I will present 3 arithmetic algebraisation Theorems

$$\overset{3}{A} \Rightarrow \overset{1}{B} \Rightarrow \overset{2}{C}$$

Theorem B is an algebraisation theorem for foliations on varieties over number fields, and theorem C is its corollary for algebraic groups, where the foliation is given by a Lie-subalgebra. Theorem A is an algebraisation theorem for formal subschemes.

Examples come along the way, and I try to explain the proof of ~~th~~ theorem A in a special, simplified setup. All theorems are in the reference BOS01.

Theorem B Let X be a smooth variety over a number field k, let $F \subseteq T_X$ be an involutive subbundle and let $P \in X(k)$ be a point. The formal leaf of F through P is algebraic if the following conditions are satisfied.

$\boxed{B1}$ Choose a model $(\mathcal{X}, \mathcal{F}, \mathcal{P})$ of (X, F, P) over some open $U \subseteq \operatorname{spec} \mathcal{O}_k$. For almost all $\mathfrak{p} \in U$ the subbundle $\mathcal{F}_{\mathfrak{p}}$ of $T_{\mathcal{X}_{\mathfrak{p}}}$ is stable under p-th powers ($p = \operatorname{char} \mathcal{O}_k/\mathfrak{p}$)

$\boxed{B2}$ For some $k \hookrightarrow \mathbb{C}$, the analytic leaf of F_σ through P_σ contains an open nbhood of P_σ which has the Liouville property.

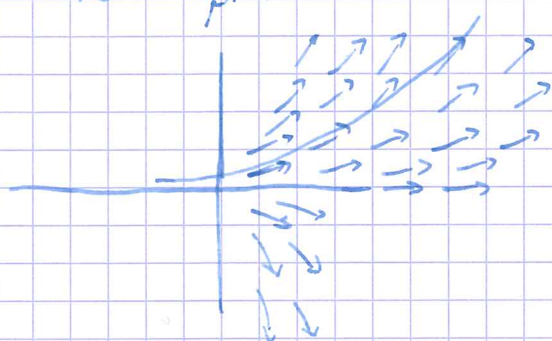
If you do not like the Liouville property, B2 can be replaced by

$\boxed{B2'}$ For some $k \xhookrightarrow{\sigma} \mathbb{C}$ there exists a complex algebraic Variety $\mathcal{U}$, a point $o \in \mathcal{U}$ and a ~~holomorphic~~ meromorphic map $\mathcal{U}(\mathbb{C}) \xrightarrow{u} X_\sigma(\mathbb{C})$ with $u(0) = P_\sigma$, mapping a nbhood of o biholomorphically onto the germ of F_σ at P_σ.

As an illustration, and as a review of material, let us consider the case k = any number field, $X = \mathbb{A}^2_k$ with coordinates x, y and the Vectorfield

$$F(x,y) = (1, ay) \qquad a \in k \text{ fixed.}$$

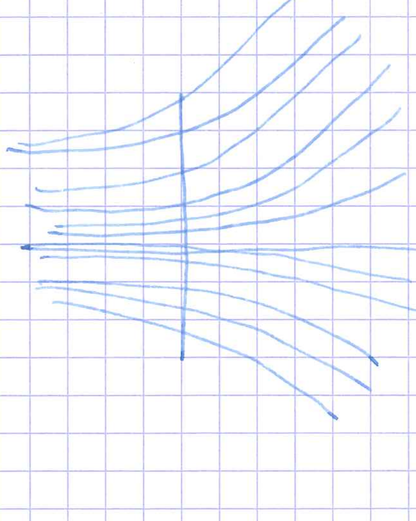
The "real picture"



Characteristic curves:

$$\gamma_c(z) : z \mapsto (z, ce^{az}) \qquad c \neq 0$$
$$\gamma_0(z) : \qquad (z, 0)$$

The foliation



Leaves of the analytic Foliation are (images of) Characteristic curves

$$\dot{\gamma}_c(z) = (1, ac\, e^{az})$$
$$F(\gamma_c(z)) = (1, ac \cdot e^{az})$$

Let us choose $P = (0,1)$ - the analytic leaf of F through P is the image of γ_1, that is, for $\sigma : k \hookrightarrow \mathbb{C}$

$$\gamma_1(\mathbb{C}) = \{ (x,y) \in \mathbb{C}^2 \mid y = \exp(\sigma(a) \cdot x) \}$$

