_D$ defined in Section 1.1.

The metric $\| \cdot \|_{\mathcal{M}_D,\sigma}$ on $\mathcal{M}_D$ is invariant under the action of $\mathbf{U}_\sigma$: let $\| \cdot \|'_{\mathcal{M}_D,\mathcal{E},\sigma}$ be the metric on $\mathcal{M}_{D,\mathcal{E}}$ obtained by twisting the hermitian line bundle $\overline{\mathcal{M}}_D$ by $\overline{\mathcal{E}}$.

Proposition 3.7. — *With the notations introduced above, the metrics $\| \cdot \|_{\mathcal{M}_D,\mathcal{E},\sigma}$ and $\| \cdot \|'_{\mathcal{M}_D,\mathcal{E},\sigma}$ coincide.*

Proof. — This is seen picking isometries $\varepsilon_i \colon \mathbf{C}^{e_i} \to \mathcal{E}_i \otimes_\sigma \mathbf{C}$ for all $i = 1, \ldots, N$. This verification is left to the reader. $\square$

3.2.3. Statement. — Keep the notations introduced before. By definition the representation ρ is *homogeneous of weight* $a = (a_1, \ldots, a_N) \in \mathbf{Z}^N$ if for every $\mathfrak{o}_K$-scheme T and for every $t_1, \ldots, t_N \in \mathbf{G}_m(T)$,

$$\rho(t_1 \cdot \mathrm{id}_{\mathcal{E}_1}, \ldots, t_N \cdot \mathrm{id}_{\mathcal{E}_N}) = t_1^{a_1} \cdots t_N^{a_N} \cdot \mathrm{id}_{\mathcal{F}}.$$

Theorem 3.8. — *With the notations introduced above, let $\overline{\mathcal{E}} = (\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_N)$ be a N-uple of hermitian vector bundles over $\mathfrak{o}_K$ such that $\operatorname{rk} \mathcal{E}_i = e_i$ for every i.*

If the representation ρ is homogeneous of weight $a = (a_1, \ldots, a_N) \in \mathbf{Z}^N$ and the subset of semi-stable points $\mathcal{X}^{\mathrm{ss}}$ is not empty, then:

(1) there exists an isomorphism $\alpha_{\mathcal{E}} \colon \mathcal{Y}_{\mathcal{E}} \to \mathcal{Y}$;

(2) for every $D \geq 0$ divisible enough there exists an isomorphism of hermitian line bundles, that is an isometric isomorphism of line bundles,

$$\beta_{\overline{\mathcal{E}}} \colon \overline{\mathcal{M}}_{D,\overline{\mathcal{E}}} \xrightarrow{\sim} \alpha_{\mathcal{E}}^* \overline{\mathcal{M}}_D \otimes \bigotimes_{i=1}^N f_{\mathcal{E}}^* (\det \overline{\mathcal{E}}_i)^{\vee \otimes a_i D_i / e_i},$$

where $f_{\mathcal{E}}\colon \mathcal{Y}_{\mathcal{E}} \to \operatorname{Spec} \mathfrak{o}_K$ is the structural morphism;

(3) $h_{\min}((\mathcal{X}_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}}) /\!\!/ \mathcal{S}_{\mathcal{E}}) = h_{\min}((\mathcal{X}, \mathcal{L}) /\!\!/ \mathcal{S}) - \sum_{i=1}^{N} a_i \, \hat{\mu}(\mathcal{E}_i)$.

Corollary 3.9. — *With the notation of Theorem 3.8, for every K-point P of $\mathcal{X}_{\mathcal{E}}$ which is semi-stable under the action of $\mathcal{S}_{\mathcal{E}}$,*

$$h_{\overline{\mathcal{L}_{\overline{\mathcal{E}}}}}(P) \geq -\sum_{i=1}^{N} a_i \, \hat{\mu}(\mathcal{E}_i) + h_{\min}((\mathcal{X}, \mathcal{L}) /\!\!/ \mathcal{S}).$$

3.2.4. Proof of Theorem 3.8. — Let us begin with the following basic fact concerning homogeneous representations.

Proposition 3.10. — *Let $\mathcal{V}$ be non-zero $\mathfrak{o}_K$-module which is flat and of finite type. Let $r\colon \mathcal{G} \to \mathbf{GL}(\mathcal{V})$ be a homogeneous representation of weight $b = (b_1, \ldots, b_N)$. If the submodule $\mathcal{V}^{\mathcal{S}}$ of $\mathcal{S}$-invariant elements of $\mathcal{V}$ is non-zero, then:*

(1) e_i divides b_i for every $i = 1, \ldots, N$;

(2) the induced representation $r\colon \mathcal{G} \to \mathbf{GL}(\mathcal{V}^{\mathcal{S}})$ is given by

$$r(g_1, \ldots, g_N) = \prod_{i=1}^{N} (\det g_i)^{b_i/e_i}$$

for every $\mathfrak{o}_K$-scheme T and for every $(g_1, \ldots, g_N) \in \mathcal{G}(T)$.

Proof. — Consider the induced representation $r\colon \mathcal{G} \to \mathbf{GL}(\mathcal{V}^{\mathcal{S}})$. Since the action of $\mathcal{S}$ on $\mathcal{F}^{\mathcal{S}}$ is trivial by definition, the map r factors through a morphism of $\mathfrak{o}_K$-group schemes

$$\widetilde{r}\colon (\mathcal{G}/\mathcal{S}) = \mathbf{G}_m^N \longrightarrow \mathbf{GL}(\mathcal{V}^{\mathcal{S}}).$$

The representation r is homogeneous of weight $b = (b_1, \ldots, b_N)$, thus, for every $\mathfrak{o}_K$-scheme T and every $t_1, \ldots, t_N \in \mathbf{G}_m(T)$,

$$r(t_1 \cdot \mathrm{id}_{\mathcal{E}_1}, \ldots, t_N \cdot \mathrm{id}_{\mathcal{E}_1}) = t_1^{b_1} \cdots t_N^{b_N} \cdot \mathrm{id}\,.$$

On the other hand,

$$r(t_1 \cdot \mathrm{id}_{\mathcal{E}_1}, \ldots, t_N \cdot \mathrm{id}_{\mathcal{E}_1}) = \widetilde{r}(t_1^{e_1}, \ldots, t_N^{e_N}),$$

thus

$$\widetilde{r}(t_1^{e_1}, \ldots, t_N^{e_N}) = t_1^{b_1} \cdots t_N^{b_N} \cdot \mathrm{id}\,.$$

Statements (1) and (2) are then clear. □

Corollary 3.11. — *Under the hypotheses of Theorem 3.8:*

(1) The action of $\mathcal{G}$ on $\mathcal{Y}$ is trivial;

(2) For every integer $D \geq 0$ divisible enough the $\mathfrak{o}_K$-group scheme $\mathcal{G}$ acts on the fibres of $\mathcal{M}_D$ through the character

$$(g_1, \ldots, g_N) \mapsto \prod_{i=1}^{N} (\det g_i)^{-a_i D/e_i}.$$

More precisely, for every $\mathfrak{o}_K$-scheme T, every $(g_1, \dots, g_N) \in \mathcal{G}(T)$, every point $y \in \mathcal{Y}(T)$ and every section $s \in \Gamma(T, y^*\mathcal{M}_D)$,

$$(g_1, \dots, g_N) \cdot (y, s) = \left(y, \prod_{i=1}^{N} (\det g_i)^{-a_i D/e_i} \cdot s \right).$$

Proof. — Pick D such that $\mathcal{M}_D$ is very ample. Then, the associated closed embedding $j_D \colon \mathcal{Y} \to \mathbf{P}(\Gamma(\mathcal{Y}, \mathcal{M}_D)^\vee)$ and the isomorphism $j_D^*\mathcal{O}(1) \simeq \mathcal{M}_D$ are $\mathcal{G}$-equivariant. The global sections $\Gamma(\mathcal{Y}, \mathcal{M}_D)$ are identified with

$$\Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})^{\mathcal{S}} = (\mathrm{Sym}^D_{\mathfrak{o}_K}(\mathcal{F}^\vee))^{\mathcal{S}}.$$

Since the representation ρ is homogeneous of weight $a = (a_1, \dots, a_N)$, the induced representation on $\mathrm{Sym}^D_{\mathfrak{o}_K}(\mathcal{F}^\vee)$ is homogeneous of weight

$$-Da = (-Da_1, \dots, -Da_N).$$

It follows from Proposition 3.10 (2) applied to $\mathcal{V} = \mathrm{Sym}^D_{\mathfrak{o}_K}(\mathcal{F}^\vee)$ that the action of $\mathcal{G}$ on $\Gamma(\mathcal{Y}, \mathcal{M}_D)$ is given by the representation

$$(g_1, \dots, g_N) \mapsto \prod_{i=1}^{N} (\det g_i)^{-a_i D/e_i} \cdot \mathrm{id}\,.$$

Assertions (1) and (2) are now straightforward. □

Theorem 3.8 follows from the previous Corollary:

Proof of Theorem 3.8. — According to Proposition 3.5 (2), the quotient $\mathcal{Y}_\mathcal{E}$ is the twist of the quotient $\mathcal{Y}$ by $\mathcal{E}$: since $\mathcal{G}$ is acting trivially, one gets an isomorphism

$$\alpha_\mathcal{E} \colon \mathcal{Y}_\mathcal{E} \xrightarrow{\sim} \mathcal{Y}.$$

Similarly, according to Proposition 3.5 (3) and Proposition 3.7, the hermitian line bundle $\overline{\mathcal{M}}_{D,\overline{\mathcal{E}}}$ is obtained twisting the hermitian line bundle $\overline{\mathcal{M}}_D$ by $\overline{\mathcal{E}}$. Since the action of $\mathcal{G}$ on the fibres of $\mathcal{M}_D$ is given by the character

$$(g_1, \dots, g_N) \mapsto \prod_{i=1}^{N} (\det g_i)^{-a_i D/e_i},$$

we get a canonical isomorphism of hermitian line bundles

$$\beta_{\overline{\mathcal{E}}} \colon \overline{\mathcal{M}}_{D,\overline{\mathcal{E}}} \xrightarrow{\sim} \alpha_\mathcal{E}^* \overline{\mathcal{M}}_D \otimes \bigotimes_{i=1}^{N} f^*(\det \overline{\mathcal{E}}_i)^{\vee \otimes a_i D_i/e_i},$$

which concludes the proof. □

4. Lower bound of the height on the quotient

4.1. Statement. — In this section we prove Theorem 1.11. Let us recall here the notations introduced in paragraph 1.5.

Let $N \geq 1$ be a positive integer and let $\overline{\mathcal{E}} = (\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_N)$ be a N-uple of hermitian vector bundles over $\mathfrak{o}_K$ of positive rank. Consider the following $\mathfrak{o}_K$-reductive groups