Condition $\boxed{B2'}$ is satisfied! Good hope! Lets verify $\boxed{B1}$!
We must compute p-th powers modulo $\mathfrak{p}$, so lets look at F as a derivation:

$$\partial_F : f(x,y) \longmapsto \frac{\partial f}{\partial x}(x,y) + ay \frac{\partial f}{\partial y}(x,y)$$
$$x \longmapsto 1$$
$$y \longmapsto ay$$

So, modulo $\mathfrak{p}$, ∂_F^p is the derivation given by

$$\partial_F^p \quad x \longmapsto 0$$
$$y \longmapsto a^p y$$

Condition $\boxed{B1}$ wants that ∂_F^p is a scalar multiple of ∂_F mod $\mathfrak{p}$

$$(0, a^p y) = \lambda_{\mathfrak{p}} \cdot (1, ay) \quad \text{mod } \mathfrak{p}$$

$$\Rightarrow \lambda_{\mathfrak{p}} = 0 \quad \text{only choice}$$
$$\Rightarrow a^p \equiv 0 \quad \text{mod } \mathfrak{p} \qquad = a = 0 \text{ mod } \mathfrak{p}$$

So $\boxed{B1}$ is satisfied iff $a \in \mathfrak{p}$ for almost all $\mathfrak{p}$, but that happens only if $a = 0$.
Conclusion: The formal leaf of F through $P = (0,1)$, i.e. the graph of the exponential function $z \mapsto \exp(az)$ is algebraic iff $a = 0$. Big surprise.

Let us specialise to the case where X is an algebraic group (not affine, without loosing a lot of generality commutative)

<u>Theorem C</u> Let G be an alg. group over k, and let h be a Lie-subalgebra of $g = Lie(G)$. If the following two conditions are satisfied, then there exists an alg. subgroup $H \subseteq G$ with $Lie(H) = h$.

$\boxed{C1}$ For almost all $\mathfrak{p} \in \operatorname{spec} \mathcal{O}_k$, the Lie-algebra $(h \bmod \mathfrak{p}) \subseteq (g \bmod \mathfrak{p})$ is stable under the p-th power map.

$\boxed{C2}$ Sea gulls are noisy.

(and conversely, if ~~h~~ $h = Lie(H)$, then $\boxed{C1}$ and $\boxed{C2}$ hold)

I give an other, somewhat more satisfactory illustration. Consider the group $G = \mathbb{G}_m \times \mathbb{G}_m$ over k. With coordinates x, y, the Lie-algebra of G is the k-vsp. gen. by $x\,dx$ and $y\,dy$. Consider a linear subspace h, generated by $x\,dx + ay\,dy$, $a \in k$. As a derivation, $x \mapsto x$, $y \mapsto ay$. Stability under p-th powers: $a^p = a$ i.e. $a \in \mathbb{F}_p \subseteq \mathcal{O}_k/\mathfrak{p}$ for all $\mathfrak{p}$. The 'leaf' through $(1,1) \in G$ is the image of $z \mapsto (\exp(z), \exp(az))$ which is algebraic iff $a \in \mathbb{Q}$. Consequence (Kronecker, 1880) An alg. number $a \in k \subseteq \overline{\mathbb{Q}}$ is rational iff ~~a∈𝔽p~~ $[a] \in \mathcal{O}_k/\mathfrak{p}$ is in $\mathbb{F}_p$ for a. all $\mathfrak{p}$.

After this, let us set up the stage for Theorem A. We need the notion of ~~[illegible]~~ p-adic size of a smooth formal subscheme of a variety X/k. Fix a prime number p.

Let $K/\mathbb{Q}_p$ be a finite field extension. We want to define the size of a formal subscheme $\hat{V} \subseteq \hat{\mathbb{A}}^d_K$. If $\hat{V}$ is given by the image of

$$u_g : \hat{\mathbb{A}}^v_K \xrightarrow[(g_1,\dots,g_d)]{} \hat{\mathbb{A}}^d_K \qquad u_g = d \text{ power series in } v \text{ variables}$$

then the size of $\hat{V}$ should be large if $g_1, \dots, g_d$ have large radii of convergence, and small (even zero) if not. Make this intrinsic.