$$\mathcal{G} = \mathbf{GL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{GL}(\mathcal{E}_N),$$
$$\mathcal{S} = \mathbf{SL}(\mathcal{E}_1) \times_{\mathfrak{o}_K} \cdots \times_{\mathfrak{o}_K} \mathbf{SL}(\mathcal{E}_N),$$

and for every complex embedding $\sigma \colon K \to \mathbf{C}$ consider the maximal compact subgroups,

$$\mathbf{U}_\sigma = \mathbf{U}(\| \cdot \|_{\mathcal{E}_1, \sigma}) \times \cdots \times \mathbf{U}(\| \cdot \|_{\mathcal{E}_N, \sigma}) \subset \mathcal{G}_\sigma(\mathbf{C}),$$
$$\mathbf{SU}_\sigma = \mathbf{SU}(\| \cdot \|_{\mathcal{E}_1, \sigma}) \times \cdots \times \mathbf{SU}(\| \cdot \|_{\mathcal{E}_N, \sigma}) \subset \mathcal{S}_\sigma(\mathbf{C}).$$

Let $\overline{\mathcal{F}}$ be a hermitian vector bundle over $\mathfrak{o}_K$ and let $\rho \colon \mathcal{G} \to \mathbf{GL}(\mathcal{F})$ be a representation which respects the hermitian structures, that is, for every embedding $\sigma \colon K \to \mathbf{C}$ the norm $\| \cdot \|_{\mathcal{F}, \sigma}$ is fixed under the action of the maximal compact subgroup $\mathbf{U}_\sigma$.

Consider the induced action of $\mathcal{S}$ on $\mathcal{F}$ and borrow notation from paragraph 1.4.

Theorem 4.1. — *Let $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ be N-uples of integers. With the notations introduced above, let*

$$\varphi \colon \bigotimes_{i=1}^N \left[\operatorname{End}(\overline{\mathcal{E}}_i)^{\otimes a_i} \otimes \overline{\mathcal{E}}_i^{\otimes b_i} \right] \longrightarrow \overline{\mathcal{F}}$$

be a $\mathcal{G}$-equivariant and generically surjective homomorphism of hermitian vector bundles. Then,

$$h_{\min}((\mathbf{P}(\mathcal{F}), \mathcal{O}_{\overline{\mathcal{F}}}(1)) /\!\!/ \mathcal{S}) \geq - \sum_{i=1}^N b_i \, \hat{\mu}(\overline{\mathcal{E}}_i) - \sum_{i : \operatorname{rk} \mathcal{E}_i \geq 3} \frac{|b_i|}{2} \ell(\operatorname{rk} \mathcal{E}_i),$$

with equality if $b_1, \ldots, b_N = 0$.

The homomorphism φ is $\mathcal{G}$-equivariant and it decreases the v-adic norms at all places v of K (the archimedean ones by hypothesis, the non-archimedean ones because φ is defined at the level of $\mathfrak{o}_K$-modules). For this reason, it suffices to prove Theorem 4.1 in the case $\varphi = \mathrm{id}$, that is:

Theorem 4.2. — *Let $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ be N-uples of integers and set*

$$\overline{\mathcal{F}} := \bigotimes_{i=1}^N \left[\operatorname{End}(\overline{\mathcal{E}}_i)^{\otimes a_i} \otimes \overline{\mathcal{E}}_i^{\otimes b_i} \right].$$

With the notations introduced above,

$$h_{\min}((\mathbf{P}(\mathcal{F}), \mathcal{O}_{\overline{\mathcal{F}}}(1)) /\!\!/ \mathcal{S}) \geq - \sum_{i=1}^N b_i \, \hat{\mu}(\overline{\mathcal{E}}_i) - \sum_{i : \operatorname{rk} \mathcal{E}_i \geq 3} \frac{|b_i|}{2} \ell(\operatorname{rk} \mathcal{E}_i),$$

with equality if $b_1, \ldots, b_N = 0$.

The remainder of Section 4 is devoted to the proof of Theorem 4.2.

4.2. Tensor products of endomorphisms algebras. —

4.2.1. Notation. — In this section we are going to prove Theorem 4.2 in the case $b_i = 0$ for all $i = 1, \ldots, N$, that is, in the case

$$\overline{\mathcal{F}} = \bigotimes_{i=1}^{N} \mathrm{End}(\overline{\mathcal{E}}_i^{\otimes a_i}),$$

for a N-uple of integers $a = (a_1, \ldots, a_N)$.

Theorem 4.3. — *With the notation introduced above,*

$$h_{\min}((\mathbf{P}(\mathcal{F}), \overline{\mathcal{O}(1)}) /\!\!/ \mathcal{S}) = 0.$$

Begin with the easy inequality:

Proposition 4.4. — *With the notation introduced above,*

$$h_{\min}((\mathbf{P}(\mathcal{F}), \overline{\mathcal{O}(1)}) /\!\!/ \mathcal{S}) \leq 0.$$

Proof. — Thanks to the canonical isomorphism $\mathrm{End}(\mathcal{E}_i^{\otimes a_i}) \simeq \mathrm{End}(\mathcal{E}_i^{\vee \otimes a_i})$ the integers a_i may be supposed non-negative.