For a power series $g \in K[[x_1, \dots, x_d]]$, say $g = \sum_{\underline{n} \in \mathbb{N}^d} a_{\underline{n}} x^{\underline{n}}$, set

$$\|g\|_r := \sup \left\{ |a_{\underline{n}}| \cdot r^{|\underline{n}|} \;\middle|\; \underline{n} \in \mathbb{N}^d \right\}$$
$$\in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

An automorphism of $\hat{\mathbb{A}}^d_K$ consists of d series $(g_1, \dots, g_d) = \underline{g}$ with $g(0) = 0$ and $Dg(0) \in GL_d(K)$. Set, for $r > 0$,

$$G(r) := \left\{ \underline{g} \in \mathrm{Aut}(\hat{\mathbb{A}}^d_K) \;\middle|\; \begin{array}{l} Dg(0) \in GL_d(\mathcal{O}_K) \text{ and} \\ \|g_i\|_r \leq r \quad \text{for } 1 \leq i \leq d \end{array} \right\}$$

= "Automorphisms of the open ball of radius r fixing the origin"

Have $G(r) \subseteq G(r') \dots \subseteq G(0)$.
$r \geq r'$

← There is a mistake with $G(0)$. Better look at Bost's original

For a formal smooth subscheme $\hat{V}$ of $\hat{\mathbb{A}}^d_K$ there exists $g \in G(0)$ with $g^*(\hat{V}) = \hat{\mathbb{A}}^v_K \times \{0\}$, $v = \dim \hat{V}$.

Definition The size of $\hat{V} \subseteq \hat{\mathbb{A}}^d_K$, smooth formal subscheme, is

$$R(\hat{V}) := \sup\left\{ r \in [0,1] \;\middle|\; \begin{array}{l} \text{there exists } g \in G(r) \text{ with} \\ g^*(\hat{V}) = \hat{\mathbb{A}}^{\nu}_K \times 0 \end{array} \right\}$$

Extend this to formal subschemes of varieties, so far: Let $\mathcal{X}/\operatorname{spec}\mathcal{O}_K$ be an $\mathcal{O}_K$-scheme o.f.t., $\mathcal{P} \in \mathcal{X}(\mathcal{O}_K)$, and $\hat{V} \subseteq \hat{X}_P$ a smooth formal subscheme. Choose an open nbhd $\mathcal{U}$ of $\mathcal{P}$ in $\mathcal{X}$ and $\mathcal{U} \xrightarrow{i} \hat{\mathbb{A}}^d_{\mathcal{O}_K}$ mapping $\mathcal{P}$ to 0. Then define the size of $\hat{V}$ as the size of $i(\hat{V})$. This is independent of choices. (but $\mathcal{P}, \mathcal{X}$ fixed, not just X, P).

<u>Note</u>: If $\hat{V} \subseteq \hat{X}_P$ extends to a formal subscheme $\hat{\mathcal{V}}$ of $\hat{\mathcal{X}}_{\mathcal{P}}$, smooth along $\mathcal{P}$, then $R(\hat{V}) = 1$.

<u>Note</u>: Suppose $\hat{V}$ is given by the graph of a power series. The convergence radius of the series is only an upper bound for the size of $\hat{V}$.

<u>Note</u> Although $R(\hat{V})$ depends on $\mathcal{X}, \mathcal{P}$, the fact of $R(\hat{V}) > 0$ or $= 0$ only depends on $(X, \hat{V}, P)$.

<u>Theorem A</u> Let X/k be an alg. variety, $P \in X(k)$, and let $\hat{V}$ be a formal subscheme of $\hat{X}$ at P.
There exists an algebraic subvariety $Y \subseteq X$ such that $P \in Y(k)$ and $\hat{V}$ is a branch of Y at P if the following conditions hold.

$\boxed{A1}$ For <u>every</u> $\mathfrak{p} \in \operatorname{spec} \mathcal{O}_k$, the size $R_{\mathfrak{p}}(\hat{V})$ is >0 and the sum

$$\sum_{\underline{\text{almost all}}\ \mathfrak{p}} -\log(R_{\mathfrak{p}}(\hat{V})) \quad \text{is finite.}$$

$\boxed{A2}$ For <u>every</u> $k \overset{\sigma}{\hookrightarrow} \mathbb{C}$, $\hat{V}_\sigma \subseteq X_\sigma$ is the germ of an analytic submanif. of $X_\sigma(\mathbb{C})$ and for one $k \overset{\sigma_0}{\hookrightarrow} \mathbb{C}$, there exists a $\mathbb{C}$-manifold M having the Liouville property, $o \in M$ and a holomorphic map $u: M \to X_{\sigma_0}(\mathbb{C})$, $u(o) = P_{\sigma_0}$, sending a nbh. of o bihol. to the germ of leaf of $\hat{V}_{\sigma_0}$ through P_{σ_0}.