According to Theorem 3.8 it suffices to show this when $K = \mathbf{Q}$ and the hermitian vector bundles $\overline{\mathcal{E}}_i$ are trivial, that is, for all $i = 1, \ldots, N$, the hermitian vector bundle $\overline{\mathcal{E}}_i$ is the $\mathbf{Z}$-module $\mathcal{E}_i = \mathbf{Z}^{e_i}$ endowed with the standard hermitian norm. Consider the hermitian vector bundle $\overline{\mathcal{E}}' := \overline{\mathcal{E}}_1^{\otimes a_1} \otimes \cdots \otimes \overline{\mathcal{E}}_N^{\otimes a_N}$. Identify it with the trivial vector bundle given by $\mathcal{E}' = \mathbf{Z}^{a_1 e_1 + \cdots + a_N e_N}$ endowed with the standard hermitian norm. With this notation one has a canonical $\mathcal{S}$-equivariant isomorphism of hermitian vector bundles $\overline{\mathcal{F}} \simeq \mathrm{End}(\overline{\mathcal{E}}')$.

Consider the endomorphism φ of $\mathcal{E}'$ given by the matrix (with respect the canonical basis of $\mathcal{E}'$) whose $(1,1)$-entry is 1 and the other entries are 0:

$$\varphi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The point $[\varphi]$ of $\mathbf{P}(\mathcal{F})$ is semi-stable under the action by conjugation of $\mathbf{SL}(\mathcal{E}')$ (hence with respect to the action of $\mathcal{S}$) because it is not nilpotent. If $h_{\overline{\mathcal{M}}}$ denotes the height on the quotient $\mathcal{Y}$ of $\mathbf{P}(\mathcal{F})^{\mathrm{ss}}$ by $\mathcal{S}$ and $\pi \colon \mathbf{P}(\mathcal{F})^{\mathrm{ss}} \to \mathcal{Y}$ is the quotient map, the Fundamental Formula for projective spaces (Corollary 1.6) gives

$$h_{\overline{\mathcal{M}}}(\pi([\varphi])) \leq h_{\mathcal{O}_{\overline{\mathcal{F}}}(1)}([\varphi]) = 0,$$

which concludes the proof. □

It remains to prove the converse inequality. It suffices to show:

Theorem 4.5. — *Let $\varphi \in \mathcal{F} \otimes_{\mathfrak{o}_K} K$ be a non-zero vector such that the associated K-point $[\varphi]$ of $\mathbf{P}(\mathcal{F})$ is semi-stable. Then,*

$$\sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{S}(\mathbf{C}_v)} \frac{\|g \cdot \varphi\|_{\mathcal{F},v}}{\|\varphi\|_{\mathcal{F},v}} \geq 0.$$

Indeed, Theorem 4.3 is deduced applying Theorem 4.5 and the Fundamental Formula (in the form given by Corollary 1.6) to every finite extension K' of K and to every semi-stable point of $\mathbf{P}(\mathcal{F})$ defined over K'.

The remainder of this section is devoted to the proof of Theorem 4.5.

4.2.2. The case of a non-nilpotent endomorphism. — Consider φ as an endomorphism of the K-vector space $\bigotimes_{i=1}^N \mathcal{E}_i^{\otimes a_i} \otimes_{\mathfrak{o}_K} K$ thanks to the canonical isomorphism

$$\alpha \colon \bigotimes_{i=1}^{N} \mathrm{End}(\mathcal{E}_i^{\otimes a_i}) \simeq \mathrm{End}\left(\bigotimes_{i=1}^{N} \mathcal{E}_i^{\otimes a_i}\right).$$

With this identification assume that φ is not nilpotent. Then the point $[\varphi]$ is semi-stable under the action $\mathcal{S}$. Actually, something more is true: consider the $\mathfrak{o}_K$-reductive group

$$\mathcal{H} := \mathbf{SL}(\mathcal{E}_1^{\otimes a_1} \otimes \cdots \otimes \mathcal{E}_N^{\otimes a_N})$$

and its action by conjugation on $\mathcal{F}$ (through the isomorphism α). According to Corollary 2.3, the point $[\varphi]$ is semi-stable under the action of $\mathcal{H}$. Thus $[\varphi]$ is semi-stable with respect to $\mathcal{S}$ and, for every place v of K,

$$\inf_{g \in \mathcal{S}(\mathbf{C}_v)} \|g \cdot \varphi\|_{\mathcal{F},v} \geq \inf_{h \in \mathcal{H}(\mathbf{C}_v)} \|h\varphi h^{-1}\|_{\mathcal{F},v}. \tag{4.2.1}$$

The isomorphism α is an isometry as soon as endow $\mathcal{F}$ with the hermitian norms deduced from the identification

$$\mathrm{End}\left(\bigotimes_{i=1}^{N} \mathcal{E}_i^{\otimes a_i}\right) = \left(\bigotimes_{i=1}^{N} \mathcal{E}_i^{\otimes a_i}\right)^{\vee} \otimes_{\mathfrak{o}_K} \left(\bigotimes_{i=1}^{N} \mathcal{E}_i^{\otimes a_i}\right).$$

Therefore one can apply Theorem 2.4 to $[\varphi]$ and obtain that

$$[\Omega : K]\left(\sum_{v \in \mathrm{V}_K} \log \inf_{h \in \mathcal{H}(\mathbf{C}_v)} \|h\varphi h^{-1}\|_{\mathcal{F},v}\right)$$

is equal to

$$\sum_{\substack{v \in \mathrm{V}_\Omega \\ \text{non-arch.}}} \log \max\{|\lambda_1|_v, \ldots, |\lambda_n|_v\} + \sum_{\sigma \colon \Omega \to \mathbf{C}} \log \sqrt{|\lambda_1|_\sigma^2 + \cdots + |\lambda_n|_\sigma^2},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of φ (counted with multiplicities) and Ω is a number field containing them. Since the latter quantity is non-negative, taking the sum of (4.2.1) over all places,

$$\sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{S}(\mathbf{C}_v)} \|g \cdot \varphi\|_{\mathcal{F},v} \geq \sum_{v \in \mathrm{V}_K} \log \inf_{h \in \mathcal{H}(\mathbf{C}_v)} \|h\varphi h^{-1}\|_{\mathcal{F},v} \geq 0,$$

this gives Theorem 4.5 in this case.

4.2.3. The case of a non-vanishing invariant linear form. — Suppose that there exists a $\mathcal{S}$-invariant linear form $f \in \Gamma(\mathbf{P}(\mathcal{F}), \mathcal{O}(1)) = \mathcal{F}^\vee$ that does not vanish at $[\varphi]$. To treat this case one needs some information describing the form of this invariants given by the First Main Theorem of Invariant Theorem. Thus let us recall it here again.

For every $i = 1, \ldots, N$ let $\mathfrak{S}_{|a_i|}$ be the permutation group on $|a_i|$ elements. For a permutation σ let $\varepsilon_{i,\sigma}$ be the automorphism of $\mathcal{E}_i^{\otimes a_i}$ permuting factors by σ. Seen as an element of $\mathrm{End}(\mathcal{E}_i^{\otimes a_i})$, the endomorphism $\varepsilon_{i,\sigma}$ is invariant under conjugation by $\mathbf{SL}(\mathcal{E}^{\otimes a_i})$. Therefore, for every N-uple of permutations,

$$\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_{|a_1|} \times \cdots \times \mathfrak{S}_{|a_N|}$$

the endomorphism $\varepsilon_\sigma = \varepsilon_{1,\sigma_1} \otimes \cdots \otimes \varepsilon_{N,\sigma_N} \in \mathcal{F}$ is invariant under the action of $\mathcal{S}$.

We can now state the First Main Theorem of Invariant Theory (*cf.* [**Wey39**, Chapter III], [**Che09**, Theorem 3.1, Corollary] and [**ABP73**, Appendix 1]):

Theorem 4.6 **(First Main Theorem of Invariant Theory)**
The subspace of elements of $\mathcal{F} \otimes_{\mathfrak{o}_K} K$ which are invariant under the action of $\mathcal{S} \times_{\mathfrak{o}_K} K$ is generated, as a K-linear space, by the elements ε_σ, while σ ranges in $\mathfrak{S}_{|a_1|} \times \cdots \times \mathfrak{S}_{|a_N|}$.

For every N-uple of permutations $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_{|a_1|} \times \cdots \times \mathfrak{S}_{|a_N|}$ denote by $\varepsilon_\sigma^\vee$ its image by the canonical isomorphism

$$\mathcal{F} = \bigotimes_{i=1}^{N} \mathrm{End}(\mathcal{E}_i^{\otimes a_i}) \simeq \mathcal{F}^\vee = \bigotimes_{i=1}^{N} \mathrm{End}(\mathcal{E}_i^{\vee \otimes a_i}).$$

Let us resume the proof of Theorem 4.5. Since there is a $\mathcal{S}$-invariant linear form non-vanishing on $[\varphi]$, according to Theorem 4.6, there exists a suitable N-uple of permutations $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_{|a_1|} \times \cdots \times \mathfrak{S}_{|a_N|}$ such that $\varepsilon_\sigma^\vee(\varphi) \neq 0$. By definition,

$$\varepsilon_\sigma^\vee(\varphi) = \mathrm{Tr}(\varphi \circ \varepsilon_{\sigma^{-1}}),$$

Since the trace of the endomorphism $\varphi \circ \varepsilon_{\sigma^{-1}}$ is non-zero it is not nilpotent. Therefore, the preceding case implies:

$$\sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{S}(\mathbf{C}_v)} \|g \cdot (\varphi \circ \varepsilon_{\sigma^{-1}})\|_{\mathcal{F},v} \geq 0. \tag{4.2.2}$$

Remark the following facts:

(1) For every place v of K and every non-zero vector $\psi \in \mathcal{F} \otimes_{\mathfrak{o}_K} \mathbf{C}_v$,

$$\|\psi \circ \varepsilon_{\sigma^{-1}}\|_{\mathcal{F},v} = \|\psi\|_{\mathcal{F},v}.$$

(2) The endomorphism $\varepsilon_{\sigma^{-1}}$ commutes with the action of $\mathcal{S}$ (it is the definition of $\mathcal{S}$-invariance).

As a consequence of these considerations, for every $g \in \mathcal{S}(\mathbf{C}_v)$,

$$\|g \cdot \varphi\|_{\mathcal{F},v} = \|g \cdot (\varphi \circ \varepsilon_{\sigma^{-1}})\|_{\mathcal{F},v}.$$

Summing over all places, the preceding equality together with (4.2.2) entails

$$\sum_{v\in \mathrm{V}_{K'}} \log \inf_{g\in\mathcal{S}(\mathbf{C}_v)} \|g\cdot\varphi\|_{\mathcal{F},v} \geq 0,$$

which proves Theorem 4.5 in this case.