The conditions $\boxed{A1}$ and $\boxed{A2}$ can be summarised in a positivity statement ($\widehat{\deg}((\dots)) > 0$). The conditions "for every" in $\boxed{A1}$ and $\boxed{A2}$ can be reformulated as: $\hat{V}$ is analytic at every place. This is necessary for $\hat{V}$ to be algebraic.

Elements of the proof, under simplifying hypotheses.

* $\hat{V} \subseteq \hat{X}_{\mathcal{P}} \subseteq X$ is a smooth formal germ of <u>curve</u>. This is the most serious assumption – avoids technicalities of symmetric powers

* X is projective, and comes with an ample line bundle L, and $\hat{V}$ is dense in X. (replace X by the Zariski-closure of $\hat{V}$ in $\mathbb{P}^N$, check that Hyp. remain intact)

* $k = \mathbb{Q}$. Simplifies language. $\widehat{\deg} = \widehat{\deg}_{\text{normalized}}$

As previously in other contexts, set:

$$E_D := \Gamma(X, L^{\otimes D}) \qquad \text{– a f.d. } \mathbb{Q}\text{-vsp}$$

$$\eta_D := E_D \xrightarrow{\text{eval.}} \Gamma(V_i, L^{\otimes D})$$

$$E_D^i := \ker(\eta_D^{i+1})$$

We get a filtration $E_D^{i+1} \subseteq E_D^i \subseteq \cdots$ of E_D. The map η_D^i is injective for large D, i, so its an exhausting filtration.

$$\chi_D^i \quad \cancel{\eta_D^i} : E_D^i / \cancel{E_D^{i+1}} \longrightarrow \ker\left(\Gamma(V_i, L^{\otimes D}) \to \Gamma(V_{i-1}, L^{\otimes D})\right)$$

$$\cong T_{\hat{V}}^{\otimes -i} \otimes L_{\mathcal{P}}^{\otimes D}.$$

Lemma 1 If $\hat{V}$ is not algebraic, then

$$\lim_{D\to\infty} \sum_{i\geq 0} \left(\frac{i}{D}\right) \cdot \frac{\mathrm{rank}(E_D^i / E_D^{i+1})}{\mathrm{rank}(E_D)} \longrightarrow +\infty$$

Note: $\mathrm{rank}(E_D) = \mathrm{rank}\,\Gamma(X, L^{\otimes D}) \sim D^{\dim X}$ as $D \to \infty$. If we manage to show that the average value of i/D in the $\neq 0$ terms stays bounded, then $\hat{V}$ is algebraic.

This Lemma is "only" algebraic geometry. As usually, an inequality $\dim(\hat{V}^{\mathrm{Zar}}) \leq \dim(\hat{V})$ ($\rightsquigarrow$ an equality) follows from boundedness of the RHS in the statement

To link hypotheses of thm A and ranks, use Arakelov setting.
We choose :

* A model $\mathcal{X}, \mathcal{L}, \mathcal{P}$ of X, L, P over $\operatorname{spec} \mathbb{Z}$,
 a metric on $\mathcal{L}$, so $\overline{\mathcal{L}}$ = hermitian line bundle

* A positive Lebesgue-measure on $X(\mathbb{C})$, inv. under conjugation.

* A norm $\|\cdot\|_0$ on the tangent line $T_{\mathcal{P}}\hat{V}_{\mathbb{C}} \cong \mathbb{C}$. $\rightsquigarrow \overline{t} = (\mathbb{Z}, \|\cdot\|_0)$

Input from analysis: On $T_P \hat{V}_{\mathbb{C}}$ we may define the 'canonical seminorm'

$$\|\cdot\|_{can} = \exp\left(\limsup_{i/D \to \infty} \frac{1}{i} \log \|\gamma_D^i\| \right) \cdot \|\cdot\|_0$$

where $\|\gamma^i_{D,\mathbb{C}}\|$ is the operator norm of $\gamma^i_D : E^i_{D,\mathbb{C}} \to T_P \hat{V}_{\mathbb{C}}^{\otimes -i} \otimes L^{\otimes D}_{P,\mathbb{C}}$,

$\to$ on $E^i_{D,\mathbb{C}}$ take L^2-norm.

$\to$ on $T_P \hat{V}_{\mathbb{C}}^{\otimes -i}$ and $L^{\otimes D}_{P,\mathbb{C}}$ take what was given.

Hypothesis A2: limsup (↑.) is very negative. Precisely, there exist $\lambda > 0$ and $d > 0$ s.t. for any (D, i) with $i > \lambda D$

$$\widehat{\deg}\, \overline{t} + \sum_{p \text{ prime}} \log(R_p(\hat{V})) - \frac{1}{i} \log \|\gamma^i_{D,\mathbb{C}}\| \geq d.$$

Fix such d, λ.