4.2.4. The general case. — Let us finally treat the general case. By definition of semi-stability there exist a positive integer $D\geq 1$ and a $\mathcal{S}$-invariant global section

$$f\in\Gamma(\mathbf{P}(\mathcal{F}),\mathcal{O}(D)) = \mathrm{Sym}^D(\mathcal{F}^\vee)$$

that does not vanish at the point $[\varphi]$. Consider the D-fold Veronese embedding

$$\begin{array}{ccc} \mathbf{P}(\mathcal{F}) & \longrightarrow & \mathbf{P}(\mathcal{F}^{\otimes D}) \\ [\varphi] & \longmapsto & [\varphi^{\otimes D}]. \end{array}$$

The point $[\varphi^{\otimes D}]$ is a semi-stable point of $\mathbf{P}(\mathcal{F}^{\otimes D})$. Since the homomorphism $\mathcal{F}^{\vee\otimes D}\to\mathrm{Sym}^D(\mathcal{F}^\vee)$ is surjective and $\mathcal{S}$-equivariant, and since the point P is defined on a field of characteristic 0, f can be supposed being the image of a $\mathcal{S}$-invariant element f' of $\mathcal{F}^{\vee\otimes D}\otimes_{\mathfrak{o}_K} K$. [5]

Up to rescaling f' one may assume that there exists a $\mathcal{S}$-invariant linear form $f'\in\Gamma(\mathbf{P}(\mathcal{F}^{\otimes D}),\mathcal{O}(1))$ which does not vanish at $[\varphi^{\otimes D}]$. Therefore one may apply the preceding case to $\varphi^{\otimes D}$ and obtain

$$\sum_{v\in \mathrm{V}_K} \log \inf_{g\in\mathcal{S}(\mathbf{C}_v)} \|g\cdot\varphi^{\otimes D}\|_{\mathcal{F}^{\otimes D},v} \geq 0.$$

For every place v of K and for every $g\in\mathcal{S}(\mathbf{C}_v)$,

$$\|g\cdot\varphi^{\otimes D}\|_{\mathcal{F}^{\otimes D},v} = \left(\|g\cdot\varphi\|_{\mathcal{F},v}\right)^D,$$

which concludes the proof of Theorem 4.5. □

4.3. The general case. —

4.3.1. Notation. — In this section we will prove the general case of Theorem 4.2. As for Theorem 4.3 this is deduced from the following:

Theorem 4.7. — *Let $\varphi\in\mathcal{F}\otimes_{\mathfrak{o}_K} K$ be a non-zero vector such that the associated K-point $P=[x]$ of $\mathbf{P}(\mathcal{F})$ is semi-stable. Then,*

$$\sum_{v\in \mathrm{V}_K} \log \inf_{g\in\mathcal{S}(\mathbf{C}_v)} \frac{\|g\cdot x\|_{\mathcal{F},v}}{\|x\|_{\mathcal{F},v}} \geq 0.$$

The rest of this section is devoted to the proof of Theorem 4.7.

5. Let k be a field and let V,W be representation of a reductive k-group G. A G-equivariant homomorphism $\varphi\colon V\to W$ induces a linear homomorphism$\varphi\colon V^G\to W^G$. If φ is surjective and k is of characteristic 0 then the homomorphism $V^G\to W^G$ is surjective [**MS72**, pages 181-182].

4.3.2. The case of a non-vanishing invariant linear form. — Suppose that there is a $\mathcal{S}$-invariant linear form which does not vanish at the point P. In particular, the submodule of $\mathcal{S}$-invariant elements of

$$\mathcal{F}^\vee = \bigotimes_{i=1}^{N} \left[\mathrm{End}(\mathcal{E}_i)^{\vee\otimes a_i} \otimes \mathcal{E}_i^{\vee\otimes b_i}\right]$$

is non-zero. Therefore Proposition 3.10 (1) implies that $e_i = \mathrm{rk}\,\mathcal{E}_i$ divides b_i for every $i = 1, \ldots, N$. The idea is to embed conveniently $\mathbf{P}(\mathcal{F})$ and deduce Theorem 4.7 in this case from Theorem 4.3.

Definition 4.8. — Fix an integer $i \in \{1, \ldots, N\}$.

(1) For $e_i = 2$, let ε_i the isomorphism of $\mathfrak{o}_K$-modules

$$\varepsilon_i \colon \mathcal{E}_i^{\otimes 2} \longrightarrow \mathrm{End}(\mathcal{E}_i) \otimes \det \mathcal{E}_i$$

whose inverse is given, for every $\varphi \in \mathrm{End}\,\mathcal{E}_i$ and every $v_1, v_2 \in \mathcal{E}_i$, by the map

$$\varphi \otimes (v_1 \wedge v_2) \mapsto \varphi(v_1) \otimes v_2 - \varphi(v_2) \otimes v_1.$$

(2) For $e_i \neq 2$, let ε_i be the homomorphism of $\mathfrak{o}_K$-modules

$$\varepsilon_i \colon \mathcal{E}_i^{\otimes e_i} \longrightarrow \mathrm{End}(\mathcal{E}_i^{\otimes e_i}) \otimes \det \mathcal{E}_i,$$

whose dual map,

$$\varepsilon_i^\vee \colon \mathrm{End}(\mathcal{E}_i^{\vee\otimes e_i}) \otimes \det \mathcal{E}_i^\vee \longrightarrow \mathcal{E}_i^{\vee\otimes e_i},$$

is defined as follows: for every $\varphi \in \mathrm{End}(\mathcal{E}_i^{\vee\otimes e_i})$ and every $v_1, \ldots, v_{e_i} \in \mathcal{E}_i^\vee$, the image of the element $\varphi \otimes (v_1 \wedge \cdots \wedge v_{e_i})$ is

$$\sum_{\gamma \in \mathfrak{S}_{e_i}} \mathrm{sign}(\gamma) \varphi(v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(e_i)}) \otimes (v_1 \wedge \cdots \wedge v_{e_i}).$$

Endow the $\mathfrak{o}_K$-modules $\mathrm{End}(\mathcal{E}_i) \otimes \det \mathcal{E}_i$ and $\mathrm{End}(\mathcal{E}_i^{\otimes e_i}) \otimes \det \mathcal{E}_i$ with the hermitian norms deduced (by taking tensor products, dual and the determinant) from the hermitian norm $\| \cdot \|_{\mathcal{E}_i}$.

Proposition 4.9. — *Fix an integer $i \in \{1, \ldots, N\}$. With the notations introduced above:*

(1) for $e_i = 2$, ε_i is a $\mathbf{GL}(\mathcal{E}_i)$-equivariant isomorphism of hermitian vector bundles;

(2) for $e_i \neq 2$, ε_i is $\mathbf{GL}(\mathcal{E}_i)$-equivariant and, for an embedding $\sigma \colon K \to \mathbf{C}$,

$$\sup_{x \neq 0} \frac{\|\varepsilon_i(x)\|_\sigma}{\|x\|_\sigma} \leq \sqrt{e_i!},$$

where the supremum is ranging on the elements of $\mathcal{E}_i^{\otimes e_i} \otimes_\sigma \mathbf{C}$, the norm in the numerator is the one of $\mathrm{End}(\mathcal{E}_i^{\otimes e_i}) \otimes \det \mathcal{E}_i$ and the norm in the denominator is the one of $\mathcal{E}_i^{\otimes e_i}$.

Proof. — The fact that the map ε_i is $\mathbf{GL}(\mathcal{E}_i)$-equivariant is clear in both cases. Let $\sigma\colon K \to \mathbf{C}$ be a complex embedding and write

$$W := \mathcal{E}_i \otimes_\sigma \mathbf{C}, \quad w := \dim_{\mathbf{C}} W = e_i, \quad \|\cdot\|_W := \|\cdot\|_{\mathcal{E}_i,\sigma}, \quad f := \varepsilon_i.$$

Let $x_1, \dots, x_w$ be an orthonormal basis of W.

(1) If $w = 2$ for every endomorphism φ of W:

$$\begin{aligned}\|f^{-1}(\varphi \otimes (x_1 \wedge x_2))\|_{W^{\otimes 2}} &= \|\varphi(x_1) \otimes x_2 - \varphi(x_2) \otimes x_1\|_{W^{\otimes 2}} \\ &= \sqrt{\|\varphi(x_1)\|_W^2 + \|\varphi(x_2)\|_W^2} \\ &= \|\varphi\|_{\mathrm{End}(W)} \\ &= \|\varphi \otimes (x_1 \wedge x_2)\|_{\mathrm{End}(W)\otimes \det W},\end{aligned}$$

which shows that f^{-1} (thus f) is an isometry.

(2) Suppose $w \neq 2$. For every w-uple $R = (r_1, \dots, r_w)$ made of integers such that $r_\alpha \in \{1, \dots, w\}$ ($\alpha = 1, \dots, w$) set $x_R := x_{r_1} \otimes \cdots \otimes x_{r_w}$. While R ranges in the set $\{1, \dots, w\}^w$ the vectors x_R form an orthonormal basis of the vector space $W^{\otimes w}$. For every element $t \in W^{\otimes w}$ write

$$t = \sum_{R \in \{1,\dots,w\}^w} t_R x_R.$$

Let $x_1^\vee, \dots, x_w^\vee$ be the basis of $W^\vee$ dual to $x_1, \dots, x_w$ and for every permutation $\gamma \in \mathfrak{S}_w$ write $x_\gamma^\vee = x_{\gamma(1)} \otimes \cdots \otimes x_{\gamma(w)}$. With this notation, for every $t \in W^{\otimes w}$, the map f is expressed as follows:

$$f(t) = \sum_{R \in \{1,\dots,w\}^w} \sum_{\gamma \in \mathfrak{S}_w} \mathrm{sign}(\gamma) t_R x_R \otimes x_\gamma^\vee \otimes (x_1 \wedge \cdots \wedge x_w).$$

Taking the norm:

$$\|f(t)\|^2_{\mathrm{End}(W)^{\otimes w} \otimes \det W} = \sum_{R \in \{1,\dots,w\}^w} |t_R|^2 \leq w! \cdot \|t\|^2_{W^{\otimes w}},$$

which gives the result. □

For every $i = 1, \dots, N$ the homomorphism $\varepsilon_i^{\otimes b_i/e_i}$ induces, through the identification $\mathcal{E}^{b_i} \simeq (\mathcal{E}^{\otimes e_i})^{\otimes b_i/e_i}$, the following homomorphisms:

$$(e_i = 2) \qquad \varepsilon_i^{\otimes b_i/2}\colon \mathcal{E}_i^{\otimes b_i} \longrightarrow \mathrm{End}(\mathcal{E}_i)^{\otimes b_i/2} \otimes (\det \mathcal{E}_i)^{\otimes b_i/2},$$

$$(e_i \neq 2) \qquad \varepsilon_i^{\otimes b_i/e_i}\colon \mathcal{E}_i^{\otimes b_i} \longrightarrow \mathrm{End}(\mathcal{E}_i)^{\otimes b_i} \otimes (\det \mathcal{E}_i)^{\otimes b_i/e_i}.$$