<u>Aside</u>. The introduction $\|\cdot\|_{can}$ was uncalled for. However, we could stress now analogies with Andreotti – Hartshorne: Equipping $t = \mathcal{O}_{\mathbb{Z}}$ with metrics $\|\cdot\|_{can}$ and $R_p(V)^{-1}|\cdot|_p$ the hypothesis $\boxed{A1}$, $\boxed{A2}$ summarises to, essentially,

$$\widehat{\deg}\,(\underbrace{t,\ \|\cdot\|_{can},\ R_p^{-1}(V)|\cdot|_p}_{\text{this is not a Herm. v.b.}}) > 0.$$

I write down the slope estimation: $\bar{\mathcal{E}}_D = \mathcal{E}_D$ with L^2-norm.

$$-c.D \leq \hat{\mu}(\bar{\mathcal{E}}_D) \leq \frac{1}{\operatorname{rank}(E_D)} \sum_{i\geq 0} \operatorname{rank}\left(\frac{E_D^i}{E_D^{i+1}}\right) \cdot \left[\widehat{\deg}(\bar{t}^{\otimes -i} \otimes \bar{\mathcal{L}}_p^{\otimes D}) + \operatorname{height}(\gamma_D^i)\right]$$

$$= \frac{1}{\operatorname{rank}(E_D)} \sum_{i\geq 0} \operatorname{rank}\left(\frac{E_D^i}{E_D^{i+1}}\right) \left[-i\widehat{\deg}(t) + \underbrace{D\cdot\widehat{\deg}(\bar{\mathcal{L}}_p)}_{\text{can take this out}} + \operatorname{height}(\gamma_D^i)\right]$$

Where γ_D^i is the morphism of hermitian v.b. $\bar{\mathcal{E}}_D^i \to \bar{t}^{\otimes -i} \otimes \bar{\mathcal{L}}_p^{\otimes D}$ "giving i-th Taylor coefficients".

$$\operatorname{height}(\gamma_D^i) = \sum_p \log\big(\underbrace{\text{Operatornorm of } \gamma_D^i \otimes \mathbb{Q}_p}_{\leq R_p(\hat{V})^{-i}}\big) + \underbrace{\log \|\gamma_{D,\mathbb{C}}^i\|}_{\leq \alpha i + \beta D}.$$

hence

$$\frac{1}{\operatorname{rank}(E_D)} \sum_{0\leq i\leq \lambda D} \operatorname{rank}\left(\frac{E_D^i}{E_D^{i+1}}\right) \left[-i\widehat{\deg}(t) + \cancel{D\cdot\widehat{\deg}(\bar{\mathcal{L}}_p)} + \operatorname{height}(\gamma_D^i)\right]$$

$$\leq \frac{1}{\operatorname{rank}(E_D)} \sum_{0\leq i\leq \lambda D} \operatorname{rank}\left(\frac{E_D^i}{E_D^{i+1}}\right) \left[-i\widehat{\deg}(t) + \cancel{D\widehat{\deg}(\bar{\mathcal{L}}_p)} - i\left(\sum_p R_p(\hat{V})^{-1}\right) + \alpha i + \beta D\right] \leq C(\lambda) D$$

Take that part "$i \leq \lambda D$" away from the slope inequality:

$$-cD \leq D \cdot \widehat{\deg}(\bar{\mathcal{L}}_p) + c(\lambda)\cdot D + \frac{1}{\operatorname{rank}(E_D)} \sum_{i > \lambda D} \operatorname{rank}\left(\frac{E_D^i}{E_D^{i+1}}\right)$$

$$\cdot \left[-i \widehat{\deg}(\bar{t}) + \operatorname{height}(\gamma_D^i) \right]$$

Our choice of λ, and estimation of height by R_p

$$-i \widehat{\deg}(\bar{t}) + \operatorname{height}(\gamma_D^i) \leq -id$$

Hence

$$\frac{1}{\operatorname{rank}(E_D)} \sum_{i \geq \lambda D} \left(\frac{i}{D}\right) \cdot \operatorname{rank}\left(\frac{E_D^i}{E_D^{i+1}}\right) \leq \frac{c + c(\lambda) + \widehat{\deg}(\bar{\mathcal{L}}_p)}{d}$$

(≥ 0) indep. on D! $(+\lambda)$

So we are back to algebraic geometry.