For every $i = 1, \dots, N$ consider the $\mathfrak{o}_K$-module

$$\mathcal{F}_i' := \begin{cases} \mathrm{End}(\mathcal{E}_i)^{\otimes a_i + b_i/2} & \text{if } e_i = 2, \\ \mathrm{End}(\mathcal{E}_i)^{\otimes a_i + b_i} & \text{otherwise} \end{cases}$$

and the homomorphism of $\mathfrak{o}_K$-modules

$$\eta_i = \mathrm{id} \otimes \varepsilon_i^{\otimes b_i/e_i}\colon \mathrm{End}(\mathcal{E}_i)^{\otimes a_i} \otimes \mathcal{E}_i^{\otimes b_i} \longrightarrow \mathcal{F}_i' \otimes \det \mathcal{E}_i^{\otimes b_i/e_i}.$$

Set $\mathcal{F}' := \mathcal{F}'_1 \otimes \cdots \otimes \mathcal{F}'_N$. The homomorphism $\eta = \eta_1 \otimes \cdots \otimes \eta_N$ gives rise to an injective $\mathcal{G}$-equivariant homomorphism of $\mathfrak{o}_K$-modules $\eta\colon \mathcal{F} \to \mathcal{F}' \otimes \mathcal{D}$, where

$$\mathcal{D} := \bigotimes_{i=1}^{N} \det \mathcal{E}_i^{\otimes b_i/e_i}.$$

Endow $\mathcal{F}'$ and $\mathcal{D}$ of the hermitian norms deduced from the hermitian norms on $\mathcal{E}_i$. Passing to the projective spaces, it is induces a $\mathcal{G}$-equivariant closed embedding

$$\eta\colon \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{F}' \otimes \mathcal{D}).$$

Therefore, since the point P is defined on a field of characteristic 0 (see footnote 5), the image $\eta(P)$ is a semi-stable K-point of $\mathbf{P}(\mathcal{F}' \otimes \mathcal{D})$ with respect to the action of $\mathcal{S}$. If $x \in \mathcal{F} \otimes_{\mathfrak{o}_K} K$ is a non-zero representative of P, the Fundamental Formula for projective spaces (Corollary 1.6) and Proposition 4.9 entail:

$$\begin{aligned} h_{\overline{\mathcal{M}}}(\pi(P)) &= \sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{S}(\mathbf{C}_v)} \frac{\|g \cdot x\|_{\mathcal{F},v}}{\|x\|_{\mathcal{F},v}} \\ &\geq \sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{S}(\mathbf{C}_v)} \frac{\|g \cdot \eta(x)\|_{\mathcal{F}'\otimes\mathcal{D},v}}{\|\eta(x)\|_{\mathcal{F}\otimes\mathcal{D},v}} - \sum_{i : e_i \geq 3} \frac{|b_i|}{2} \ell(e_i). \end{aligned}$$

Since the action of $\mathcal{S}$ is trivial on the line bundle $\mathcal{D}$, the canonical isomorphism $\alpha\colon \mathbf{P}(\mathcal{F}' \otimes \mathcal{D}) \to \mathbf{P}(\mathcal{F}')$ is $\mathcal{S}$-equivariant. Moreover, it induces an isomorphism of hermitian line bundles

$$\alpha^* \mathcal{O}_{\overline{\mathcal{F}}'}(1) \simeq \mathcal{O}_{\overline{\mathcal{F}}' \otimes \overline{\mathcal{D}}}(1) \otimes f^* \overline{\mathcal{D}}^{\vee},$$

where $f\colon \mathbf{P}(\mathcal{F}') \to \operatorname{Spec} \mathfrak{o}_K$ is the structural morphism.

Let $\mathcal{Y}'$ be categorical quotient of $\mathbf{P}(\mathcal{F})^{\mathrm{ss}}$ by $\mathcal{S}$ and let $\pi'\colon \mathbf{P}(\mathcal{F}')^{\mathrm{ss}} \to \mathcal{Y}'$ be the quotient map. Denote by $h_{\overline{\mathcal{M}}'}$ is the height on the quotient $\mathcal{Y}'$ (with respect to $\mathcal{S}$ and $\mathcal{O}_{\overline{\mathcal{F}}'}(1)$). Applying again the Fundamental Formula, one finds

$$\sum_{v \in \mathrm{V}_K} \log \inf_{g \in \mathcal{S}(\mathbf{C}_v)} \frac{\|g \cdot \eta(x)\|_{\mathcal{F}'\otimes\mathcal{D},v}}{\|\eta(x)\|_{\mathcal{F}\otimes\mathcal{D},v}} = h_{\overline{\mathcal{M}}'}(\pi'(\alpha \circ \eta(P))) - \sum_{i=1}^{N} b_i \,\hat{\mu}(\overline{\mathcal{E}}_i),$$

so that, putting all together, one obtains

$$h_{\overline{\mathcal{M}}}(\pi(P)) \geq h_{\overline{\mathcal{M}}'}(\pi'(\alpha \circ \eta(P))) - \sum_{i=1}^{N} b_i \,\hat{\mu}(\overline{\mathcal{E}}_i) - \sum_{i : e_i \geq 3} \frac{|b_i|}{2} \ell(e_i).$$

Thanks to Theorem 4.3 the height $h_{\overline{\mathcal{M}}'}$ is non-negative, which concludes the proof of Theorem 4.7 in this case.

4.3.3. The general case. — Suppose that there exists a $\mathcal{S}$-invariant global section $s \in \Gamma(\mathbf{P}(\mathcal{F}), \mathcal{O}(D))$ that does not vanish at P. One argues as in paragraph 4.2.4 — namely, taking the D-uple embedding $\mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{F}^{\otimes D})$ and applying the preceding case. These details are left to the reader. This concludes of the proof of Theorem 4.7, hence of Theorem 4.2. □

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